EQUIVARIANT ETA FORMS AND EQUIVARIANT DIFFERENTIAL K-THEORY

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Abstract. In this paper, we prove some properties of the equivariant eta forms and use them to construct a geometric model of equivariant differential K-theory for finite group.

Contents

0. Introduction

By de Rham theory, the de Rham cohomology of a smooth manifold can be represented by differential forms, thus getting the global information by means of local data. In a similar way, a generalized differential cohomology theory gives a way to combine the cohomological information with differential geometric objects. An important case is the differential K-theory.

The differential K-theory is partly motivated by Type II superstring theory in theoretical physics, in which a Ramond-Ramond field carries the global information of a K-theory class together with the locality of a field [8]. Various definitions of differential K-theory have been proposed (see Bunke-Schick [3], Freed-Lott [10], Hopkins-Singer [11] and Simons-Sullivan [20]). Axioms for differntial extensions of generalized cohomology theories are given in [4].

For a finite group G, the equivariant differential K-theory was defined in Szabo-Valentino [21] and Ortiz [18]. In [5], Bunke and Schick constructed a cycle model for differential K-theory for orbifold. In fact, it gives a new model for the equivariant differential K-theory for finite group. Inspired by the model of Bunke and Schick, as a parallel version, in this paper, we construct a geometric model of equivariant differential K-theory for finite group. The construction relies on the properties of the equivariant eta forms. Comparing the models of Szabo-Valentino [21], Ortiz [18] and Bunke-Schick [5], the construction of our model of equivariant differential K-theory is purely geometrical.

This paper is organized as follows.

In Section 1, we study the properties of equivariant eta form of Melrose-Piazza. In Section 2, we construct a geometric model for equivariant differential K-theory for finite group and define the push-forward map.

To simplify the notations, we use the Einstein summation convention in this paper.

In the whole paper, we use the superconnection formalism of Quillen [19]. If A is a $\mathbb{Z}_2$-graded algebra, and if $a, b \in A$, then we will note $[a, b]$ as the supercommutator of a, b. If B is another $\mathbb{Z}_2$-graded algebra, we will note $A\widehat{\otimes}B$ as the $\mathbb{Z}_2$-graded tensor product. If A, B are not $\mathbb{Z}_2$-graded, sometimes, we also denote $A\widehat{\otimes}B$ by considering the whole algebra as the even part.

For a trace class operator P acting on a space E, if $E = E_+ \oplus E_-$ is a $\mathbb{Z}_2$-graded space, we denote by

$$\mathrm{Tr}_s[P] = \mathrm{Tr}|_{E_+}[P] - \mathrm{Tr}|_{E_-}[P]. \tag{0.1}$$

If $\mathrm{Tr}[P]$ takes value in differential forms, we denote by $\mathrm{Tr}^{\mathrm{odd/even}}[P]$ the part of $\mathrm{Tr}[P]$ which takes value in odd or even forms. We denote by

$$\widetilde{\mathrm{Tr}}[P] = \begin{cases} \mathrm{Tr}_s[P], & \text{if } E \text{ is } \mathbb{Z}_2\text{-graded;} \\ \mathrm{Tr}^{\mathrm{odd}}[P], & \text{if } E \text{ is not } \mathbb{Z}_2\text{-graded.} \end{cases} \tag{0.2}$$

For a vector bundle $\pi : W \to S$, we will often use the integration of the differential forms along the fiber Z in this paper. Since the fibers may be odd dimensional, we must make precise our sign conventions. If α is a differential form on W which in local coordinates is given by

$$\alpha = dy^{p_1} \wedge \cdots \wedge dy^{p_q} \wedge \beta(x) dx^1 \wedge \cdots \wedge dx^n, \tag{0.3}$$

we set

$$\int_Z \alpha = dy^{p_1} \wedge \cdots \wedge dy^{p_q} \int_Z \beta(x) dx^1 \wedge \cdots \wedge dx^n. \tag{0.4}$$

1. Equivariant eta form

1.1. **Equivariant geometric family.** Let $\pi : W \to B$ be a smooth submersion of smooth closed manifolds with closed fiber Z. Let $TZ = TW/B$ be the relative tangent bundle to the fibers Z. We assume that TZ is orientable with an orientation $o(TZ)$.

Let $T^H_\pi W$ be a horizontal subbundle of TW such that

$$TW = T^H_\pi W \oplus TZ. \tag{1.1}$$

The splitting (1.1) gives an identification

$$T_\pi^H W \cong \pi^* TB. \tag{1.2}$$

Let P^{TZ} be the projection

$$P^{TZ} : TW = T_\pi^H W \oplus TZ \to TZ. \tag{1.3}$$

Let g^{TZ}, g^{TB} be Riemannian metrics on TZ, TB. We equip $TW = T_\pi^H W \oplus TZ$ with the Riemannian metric

$$g^{TW} = \pi^* g^{TB} \oplus g^{TZ}. \tag{1.4}$$

Let ∇^{TW}, ∇^{TB} be the Levi-Civita connections on (W, g^{TW}), (B, g^{TB}). Set

$$\nabla^{TZ} = P^{TZ} \nabla^{TW} P^{TZ}. \tag{1.5}$$

Then ∇^{TZ} is a Euclidean connection on TZ. By [2, Theorem 1.9], we know that ∇^{TZ} only depends on $(T_\pi^H W, g^{TZ})$. Let ${}^0\nabla^{TW}$ be the connection on $TW = T_\pi^H W \oplus TZ$ defined by

$${}^0\nabla^{TW} = \pi^* \nabla^{TB} \oplus \nabla^{TZ}. \tag{1.6}$$

Then ${}^0\nabla^{TW}$ preserves the metric g^{TW} in (1.4). Set

$$S = \nabla^{TW} - {}^0\nabla^{TW}. \tag{1.7}$$

Then S is a 1-form on W with values in antisymmetric elements of $\mathrm{End}(TW)$.

Let $C(TZ)$ be the Clifford algebra bundle of (TZ, g^{TZ}), whose fiber at $x \in W$ is the Clifford algebra $C(T_xZ)$ of the Euclidean space (T_xZ, g^{T_xZ}). Recall that by a Clifford module of $C(TZ)$, we shall mean a vector bundle over W with a smooth action of $C(TZ)$ on it and in case TZ is even-dimensional, we demand that the module is $\mathbb{Z}_2$-graded and the action of TZ exchanges the $\mathbb{Z}_2$-grading. If the Clifford module is equipped with a Hermitian metric, we say that the Clifford module is self-adjoint if for any $V \in TZ$, the action $c(V)$ is skew-adjoint.

Let $\mathcal{E}$ be a self-adjoint Clifford module of $C(TZ)$ with Hermitian metric $h^{\mathcal{E}}$. Let $\nabla^{\mathcal{E}}$ be a Hermitian Clifford connection on $(\mathcal{E}, h^{\mathcal{E}})$ associated to ∇^{TZ}, that is, $\nabla^{\mathcal{E}}$ preserves $h^{\mathcal{E}}$ and for any $U \in TW$, $V \in \mathscr{C}^\infty(W, TZ)$,

$$\left[\nabla_U^{\mathcal{E}}, c(V)\right] = c\left(\nabla_U^{TZ} V\right). \tag{1.8}$$

Let G be a compact Lie group (maybe non-connected). We assume that $\pi : W \to B$ is a G-equivariant fiber bundle and the action of G preserves the orientation $o(TZ)$ and the metric g^{TZ} and $T^H W$. Then the connection ∇^{TZ} is G-invariant and there is a naturally induced G-action on $C(TZ)$, which preserves the product. We assume that $(\mathcal{E}, h^{\mathcal{E}})$ is a G-equivariant self-adjoint Clifford module of $C(TZ)$, that is, $\mathcal{E}$ is a G-equivariant vector bundle over W, $h^{\mathcal{E}}$ is G-invariant and the Clifford module structure preserved by the group action. We assume that $\nabla^{\mathcal{E}}$ is G-invariant and if TZ is even-dimensional, the G-action preserves the $\mathbb{Z}_2$-grading of $\mathcal{E}$.

Definition 1.1. An equivariant geometric family $\mathcal{F}$ over B is a family of G-equivariant geometric data

$$\mathcal{F} = (W, \mathcal{E}, o(TZ), T^H_\pi W, g^{TZ}, h^{\mathcal{E}}, \nabla^{\mathcal{E}}) \tag{1.9}$$

described as above.

Let $\{e_i\}$ be a local orthonormal frame of TZ. Let $D(\mathcal{F})$ be the fiberwise Dirac operator

$$D(\mathcal{F}) = c(e_i)\nabla^{\mathcal{E}}_{e_i} \tag{1.10}$$

associated to the equivariant geometric family $\mathcal{F}$. Then the G-action commutes with $D(\mathcal{F})$. Thus the classical construction of Atiyah-Singer assigns to this family its equivariant (analytic) index $\mathrm{ind}(D(\mathcal{F})) \in K^*_G(B)$, where $*$ is equal to the parity of the dimension of TZ.

For $b \in B$, let $\mathscr{E}_b$ be the set of smooth sections over Z_b of $\mathcal{E}_b$. As in [2], we will regard $\mathscr{E}$ as an infinite dimensional fiber bundle over B.

If $V \in TB$, let $V^H \in T^H_\pi W$ be its horizontal lift in $T^H_\pi W$ so that $\pi_* V^H = V$. For any $V \in TB$, $s \in \mathscr{C}^\infty(B, \mathscr{E}) = \mathscr{C}^\infty(W, \mathcal{E})$, set

$$\nabla^{\mathscr{E}, \mathcal{U}}_V s := \nabla^{\mathcal{E}}_{V^H} s - \frac{1}{2}\langle S(e_i)e_i, V^H\rangle\, s. \tag{1.11}$$

Let $\{f_p\}$ be a local orthonormal frame of TB and $\{f^p\}$ be its dual. We denote by $\nabla^{\mathscr{E},\mathcal{U}} = f^p \wedge \nabla^{\mathscr{E},\mathcal{U}}_{f_p}$. Let T be the torsion of ${}^0\nabla^{TW}$. Then $T(f^H_p, f^H_q) \in TZ$. We denote by

$$c(T) = \frac{1}{2}\, c\left(T(f^H_p, f^H_q)\right) f^p \wedge f^q \wedge . \tag{1.12}$$

By [2, (3.18)], the (rescaled) Bismut superconnection

$$\mathbb{B}_u : \mathscr{C}^\infty(B, \Lambda(T^*B)\widehat{\otimes}\mathscr{E}) \to \mathscr{C}^\infty(B, \Lambda(T^*B)\widehat{\otimes}\mathscr{E}) \tag{1.13}$$

is defined by

$$\mathbb{B}_u = \sqrt{u}D(\mathcal{F}) + \nabla^{\mathscr{E},\mathcal{U}} - \frac{1}{4\sqrt{u}}c(T). \tag{1.14}$$

Obviously, the Bismut superconnection $\mathbb{B}_u$ commutes with the G-action. Moreover, $\mathbb{B}_u^2$ is a 2-order elliptic differential operator along the fiber Z. Let $\exp(-\mathbb{B}_u^2)$ be the family of heat operators associated to the fiberwise elliptic operator $\mathbb{B}_u^2$. From [1, Theorem 9.50], we know that $\exp(-\mathbb{B}_u^2)$ is a smooth family of smoothing operators.

For $\alpha \in \Omega^i(B)$, set

$$\psi_B(\alpha) = \begin{cases} \left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{i}{2}} \cdot \alpha, & \text{if } i \text{ is even;} \\ \frac{1}{\sqrt{\pi}}\left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{i-1}{2}} \cdot \alpha, & \text{if } i \text{ is odd.} \end{cases} \tag{1.15}$$

Let B^g be the fixed point set of g on B. The B^g is a submanifold of B. For any $\mathscr{K} \in K^*_G(B)$ and $g \in G$, we define the equivariant Chern character map

$$\mathrm{ch}_g : K^*_G(B) \longrightarrow \Omega^*_{d=0}(B^g, \mathbb{C})/\mathrm{Im}\, d \tag{1.16}$$

by

$$\mathrm{ch}_g(\mathscr{K}) := \mathrm{ch}_g(\mathscr{K}|_{B^g}). \tag{1.17}$$

Here $\Omega^*_{d=0}(B^g, \mathbb{C})$ denotes the set of closed forms.

Let $\mathbb{B}|_{B^g}$ be the Bismut superconnection of $\pi : \pi^{-1}(B^g) \to B^g$, where $\mathcal{F}_{B^g}$ is the restriction of the equivariant geometric family $\mathcal{F}$ to B^g. We state the equivariant family local index theorem as follows (see [13, Theorem 1.2]).

Theorem 1.2. *For any $u > 0$ and $g \in G$, restricted to the fixed point set B^g, the differential form $\psi_{B^g}\widetilde{\mathrm{Tr}}|_{B^g}[g\exp(-(\mathbb{B}_u|_{B^g})^2)] \in \Omega^*(B^g, \mathbb{C})$ is closed and its cohomology class represents $[\mathrm{ch}_g(\mathrm{ind}(D(\mathcal{F})))] \in H^*(B^g, \mathbb{C})$. As $u \to 0$, the limit of $\psi_{B^g}\widetilde{\mathrm{Tr}}|_{B^g}[g\exp(-(\mathbb{B}_u|_{B^g})^2)]$ exists.*

Set

$$\Omega_g(\mathcal{F}) = \lim_{u\to 0} \psi_{B^g}\widetilde{\mathrm{Tr}}|_{B^g}[g\exp(-(\mathbb{B}_u|_{B^g})^2)] \in \Omega^*(B^g, \mathbb{C}). \tag{1.18}$$

Then modulo exact forms, we have

$$\mathrm{ch}_g(\mathrm{ind}(D(\mathcal{F}))) = \Omega_g(\mathcal{F}). \tag{1.19}$$

Let W^g be the fixed point set of g on W. Then $\pi : W^g \to B^g$ is a fibration with fiber Z^g. Note that if Z^g is oriented, we can write $\Omega_g(\mathcal{F})$ as an integral of a characteristic form on W^g along the fiber Z^g (see [13, (1.57)]).

From the equivariant family local index theorem, $\Omega_g(\mathcal{F})$ only depends on $\nabla^{\mathcal{E}}$. We will also denote it by $\Omega_g(\nabla^{\mathcal{E}})$. In fact, if ∇^{TZ} is only a G-invariant connection on TZ and $\nabla^{\mathcal{E}}$ is a G-invariant Clifford connection associated to ∇^{TZ}, the local index form $\Omega_g(\nabla^{\mathcal{E}}) \in \Omega^*(B^g, \mathbb{C})$ is still well-defined. Let ∇^{TZ}, ∇'^{TZ} be two G-invariant connections on TZ and $\nabla^{\mathcal{E}}$, $\nabla'^{\mathcal{E}}$ be G-invariant Clifford connections associated to ∇^{TZ}, ∇'^{TZ}. From [15, Theorem B.5.4], we can construct the Chern-Simons form $\widetilde{\Omega}_g(\nabla^{\mathcal{E}}, \nabla'^{\mathcal{E}}) \in \Omega^*(B^g, \mathbb{C})$ such that

$$d\widetilde{\Omega}_g(\nabla^{\mathcal{E}}, \nabla'^{\mathcal{E}}) = \Omega_g(\nabla'^{\mathcal{E}}) - \Omega_g(\nabla^{\mathcal{E}}). \tag{1.20}$$

1.2. **Equivariant spectral section.** In [16, 17], Melrose and Piazza defined a spectral section for a geometric family when the index vanishes. In this section, we explain that the spectral section can be naturally extended to G-equivariant case.

Definition 1.3. (compare with [16, Definition 1] and [17, Definition 1]) For an equivariant geometric family $\mathcal{F}$, an equivariant spectral section is a family of self-adjoint pseudodifferential projections P, which commutes with the G-action, such that for some smooth function $R : B \to \mathbb{R}$ (depending on P) and every $b \in B$,

$$D_b(\mathcal{F})u = \lambda u \Longrightarrow \begin{cases} P_b u = u, & \text{if } \lambda > R(b); \\ P_b u = 0, & \text{if } \lambda < -R(b), \end{cases} \tag{1.21}$$

and if $\dim TZ$ is even, we have

$$\sigma P + P\sigma = \sigma, \tag{1.22}$$

where σ is the chirality operator on $\mathcal{E}$, i.e., $\sigma = \pm 1$ on $\mathcal{E}_\pm$.

As in [16, 17], we show that an equivariant geometric family which has an equivariant spectral section has a finite rank equivariant homotopy to an invertible family.

Proposition 1.4. *(compare with* [16, Lemma 8] *and* [17, Lemma 1]*) If the equivariant geometric family $\mathcal{F}$ has an equivariant spectral section P, then there is a family of self-adjoint equivariant smoothing operators A_P with range in a finite sum of eigenspaces of $D(\mathcal{F})$ such that*

$$D(\mathcal{F}, P) := D(\mathcal{F}) + A_P \tag{1.23}$$

is invertible and P is the Atiyah-Patodi-Singer projection onto the positive part of the spectrum of $D(\mathcal{F}, P)$. If $\dim TZ$ is even, A_P is $\mathbb{Z}_2$-graded.

Proof. Observe that all the operators in the proofs of [16, Lemma 8] and [17, Lemma 1] commute with the G-action. □

We list in the following the basic properties of equivariant spectral sections, which are the natural extension of the results in [16, 17].

Proposition 1.5. *Let $\mathcal{F}$ be an equivariant geometric family.*

(A) (compare with [16, Proposition 1] *and* [17, Proposition 2]*) There exists an equivariant spectral section for $\mathcal{F}$ if and only if* $\operatorname{ind} D(\mathcal{F}) = 0 \in K_G^*(B)$.

(B) (compare with [16, Proposition 2]*) Given equivariant spectral sections P, Q, there exists an equivariant spectral section R such that $PR = R$ and $QR = R$. Such an equivariant spectral section will be called a majorizing equivariant spectral section.*

(C) (compare with [16, Lemma 7] *and* [17, Proposition 4]*) If* $\dim TZ$ *is odd and R majorizes P: $PR = R$, then* $\ker\{P_bR_b : \mathrm{Im}(R_b) \to \mathrm{Im}(P_b)\}_{b\in B}$ *forms an equivariant vector bundle over B, denoted by $[R-P]$. Hence for any two equivariant spectral sections P, Q, the difference element $[P-Q]$ can be defined as an element in $K_G^0(B)$, as follows:*

$$[P - Q] := [R - Q] - [R - P] \in K_G^0(B), \tag{1.24}$$

for any majorizing equivariant spectral section R and the class in (1.24) is independent of the choice of the majorizing equivariant spectral section.

If $\dim TZ$ *is even, after a suspension argument, we can define the difference element*

$$[P - Q] \in K_G^1(B) \tag{1.25}$$

using the result of odd case.

(D) If P_1 , P_2 , P_3 are equivariant spectral sections for $\mathcal{F}$, then

$$[P_3 - P_1] = [P_3 - P_2] + [P_2 - P_1]. \tag{1.26}$$

(E) (compare with [17, Proposition 12]*) The equivariant K-group $K_G^*(B)$ is generated by all these difference elements.*

1.3. **Equivariant eta form.** In this subsection, we assume that $\operatorname{ind}(D(\mathcal{F}))$ vanishes. From Proposition 1.5 (A), there exists an equivariant spectral section P for $\mathcal{F}$.

Let $\chi \in \mathscr{C}^\infty(\mathbb{R})$ be a cutoff function such that

$$\chi(u) = \begin{cases} 0, & \text{if } u < 1; \\ 1, & \text{if } u > 2. \end{cases} \tag{1.27}$$

From Proposition 1.4, there exists a family of self-adjoint equivariant smoothing operators A_P, such that $D(\mathcal{F}) + A_P$ is invertible. For any $g \in G$, set

$$\mathbb{B}'_u|_{B^g} = \mathbb{B}_u|_{B^g} + \sqrt{u}\chi(\sqrt{u})A_P|_{B^g}. \tag{1.28}$$

Since $\chi(u) = 0$ when $u \to 0$, by (1.18),

$$\lim_{u\to 0} \psi_{B^g}\widetilde{\mathrm{Tr}}|_{B^g}[g\exp(-(\mathbb{B}'_u|_{B^g})^2)] \in \Omega^*(B^g, \mathbb{R}) = \Omega_g(\mathcal{F}). \tag{1.29}$$

Since $\chi(u) = 1$ when $u \to +\infty$, from [1, Theorem 9.19], we have

$$\lim_{u\to +\infty} \psi_{B^g}\widetilde{\mathrm{Tr}}|_{B^g}\left[g\exp\left(-(\mathbb{B}'_u|_{B^g})^2\right)\right] = 0. \tag{1.30}$$

Definition 1.6. Assume that $\mathrm{ind}(D(\mathcal{F})) = 0 \in K^*_G(B)$. For any $g \in G$, the equivariant eta form with spectral section of Melrose-Piazza is defined by

(1.31)
$$\tilde{\eta}_g(\mathcal{F}, P) := \begin{cases} \displaystyle\int_0^\infty \frac{1}{\sqrt{\pi}}\psi_{B^g}\,\mathrm{Tr}_s\Big|_{B^g}\left[g\,\frac{\partial \mathbb{B}'_u}{\partial u}\Big|_{B^g}\exp(-(\mathbb{B}'_u|_{B^g})^2)\right]du \in \Omega^*(B^g, \mathbb{C}), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \dim Z \text{ is even;} \\ \displaystyle\int_0^\infty \frac{1}{2\sqrt{-1}\sqrt{\pi}}\psi_{B^g}\,\mathrm{Tr}^{\mathrm{even}}\Big|_{B^g}\left[g\,\frac{\partial \mathbb{B}'_u}{\partial u}\Big|_{B^g}\exp(-(\mathbb{B}'_u|_{B^g})^2)\right]du \in \Omega^*(B^g, \mathbb{C}), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \dim Z \text{ is odd.} \end{cases}$$

The regularities of the integral in the right hand side of (1.35) are proved in [13, Section 1.4]. As in [13, (1.81)], we have

$$d\tilde{\eta}_g(\mathcal{F}, P) = \Omega_g(\mathcal{F}). \tag{1.32}$$

If $\alpha \in \Lambda(T^*(\mathbb{R}_+ \times B))$, we can expand α in the form

$$\alpha = du \wedge \alpha_0 + \alpha_1, \quad \alpha_0, \alpha_1 \in \Lambda(T^*B). \tag{1.33}$$

Set

$$[\alpha]^{du} = \alpha_0. \tag{1.34}$$

Then by [13, Remark 1.4], we have

$$\tilde{\eta}_g(\mathcal{F}, P) = -\int_0^\infty \left\{\psi_{B^g}\,\widetilde{\mathrm{Tr}}\Big|_{B^g}\left[g\exp\left(-\left(\mathbb{B}'_{u^2}|_{B^g} + du \wedge \frac{\partial}{\partial u}\right)^2\right)\right]\right\}^{du} du. \tag{1.35}$$

1.4. **Anomaly formula.** Let P, Q be two equivariant spectral sections for $\mathcal{F}$. As a natural extension of [16, Proposition 17], we can get

$$\tilde{\eta}_g(\mathcal{F}, P) - \tilde{\eta}_g(\mathcal{F}, Q) = 2\,\mathrm{ch}_g([P - Q]) \in \Omega^*_{d=0}(B^g, \mathbb{C})/\mathrm{Im}\, d. \tag{1.36}$$

Let $\mathcal{F}$ and $\mathcal{F}'$ be two equivariant geometric families over B. An isomorphism $\mathcal{F} \xrightarrow{\sim} \mathcal{F}'$ consists of the following data:

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "F"] \arrow[d] & & \mathcal{E}' \arrow[d] \\
W \arrow[rr, "f"] \arrow[dr, "\pi"'] & & W' \arrow[dl, "\pi'"] \\
& B &
\end{tikzcd}$$

where

1. f is a diffeomorphism over B commuting with the G-action,
2. F is an equvariant bundle isomorphism over f,
3. f preserves the G-invariant orientation of the relative tangent bundle,
4. f preserves the horizontal subbundle and the vertical metric,
5. F preserves the metric, connection, Clifford multiplication and the grading of the Clifford module.

If only the first three conditions hold, we say that $\mathcal{F}$ and $\mathcal{F}'$ have the same topological structure.

Now we study the anomaly formula for two equivariant geometric families $\mathcal{F}$ and $\mathcal{F}'$ with the same topological structure. First, a horizontal subbundle on W is simply a splitting of the exact sequence

$$0 \to TZ \to TW \to \pi^*TB \to 0. \tag{1.37}$$

As the space of the splitting map is affine, it follows that any pair of horizontal subbundles can be connected by a smooth path of horizontal distributions. Let $s \in [0,1]$ parametrize a smooth path $\{T^H_{\pi,s}W\}_{s\in[0,1]}$ such that $T^H_{\pi,0}W = T^H_\pi W$ and $T^H_{\pi,1}W = T'^H_{\pi,s}W$. Similarly, let g^{TZ}_s and $h^{\mathcal{E}}_s$ be the G-invariant metrics on TZ and $\mathcal{E}$, depending smoothly on $s \in [0,1]$, which coincide with g^{TZ} and $h^{\mathcal{E}}$ at $s = 0$ and with g'^{TZ} and $h'^{\mathcal{E}}$ at $s = 1$. By the same reason, we can choose G-invariant Clifford connection $\nabla^{\mathcal{E}}_s$ on $\mathcal{E}$ preserving $h^{\mathcal{E}}_s$. such that $\nabla^{\mathcal{E}}_0 = \nabla^{\mathcal{E}}$, $\nabla^{\mathcal{E}}_1 = \nabla'^{\mathcal{E}}$.

Let $\widetilde{B} = [0,1] \times B$ and $\mathrm{pr} : \widetilde{B} \to B$ be the projection. We consider the bundle $\widetilde{\pi} : \widetilde{W} := [0,1] \times W \to \widetilde{B}$ together with the canonical projection $\mathrm{Pr} : \widetilde{W} \to W$. Then $T^H_{\widetilde{\pi}}\widetilde{W}_{(s,\cdot)} = \mathbb{R} \times T^H_{\pi,s}W$ defines a horizontal subbundle of $T\widetilde{W}$, and $T\widetilde{Z} := \mathrm{Pr}^*TZ$, $\widetilde{\mathcal{E}} := \mathrm{Pr}^*\mathcal{E}$ are naturally equipped with metrics $g^{T\widetilde{Z}}$, $h^{\widetilde{\mathcal{E}}}$ and connection $\nabla^{\widetilde{\mathcal{E}}}$. Then the fiberwise G-action can be naturally extended to $\widetilde{\pi} : \widetilde{W} \to \widetilde{B}$ such that G acts as identity on $[0,1]$ and $g^{T\widetilde{Z}}$, $h^{\widetilde{\mathcal{E}}}$, $\nabla^{\widetilde{\mathcal{E}}}$ are G-invariant. Thus, we get equivariant geometric families $\mathcal{F}_s = (W, \mathcal{E}, o(TZ), T^H_{\pi,s}W, g^{TZ}_s, h^{\mathcal{E}}_s, \nabla^{\mathcal{E}}_s)$ and $\widetilde{\mathcal{F}} = (\widetilde{W}, \widetilde{\mathcal{E}}, o(TZ), T^H_{\widetilde{\pi}}\widetilde{W}, g^{T\widetilde{Z}}, h^{\widetilde{\mathcal{E}}}, \nabla^{\widetilde{\mathcal{E}}})$.

Since the equivariant index of $\mathcal{F}$ vanishes, the homotopy invariance of the equivariant index bundle implies that the equivariant indices of each $\mathcal{F}_s$ and $\widetilde{\mathcal{F}}$ vanish. Let P , P' be equivariant spectral sections of $\mathcal{F}$, $\mathcal{F}'$ respectively. If we consider the total family $\widetilde{\mathcal{F}}$, then there exists a total spectral section $\widetilde{P}$. Let P_s be the restriction of $\widetilde{P}$ over $\{s\} \times B$.

As in [6, Definition 1.5], we denote by

$$\mathrm{sf}\{(\mathcal{F}, P), (\mathcal{F}', P')\} := [P - P_0] - [P' - P_1] \in K^*_G(B). \tag{1.38}$$

From (1.26), we know that $\mathrm{sf}\{(\mathcal{F}, P), (\mathcal{F}', P')\}$ does not depend on the choice of $(\widetilde{\mathcal{F}}, \widetilde{P})$.

Proposition 1.7. *Let $\mathcal{F}$, $\mathcal{F}'$ be two equivariant geometric families with vanishing equivariant indices which have the same topological structure. Let P, P' be two equivariant spectral sections for $\mathcal{F}$, $\mathcal{F}'$ respectively. For any $g \in G$, modulo exact forms, we have*

(1.39)
$$\tilde{\eta}_g(\mathcal{F}', P') - \tilde{\eta}_g(\mathcal{F}, P) = \widetilde{\Omega}_g(\mathcal{F}, \mathcal{F}') + 2\ \mathrm{ch}_g\left(\mathrm{sf}\{(\mathcal{F}, P), (\mathcal{F}', P')\}\right) \in \Omega^*(B^g, \mathbb{R})/\mathrm{Im}d.$$

Proof. Following the proof of [13, Theorem 1.7], we can get

$$\tilde{\eta}_g(\mathcal{F}', P_1) - \tilde{\eta}_g(\mathcal{F}, P_0) = \widetilde{\Omega}_g(\mathcal{F}, \mathcal{F}'). \tag{1.40}$$

Then Proposition 1.7 follows from (1.36), (1.38) and (1.40). □

1.5. **Functoriality of equivariant eta forms.** Let W, V, B be smooth closed manifolds. Let $\pi_1 : W \to V$, $\pi_2 : V \to B$ be smooth fibrations with closed oriented fibers X, Y. Then $\pi_3 = \pi_2 \circ \pi_1 : W \to S$ is a smooth fibration with closed oriented fiber Z. Then we have the diagram of smooth fibrations:

$$\begin{tikzcd}
X \arrow[r] & Z \arrow[r] \arrow[d] & W \arrow[d, "\pi_1"'] \arrow[dr, "\pi_3"] & \\
 & Y \arrow[r] & V \arrow[r, "\pi_2"] & B.
\end{tikzcd}$$

Let TX, TY, TZ be the relative tangent bundles. Then we can choose the geometric data $(T^H_{\pi_1}W, g^{TX})$, $(T^H_{\pi_2}V, g^{TY})$, $(T^H_{\pi_3}W, g^{TZ})$ associated with π_1, π_2, π_3 respectively.

We make the assumption that TY has a Spin^c structure. Then there exists a complex line bundle L_Y over V such that $\omega_2(TY) = c_1(L_Y) \mod (2)$. Let $\mathcal{S}(TY, L_Y)$ be the fundamental complex spinor bundle for (TY, L_Y), which has a smooth action of $C(TY)$ (cf. [12, Appendix D.9]). Let h^{L_Y} be the Hermitian metric on L_Y and ∇^{L_Y} be the Hermitian connection on (L_Y, h^{L_Y}). Let $h^{\mathcal{S}_Y}$ be the Hermitian metric on $\mathcal{S}(TY, L_Y)$ induced by g^{TY} and h^{L_Y} and $\nabla^{\mathcal{S}_Y}$ be the connection on $\mathcal{S}(TY, L_Y)$ induced by ∇^{TY} and ∇^{L_Y}. Then $\nabla^{\mathcal{S}_Y}$ is a Hermitian Clifford connection on $(\mathcal{S}(TY, L), h^{\mathcal{S}_Y})$.

Let G be a compact Lie group which acts on W such that the fibrations π_1, π_2, π_3 are all G-equivariant. We assume that the action of G preserves the Spin^c structure TY and g^{TX}, g^{TY}, g^{TZ}, h^{L_Y}, ∇^{L_Y} are G-invariant.

Let $\mathcal{E}_X$ be a G-equivariant Clifford module of $C(TX)$ over W with G-equivariant Hermitian metric $h^{\mathcal{E}_X}$ and G-equivariant Hermitian Clifford connection $\nabla^{\mathcal{E}_X}$. Then

$$\mathcal{F}_X := (W, \mathcal{E}_X, o(TX), T^H_{\pi_1}W, g^{TX}, h^{\mathcal{E}_X}, \nabla^{\mathcal{E}_X}) \tag{1.41}$$

is an equivariant geometric family over V. Let $o(TY)$ be the orientation of TY induced by the Spin^c structure of TY. Set $T^H_{\pi_1}Z := T^H_{\pi_1}W \cap TZ$. Then we have the splitting of smooth vector bundles over W,

$$TZ = T^H_{\pi_1}Z \oplus TX, \tag{1.42}$$

and

$$T^H_{\pi_1}Z \cong \pi_1^* TY. \tag{1.43}$$

Then we can get a G-invariant orientation $o(TZ)$ of TZ induced by $o(TX)$ and $o(TY)$. Since $\mathcal{S}(TY,L)$ is an equivariant Clifford module of $C(TY)$, $\pi_1^*\mathcal{S}(TY,L)\widehat{\otimes}\mathcal{E}_X$ is an equivariant Clifford module of $C(TZ)$. Let ∇^{TZ} be the Euclidean connection defined in (1.5). Let $\nabla^{\mathcal{S}_Y\widehat{\otimes}\mathcal{E}_X}$ be an equivariant Hermitian Clifford connection on $(\pi_1^*\mathcal{S}(TY,L)\widehat{\otimes}\mathcal{E}_X, h^{\mathcal{S}_Y}\otimes h^{\mathcal{E}_X})$ associated to ∇^{TZ}. Then

$$\mathcal{F}_Z := (W, \pi_1^*\mathcal{S}(TY,L)\widehat{\otimes}\mathcal{E}_X, o(TZ), T^H_{\pi_3}W, g^{TZ}, h^{\mathcal{S}_Y}\otimes h^{\mathcal{E}_X}, \nabla^{\mathcal{S}_Y\widehat{\otimes}\mathcal{E}_X}) \tag{1.44}$$

is an equivariant geometric family over B.

Let $\nabla^{TY,TX}$ be the connection on $TZ = T^HZ\oplus TX$ defined by

$$\nabla^{TY,TX} = \pi^*\nabla^{TY}\oplus\nabla^{TX} \tag{1.45}$$

as in (1.6). Let

$$\nabla^{\mathcal{S}_Y,\mathcal{E}_X} := \pi_1^*\nabla^{\mathcal{S}_Y}\otimes 1 + 1\otimes\nabla^{\mathcal{E}_X}. \tag{1.46}$$

Then $\nabla^{\mathcal{S}_Y,\mathcal{E}_X}$ is an equivariant Hermitian Clifford connection on $(\pi_1^*\mathcal{S}(TY,L)\widehat{\otimes}\mathcal{E}_X, h^{\mathcal{S}_Y}\otimes h^{\mathcal{E}_X})$ associated to $\nabla^{TY,TX}$.

Lemma 1.8. *If* $\mathrm{ind}(D(\mathcal{F}_X)) = 0\in K_G^*(V)$, *then* $\mathrm{ind}(D(\mathcal{F}_Z)) = 0\in K_G^*(B)$.

The proof of Lemma 1.8 is left to the next subsection.

Let R^L be the curvature of ∇^L. For any $g\in G$, we write

$$\mathrm{Td}_g(TY,\nabla^{TY,L_Y}) := \widehat{\mathrm{A}}_g(TY,\nabla^{TY})\wedge\left[g\cdot\exp\left(\frac{\sqrt{-1}}{4\pi}R^L|_{W^g}\right)\right]\in\Omega^*(W^g). \tag{1.47}$$

For the definition of $\widehat{\mathrm{A}}_g(TY,\nabla^{TY})$, see [14].

From [13, Proposition 1.1], since the G-action preserves the Spin^c structure of TY, Y^g is naturally oriented.

Theorem 1.9. *Assume that* $\mathrm{ind}(D(\mathcal{F}_X)) = 0\in K_G^*(V)$. *Let P_Z and P_X be two equivariant spectral sections for $\mathcal{F}_Z$ and $\mathcal{F}_X$. Then there exists $\mathscr{K}(P_Z,P_X)\in K_G^*(B)$ depending on P_Z, P_X, such that modulo exact forms,*

$$\begin{aligned}\widetilde{\eta}_g(\mathcal{F}_Z,P_Z) = \int_{Y^g}\mathrm{Td}_g(TY,\nabla^{TY,L_Y})\wedge\widetilde{\eta}_g(\mathcal{F}_X,P_X) + \widetilde{\Omega}_g\left(\nabla^{\mathcal{S}_Y\widehat{\otimes}\mathcal{E}_X},\nabla^{\mathcal{S}_Y,\mathcal{E}_X}\right)\\ + \mathrm{ch}_g(\mathscr{K}(P_Z,P_X))\in\Omega^*(B^g,\mathbb{C})/\mathrm{Im}d.\end{aligned} \tag{1.48}$$

1.6. **Proof of Theorem 1.9.** From Proposition 1.7, We can simply assume that

$$T^H_{\pi_3}W\subset T^H_{\pi_1}W,\quad g^{TZ} = \pi_1^*g^{TY}\oplus g^{TX}. \tag{1.49}$$

Let

$$g^{TZ}_T = \pi_1^*g^{TY}\oplus\frac{1}{T^2}g^{TX}. \tag{1.50}$$

We denote the Clifford algebra bundle of TZ with respect to g^{TZ}_T by $C_T(TZ)$. If $U\in TV$, let $U_1^H\in T^H_{\pi_1}W$ be the horizontal lift of U, so that $\pi_{1,*}(U_1^H) = U$. Let $\{e_i\}$, $\{f_p\}$ be local orthonormal frames of (TX,g^{TX}), (TY,g^{TY}). Then $\{Te_i\}\cup\{f^H_{p,1}\}$ is a local orthonormal frame of (TZ,g^{TZ}_T). We define a Clifford algebra isomorphism

$$\mathcal{G}_T : C_T(TZ)\to C(TZ) \tag{1.51}$$

by

$$\mathcal{G}_T(c(f_{p,1}^H)) = c(f_{p,1}^H), \quad \mathcal{G}_T(c_T(Te_i)) = c(e_i). \tag{1.52}$$

Under this isomorphism, we can consider $(\pi_1^*\mathcal{S}(TY, L_Y)\widehat{\otimes}\mathcal{E}_X, \pi_1^* h^{\mathcal{S}_Y} \otimes h^{\mathcal{E}_X})$ as a self-adjoint Hermitian equivariant Clifford module of $C_T(TZ)$. Let ∇_T^{TZ} be the connection associated to $(T_{\pi_3}^H W, g_T^{TZ})$ as in (1.5). Let $\nabla_T^{\mathcal{S}_Y\widehat{\otimes}\mathcal{E}_X}$ be a Hermitian Clifford connection on $(\pi_1^*\mathcal{S}(TY, L_Y)\widehat{\otimes}\mathcal{E}_X, \pi_1^* h^{\mathcal{S}_Y} \otimes h^{\mathcal{E}_X})$ associated to ∇_T^{TZ}.

Then

$$\mathcal{F}_{Z,T} = (W, \pi_1^*\mathcal{S}(TY, L_Y)\widehat{\otimes}\mathcal{E}_X, o(TZ), T_{\pi_3}^H W, g_T^{TZ}, \pi_1^* h^{\mathcal{S}_Y} \otimes h^{\mathcal{E}_X}, \nabla_T^{\mathcal{S}_Y\widehat{\otimes}\mathcal{E}_X}) \tag{1.53}$$

is an equivariant geometric family over B and $\mathcal{F}_Z = \mathcal{F}_{Z,1}$.

If $\mathrm{ind}(D(\mathcal{F}_X)) = 0 \in K_G^*(V)$, then from Proposition 1.4, there exists an equivariant spectral section P_X and a family of self-adjoint equivariant smoothing operators A_P such that $\ker(D(\mathcal{F}_X) + A_P) = 0$. As in [13, Lemma 2.2], when T large, we can get $\ker(D(\mathcal{F}_{Z,T}) + TA_P) = 0$. So by the homotopy invariance of the equivariant index, for any $T \geq 1$, we have $\mathrm{ind}(D(\mathcal{F}_{Z,T})) = 0$.

We get the proof of Lemma 1.8.

After using (1.36), the proof of Theorem 1.9 is almost the same as the proof of [13, Theorem 2.4] and Assumption 2.1 and 2.3 in [13] naturally hold in our case.

Let $\mathbb{B}_{u,T}$ be the Bismut superconnection associated to equivariant geometric family $\mathcal{F}_{Z,T}$. Let

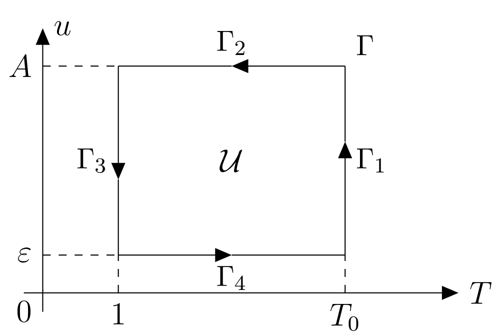
$$\widehat{\mathbb{B}}|_{(T,u),B^g} = \mathbb{B}_{u^2,T}|_{B^g} + uT\chi(uT)A_P|_{B^g} + dT \wedge \frac{\partial}{\partial T} + du \wedge \frac{\partial}{\partial u}. \tag{1.54}$$

We define $\beta_g = du \wedge \beta_g^u + dT \wedge \beta_g^T$ to be the part of $\psi_{B^g}\widetilde{\mathrm{Tr}}[g\exp(-(\widehat{\mathbb{B}}|_{B^g})^2)]$ of degree one with respect to the coordinates (T, u), with functions β_g^u, $\beta_g^T : \mathbb{R}_{+,T} \times \mathbb{R}_{+,u} \to \Omega^*(B^g)$.

Comparing with [13, Proposition 3.2], there exists a smooth family $\alpha_g : \mathbb{R}_{+,T} \times \mathbb{R}_{+,u} \to \Omega^*(B^g)$ such that

$$\left(du \wedge \frac{\partial}{\partial u} + dT \wedge \frac{\partial}{\partial T}\right)\beta_g = dT \wedge du\ d^S\alpha_g. \tag{1.55}$$

Take ε, A, T_0, $0 < \varepsilon \leq 1 \leq A < \infty$, $1 \leq T_0 < \infty$. Let $\Gamma = \Gamma_{\varepsilon,A,T_0}$ be the oriented contour in $\mathbb{R}_{+,T} \times \mathbb{R}_{+,u}$.



The contour Γ is made of four oriented pieces $\Gamma_1, \cdots, \Gamma_4$ indicated in the above picture. For $1 \le k \le 4$, set $I_k^0 = \int_{\Gamma_k} \beta_g$. Then by Stocks' formula and (1.55),

$$\sum_{k=1}^4 I_k^0 = \int_{\partial\mathcal{U}} \beta_g = \int_{\mathcal{U}} \left(du \wedge \frac{\partial}{\partial u} + dT \wedge \frac{\partial}{\partial T} \right) \beta_g = d^S \left(\int_{\mathcal{U}} \alpha_g dT \wedge du \right). \tag{1.56}$$

The following theorems are the analogues of Theorem 3.3-3.6 in [13]. Since $\ker(D(\mathcal{F}_X)+A_P) = 0$, the proofs in our case are much easier. From the the proof of Lemma 1.8, we can simply assume that $\ker(D(\mathcal{F}_Z) + A_P) = 0$. We only need to replace D^X and D_T^Z in [13] by $D(\mathcal{F}_X) + A_P$ and $D(\mathcal{F}_{Z,T}) + TA_P$ and notice that A_P is a bounded operator along the fiber X with smooth kernel.

Theorem 1.10. *i) For any $u > 0$, we have*

$$\lim_{T\to\infty} \beta_g^u(T, u) = 0. \tag{1.57}$$

ii) For $0 < u_1 < u_2$ fixed, there exists $C > 0$ such that, for $u \in [u_1, u_2]$, $T \ge 1$, we have

$$|\beta_g^u(T, u)| \le C. \tag{1.58}$$

iii) We have the following identity:

$$\lim_{T\to+\infty} \int_1^\infty \beta_g^u(T, u)du = 0. \tag{1.59}$$

Theorem 1.11. *We have the following identity:*

$$\lim_{u\to+\infty} \int_1^\infty \beta_g^T(T, u)dT = 0. \tag{1.60}$$

In fact, $\Omega_g(\nabla^{\mathcal{S}_Y\widehat{\otimes}\mathcal{E}_X})$ only depends on $g \in G$ and $R^{\mathcal{S}_Y\widehat{\otimes}\mathcal{E}_X} := (\nabla^{\mathcal{S}_Y\widehat{\otimes}\mathcal{E}_X})^2$. So we can denote it by $\Omega_g(R^{\mathcal{S}_Y\widehat{\otimes}\mathcal{E}_X})$. Let $R_T^{\mathcal{S}_Y\widehat{\otimes}\mathcal{E}_X} := (\nabla_T^{\mathcal{S}_Y\widehat{\otimes}\mathcal{E}_X})^2$. Set

$$\gamma_\Omega(T) = -\left.\frac{\partial}{\partial b}\right|_{b=0} \Omega_g \left(R_T^{\mathcal{S}_Y\widehat{\otimes}\mathcal{E}_X} + b\frac{\partial \nabla_T^{\mathcal{S}_Y\widehat{\otimes}\mathcal{E}_X}}{\partial T} \right). \tag{1.61}$$

By a standard argument in Chern-Weil theory, we know that

$$\frac{\partial}{\partial T}\widetilde{\Omega}_g(\nabla^{\mathcal{S}_Y\widehat{\otimes}\mathcal{E}_X}, \nabla_T^{\mathcal{S}_Y\widehat{\otimes}\mathcal{E}_X}) = -\gamma_\Omega(T). \tag{1.62}$$

Proposition 1.12. *When $T \to +\infty$, we have $\gamma_\Omega(T) = O(T^{-2})$. Moreover, modulo exact forms on W^g, we have*

$$\widetilde{\Omega}_g(\nabla^{\mathcal{S}_Y\widehat{\otimes}\mathcal{E}_X}, \nabla^{\mathcal{S}_Y,\mathcal{E}_X}) = -\int_1^{+\infty} \gamma_\Omega(T)dT. \tag{1.63}$$

Let $\mathbb{B}_{X,t}$ be the Bismut superconnection associated to the equivariant geometric family $\mathcal{F}_X$. Set

$$\gamma_1(t) = \left\{ \psi_{V^g} \widetilde{\mathrm{Tr}}|_{V^g} \left[g \exp\left(-\left(\mathbb{B}_{X,t^2}|_{V^g} + \chi(t)A_P|_{V^g} + dt \wedge \frac{\partial}{\partial t} \right)^2 \right) \right] \right\}^{dt}. \tag{1.64}$$

Then

$$\widetilde{\eta}_g(\mathcal{F}_X, P_X) = -\int_0^\infty \gamma_1(t)dt. \tag{1.65}$$

Theorem 1.13. *i) For any $u > 0$, there exist $C > 0$ and $\delta > 0$ such that, for $T \geq 1$, we have*

$$|\beta_g^T(T, u)| \leq \frac{C}{T^{1+\delta}}. \tag{1.66}$$

ii) For any $T > 0$, we have

$$\lim_{\varepsilon \to 0} \varepsilon^{-1}\beta_g^T(T\varepsilon^{-1}, \varepsilon) = \int_{Y^g} \mathrm{Td}_g(TY, \nabla^{TY, L_Y}) \wedge \gamma_1(T). \tag{1.67}$$

iii) There exists $C > 0$ such that for $\varepsilon \in (0, 1]$, $\varepsilon \leq T \leq 1$,

$$\varepsilon^{-1}\left|\beta_g^T(T\varepsilon^{-1}, \varepsilon) - \gamma_\Omega(T\varepsilon^{-1})\right| \leq C. \tag{1.68}$$

iv) There exist $\delta \in (0, 1]$, $C > 0$ such that, for $\varepsilon \in (0, 1]$, $T \geq 1$,

$$\varepsilon^{-1}|\beta_g^T(T\varepsilon^{-1}, \varepsilon)| \leq \frac{C}{T^{1+\delta}}. \tag{1.69}$$

By (1.56), we know that

$$\begin{aligned}\int_\varepsilon^A \beta_g^u(T_0, u)du - \int_1^{T_0} \beta_g^T(T, A)dT - \int_\varepsilon^A \beta_g^u(1, u)du + \int_1^{T_0} \beta_g^T(T, \varepsilon)dT \\ = I_1 + I_2 + I_3 + I_4\end{aligned} \tag{1.70}$$

is an exact form. We take the limits $A \to \infty$, $T \to \infty$ and then $\varepsilon \to 0$ in the indicated order. Let I_j^k, $j = 1, 2, 3, 4$, $k = 1, 2, 3$ denote the value of the part I_j after the kth limit. By [7, §22, Theorem 17], $d\Omega(B^g)$ is closed under uniformly convergence on B^g. Thus,

$$\sum_{j=1}^4 I_j^3 \equiv 0 \bmod d\Omega^*(B^g). \tag{1.71}$$

Since $\ker(D(\mathcal{F}_Z) + A_P) = 0$, the Atiyah-Patodi-Singer projection P_Z onto the positive part of the spectrum of $D(\mathcal{F}_Z) + A_P$ is an equivariant spectral section for $\mathcal{F}_Z$ and A_P is the corresponding self-adjoint smoothing operators of P_Z From (1.35), we obtain that

$$I_3^3 = \tilde{\eta}_g(\mathcal{F}_Z, P_Z). \tag{1.72}$$

Furthermore, by Theorem 1.11, we get

$$I_2^2 = I_2^3 = 0. \tag{1.73}$$

From Theorem 1.10, we have

$$I_1^3 = 0. \tag{1.74}$$

Finally, using Theorem 1.13, we get

$$I_4^3 = -\int_{Y^g} \mathrm{Td}_g(TY, \nabla^{TY, L_Y}) \wedge \widetilde{\eta}_g(\mathcal{F}_X, P_X) - \widetilde{\Omega}_g\left(\nabla^{\mathcal{S}_Y \widehat{\otimes} \mathcal{E}_X}, \nabla^{\mathcal{S}_Y, \mathcal{E}_X}\right) \tag{1.75}$$

as follows: We write

$$\int_1^{+\infty} \beta_g^T(T,\varepsilon)dT = \int_\varepsilon^{+\infty} \varepsilon^{-1}\beta_g^T(T\varepsilon^{-1},\varepsilon)dT. \tag{1.76}$$

Convergence of the integrals above is granted by (1.66). Using (1.67), (1.69) and Proposition 1.12, we get

$$\lim_{\varepsilon\to 0}\int_1^{+\infty} \varepsilon^{-1}\beta_g^T(T\varepsilon^{-1},\varepsilon)dT = \int_{Y^g} \mathrm{Td}_g(TY,\nabla^{TY,L_Y}) \wedge \int_1^{+\infty} \gamma_1(T)dT \tag{1.77}$$

and

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_\varepsilon^1 \varepsilon^{-1}\left[\beta_g^T(T\varepsilon^{-1},\varepsilon)dT - \gamma_\Omega(T\varepsilon^{-1})\right]dT \\ = \int_{Y^g} \mathrm{Td}_g(TY,\nabla^{TY,L_Y}) \wedge \int_0^1 \gamma_1(T)dT.\end{aligned} \tag{1.78}$$

The remaining part of the integral yields by (1.68)

$$\int_\varepsilon^1 \varepsilon^{-1}\gamma_\Omega(T\varepsilon^{-1})dT = \int_1^{+\infty} \gamma_\Omega(T)dT = -\widetilde{\Omega}_g\left(\nabla^{\mathcal{S}_Y\widehat{\otimes}\mathcal{E}_X}, \nabla^{\mathcal{S}_Y,\mathcal{E}_X}\right). \tag{1.79}$$

These four equations for I_k^3, $k=1,2,3,4$, and (1.36) imply Theorem 1.9.

2. Equivariant differntial K-theory

In this section, we assume that G is a finite group.

2.1. **Axioms of equivariant differntial K-theory.** Let G be a finite group. For $\mathscr{K} \in K_G^*(B)$, we define the equivariant Chern character map

$$\mathrm{ch}_G(\mathscr{K}) := \bigoplus_{g\in G} \mathrm{ch}_g(\mathscr{K}|_{B^g}) \in \left(\bigoplus_{g\in G}\bigoplus_{k\in\mathbb{Z}} \Omega^{*+2k}(B^g,\mathbb{C})/\mathrm{Im}\, d\right)^G. \tag{2.1}$$

Denote by $\Omega^*(B,\mathbb{C})_G := \left(\bigoplus_{g\in G}\bigoplus_{k\in\mathbb{Z}} \Omega^{*+2k}(B^g,\mathbb{C})\right)^G$, $*=0,1$. Let $H^*(B,\mathbb{R})_G$ be the cohomology of the differential complex $(\Omega^*(B,\mathbb{R})_G, d_G)$, where

$$(d_G\omega)_g = d\omega_g. \tag{2.2}$$

From [18, Theorem 2.1], the equivariant Chern character map extends to an isomorphism

$$\mathrm{ch}_G : K_G^*(B)\otimes\mathbb{C} \xrightarrow{\sim} H^*(B,\mathbb{C})_G. \tag{2.3}$$

The complex algebra $K_G^*(B)\otimes\mathbb{C}$ has a canonical real structure given by conjugation on $\mathbb{C}$, and this induces a real structure on $H^*(B,\mathbb{C})_G$. We denote the real subspace of $H^*(B,\mathbb{C})_G$ by $H^*(B,\mathbb{R})_G$. in fact,

$$H^*(B,\mathbb{R})_G = \{c\in H^*(B,\mathbb{C})_G : \forall g\in G, c|_{B^g} = \overline{c|_{B^{g^{-1}}}}\}. \tag{2.4}$$

Let $\Omega^*(B,\mathbb{R})_G \subset \Omega^*(B,\mathbb{C})_G$ be the ring of forms ω, such that $\omega|_{B^g} = \overline{\omega|_{B^{g^{-1}}}}$. Then $H^*(B,\mathbb{R})_G$ is the cohomology of the differential complex $(\Omega^*(B,\mathbb{R})_G, d_G)$.

Definition 2.1. An equivariant differential K-theory $\widehat{K}_G$ is a functor $B \to \widehat{K}_G(B)$ from the category of closed smooth manifolds to $\mathbb{Z}$-graded groups together with natural transformations

(1) $R : \widehat{K}^*_G(B) \to \Omega^*_{d=0}(B, \mathbb{R})_G$ (curvature)
(2) $I : \widehat{K}^*_G(B) \to K^*_G(B)$ (underlying K_G group)
(3) $a : \Omega(B, \mathbb{R})_G/\mathrm{Im}\, d_G \to \widehat{K}_G(X)$ (action of forms).

Here $\Omega_{d=0}(B, \mathbb{R})_G \subset \Omega(B, \mathbb{R})_G$ denote the subring of closed forms.

The transformations I, a, R are required to satisfy the following axioms:

(1) The following diagram commutes

$$\begin{array}{ccc} \widehat{K}^*_G(B) & \xrightarrow{\quad I \quad} & K^*_G(B) \\ \big\downarrow R & & \big\downarrow \mathrm{ch}_G \\ \Omega^*_{d=0}(B, \mathbb{R})_G & \xrightarrow{\text{de Rham}} & H^*(B, \mathbb{R})_G. \end{array}$$

(2)

$$R \circ a = d. \tag{2.5}$$

(3) a is of degree 1.
(4) The following sequence is exact:

$$K^{*-1}_G(B) \xrightarrow{\mathrm{ch}_G} \Omega^{*-1}_{d=0}(B, \mathbb{R})_G/\mathrm{Im}\, d_G \xrightarrow{a} \widehat{K}^*_G(B) \xrightarrow{I} K^*_G(B) \longrightarrow 0. \tag{2.6}$$

Note that 2.6 is just [18, (15)].

2.2. **Geometric model of equivariant differential K-theory.** In the following subsections, we explain the geometric model of equivariant differential K-theory of Bunke-Schick. All definitions and properties are parallel with [5] by replacing the taming in [3, 5] the spectral section of Melrose-Piazza and using the analytical results in Section 1 to prove the properties.

Definition 2.2. Let B be a closed G-manifold. A cycle for an equivariant differential K-theory class over B is a pair $(\mathcal{F}, \rho)$, where $\mathcal{F} = (W, \mathcal{E}, o(TZ), T^H_\pi W, g^{TZ}, h^{\mathcal{E}}, \nabla^{\mathcal{E}})$ is an equivariant geometric family and $\rho \in \Omega(B, \mathbb{R})_G/\mathrm{Im}\, d_G$. The cycle $(\mathcal{F}, \rho)$ is called even (resp. odd) if $\dim TZ$ is even (resp. odd) and $\rho \in \Omega^{\mathrm{odd}}(B, \mathbb{R})_G/\mathrm{Im}\, d_G$ (resp. $\rho \in \Omega^{\mathrm{even}}(B, \mathbb{R})_G/\mathrm{Im}\, d_G$).

If $\mathrm{ind}(D(\mathcal{F}))$ vanishes, for an equivariant spectral section P, the equivariant eta form $\widetilde{\eta}_G(\mathcal{F}, P)$ is defined by

$$\widetilde{\eta}_G(\mathcal{F}, P) = \bigoplus_{g \in G} \widetilde{\eta}_g(\mathcal{F}, P) \in \Omega(B, \mathbb{R})_G. \tag{2.7}$$

Definition 2.3. The opposite $\mathcal{F}^{\mathrm{op}}$ of an equivariant geometric family $\mathcal{F}$ is obtained by reversing the signs of the Clifford multiplication and the grading (in the even sense) of the underlying family of Clifford modules, and of the orientation of the relative tangent bundle.

By definition, we have

$$\operatorname{ind}(D(\mathcal{F}^{\mathrm{op}})) = -\operatorname{ind}(D(\mathcal{F})). \tag{2.8}$$

Definition 2.4. Two cycles $(\mathcal{F},\rho)$ and $(\mathcal{F}',\rho')$ are called isomorphic if $\mathcal{F}$ and $\mathcal{F}'$ are isomorphic and $\rho=\rho'$. Let $\widehat{G}^*(B)$ denote the set of isomorphism classes of cycles over B of parity $*\in\{\mathrm{even},\mathrm{odd}\}$.

Given two equivariant geometric families $\mathcal{F}$ and $\mathcal{F}'$ we can form their sum $\mathcal{F}\sqcup_B\mathcal{F}'$ over B. The underlying fibration with closed fibers of the sum is $\pi\sqcup\pi' : W\sqcup W'\to B$, where $\sqcup$ is the disjoint union. The remaining structures of $\mathcal{F}\sqcup_B\mathcal{F}'$ are induced in the obvious way.

Definition 2.5. The sum of two cycles $(\mathcal{F},\rho)$ and $(\mathcal{F}',\rho')$ is defined by

$$(\mathcal{F},\rho)+(\mathcal{F}',\rho') := (\mathcal{F}\sqcup_B\mathcal{F}', \rho+\rho'). \tag{2.9}$$

The sum of of cycles induces on $\widehat{G}^*(B)$ the structure of a graded abelian semigroup. The identity element of $\widehat{G}^*(B)$ is the cycle $0 := (\emptyset, 0)$, where $\emptyset$ is the empty geometric family.

Definition 2.6. We call two cycles $(\mathcal{F},\rho)$ and $(\mathcal{F}',\rho')$ paired if $\operatorname{ind}(D(\mathcal{F}))=\operatorname{ind}(D(\mathcal{F}'))$ and there exists an equivariant spectral section P such that

$$\rho-\rho' = \widetilde{\eta}_G(\mathcal{F}\sqcup_B\mathcal{F}'^{\mathrm{op}}, P). \tag{2.10}$$

Let $\sim$ denote the equivalence relation generated by the relation "paired".

If $\operatorname{ind} D(\mathcal{F})=0$, for an equivariant spectral section P, we can get that

$$(\mathcal{F}, \widetilde{\eta}_G(\mathcal{F},P)) \sim (\emptyset, 0). \tag{2.11}$$

As in [3, Lemma 2.11-2.13], we can get that $\widehat{G}^*(B)/\sim$ is an abelian semigroup.

Definition 2.7. We define the equivariant differential K-theory $\widehat{K}_G^0(B)$ (resp. $\widehat{K}_G^{-1}(B)$) of B to be the group completion of the abelian semigroup $\widehat{G}^{\mathrm{even}}(B)/\sim$ (resp. $\widehat{G}^{\mathrm{odd}}(B)/\sim$).

If $(\mathcal{F},\rho)$ is a cycle, we denote by $[\mathcal{F},\rho]\in\widehat{K}_G^*(B)$ the corresponding class in equivariant differential K-theory. We collect some simple facts as a natural extension of [3, Lemma 2.15-2.17].

Proposition 2.8. *(A) We have* $[\mathcal{F},\rho]+[\mathcal{F}^{\mathrm{op}},-\rho]=0$.

(B) Every element of $\widehat{K}_G^*(B)$ *can be represented in the form* $[\mathcal{F},\rho]$.

(C) If $[\mathcal{F}_0,\rho_0]=[\mathcal{F}_1,\rho_1]$, *then there exists a cycle* $(\mathcal{F}',\rho')$ *such that* $(\mathcal{F}_0,\rho_0)+(\mathcal{F}',\rho')$ *is paired with* $(\mathcal{F}_1,\rho_1)+(\mathcal{F}',\rho')$.

In the following, we will prove that $\widehat{K}_G^*(B)$ satisfies all the conditions in Definition 2.1.

We first define the natural transformation

$$I : \widehat{K}_G^*(B)\to K_G^*(B) \tag{2.12}$$

by

$$I([\mathcal{F},\rho]) := \operatorname{ind}(D(\mathcal{F})). \tag{2.13}$$

From Definition 2.6, the transformation I is well defined.

We construct a parity-reversing natural transformation

$$a:\Omega(B,\mathbb{R})_G/\mathrm{Im}\, d_G\to \widehat{K}_G^*(B) \tag{2.14}$$

by

$$a(\rho):=[\emptyset,-\rho]. \tag{2.15}$$

Let

$$\Omega_G(\mathcal{F})=\bigoplus_{g\in G}\Omega_g(\mathcal{F})\in\Omega_{d=0}(B,\mathbb{R})_G. \tag{2.16}$$

We define a transformation

$$R:\widehat{K}_G^*(B)\to\Omega_{d=0}(B,\mathbb{R})_G \tag{2.17}$$

by

$$R([\mathcal{F},\rho]):=\Omega_G(\mathcal{F})-d\rho. \tag{2.18}$$

The transformation R is well-defined. In fact, if $(\mathcal{F}',\rho')$ is a cycle paired with $(\mathcal{F},\rho)$, we have $\mathrm{ind}(D(\mathcal{F}\sqcup_B\mathcal{F}^{'\mathrm{op}}))=0$. So there exists an equivariant spectral section P, such that $\rho-\rho'=\widetilde{\eta}_G(\mathcal{F}\sqcup_B\mathcal{F}^{'\mathrm{op}},P)$. From (1.32), we have

$$\begin{aligned}R([\mathcal{F},\rho])-R([\mathcal{F}',\rho'])&=\Omega_G(\mathcal{F})-\Omega_G(\mathcal{F}')-d(\rho-\rho')\\&=d\widetilde{\eta}_G(\mathcal{F}\sqcup_B\mathcal{F}^{'\mathrm{op}},P)-d\widetilde{\eta}_G(\mathcal{F}\sqcup_B\mathcal{F}^{'\mathrm{op}},P)=0.\end{aligned} \tag{2.19}$$

From (2.15) and (2.18), we have

$$R\circ a=d. \tag{2.20}$$

By (1.19), the diagram in Definition 2.1 (1) commutes.

Proposition 2.9. *The following sequence is exact:*

$$K_G^{*-1}(B)\xrightarrow{\mathrm{ch}_G}\Omega_{d=0}^{*-1}(B,\mathbb{R})_G/\mathrm{Im}\, d_G\xrightarrow{a}\widehat{K}_G^*(B)\xrightarrow{I}K_G^*(B)\longrightarrow 0. \tag{2.21}$$

Proof. The surjectivity of I is a natural extension of [3, Lemma 2.18].

Next, we show the exactness at $\widehat{K}_G^*(B)$. It is obvious that $I\circ a=0$. For a cycle $(\mathcal{F},\rho)$, if $I([\mathcal{F},\rho])=0$, we have $\mathrm{ind}(D(\mathcal{F}))=0$. So there exists an equivariant spectral section P for $\mathcal{F}$. By (2.11), we have

$$[\mathcal{F},\rho]=a(\widetilde{\eta}_G(\mathcal{F},P)-\rho). \tag{2.22}$$

Finally, We prove the exactness at $\Omega_{d=0}^{*-1}(B,\mathbb{R})_G/\mathrm{Im}\, d_G$. By Proposition 1.5 (E), for any $\mathscr{K}\in K_G^*(B)$, it can be generated by the differences of the equivariant spectral sections for some equivariant geometric family $\mathcal{F}$ with $\mathrm{ind}(D(\mathcal{F}))=0$. We simply assume that there exist two equivariant spectral sections P_1,P_2 such that $[P_1-P_2]=\mathscr{K}$. By (1.36) and (2.11), we have

$$2a\circ\mathrm{ch}_G(\mathscr{K})=[\emptyset,-2\,\mathrm{ch}_G(\mathscr{K})]=[\mathcal{F},\tilde{\eta}_G(\mathcal{F},P_2)]-[\mathcal{F},\tilde{\eta}_G(\mathcal{F},P_1)]=0. \tag{2.23}$$

So $a \circ \mathrm{ch}_G(\mathscr{K}) = 0$. From Proposition 2.8 (C), if $a(\rho) = 0$, for any equivariant geometric family $\mathcal{F}$ with vanishing index and equivariant spectral section P, by (2.11) we have

$$(\emptyset, 0) \sim (\mathcal{F}, \tilde{\eta}_G(\mathcal{F}, P)) \sim (\mathcal{F}, \tilde{\eta}_G(\mathcal{F}, P) - \rho). \tag{2.24}$$

So there exists another equivariant spectral section P', such that $\tilde{\eta}_G(\mathcal{F}, P') = \tilde{\eta}_G(\mathcal{F}, P) - \rho$. By (1.36), we have

$$\rho = \mathrm{ch}_G(2[P' - P]). \tag{2.25}$$

□

2.3. **Push-forward map.** In this subsection, we use the notations in Section 1.5.

Let V, B be two closed smooth G-manifolds and $\pi : V \to B$ be an equivariant fiber bundle with closed fiber Y. We assume that TY has a G-equivariant Spin^c structure, with an associated complex line bundle L_Y.

We consider the set $\mathcal{O}$ of equivariant geometric data $o = (T_\pi^H V, g^{TY}, \nabla^{L_Y}, \sigma(Y))$, where the first three entries have the same meaning in Section 1.5 and $\sigma(Y) \in \Omega^{\mathrm{odd}}(V, \mathbb{R})_G / \mathrm{Im}\, d_G$.

Let

$$\mathrm{Td}_G(\nabla^{TY, L_Y}) = \bigoplus_{g \in G} \mathrm{Td}_g(\nabla^{TY, L_Y}) \in \Omega_{d=0}(B, \mathbb{R})_G. \tag{2.26}$$

Let $o' = (T'^H V, g'^{TY}, \nabla'^{L_Y}, \sigma'(Y)) \in \mathcal{O}$ be another equivariant tuple. As in (1.20), from [15, Theorem B.5.4], we can construct the Chern-Simons form $\widetilde{\mathrm{Td}}_G(\nabla^{TY, L_Y},\ \nabla'^{TY, L_Y})$ such that

$$d\, \widetilde{\mathrm{Td}}_G(\nabla^{TY, L_Y}, \nabla'^{TY, L_Y}) = \mathrm{Td}_G(\nabla'^{TY, L_Y}) - \mathrm{Td}_G(\nabla^{TY, L_Y}). \tag{2.27}$$

We introduce a relation $o \sim o'$ as follows. Two equivariant tuples $o = (T^H V, g^{TY}, \nabla^{L_Y}, \sigma(Y))$, $o' = (T'^H V, g'^{TY}, \nabla'^{L_Y}, \sigma'(Y))$ are related if and only if

$$\sigma'(Y) - \sigma(Y) = -\widetilde{\mathrm{Td}}_G(\nabla^{TY, L_Y}, \nabla'^{TY, L_Y}), \tag{2.28}$$

where we mark the objects associated to the second tuple by $'$.

Definition 2.10. The set of equivariant differential K-orientations is the set of equivalence classes $\mathcal{O}/\sim$.

We now start with the construction of the push-forward map $\widehat{\pi}_G! : \widehat{K}^*_G(V) \to \widehat{K}^*_G(B)$ for a given equivariant differential K-orientation. For a cycle $(\mathcal{F}_X, \rho)$ over V, let $\mathcal{F}_Z$ be the equivariant geometric family defined in (1.44). We denote by

$$\pi_G!(\mathcal{F}_X) = \mathcal{F}_Z. \tag{2.29}$$

We define

$$\begin{aligned} \widehat{\pi}_G!(\mathcal{F}_X, \rho) = \Bigg[\mathcal{F}_Z, \int_{\sqcup_{g \in G} Y^g} \mathrm{Td}_G(\nabla^{TY, L_Y}) \wedge \rho + \widetilde{\Omega}_G\left(\nabla^{\mathcal{S}_Y \widehat{\otimes} \mathcal{E}_X}, \nabla^{\mathcal{S}_Y, \mathcal{E}_X}\right) \\ + \int_{\sqcup_{g \in G} Y^g} \sigma(Y) \wedge (\Omega_G(\mathcal{F}_X) - d\rho) \Bigg] \in \widehat{K}^*_G(B). \end{aligned} \tag{2.30}$$

Theorem 2.11. *The map $\widehat{\pi}_G! : \widehat{G}^*(V) \to \widehat{K}^*_G(B)$ in (2.30) can be deduced to a map $\widehat{\pi}_G! : \widehat{K}^*_G(V) \to \widehat{K}^*_G(B)$.*

Proof. Let $(\mathcal{F}_X, \rho)$, $(\mathcal{F}'_X, \rho')$ be two cycles over V. By (2.30), we have

$$\widehat{\pi}_G!(\mathcal{F}_X, \rho) - \widehat{\pi}_G!(\mathcal{F}'_X, \rho') = \widehat{\pi}_G!(\mathcal{F}_X \sqcup \mathcal{F}'^{\mathrm{op}}_X, \rho - \rho'). \tag{2.31}$$

If $(\mathcal{F}_X, \rho)$ is paired with $(\mathcal{F}'_X, \rho')$, there exists an equivariant spectral section P_X, such that

$$\rho - \rho' = \widetilde{\eta}_G(\mathcal{F}_X \sqcup \mathcal{F}'^{\mathrm{op}}_X, P_X). \tag{2.32}$$

So we only need to prove $\widehat{\pi}_G!(\mathcal{F}_X, \widetilde{\eta}_G(\mathcal{F}_X, P_X)) = 0 \in \widehat{K}^*_G(B)$ when $\mathrm{ind}(D(\mathcal{F}_X)) = 0 \in K^*_G(V)$.

From (2.30), if $\mathrm{ind}(D(\mathcal{F}_X)) = 0 \in K^*_G(V)$, we have

$$\begin{aligned}\widehat{\pi}_G!(\mathcal{F}_X, \widetilde{\eta}_G(\mathcal{F}_X, P_X))) = \Bigg[\mathcal{F}_Z, \int_{\sqcup_{g\in G} Y^g} \mathrm{Td}_G(\nabla^{TY, L_Y}) \wedge \widetilde{\eta}_G(\mathcal{F}_X, P_X)) \\ + \widetilde{\Omega}_G\left(\nabla^{\mathcal{S}_Y \widehat{\otimes} \mathcal{E}_X}, \nabla^{\mathcal{S}_Y, \mathcal{E}_X}\right) + \int_{\sqcup_{g\in G} Y^g} \sigma(Y) \wedge (\Omega_G(\mathcal{F}_X) - d\widetilde{\eta}_G(\mathcal{F}_X, P_X)))\Bigg].\end{aligned} \tag{2.33}$$

Let P_Z be an equivariant spectral section of $\mathcal{F}_Z$. By (1.32), (2.11) and Theorem 1.9, we have

$$\widehat{\pi}_G!(\mathcal{F}_X, \widetilde{\eta}_G(\mathcal{F}_X, P_X))) = [\mathcal{F}_Z, \widetilde{\eta}_G(\mathcal{F}_Z, P_Z) - \mathrm{ch}_G(\mathscr{K}(P_Z, P_X))] = [\emptyset, -\mathrm{ch}_G(\mathscr{K}(P_Z, P_X))]. \tag{2.34}$$

By (2.23), we have

$$[\emptyset, -\mathrm{ch}_G(\mathscr{K}(P_Z, P_X))] = 0 \in \widehat{K}^*_G(B). \tag{2.35}$$

The proof of Theorem 2.11 is complete. □

Our construction of $\widehat{\pi}_G!$ involve an explicit choice of a representative $o = (T^H V, g^{TY}, \nabla^{L_Y}, \sigma(Y))$ of the equivariant differential K-orientation.

Lemma 2.12. *The homomorphism $\widehat{\pi}_G! : \widehat{K}^*_G(V) \to \widehat{K}^*_G(B)$ only depend on the equivariant differential K-orientation represented by o.*

Proof. Let $o = (T^H_\pi V, g^{TY}, \nabla^{L_Y}, \sigma(Y))$, $o' = (T'^H_\pi V, g'^{TY}, \nabla'^{L_Y}, \sigma'(Y))$ be two representative of an equivariant differential K-orientation. We will mark the objects associated to the second representative by $'$.

Then from (2.27) and (2.28), we have

$$\begin{aligned}(2.36)\quad &\widehat{\pi}'_G!(\mathcal{F}_X,\rho)-\widehat{\pi}_G!(\mathcal{F}_X,\rho)=[\mathcal{F}'_Z\sqcup\mathcal{F}_Z^{\mathrm{op}},\\ &\int_{\sqcup_{g\in G}Y^g}\left(\mathrm{Td}_G(\nabla'^{TY,L_Y})-\mathrm{Td}_G(\nabla^{TY,L_Y})\right)\wedge\rho-\widetilde{\Omega}_G\left(\nabla^{\mathcal{S}_Y\widehat{\otimes}\mathcal{E}_X},\nabla'^{\mathcal{S}_Y\widehat{\otimes}\mathcal{E}_X}\right)\\ &+\widetilde{\Omega}_G\left(\nabla^{\mathcal{S}_Y,\mathcal{E}_X},\nabla'^{\mathcal{S}_Y,\mathcal{E}_X}\right)+\int_{\sqcup_{g\in G}Y^g}(\sigma'(Y)-\sigma(Y))\wedge(\Omega_G(\mathcal{F}_X)-d\rho)\Big]\\ &=\Big[\mathcal{F}'_Z\sqcup\mathcal{F}_Z^{\mathrm{op}},\int_{\sqcup_{g\in G}Y^g}d\,\widetilde{\mathrm{Td}}_G(\nabla^{TY,L_Y},\nabla'^{TY,L_Y})\wedge\rho\\ &-\int_{\sqcup_{g\in G}Y^g}\widetilde{\mathrm{Td}}_G(\nabla^{TY,L_Y},\nabla'^{TY,L_Y})\wedge d\rho+\int_{\sqcup_{g\in G}Y^g}d\,\widetilde{\mathrm{Td}}_G(\nabla^{TY,L_Y},\nabla'^{TY,L_Y})\wedge\Omega_G(\mathcal{F}_X)\\ &-\widetilde{\Omega}_G\left(\nabla^{\mathcal{S}_Y\widehat{\otimes}\mathcal{E}_X},\nabla'^{\mathcal{S}_Y\widehat{\otimes}\mathcal{E}_X}\right)+\widetilde{\Omega}_G\left(\nabla^{\mathcal{S}_Y,\mathcal{E}_X},\nabla'^{\mathcal{S}_Y,\mathcal{E}_X}\right)\Big]\\ &=[\mathcal{F}'_Z\sqcup\mathcal{F}_Z^{\mathrm{op}},\widetilde{\Omega}_G\left(\mathcal{F}_Z,\mathcal{F}'_Z\right)].\end{aligned}$$

From Definition 1.3, for equivariant geometric family $\mathcal{F}_Z\sqcup\mathcal{F}_Z^{\mathrm{op}}$, $\ker(D(\mathcal{F}_Z\sqcup\mathcal{F}_Z^{\mathrm{op}}))=0$. So we can take the equivariant spectral section P to be the Atiyah-patodi-Singer projection. In this case, $A_P=0$. From Definition 1.6 and 2.3, we have

$$(2.37)\qquad \widetilde{\eta}_G(\mathcal{F}_Z\sqcup\mathcal{F}_Z^{\mathrm{op}},P)=0.$$

By (2.37) and Proposition 1.7, there exist equivariant spectral section P' for $\mathcal{F}'_Z\sqcup\mathcal{F}_Z^{\mathrm{op}}$ and $\mathcal{K}(P')\in K_G^*(S)$, such that

$$(2.38)\quad \widetilde{\Omega}_G\left(\mathcal{F}_Z,\mathcal{F}'_Z\right)=\widetilde{\Omega}_G\left(\mathcal{F}_Z\sqcup\mathcal{F}_Z^{\mathrm{op}},\mathcal{F}'_Z\sqcup\mathcal{F}_Z^{\mathrm{op}}\right)=-\widetilde{\eta}_G(\mathcal{F}'_Z\sqcup\mathcal{F}_Z^{\mathrm{op}},P')+\mathrm{ch}_G(\mathcal{K}(P')).$$

Then Lemma 2.12 follows from (2.11), (2.23), (2.36) and (2.38). □

2.4. **Functoriality of the push-forward maps.** We now discuss the functoriality of the push-forward maps with respect to the composition of fiber bundles. Let $\pi:V\to B$ with fiber Y be as in the above subsection together with a representative of an equivariant differential K-orientation $o_Y=(T_\pi^HV,g^{TY},\nabla^{L_Y},\sigma(Y))$. Let $r:B\to S$ be another proper submersion with closed fiber U together with a representative of an equivariant differential K-orientation $o_U=(T_r^HB,g^{TU},\nabla^{L_U},\sigma(U))$.

Let $q:=r\circ\pi:V\to B$ be the composition of two fibrations with fiber A. Let T_q^HV be a horizontal subbundle associated to q. We assume that $T_q^HV\subset T_\pi^HV$. Set $g^{TA}=r^*g^{TU}\oplus g^{TY}$, $\nabla^{L_A}=r^*\nabla^{L_U}\otimes\nabla^{L_Y}$.

Definition 2.13. We define the composite $o(A)=o(U)\circ o(Y)$ of the representatives of equivariant differential K-orientations of π and r by

$$(2.39)\qquad o(A):=(T_q^HV,g^{TA},\nabla^{L_A},\sigma(A)),$$

where

$$\begin{aligned}(2.40)\quad \sigma(A):=\;&\sigma(Y)\wedge\pi^*\,\mathrm{Td}_G(\nabla^{TU,L_U})+\mathrm{Td}_G(\nabla^{TY,L_Y})\wedge\pi^*\sigma(U)\\ &+\widetilde{\mathrm{Td}}_G(\nabla^{\mathcal{S}_U\widehat{\otimes}\mathcal{S}_Y},\nabla^{\mathcal{S}_U,\mathcal{S}_Y})-d\sigma(Y)\wedge\sigma(U).\end{aligned}$$

Theorem 2.14. *We have the equality of homomorphisms* $\widehat{K}^*_G(V) \to \widehat{K}^*_G(S)$

$$\widehat{q}_G! = \widehat{r}_G! \circ \widehat{\pi}_G!. \tag{2.41}$$

Proof. The theorem follows from a direct calculation using (2.30) and (2.40). □

References

[1] N. Berline, E. Getzler, and M. Vergne. *Heat kernels and Dirac operators.* Grundlehren Text Editions. Springer-Verlag, Berlin, 2004. Corrected reprint of the 1992 original.

[2] J.-M. Bismut. The Atiyah-Singer index theorem for families of Dirac operators: two heat equation proofs. *Invent. Math.*, 83(1):91–151, 1985.

[3] U. Bunke and T. Schick. Smooth K-theory. *Astrisque*, 2009, 45-135 (2010).

[4] U. Bunke and T. Schick. Differential K-theory: a survey. *Global differential geometry*, 17: 303-357 (2012).

[5] U. Bunke and T. Schick. Differential orbifold K-theory. *J. Noncommut. Geom.*, 7(4):1027–1104, 2013.

[6] X. Dai and W. Zhang. Higher spectral flow. *J. Funct. Anal.*, 157(2):432–469, 1998.

[7] G. de Rham. *Variétés différentiables. Formes, courants, formes harmoniques.* Hermann, Paris, 1973. Troisième édition revue et augmentée, Publications de l'Institut de Mathématique de l'Université de Nancago, III, Actualités Scientifiques et Industrielles, No. 1222b.

[8] D. S. Freed. Dirac charge quantization and generalized differential cohomology. In *Surveys in differential geometry*, Surv. Differ. Geom., VII, pages 129–194. Int. Press, Somerville, MA, 2000.

[9] D. S. Freed, M. J. Hopkins, and C. Teleman. Loop groups and twisted K-theory I. *J. Topol.*, 4(4):737–798, 2011.

[10] D. S. Freed and J. Lott. An index theorem in differential K-theory. *Geom. Topol.*, 14(2):903–966, 2010.

[11] M. J. Hopkins and I. M. Singer. Quadratic functions in geometry, topology, and M-theory. *J. Differential Geom.*, 70(3):329–452, 2005.

[12] H.B. Lawson and M.-L. Michelsohn. *Spin geometry*, volume 38 of *Princeton Mathematical Series.* Princeton University Press, Princeton, NJ, 1989.

[13] B. Liu *Functoriality of equivariant eta forms* arXiv:1505.04454.

[14] K. Liu and X. Ma. On family rigidity theorems. I. *Duke Math. J.*, 102(3):451–474, 2000.

[15] X. Ma and G. Marinescu. *Holomorphic Morse inequalities and Bergman kernels*, volume 254 of *Progress in Mathematics.* Birkhäuser Verlag, Basel, 2007.

[16] R.B. Melrose and P. Piazza. Families of Dirac operators, boundaries and the b-calculus. *J. Differential Geom.*, 46(1):99–180, 1997.

[17] R.B. Melrose and P. Piazza. An index theorem for families of Dirac operators on odd-dimensional manifolds with boundary. *J. Differential Geom.*, 46(2):287–334, 1997.

[18] M.L. Ortiz. *Differential equivariant K-theory.* ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)–The University of Texas at Austin.

[19] D. Quillen. Superconnections and the Chern character. *Topology*, 24(1):89–95, 1985.

[20] J. Simons and D. Sullivan. Structured bundles define differential K-theory. *Astérisque*, (321):1–3, 2008. Géométrie différentielle, physique mathématique, mathématiques et société. I.

[21] R. J. Szabo and A. Valentino. Ramond-Ramond fields, fractional branes and orbifold differential K-theory. *Comm. Math. Phys.*, 294(3):647–702, 2010.

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