FUNCTORIALITY OF EQUIVARIANT ETA FORMS

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Abstract. In this paper, we define the equivariant eta form of Bismut-Cheeger for a compact Lie group and establish a formula about the functoriality of equivariant eta forms with respect to the composition of two submersions.

Contents

0. Introduction

In order to find a well-defined index for a first order elliptic differential operator over a compact manifold with nonempty boundary, Atiyah-Patodi-Singer [2] introduced a boundary condition which is particularly significant for applications. In this situation, an invariant of a first order self-adjoint operator called the eta invariant, η, enters into the index formula. Formally, the eta invariant is equal to the number of positive eigenvalues of the self-adjoint operator minus the number of negative eigenvalues.

Extending the work of Bismut-Freed [13], which is a rigorous proof of Witten's holonomy theorem [34], Bismut and Cheeger [9] studied the adiabatic limit for a fibration of closed Spin manifolds and found that under the invertible assumption of the Dirac family along the fibers, the adiabatic limit of the eta invariant of a Dirac operator on the total space is expressible in terms of a canonically constructed differential form, $\tilde{\eta}$, on the base space. Later, Dai [20] extended this result to the case when the kernel of the Dirac family forms a vector bundle over the base manifold.

This eta form of Bismut-Cheeger, $\tilde{\eta}$, is the higher degree version of the eta invariant η, i.e., it is exactly the boundary correction term in the family index theorem for manifolds with boundary [10, 11, 29]. When the base space is a point, the eta form of Bismut-Cheeger is just the eta invariant of Atiyah-Patodi-Singer. On the other hand, by [4, 9, 20], when the dimension of the fibers are even, the eta form serves as a canonically constructed transgression between the Chern character of the family index and Bismut's explicit local index representative [6] of it. We can also see it later by taking $g = 1$ in (0.3).

Recently, in the study of differential K-theory, the Bismut-Cheeger eta form naturally appears in the geometric model constructed by Bunke and Schick [18] as a key ingredient. Moreover, the results in [18] are highly relied on the properties of the eta form. In particular, the well-defined property of the push-forward map is based on a formula about the functoriality of eta forms proved by Bunke and Ma [16], which is a family version of [9]. In [17], Bunke and Schick extend their geometric model to the orbifold case. It can also be regarded as a geometric model for the equivariant differential K-theory for a finite group. Thus the equivariant eta form appears naturally here and this motivates us to understand systematically the equivariant eta form.

In this paper, we define first the equivariant eta form when the fibration admits a fiberwise compact Lie group action and establish a formula about the functoriality of equivariant eta forms which extends [16, Theorem 5.11] and [9] to our case. Note that Bunke-Ma in [16] worked for the eta form associated to flat vector bundles, and many analytic arguments are only sketched. Here we work on the equivariant situation, thus we need to combine the equivariant local index technique to the different functional analysis technique in analytic localization developed by Bismut and his collaborators

[5, 7, 8, 14, 15, 26, 27]. We take this opportunity to give also the details of the analytic arguments missed in Bunke-Ma [16].

Let $\pi : W \to S$ be a smooth submersion of smooth manifolds with closed oriented fiber Z, with $\dim Z = n$. Let $TZ = TW/S$ be the relative tangent bundle to the fibers Z with Riemannian metric g^{TZ} and T^HW be a horizontal subbundle of TW, such that $TW = T^HW \oplus TZ$. Let ∇^{TZ} be the Euclidean connection on TZ defined in (1.15). We assume that TZ has a Spin^c structure. Let L_Z be the complex line bundle associated to the Spin^c structure of TZ with a Hermitian metric h^{L_Z} and a Hermitian connection ∇^{L_Z} (see [22, Appendix D]).

Let G be a compact Lie group which acts fiberwisely on W and as identity on S. We assume that the action of G preserves the Spin^c structure of TZ and all metrics and connections are G-invariant. Let (E, h^E) be a G-equivariant Hermitian vector bundle over W with a G-invariant Hermitian connection ∇^E. Let D^Z be the fiberwise Dirac operator defined in (1.21) and B_t be the Bismut superconnection defined in (1.32). For $\alpha \in \Omega^i(S)$, the differential form on S with degree i, set

$$\psi_S(\alpha) = \begin{cases} \left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{i}{2}} \cdot \alpha, & \text{if } i \text{ is even;} \\ \frac{1}{\sqrt{\pi}}\left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{i-1}{2}} \cdot \alpha, & \text{if } i \text{ is odd.} \end{cases} \tag{0.1}$$

We define now the equivariant eta form (cf. (1.62) and Definition 1.3).

Definition 0.1. Assume that $\dim\ker D^Z$ is locally constant on S. For any $g \in G$, the equivariant eta form of Bismut-Cheeger is defined by

$$\begin{aligned} &\tilde{\eta}_g(T^HW, g^{TZ}, h^{L_Z}, h^E, \nabla^{L_Z}, \nabla^E) \\ &:= \begin{cases} \displaystyle\int_0^\infty \frac{1}{2\sqrt{-1}\sqrt{\pi}}\psi_S \operatorname{Tr}_s\left[g\frac{\partial B_t}{\partial t}\exp(-B_t^2)\right]dt, & \text{if } n \text{ is even;} \\ \displaystyle\int_0^\infty \frac{1}{\sqrt{\pi}}\psi_S \operatorname{Tr}^{\mathrm{even}}\left[g\frac{\partial B_t}{\partial t}\exp(-B_t^2)\right]dt, & \text{if } n \text{ is odd.} \end{cases} \end{aligned} \tag{0.2}$$

The regularities of the integral in the right hand side of (0.2) are proved in Section 1.4. Let W^g be the fixed point set of g on W. Then W^g is a submanifold of W and the restriction of π on W^g gives a fibration $\pi : W^g \to S$ with fiber Z^g. Furthermore, it verifies the following transgression.

$$\begin{aligned} &d^S\tilde{\eta}_g(T^HW, g^{TZ}, h^{L_Z}, h^E, \nabla^{L_Z}, \nabla^E) \\ &= \begin{cases} \displaystyle\int_{Z^g} \widehat{\mathrm{A}}_g(TZ, \nabla^{TZ}) \wedge \mathrm{ch}_g(L_Z^{1/2}, \nabla^{L_Z^{1/2}}) \wedge \mathrm{ch}_g(E, \nabla^E) \\ \qquad\qquad - \mathrm{ch}_g(\ker D^Z, \nabla^{\ker D^Z}), & \text{if } n \text{ is even;} \\ \displaystyle\int_{Z^g} \widehat{\mathrm{A}}_g(TZ, \nabla^{TZ}) \wedge \mathrm{ch}_g(L_Z^{1/2}, \nabla^{L_Z^{1/2}}) \wedge \mathrm{ch}_g(E, \nabla^E), & \text{if } n \text{ is odd.} \end{cases} \end{aligned} \tag{0.3}$$

For the definition of characteristic forms in (0.3), see (1.44), (1.45) and (1.56).

By (0.2), the equivariant eta form depends on the geometric data $(T^HW, g^{TZ}, h^{L_Z}, h^E, \nabla^{L_Z}, \nabla^E)$. When the geometric data vary, we have the anomaly formula for the equivariant eta forms.

Theorem 0.2. *Assume that there exists a smooth path connecting $(T^HW, g^{TZ}, h^{L_Z}, h^E, \nabla^{L_Z}, \nabla^E)$ and $(T'^HW, g'^{TZ}, h'^{L_Z}, h'^E, \nabla'^{L_Z}, \nabla'^E)$ such that the dimension of the kernel of the Dirac family is constant (see Assumption 1.6).*

i) When n is odd, modulo exact forms on S, we have

$$
\begin{aligned}
(0.4)\quad & \tilde{\eta}_g(T'^HW, g'^{TZ}, h'^{L_Z}, h'^E, \nabla'^{L_Z}, \nabla'^E) - \tilde{\eta}_g(T^HW, g^{TZ}, h^{L_Z}, h^E, \nabla^{L_Z}, \nabla^E) \\
&= \int_{Z^g} \widetilde{\widetilde{\widehat{\mathrm{A}}}}_g(TZ, \nabla^{TZ}, \nabla'^{TZ}) \wedge \mathrm{ch}_g(L_Z^{1/2}, \nabla^{L_Z^{1/2}}) \wedge \mathrm{ch}_g(E, \nabla^E) \\
&+ \int_{Z^g} \widehat{\mathrm{A}}_g(TZ, \nabla'^{TZ}) \wedge \widetilde{\mathrm{ch}}_g(L_Z^{1/2}, \nabla^{L_Z^{1/2}}, \nabla'^{L_Z^{1/2}}) \wedge \mathrm{ch}_g(E, \nabla^E) \\
&\qquad + \int_{Z^g} \widehat{\mathrm{A}}_g(TZ, \nabla'^{TZ}) \wedge \mathrm{ch}_g(L_Z^{1/2}, \nabla'^{L_Z^{1/2}}) \wedge \widetilde{\mathrm{ch}}_g(E, \nabla^E, \nabla'^E).
\end{aligned}
$$

ii) When n is even, modulo exact forms on S, we have

$$
\begin{aligned}
(0.5)\quad & \tilde{\eta}_g(T'^HW, g'^{TZ}, h'^{L_Z}, h'^E, \nabla'^{L_Z}, \nabla'^E) - \tilde{\eta}_g(T^HW, g^{TZ}, h^{L_Z}, h^E, \nabla^{L_Z}, \nabla^E) \\
&= \int_{Z^g} \widetilde{\widetilde{\widehat{\mathrm{A}}}}_g(TZ, \nabla^{TZ}, \nabla'^{TZ}) \wedge \mathrm{ch}_g(L_Z^{1/2}, \nabla^{L_Z^{1/2}}) \wedge \mathrm{ch}_g(E, \nabla^E) \\
&+ \int_{Z^g} \widehat{\mathrm{A}}_g(TZ, \nabla'^{TZ}) \wedge \widetilde{\mathrm{ch}}_g(L_Z^{1/2}, \nabla^{L_Z^{1/2}}, \nabla'^{L_Z^{1/2}}) \wedge \mathrm{ch}_g(E, \nabla^E) \\
&+ \int_{Z^g} \widehat{\mathrm{A}}_g(TZ, \nabla'^{TZ}) \wedge \mathrm{ch}_g(L_Z^{1/2}, \nabla'^{L_Z^{1/2}}) \wedge \widetilde{\mathrm{ch}}_g(E, \nabla^E, \nabla'^E) \\
&\qquad - \widetilde{\mathrm{ch}}_g(\ker D^Z, \nabla^{\ker D^Z}, \nabla'^{\ker D^Z}).
\end{aligned}
$$

For the definitions of the Chern-Simons forms $\widetilde{\widetilde{\widehat{\mathrm{A}}}}_g(TZ, \nabla^{TZ}, \nabla'^{TZ})$, $\widetilde{\mathrm{ch}}_g(L_Z^{1/2}, \nabla^{L_Z^{1/2}}, \nabla'^{L_Z^{1/2}})$ and $\widetilde{\mathrm{ch}}_g(\ker D^Z, \nabla^{\ker D^Z}, \nabla'^{\ker D^Z})$ used here, see (1.86).

For the reminder of this introduction, we shall consider the composition of two submersions.

Let W, V, S be smooth manifolds. Let $\pi_1 : W \to V$, $\pi_2 : V \to S$ be smooth submersions with closed oriented fiber X, Y. Then $\pi_3 = \pi_2 \circ \pi_1 : W \to S$ is a smooth submersion with closed oriented fiber Z. We have the diagram of fibrations:

$$
\begin{tikzcd}
X \arrow[r] & Z \arrow[r] \arrow[d, "\pi_1"'] & W \arrow[d, "\pi_1"'] \arrow[dr, "\pi_3"] & \\
& Y \arrow[r] & V \arrow[r, "\pi_2"] & S.
\end{tikzcd}
$$

Let TX, TY, TZ be the relative tangent bundles. We assume that TX and TY have the Spin^c structures with complex line bundles L_X and L_Y respectively. Then TZ have a Spin^c structure with a complex line bundle L_Z. We take the geometric data $(T_1^HW, g^{TX}, h^{L_X}, \nabla^{L_X})$, $(T_2^HV, g^{TY}, h^{L_Y}, \nabla^{L_Y})$ and $(T_3^HW, g^{TZ}, h^{L_Z}, \nabla^{L_Z})$ with respect to submersions π_1, π_2 and π_3 respectively. Let ${}^0\nabla^{TZ}$, ${}^0\nabla^{L_Z}$ be the connections on TZ,

L_Z defined in (2.4), (2.5). For any $g \in G$, let $T_1^H(W|_{V^g}) = T_1^H W|_{V^g} \cap T(W|_{V^g})$ be the horizontal subbundle of $T(W|_{V^g})$.

Let G be a compact Lie group which acts on W such that for any $g \in G$, $g \cdot \pi_1 = \pi_1 \cdot g$ and $\pi_3 \cdot g = \pi_3$. We assume that the action of G preserves the Spin^c structures of TX, TY, TZ and all metrics and connections are G-invariant.

The purpose of this paper is to establish the following result, which we state as Theorem 2.4.

Theorem 0.3. *If Assumption 2.1 and 2.3 hold, for any $g \in G$, we have the following identity in $\Omega^*(S)/d^S\Omega^*(S)$,*

$$
\begin{aligned}
(0.6)\quad \tilde{\eta}_g(T_3^H W, g^{TZ}, h^{L_Z}, \nabla^{L_Z}) &= \tilde{\eta}_g(T_2^H V, g^{TY}, h^{L_Y}, h^{\ker D^X}, \nabla^{L_Y}, \nabla^{\ker D^X}) \\
&\quad + \int_{Y^g} \widehat{\mathrm{A}}_g(TY, \nabla^{TY}) \wedge \mathrm{ch}_g(L_Y^{1/2}, \nabla^{L_Y^{1/2}}) \wedge \tilde{\eta}_g(T_1^H(W|_{V^g}), g^{TX}, h^{L_X}, \nabla^{L_X}) \\
&\quad + \int_{Z^g} \widetilde{\widehat{\mathrm{A}}}_g(TZ, \nabla^{TZ}, {}^0\nabla^{TZ}) \wedge \mathrm{ch}_g(L_Z^{1/2}, \nabla^{L_Z^{1/2}}) \\
&\quad + \int_{Z^g} \widehat{\mathrm{A}}_g(TZ, {}^0\nabla^{TZ}) \wedge \widetilde{\mathrm{ch}}_g(L_Z^{1/2}, \nabla^{L_Z^{1/2}}, \nabla'^{L_Z^{1/2}}).
\end{aligned}
$$

Note that if $\ker D^Z$ is not locally constant, we can also construct an equivariant eta form when $\mathrm{ind}(D^Z) = 0 \in K_G^*(S)$ using the spectral section technique [29]. The functoriality of equivariant eta forms in this case is almost the same as Theorem 0.3. We will construct the push-forward map for equivariant differential K-theory [17] in a comparative paper [23] as applications of the result in this paper.

This paper is organized as follows.

In Section 1, we define the equivariant eta form and prove the anomaly formula Theorem 0.2. In Section 2, we state our main result Theorem 0.3. In Section 3, we use some intermediate results, whose proofs are delayed to Section 4-8, to prove Theorem 0.3. Section 4-8 are devoted to the proofs of the intermediate results stated in Section 3.

To simplify the notations, we use the Einstein summation convention in this paper.

In the whole paper, we use the superconnection formalism of Quillen [30]. If A is a $\mathbb{Z}_2$-graded algebra, and if $a, b \in A$, then we will note $[a,b]$ as the supercommutator of a, b. If B is another $\mathbb{Z}_2$-graded algebra, we will note $A\widehat{\otimes}B$ as the $\mathbb{Z}_2$-graded tensor product. If A, B are not $\mathbb{Z}_2$-graded, sometimes, we also denote $A\widehat{\otimes}B$ by considering the whole algebra as the even part.

For a trace class operator P acting on a space E, if $E = E_+ \oplus E_-$ is a $\mathbb{Z}_2$-graded space, we denote by

$$
(0.7)\qquad \mathrm{Tr}_s[P] = \mathrm{Tr}|_{E_+}[P] - \mathrm{Tr}|_{E_-}[P].
$$

If $\mathrm{Tr}[P]$ takes value in differential forms, we denote by $\mathrm{Tr}^{\mathrm{odd/even}}[P]$ the part of $\mathrm{Tr}[P]$ which takes value in odd or even forms. We denote by

$$
(0.8)\qquad \widetilde{\mathrm{Tr}}[P] = \begin{cases} \mathrm{Tr}_s[P], & \text{if } E \text{ is } \mathbb{Z}_2\text{-graded;} \\ \mathrm{Tr}^{\mathrm{odd}}[P], & \text{if } E \text{ is not } \mathbb{Z}_2\text{-graded.} \end{cases}
$$

For a vector bundle $\pi : W \to S$, we will often use the integration of the differential forms along the fiber Z in this paper. Since the fibers may be odd dimensional, we must make precise our sign conventions. If α is a differential form on W which in local coordinates is given by

$$\alpha = dy^{p_1} \wedge \cdots \wedge dy^{p_q} \wedge \beta(x)dx^1 \wedge \cdots \wedge dx^n, \tag{0.9}$$

we set

$$\int_Z \alpha = dy^{p_1} \wedge \cdots \wedge dy^{p_q} \int_Z \beta(x)dx^1 \wedge \cdots \wedge dx^n. \tag{0.10}$$

1. Equivariant eta form

The purpose of this section is to define the equivariant eta form and prove the anomaly formula. In Section 1.1, we recall elementary results on Clifford algebras of arbitrary dimension. In Section 1.2, we describe the geometry of fibration and introduce the Bismut superconnection and Bismut's Lichnerowicz formula (cf. [4, 6]). In Section 1.3, we explain the equivariant family local index theorem. In Section 1.4, we define the equivariant eta form when the dimension of the kernel of Dirac operators is locally constant. In Section 1.5, we prove the anomaly formula. In this section, we follow mainly from [9].

1.1. **Clifford algebras.** Let $C(V^n)$ denote the complex Clifford algebra of the real inner product space, V^n. Related to an orthonormal basis, $\{e_i\}$, $C(V^n)$ is defined by the relations

$$e_ie_j + e_je_i = -2\delta_{ij}. \tag{1.1}$$

To avoid ambiguity, we denote by $c(e_i)$ the element of $C(V^n)$ corresponding to e_i. We consider the group Spin^c_n as a multiplicative subgroup of the group of units of $C(V^n)$. For the definition and the properties of the group Spin^c_n, see [22, Appendix D].

As a vector space,

$$C(V^n) \simeq \Lambda(V^n). \tag{1.2}$$

The Clifford multiplication on ΛV^n is exterior multiplication minus interior multiplication. The elements $c(e_I) = c(e_{i_1}) \cdots c(e_{i_j})$, $I = \{i_1, \cdots, i_j\} \subset \{1, \cdots, n\}$, $i_1 < \cdots < i_j$, form a basis for $C(V^n)$. Put $|I| = j$. The subspace $C_0(V^n)$, $C_1(V^n)$ spanned by those $c(e_I)$ with $|I|$ even (resp. odd) give $C(V^n)$ the structure of a $\mathbb{Z}_2$-graded algebra.

For $n = 2k$, even, up to isomorphism, $C(V^n)$ has a unique irreducible module, $\mathcal{S}_n$, which has dimension 2^k and is $\mathbb{Z}_2$-graded. In fact, $C(V^{2k}) \simeq \mathrm{End}(\mathcal{S}_{2k})$. If V is oriented, the element

$$\tau = (\sqrt{-1})^k c(e_1) \cdots c(e_{2k}) \tag{1.3}$$

is independent of the choice $\{e_i\}$ and satisfies

$$\tau^2 = 1. \tag{1.4}$$

Set $\mathcal{S}_{\pm,n} = \{s \in \mathcal{S}_n : \tau s = \pm s\}$. We write $\mathrm{Tr}_s[\cdot]$ for the supertrace of $C(V^{2k})$ on $\mathcal{S}_n$ defined as (0.7).

If $n=2k-1$ is odd, $C(V^n)$ has two inequivalent irreducible modules, each of dimension 2^{k-1}. For arbitrary n,

$$c(e_j)\to c(e_j)c(e_{n+1}) \tag{1.5}$$

defines an isomorphism, $C(V^n)\simeq C_0(V^n\oplus\mathbb{R})$. Thus, for n odd, we can regard $\mathcal{S}_{\pm,n+1}$ for $V^n\oplus\mathbb{R}$ as (inequivalent) modules over $C(V^n)$. However, they are equivalent when restricted to Spin^c_n. For V^{2k-1} oriented, the notation $\mathrm{Tr}[\,\cdot\,]$ refers to the representation $\mathcal{S}_{+,2k}$.

By [10, Lemma 1.22], if $n=2k$ is even, then

$$\mathrm{Tr}_s[c(e_I)]=\begin{cases}(-\sqrt{-1})^k2^k, & \text{if } I=\{1,\cdots,2k\};\\ 0, & \text{if } I\neq\{1,\cdots,2k\}.\end{cases} \tag{1.6}$$

If $n=2k-1$ is odd and $|I|\geq 1$,

$$\mathrm{Tr}[c(e_I)]=\begin{cases}(-\sqrt{-1})^k2^{k-1}, & \text{if } I=\{1,\cdots,2k-1\};\\ 0, & \text{if } I\neq\{1,\cdots,2k-1\}.\end{cases} \tag{1.7}$$

By (1.6) and (1.7), for n odd, the trace Tr behaves on the odd elements of $C(V^n)$ in exactly the same way as the supertrace Tr_s on the even elements of $C(V^n)$ for n even, i.e. we must saturate all the elements $c(e_1),\cdots,c(e_n)$ to get a non-zero trace or supertrace. It will be of utmost importance in the computations of the local index in Section 6. We set

$$\widetilde{c}_{V^n}=\begin{cases}\mathrm{Tr}_s[c(e_1)\cdots c(e_n)], & \text{if } n \text{ is even};\\ \mathrm{Tr}[c(e_1)\cdots c(e_n)], & \text{if } n \text{ is odd}.\end{cases} \tag{1.8}$$

Let W^m be another real inner product space with orthonormal basis $\{f_p\}$. Then as Clifford algebras,

$$C(V^n\oplus W^m)\simeq C(V^n)\widehat{\otimes}C(W^m). \tag{1.9}$$

By (1.6), (1.7) and (1.8), we have

$$\widetilde{c}_{V^n\oplus W^m}=\begin{cases}2\sqrt{-1}\,\widetilde{c}_{V^n}\cdot\widetilde{c}_{W^m}, & \text{if } n,\, m \text{ are odd};\\ \widetilde{c}_{V^n}\cdot\widetilde{c}_{W^m}, & \text{if others}.\end{cases} \tag{1.10}$$

Finally, we note the effect of scaling the inner product $\langle\cdot,\cdot\rangle$ on V. Fix any inner product, $\langle\cdot,\cdot\rangle$ and let $C_t(V)$ be the Clifford algebra associated to $t^{-1}\langle\cdot,\cdot\rangle$. Then the map $t^{1/2}V\to V$ provides a natural isomorphism $C_t(V)\simeq C(V)$. It also provides a natural isomorphism between the orthonormal frames $\{t^{1/2}e_i\}$ for $t^{-1}\langle\cdot,\cdot\rangle$ and $\{e_i\}$ for $\langle\cdot,\cdot\rangle$. Thus, the spinor $\mathcal{S}$ for $\langle\cdot,\cdot\rangle$ is also an irreducible module for $C_t(V)$ via the above isomorphism. In the sequel, if Z is a Riemannian Spin^c manifold, we will always assume that the space of spinors has been chosen independent of the scaling parameter of the metric.

1.2. **Bismut superconnection and Lichnerowicz formula.** Let $\pi : W \to S$ be a smooth submersion of smooth manifolds with closed oriented fiber Z, with $\dim Z = n$. Let $TZ = TW/S$ be the relative tangent bundle to the fibers Z.

Let T^HW be a horizontal subbundle of TW such that

$$TW = T^HW \oplus TZ. \tag{1.11}$$

The splitting (1.11) gives an identification

$$T^HW \cong \pi^* TS. \tag{1.12}$$

Let P^{TZ} be the projection

$$P^{TZ} : TW = T^HW \oplus TZ \to TZ. \tag{1.13}$$

Let g^{TZ}, g^{TS} be Riemannian metrics on TZ, TS. We equip $TW = T^HW \oplus TZ$ with the Riemannian metric

$$g^{TW} = \pi^* g^{TS} \oplus g^{TZ}. \tag{1.14}$$

Let ∇^{TW}, ∇^{TS} be the Levi-Civita connections on (W, g^{TW}), (S, g^{TS}). Set

$$\nabla^{TZ} = P^{TZ} \nabla^{TW} P^{TZ}. \tag{1.15}$$

Then ∇^{TZ} is a Euclidean connection on TZ. Let ${}^0\nabla^{TW}$ be the connection on $TW = T^HW \oplus TZ$ defined by

$${}^0\nabla^{TW} = \pi^* \nabla^{TS} \oplus \nabla^{TZ}. \tag{1.16}$$

Then ${}^0\nabla^{TW}$ preserves the metric g^{TW} in (1.14). Set

$$S = \nabla^{TW} - {}^0\nabla^{TW}. \tag{1.17}$$

Then S is a 1-form on W with values in antisymmetric elements of $\mathrm{End}(TW)$. By [6, Theorem 1.9], we know that ∇^{TZ} and the $(3,0)$ tensor $\langle S(\cdot)\cdot, \cdot\rangle$ only depend on (T^HW, g^{TZ}), where $\langle \cdot, \cdot \rangle = g^{TW}(\cdot, \cdot)$.

Let $C(TZ)$ be the Clifford algebra bundle of (TZ, g^{TZ}), whose fiber at $x \in W$ is the Clifford algebra $C(T_xZ)$ of the Euclidean space (T_xZ, g^{T_xZ}). We make the assumption that TZ has a Spin^c structure. Then there exists a complex line bundle L_Z over W such that $\omega_2(TZ) = c_1(L_Z) \mod (2)$. Let $\mathcal{S}(TZ, L_Z)$ be the fundamental complex spinor bundle for (TZ, L_Z), which has a smooth action of $C(TZ)$ (cf. [22, Appendix D.9]). Locally, the spinor $\mathcal{S}(TZ, L_Z)$ may be written as

$$\mathcal{S}(TZ, L_Z) = \mathcal{S}(TZ) \otimes L_Z^{1/2}, \tag{1.18}$$

where $\mathcal{S}(TZ)$ is the fundamental spinor bundle for the (possibly non-existent) spin structure on TZ, and $L_Z^{1/2}$ is the (possibly non-existent) square root of L_Z. Let h^{L_Z} be the Hermitian metric on L_Z and ∇^{L_Z} be the Hermitian connection on (L_Z, h^{L_Z}). Let $h^{\mathcal{S}_Z}$ be the Hermitian metric on $\mathcal{S}(TZ, L_Z)$ induced by g^{TZ} and h^{L_Z} and $\nabla^{\mathcal{S}_Z}$ be the connection on $\mathcal{S}(TZ, L_Z)$ induced by ∇^{TZ} and ∇^{L_Z}. Then $\nabla^{\mathcal{S}_Z}$ is a Hermitian connection on $(\mathcal{S}(TZ, L_Z), h^{\mathcal{S}_Z})$. Moreover, it is a Clifford connection associated to ∇^{TZ}, i.e., for any $U \in TW$, $V \in \mathscr{C}^\infty(W, TZ)$,

$$\left[\nabla_U^{\mathcal{S}_Z}, c(V)\right] = c\left(\nabla_U^{TZ} V\right). \tag{1.19}$$

If $n = \dim Z$ is even, the spinor $\mathcal{S}(TZ, L_Z)$ is $\mathbb{Z}_2$-graded and the action of TZ exchanges the $\mathbb{Z}_2$-grading. Let (E, h^E) be a Hermitian vector bundle over W, and ∇^E a Hermitian connection on (E, h^E). Set

$$\nabla^{\mathcal{S}_Z\otimes E} = \nabla^{\mathcal{S}_Z} \otimes 1 + 1 \otimes \nabla^E. \tag{1.20}$$

Then $\nabla^{\mathcal{S}_Z\otimes E}$ is a Hermitian connection on $(\mathcal{S}(TZ, L_Z) \otimes E, h^{\mathcal{S}_Z} \otimes h^E)$.

Let $\{e_i\}$, $\{f_p\}$ be local orthonormal frames of TZ, TS and $\{e^i\}$, $\{f^p\}$ be the dual. Let D^Z be the fiberwise Dirac operator

$$D^Z = c(e_i)\nabla^{\mathcal{S}_Z\otimes E}_{e_i}. \tag{1.21}$$

For $b \in S$, let $\mathscr{E}_{Z,b}$ be the set of smooth sections over Z_b of $\mathcal{S}(TZ, L_Z) \otimes E$. As in [6], we will regard $\mathscr{E}_Z$ as an infinite dimensional fiber bundle over S.

Let dv_Z be the Riemannian volume element in the fiber Z. For any $b \in S$, $s_1, s_2 \in \mathscr{E}_{Z,b}$, we can define the scalar product

$$\langle s_1, s_2\rangle_0 = \int_{Z_b} \langle s_1(x), s_2(x)\rangle dv_Z. \tag{1.22}$$

This scalar product could be naturally extended on $\Lambda(T^*S)\widehat{\otimes}\mathscr{E}_Z$. We still denote it by $\langle\cdot,\cdot\rangle_0$.

If $U \in TS$, let $U^H \in T^HW$ be its horizontal lift in T^HW so that $\pi_*U^H = U$. For any $U \in TS$, $s \in \mathscr{C}^\infty(S, \mathscr{E}_Z) = \mathscr{C}^\infty(W, \mathcal{S}(TZ, L_Z) \otimes E)$, we set

$$\nabla^{\mathscr{E}_Z}_U s = \nabla^{\mathcal{S}_Z\otimes E}_{U^H} s. \tag{1.23}$$

Then $\nabla^{\mathscr{E}_Z}$ is a connection on $\mathscr{E}_Z$, but need not preserve the scalar product $\langle\cdot,\cdot\rangle_0$ in (1.22). By [12, Proposition 1.4], for $U \in TS$, the connection

$$\nabla^{\mathscr{E}_Z,u}_U := \nabla^{\mathscr{E}_Z}_U - \frac{1}{2}\langle S(e_i)e_i, U^H\rangle \tag{1.24}$$

preserves the scalar product $\langle\cdot,\cdot\rangle_0$.

Let T be the torsion of ${}^0\nabla^{TW}$. If $U_1, U_2 \in \mathscr{C}^\infty(S, TS)$, by [6, (1.30)], we have

$$T(U_1, U_2) = -P^{TZ}[U_1^H, U_2^H]. \tag{1.25}$$

We denote by

$$c(T) = \frac{1}{2} c\left(T(f_p^H, f_q^H)\right) f^p \wedge f^q \wedge . \tag{1.26}$$

By [6, (3.18)], the Bismut superconnection

$$B : \mathscr{C}^\infty(S, \Lambda(T^*S)\widehat{\otimes}\mathscr{E}_Z) \to \mathscr{C}^\infty(S, \Lambda(T^*S)\widehat{\otimes}\mathscr{E}_Z) \tag{1.27}$$

is defined by

$$B = D^Z + \nabla^{\mathscr{E}_Z,u} - \frac{1}{4}c(T). \tag{1.28}$$

In fact, the Bismut superconnection only depends on the quadruple $(T^HW, g^{TZ}, \nabla^{L_Z}, \nabla^E)$.

In the sequel, if $A(U)$ is any 0-order operator depending linearly on $U \in TW$, we define the operator

$$\left(\nabla^{\mathcal{S}_Z\otimes E}_{e_i} + A(e_i)\right)^2 \tag{1.29}$$

as follows: if $\{e_i(x)\}_{i=1}^n$ is any (locally defined) smooth orthonormal frame of TZ, then

$$\begin{aligned}(1.30)\quad &\left(\nabla_{e_i}^{\mathcal{S}_Z\otimes E} + A(e_i)\right)^2 \\ &:= \sum_{i=1}^n \left(\nabla_{e_i(x)}^{\mathcal{S}_Z\otimes E} + A(e_i(x))\right)^2 - \nabla^{\mathcal{S}_Z\otimes E}_{\sum_{i=1}^n \nabla_{e_i}^{TZ} e_i} - A\left(\sum_{i=1}^n \nabla_{e_i}^{TZ} e_i\right).\end{aligned}$$

Let R^{TZ}, R^{L_Z}, R^E and $R^{\mathcal{S}_Z\otimes E}$ be the curvatures of ∇^{TZ}, ∇^{L_Z}, ∇^E and $\nabla^{\mathcal{S}_Z\otimes E}$ respectively. By (1.18), we have

$$R^{\mathcal{S}_Z\otimes E} = \frac{1}{4}\langle R^{TZ}e_i, e_j\rangle c(e_i)c(e_j) + \frac{1}{2}R^{L_Z} + R^E. \tag{1.31}$$

For $t > 0$, we denote δ_t the operator on $\Lambda^i(T^*S)\widehat{\otimes}\mathscr{E}_Z$ by multiplying differential forms by $t^{-i/2}$. Set

$$B_t := \sqrt{t}\,\delta_t \circ B \circ \delta_t^{-1}. \tag{1.32}$$

Then from (1.28) and (1.32), we get

$$B_t = \sqrt{t}D^X + \nabla^{\mathscr{E}_Z,u} - \frac{1}{4\sqrt{t}}c(T). \tag{1.33}$$

Let K^Z be the scalar curvature of the fibers (TZ, g^{TZ}). We have the Bismut's Lichnerowicz formula (see [4, Theorem 10.17], [6, Theorem 3.5]),

$$\begin{aligned}(1.34)\quad B_t^2 = &-\left(\sqrt{t}\nabla_{e_i}^{\mathcal{S}_Z\otimes E} + \frac{1}{2}\langle S(e_i)e_j, f_p^H\rangle c(e_j) f^p\wedge + \frac{1}{4\sqrt{t}}\langle S(e_i)f_p^H, f_q^H\rangle f^p\wedge f^q\wedge\right)^2 \\ &+ \frac{t}{4}K^Z + \frac{t}{2}\left(\frac{1}{2}R^{L_Z} + R^E\right)(e_i, e_j)c(e_i)c(e_j) + \sqrt{t}\left(\frac{1}{2}R^{L_Z} + R^E\right)(e_i, f_p^H)c(e_i)f^p\wedge \\ &+ \frac{1}{2}\left(\frac{1}{2}R^{L_Z} + R^E\right)(f_p^H, f_q^H) f^p\wedge f^q\wedge.\end{aligned}$$

In particular, B_t^2 is a 2-order elliptic differential operator along the fiber Z. Let $\exp(-B_t^2)$ be the family of heat operators associated to the fiberwise elliptic operator B_t^2 in (1.34). From [4, Theorem 9.50], we know that $\exp(-B_t^2)$ is a smooth family of smoothing operators.

1.3. **Compact Lie group action and equivariant family local index theorem.** Let G be a compact Lie group which acts on W such that for any $g \in G$, $\pi\circ g = \pi$. So it acts trivially on S. We assume that the action of G preserves the splitting (1.11), the Spin^c structure of TZ and g^{TZ}, h^{L_Z}, ∇^{L_Z} are G-invariant. We assume that E is a G-equivariant complex vector bundle and h^E, ∇^E are G-invariant. So the action of G commutes with the Bismut superconnection B in (1.28).

Take $g \in G$ and set

$$W^g = \{x \in W : gx = x\}. \tag{1.35}$$

Then W^g is a submanifold of W and $\pi : W^g \to S$ is a fiber bundle with closed fiber Z^g. Let N denote the normal bundle of W^g in W, then $N = TZ/TZ^g$. Since G preserves the orientation of TZ, the normal bundle N is even dimensional. We denote the differential of g by dg which gives a bundle isometry $dg : N \to N$. Since g lies in a compact abelian

Lie group, we know that there is an orthonormal decomposition of smooth vector bundles on W^g

$$N = N(\pi) \oplus \oplus_{0<\theta<\pi} N(\theta), \tag{1.36}$$

where $dg|_{N(\pi)} = -\mathrm{id}$ and for each θ, $0 < \theta < \pi$, $N(\theta)$ is a complex vector bundle on which dg acts by multiplication by $e^{\sqrt{-1}\theta}$, and $\dim N(\pi)$ is even. By the following proposition, Z^g and N are all naturally oriented. This proposition is a modification of [4, Theorem 6.14].

Proposition 1.1. *Let Z be a closed oriented manifold and G be a compact Lie group. If TZ has a G-equivariant* Spin^c *structure, then for each $g \in G$, Z^g is naturally oriented.*

Proof. We fix a connected component of Z^g and assume that the dimension of the normal bundle N of this connected component is $2k$. By (1.36), on N, the matrix of g has diagonal blocks

$$\begin{pmatrix} \cos(\theta_j) & -\sin(\theta_j) \\ \sin(\theta_j) & \cos(\theta_j) \end{pmatrix}, \quad j = 1, 2, \cdots, k, \quad 0 < \theta_j \leq \pi. \tag{1.37}$$

By the definition of the Spin^c group, the action of g on the spinor is given by

$$g = \alpha \cdot \prod_{j=1}^{k} (\cos(\theta_j/2) + \sin(\theta_j/2) c(e_{2j-1}) c(e_{2j})), \tag{1.38}$$

where $\alpha \in S^1$. Let $\sigma : C(N) \to \Lambda(N)$ be the isomorphism in (1.2). For $\beta \in \Lambda(N)$, let $[\beta]_{2k}$ denote the degree $2k$ part of β. Since α and θ_j are locally constant on Z^g, the term

$$\alpha^{-1} [\sigma(g)]_{2k} = \left(\prod_{j=1}^{k} \sin(\theta_j/2) \right) e^1 \wedge \cdots \wedge e^{2k} \tag{1.39}$$

gives a non-zero section of $\Lambda^{2k}(N)$. Then it gives a canonical orientation of N. The canonical orientation of Z^g can be obtained by the orientations of Z and N.

The proof of Proposition 1.1 is complete. □

Since g^{TZ} is G-invariant, the connection ∇^{TZ} preserves the decomposition of smooth vector bundles on W^g

$$TZ|_{W^g} = TZ^g \oplus \oplus_{0<\theta\leq\pi} N(\theta). \tag{1.40}$$

Let ∇^{TZ^g}, ∇^N and $\nabla^{N(\theta)}$ be the corresponding induced connections on TZ^g, N and $N(\theta)$, and let R^{TZ^g}, R^N and $R^{N(\theta)}$ be the corresponding curvatures. Here we consider $N(\theta)$ as a real vector bundle. We have the decompositions on W^g:

$$\nabla^{TZ}|_{W^g} = \nabla^{TZ^g} \oplus \nabla^N, \quad \nabla^N = \oplus_{0<\theta\leq\pi} \nabla^{N(\theta)}, \tag{1.41}$$

and

$$R^{TZ}|_{W^g} = R^{TZ^g} \oplus R^N, \quad R^N = \oplus_{0<\theta\leq\pi} R^{N(\theta)}. \tag{1.42}$$

For $0 < \theta \leq \pi$, we write

$$\widehat{\mathrm{A}}_\theta \left(N(\theta), \nabla^{N(\theta)} \right) = \left((\sqrt{-1})^{\frac{1}{2} \dim_{\mathbb{R}} N(\theta)} \det{}^{\frac{1}{2}} \left(1 - g \exp \left(\frac{\sqrt{-1}}{2\pi} R^{N(\theta)} \right) \right) \right)^{-1}. \tag{1.43}$$

Set

$$
(1.44)\qquad
\begin{aligned}
\widehat{\mathrm{A}}\left(TZ^g,\nabla^{TZ^g}\right)&=\det{}^{\frac{1}{2}}\left(\frac{\frac{\sqrt{-1}}{4\pi}R^{TZ^g}}{\sinh\left(\frac{\sqrt{-1}}{4\pi}R^{TZ^g}\right)}\right),\\
\widehat{\mathrm{A}}_g(TZ,\nabla^{TZ})&=\widehat{\mathrm{A}}\left(TZ^g,\nabla^{TZ^g}\right)\cdot\prod_{0<\theta\leq\pi}\widehat{\mathrm{A}}_\theta\left(N(\theta),\nabla^{N(\theta)}\right)\in\Omega^{4*}(W^g,\mathbb{C}).
\end{aligned}
$$

Note that for any Euclidean connection ∇ on (TZ,g^{TZ}), we can define the characteristic form $\widehat{\mathrm{A}}_g(TZ,\nabla)$ as in (1.44). Let $\widehat{\mathrm{A}}_g(TZ)\in H^{4*}(W^g,\mathbb{C})$ denote the cohomology class of $\widehat{\mathrm{A}}_g(TZ,\nabla^{TZ})$. If E is $\mathbb{Z}_2$-graded, we assume that the G-action and ∇^E preserve the $\mathbb{Z}_2$-grading. Set

$$
(1.45)\qquad \mathrm{ch}_g(E,\nabla^E)=\begin{cases}\mathrm{Tr}\left[g\exp\left(\frac{\sqrt{-1}}{2\pi}R^E|_{W^g}\right)\right], & \text{if } E \text{ is not } \mathbb{Z}_2\text{-graded;}\\ \mathrm{Tr}_s\left[g\exp\left(\frac{\sqrt{-1}}{2\pi}R^E|_{W^g}\right)\right], & \text{if } E \text{ is } \mathbb{Z}_2\text{-graded.}\end{cases}
$$

Let $\mathrm{ch}_g(E)\in H^{2*}(W^g,\mathbb{C})$ denote the cohomology class of $\mathrm{ch}_g(E,\nabla^E)$. By Chern-Weil theory [35], the classes $\widehat{\mathrm{A}}_g(TZ)$ and $\mathrm{ch}_g(E)$ are independent of ∇^{TZ} and ∇^E. Furthermore, if S is compact, the equivariant Chern character in (1.45) descends to a ring homomorphism

$$
(1.46)\qquad \mathrm{ch}_g: K_G^0(W^g)\to H^{2*}(W^g,\mathbb{C}).
$$

Assume that n is even. If S is compact, the index bundle $\mathrm{ind}(D^Z)$ is an element of $K_G^0(S)$. Under the equivariant Chern character map (1.46), for any $g\in G$, we have

$$
(1.47)\qquad \mathrm{ch}_g(\mathrm{ind}(D^Z))\in H^{2*}(S,\mathbb{C}).
$$

Since the fiber is even-dimensional, the spinor $\mathcal{S}(TZ,L_Z)$ is $\mathbb{Z}_2$-graded, i.e., $\mathcal{S}(TZ,L_Z)=\mathcal{S}_+(TZ,L_Z)\oplus\mathcal{S}_-(TZ,L_Z)$. Note that if $\dim\ker D^Z$ is locally constant,

$$
(1.48)\qquad \mathrm{ind}(D^Z)=\ker D_+^Z-\ker D_-^Z\in K_G^0(S),
$$

where $D_\pm^Z$ is the restriction of D^Z on $\mathcal{S}_\pm(TZ,L_Z)\otimes E$.

Let $\mathscr{E}_{Z,\pm}$ be the set of smooth sections of $\mathcal{S}_\pm(TZ,L_Z)\otimes E$ over W. Then $\mathscr{E}_Z=\mathscr{E}_{Z,+}\oplus\mathscr{E}_{Z,-}$ is a $\mathbb{Z}_2$-graded infinite dimensional vector bundle over S and $\Lambda(T^*S)\widehat{\otimes}\,\mathrm{End}(\mathscr{E}_Z)$ is also $\mathbb{Z}_2$-graded. We extend Tr, Tr_s to the trace class element $A\in\Lambda(T^*S)\widehat{\otimes}\,\mathrm{End}(\mathscr{E}_Z)$, which take values in $\Lambda(T^*S)$. We use the convention that if $\omega\in\Lambda(T^*S)$,

$$
(1.49)\qquad \mathrm{Tr}[\omega A]=\omega\,\mathrm{Tr}[A],\quad \mathrm{Tr}_s[\omega A]=\omega\,\mathrm{Tr}_s[A].
$$

If n is odd, the fibrewise Dirac operator D^Z is a family of equivariant self-adjoint Fredholm operators. Set

$$
(1.50)\qquad D_\theta^Z=\begin{cases} I\cos\theta+\sqrt{-1}D^Z\sin\theta, & \text{if } 0\leq\theta\leq\pi;\\ (\cos\theta+\sqrt{-1}\sin\theta)I, & \text{if } \pi\leq\theta\leq 2\pi.\end{cases}
$$

If S is compact, then $\mathrm{ind}(\{D_\theta^Z\})\in K_G^0(S^1\times S)$. Since the restriction of D_θ^Z to $\{0\}\times S$ is trivial, so it can be regarded as an element of $K_G^1(S)$. From [3] and [31], the definition of the index of D^Z is

$$\operatorname{ind}(D^Z) := \operatorname{ind}(\{D^Z_\theta\}) \in K^1_G(S). \tag{1.51}$$

Note that as an analogue of (1.46), for any $g \in G$, there is a homomorphism

$$\operatorname{ch}_g : K^1_G(S) \to H^{\mathrm{odd}}(S, \mathbb{C}) \tag{1.52}$$

defined by the suspension. In our case,

$$\operatorname{ch}_g(\operatorname{ind}(D^Z)) = \left[\frac{1}{2\pi\sqrt{-1}} \int_{S^1} \operatorname{ch}_g(\operatorname{ind}(\{D^Z_\theta\}))\right] \in H^{\mathrm{odd}}(S, \mathbb{C}). \tag{1.53}$$

Here we use the sign convention (0.10) in this integration. The constant $(2\pi\sqrt{-1})^{-1}$ here is chosen to normalize the constant in Theorem 1.2.

When the fiber is odd dimensional, the spinor $\mathcal{S}(TZ, L_Z)$ is not $\mathbb{Z}_2$-graded. For a trace class element $A \in \Lambda(T^*S) \otimes \operatorname{End}(\mathscr{E}_Z)$, we also use the convention as in (1.49) that if $\omega \in \Lambda(T^*S)$,

$$\operatorname{Tr}[\omega A] = \omega \operatorname{Tr}[A]. \tag{1.54}$$

It is compatible with the sign convention in (0.10).

For $\alpha \in \Omega^i(S)$, set

$$\psi_S(\alpha) = \begin{cases} \left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{i}{2}} \cdot \alpha, & \text{if } i \text{ is even;} \\ \frac{1}{\sqrt{\pi}} \left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{i-1}{2}} \cdot \alpha, & \text{if } i \text{ is odd.} \end{cases} \tag{1.55}$$

Comparing with (1.45), for the locally defined line bundle $L_Z^{1/2}$, we write

$$\operatorname{ch}_g(L_Z^{1/2}, \nabla^{L_Z^{1/2}}) := g \cdot \exp\left(\frac{\sqrt{-1}}{4\pi} R^{L_Z}|_{W^g}\right) \in \Omega^{2*}(W^g, \mathbb{C}) \tag{1.56}$$

and $\operatorname{ch}_g(L_Z^{1/2}) \in H^{2*}(W^g, \mathbb{C})$ as the corresponding cohomology class. Denote by $\pi_* : H^*(W^g, \mathbb{C}) \to H^*(S, \mathbb{C})$ the integration along the fiber Z^g with the sign convention (0.10). Recall that the trace operator $\widetilde{\operatorname{Tr}}$ is defined in (0.8). We give the equivariant family local index theorem as follows.

Theorem 1.2. *For any $t > 0$ and $g \in G$, the differential form $\psi_S \widetilde{\operatorname{Tr}}[g \exp(-B_t^2)] \in \Omega^*(S)$ is closed and its cohomology class is independent of t. As $t \to 0$,*

$$\lim_{t \to 0} \psi_S \widetilde{\operatorname{Tr}}[g \exp(-B_t^2)] = \int_{Z^g} \widehat{\mathrm{A}}_g(TZ, \nabla^{TZ}) \wedge \operatorname{ch}_g(L_Z^{1/2}, \nabla^{L_Z^{1/2}}) \wedge \operatorname{ch}_g(E, \nabla^E). \tag{1.57}$$

If S is compact, the differential form $\psi_S \widetilde{\operatorname{Tr}}[g \exp(-B_t^2)]$ represents $\operatorname{ch}_g(\operatorname{ind}(D^Z))$ in (1.47) or (1.53). In $H^*(S, \mathbb{C})$,

$$\operatorname{ch}_g(\operatorname{ind}(D^Z)) = \pi_* \left\{\widehat{\mathrm{A}}_g(TZ) \operatorname{ch}_g(L_Z^{1/2}) \operatorname{ch}_g(E)\right\}. \tag{1.58}$$

Proof. If n is even, the proof is the same as that of [24, Theorem 1.1]. If n is odd, the proof follows from [13, Theorem 2.10] and the even case. ∎

1.4. **Equivariant eta form.** In this subsection, we define the equivariant eta form when $\dim\ker D^Z$ is locally constant. We will proceed as the proof of [4, Theorem 10.32] as follows.

Let $\widehat{S}=\mathbb{R}_+\times S$ and $\mathrm{pr}:\widehat{S}\to S$ be the projection. We consider the bundle $\widehat{\pi}:\widehat{W}:=\mathbb{R}_+\times W\to\widehat{S}$ together with the canonical projection $\mathrm{Pr}:\widehat{W}\to W$. Set $T^H\widehat{W}=T(\mathbb{R}_+)\oplus\mathrm{Pr}^*(T^HW)$. Then $T^H\widehat{W}$ is a horizontal subbundle of $T\widehat{W}$ as in (1.11). We fix the vertical metric $\widehat{g}^{TZ}$ which restricts to $t^{-1}g^{TZ}$ over $\{t\}\times W$. Let $\widehat{C}(TZ)$ be the Clifford algebra bundle associated to $\widehat{g}^{TZ}$. Then $\widehat{\mathcal{S}}(TZ,\mathrm{Pr}^*L_Z)=\mathrm{Pr}^*\mathcal{S}(TZ,L_Z)$ is the spinor of $\widehat{C}(TZ)$ by the assumption in the end of Section 1.1. Let $h^{\widehat{L}_Z}=\mathrm{Pr}^*h^{L_Z}$ and $\nabla^{\widehat{L}_Z}=\mathrm{Pr}^*\nabla^{L_Z}$. Let $\widehat{E}=\mathrm{Pr}^*E$, $h^{\widehat{E}}=\mathrm{Pr}^*h^E$ and $\nabla^{\widehat{E}}=\mathrm{Pr}^*\nabla^E$. We naturally extend the G-actions to this case such that the G-action is identity on $\mathbb{R}_+\times S$. We will mark the objects associated to $(T^H\widehat{W},\widehat{g}^{TZ},h^{\widehat{L}_Z},h^{\widehat{E}},\nabla^{\widehat{L}_Z},\nabla^{\widehat{E}})$ by $\widehat{\ }$.

For $t\in\mathbb{R}_+$, the fiberwise Dirac operator $D^{\widehat{Z}}$ on $\{t\}\times Z$ is $t^{1/2}D^Z$. By (1.24), $\nabla^{\widehat{\mathscr{E}_Z},u}=\nabla^{\mathscr{E}_Z,u}-\frac{n}{4t}\frac{\partial}{\partial t}$. Since B_t in (1.33) is just the Bismut superconnection associated to $(T^HW,t^{-1}g^{TZ},\nabla^{L_Z},\nabla^E)$, from (1.28) and (1.33), the Bismut superconnection associated to $(T^H\widehat{W},\widehat{g}^{TZ},\nabla^{\widehat{L}_Z},\nabla^{\widehat{E}})$ is

$$\widehat{B}|_{(t,b)}=B_t+dt\wedge\frac{\partial}{\partial t}-\frac{n}{4t}dt, \tag{1.59}$$

for $(t,b)\in\widehat{S}$. Note that the extended G-action commutes with the Bismut superconnection $\widehat{B}$.

If $\alpha\in\Lambda(T^*(\mathbb{R}_+\times S))$, we can expand α in the form

$$\alpha=dt\wedge\alpha_0+\alpha_1,\quad \alpha_0,\alpha_1\in\Lambda(T^*S). \tag{1.60}$$

Set

$$[\alpha]^{dt}=\alpha_0. \tag{1.61}$$

For any $g\in G$, set

$$\gamma(t)=\begin{cases}-\dfrac{1}{2\sqrt{-1}\sqrt{\pi}}\psi_S\,\mathrm{Tr}_s\left[g\dfrac{\partial B_t}{\partial t}\exp(-B_t^2)\right], & \text{if } n \text{ is even;}\\[2ex] -\dfrac{1}{\sqrt{\pi}}\psi_S\,\mathrm{Tr}^{\mathrm{even}}\left[g\dfrac{\partial B_t}{\partial t}\exp(-B_t^2)\right], & \text{if } n \text{ is odd,}\end{cases} \tag{1.62}$$

and

$$r(t)=\psi_S\widetilde{\mathrm{Tr}}[g\exp(-B_t^2)]. \tag{1.63}$$

Then from Duhamel's principle, (1.55) and (1.59), we have

$$\psi_S\widetilde{\mathrm{Tr}}[g\exp(-\widehat{B}^2)]=dt\wedge\gamma(t)+r(t). \tag{1.64}$$

So

$$\gamma(t)=\psi_S\widetilde{\mathrm{Tr}}[g\exp(-\widehat{B}^2)]^{dt}. \tag{1.65}$$

For $u\in(0,+\infty)$, set $\widehat{B}_u=\sqrt{u}\delta_u\widehat{B}\delta_u^{-1}$. Similarly as in (1.64), we decompose

$$\psi_S\widetilde{\mathrm{Tr}}[g\exp(-\widehat{B}_u^2)]=dt\wedge\gamma(u,t)+r(u,t). \tag{1.66}$$

Take $t = 1$. Then

$$\left.\frac{\partial B_{ut}}{\partial t}\right|_{t=1} = u\frac{\partial B_u}{\partial u}. \tag{1.67}$$

So from (1.62), (1.63) and (1.67), we have

$$\gamma(u,1) = u\gamma(u), \quad r(u,1) = r(u). \tag{1.68}$$

From the asymptotic expansion of the heat kernel, when $u \to 0$, there exist $a_i(t) \in \Lambda(T^*(\mathbb{R}_+ \times S))$, $i \in \mathbb{N}$, such that

$$\psi_S \widetilde{\mathrm{Tr}}[g\exp(-\widehat{B}_u^2)] \sim \sum_{i=0}^{+\infty} a_i(t) u^{i/2}. \tag{1.69}$$

By Theorem 1.2, $r(0,t)$ exists and $a_0(t) = r(0,t)$. Take $t = 1$ in (1.69). By Theorem 1.2 and (1.63), we have

$$r(0) = \int_{Z^g} \widehat{\mathrm{A}}_g(TZ, \nabla^{TZ}) \wedge \mathrm{ch}_g(L_Z^{1/2}, \nabla^{L_Z^{1/2}}) \wedge \mathrm{ch}_g(E, \nabla^E). \tag{1.70}$$

From (1.66) and (1.68), we have

$$dt \wedge u\gamma(u) + r(u) - r(0) \sim \sum_{i=1}^{+\infty} a_i(1) u^{i/2}, \tag{1.71}$$

that is, when $u \to 0$,

$$\gamma(u) = O(u^{-1/2}). \tag{1.72}$$

Assume that $\dim\ker D^Z$ is locally constant, then $\ker D^Z$ forms a vector bundle over S. Let $P^{\ker D^Z} : \mathscr{E}_Z \to \ker D^Z$ be the orthogonal projection with respect to the scalar product in (1.22). Let

$$\nabla^{\ker D^Z} = P^{\ker D^Z} \nabla^{\mathscr{E},u} P^{\ker D^Z} \tag{1.73}$$

be a connection on the vector bundle $\ker D^Z$. For $b \in S$, $t \in (0,+\infty)$, $\ker(t^{1/2}D_b^Z) = \ker D_b^Z$. So $\ker D^{\widehat{Z}}$ forms a vector bundle over $\mathbb{R}_+ \times S$. As in (1.73), we can define the connection $\nabla^{\ker D^{\widehat{Z}}}$ on the vector bundle $\ker D^{\widehat{Z}}$. If n is even, $\ker D^Z$ and $\ker D^{\widehat{Z}}$ are $\mathbb{Z}_2$-graded. Since the curvature of $\nabla^{\widehat{\mathscr{E}},u}$ is trivial along $\mathbb{R}_+$, the equivariant Chern character $\mathrm{ch}_g(\ker D^{\widehat{Z}}, \nabla^{\ker D^{\widehat{Z}}})$ does not involve dt.

From [4, Theorem 9.19], which is also valid in odd dimensional fiber case, we know that when $u \to +\infty$,

$$\psi_S \widetilde{\mathrm{Tr}}[g\exp(-\widehat{B}_u^2)] = \begin{cases} \mathrm{ch}_g(\ker D^{\widehat{Z}}, \nabla^{\ker D^{\widehat{Z}}}) + O(u^{-1/2}), & \text{if } n \text{ is even;} \\ O(u^{-1/2}), & \text{if } n \text{ is odd,} \end{cases} \tag{1.74}$$

and

$$r(\infty) := \lim_{u\to\infty} r(u,1) = \begin{cases} \mathrm{ch}_g(\ker D^Z, \nabla^{\ker D^Z}), & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \tag{1.75}$$

Take $t = 1$ in (1.74). From (1.66), (1.68) and (1.75) we have

$$dt \wedge u\gamma(u) + r(u) - r(\infty) = O(u^{-1/2}). \tag{1.76}$$

By (1.63), (1.74) and (1.76), when $u \to +\infty$,

$$\gamma(u) = O(u^{-3/2}). \tag{1.77}$$

Definition 1.3. Assume that $\dim \ker D^Z$ is locally constant on S. For any $g \in G$, the equivariant eta form of Bismut-Cheeger $\tilde{\eta}_g(T^HW, g^{TZ}, h^{L_Z}, h^E, \nabla^{L_Z}, \nabla^E) \in \Omega^*(S)$ is defined by

$$\tilde{\eta}_g(T^HW, g^{TZ}, h^{L_Z}, h^E, \nabla^L, \nabla^E) := -\int_0^\infty \gamma(t)dt. \tag{1.78}$$

Note that by (1.72) and (1.77), the integral on the right hand side of (1.78) is convergent.

When $g = 1$, TZ is Spin, this equivariant eta form is just the usual eta form of Bismut-Cheeger defined in [9] and [20]. Note that the equivariant eta form here was also defined in [33] when TZ is Spin and n is odd.

From [6], we know that $\widetilde{\mathrm{Tr}}[g\exp(-B^2)]$ is a closed differential form. So

$$\left(dt \wedge \frac{\partial}{\partial t} + d^S\right)\psi_S\widetilde{\mathrm{Tr}}[g\exp(-\widehat{B}^2)] = 0, \quad d^S\psi_S\widetilde{\mathrm{Tr}}[g\exp(-B_t^2)] = 0. \tag{1.79}$$

By (1.63), (1.64) and (1.79), we have

$$d^S\gamma(t) = \frac{\partial r(t)}{\partial t}. \tag{1.80}$$

Then from (1.63), (1.70), (1.80) and Definition 1.3, we have

$$\begin{aligned} d^S\tilde{\eta}_g(T^HW, g^{TZ}, h^{L_Z}, h^E, \nabla^{L_Z}, \nabla^E) &= -\int_0^{+\infty} \frac{\partial r(t)}{\partial t}dt = r(0) - r(\infty) \\ &= \begin{cases} \displaystyle\int_{Z^g} \widehat{\mathrm{A}}_g(TZ, \nabla^{TZ}) \wedge \mathrm{ch}_g(L_Z^{1/2}, \nabla^{L_Z^{1/2}}) \wedge \mathrm{ch}_g(E, \nabla^E) \\ \qquad\qquad - \mathrm{ch}_g(\ker D^Z, \nabla^{\ker D^Z}), & \text{if } n \text{ is even;} \\ \displaystyle\int_{Z^g} \widehat{\mathrm{A}}_g(TZ, \nabla^{TZ}) \wedge \mathrm{ch}_g(L_Z^{1/2}, \nabla^{L_Z^{1/2}}) \wedge \mathrm{ch}_g(E, \nabla^E), & \text{if } n \text{ is odd.} \end{cases} \end{aligned} \tag{1.81}$$

Remark 1.4. If we fix the vertical metric $\widehat{g}^{TZ}$ which restricts to $t^{-2}g^{TZ}$ over $\{t\} \times W$ in the beginning of this subsection, as in (1.59), we have

$$\widehat{B}'|_{(t,b)} = B_{t^2} + dt \wedge \frac{\partial}{\partial t} - \frac{n}{4t^2}dt, \tag{1.82}$$

and

$$\begin{aligned} \gamma'(t) &= \psi_S\widetilde{\mathrm{Tr}}[g\exp(-\widehat{B}'^2)]^{dt} \\ &= \begin{cases} -\dfrac{1}{2\sqrt{-1}\sqrt{\pi}}\psi_S \mathrm{Tr}_s\left[g\dfrac{\partial B_{t^2}}{\partial t}\exp(-B_{t^2}^2)\right], & n \text{ is even;} \\ -\dfrac{1}{\sqrt{\pi}}\psi_S \mathrm{Tr}^{\mathrm{even}}\left[g\dfrac{\partial B_{t^2}}{\partial t}\exp(-B_{t^2}^2)\right], & n \text{ is odd.} \end{cases} \end{aligned} \tag{1.83}$$

After changing the variable, we still have

$$\tilde{\eta}_g(T^HW, g^{TZ}, h^L, h^E, \nabla^L, \nabla^E) := -\int_0^\infty \gamma'(t)dt. \tag{1.84}$$

Remark 1.5. The Spin^c condition used here is just to get an explicit local index representative in Theorem 1.2. In fact, Definition 1.3 can be extended to equivariant Clifford module case.

1.5. **Anomaly formula.** From the construction in Section 1.4, the equivariant eta form only depends on the sextuple $(T^HW, g^{TZ}, h^{L_Z}, h^E, \nabla^{L_Z}, \nabla^E)$. We now describe how $\tilde{\eta}_g(T^HW, g^{TZ}, h^{L_Z}, h^E, \nabla^{L_Z}, \nabla^E)$ depends on its arguments. Let $(T^HW, g^{TZ}, h^{L_Z}, h^E, \nabla^{L_Z}, \nabla^E)$ and $(T'^HW, g'^{TZ}, h'^{L_Z}, h'^E, \nabla'^{L_Z}, \nabla'^E)$ be two sextuples of geometric data. We will mark the objects associated to the second sextuple by $'$.

First, a horizontal subbundle on W is simply a splitting of the exact sequence

$$0 \to TZ \to TW \to \pi^*TS \to 0. \tag{1.85}$$

As the space of the splitting map is affine and G is compact, it follows that any pair of equivariant horizontal subbundles can be connected by a smooth path of horizontal distributions. Let $s \in [0,1]$ parametrize a smooth path $\{T^H_sW\}_{s\in[0,1]}$ such that $T^H_0W = T^HW$ and $T^H_1W = T'^HW$. Similarly, let g^{TZ}_s, $h^{L_Z}_s$ and h^E_s be the G-invariant metrics on TZ, L_Z and E, depending smoothly on $s \in [0,1]$, which coincide with g^{TZ}, h^{L_Z} and h^E at $s=0$ and with g'^{TZ}, h'^{L_Z} and h'^E at $s=1$. Let ∇ and ∇' be equivariant Euclidean connections on (TZ, g^{TZ}) and (TZ, g'^{TZ}). By the same reason, we can choose G-invariant connections ∇_s, $\nabla^{L_Z}_s$ and ∇^E_s on TZ, L_Z and E preserving g^{TZ}_s, $h^{L_Z}_s$ and h^E_s such that $\nabla_0 = \nabla$, $\nabla_1 = \nabla'$, $\nabla^{L_Z}_0 = \nabla^{L_Z}$, $\nabla^{L_Z}_1 = \nabla'^{L_Z}$, $\nabla^E_0 = \nabla^E$, $\nabla^E_1 = \nabla'^E$.

Let $\widetilde{S} = [0,1] \times S$ and $\mathrm{pr}' : \widetilde{S} \to S$ be the projection. We consider the bundle $\widetilde{\pi} : \widetilde{W} := [0,1] \times W \to \widetilde{S}$ together with the canonical projection $\mathrm{Pr}' : \widetilde{W} \to W$. Then $T^H\widetilde{W}_{(s,\cdot)} = \mathbb{R} \times T^H_sW$ defines a horizontal subbundle of $T\widetilde{W}$, and $T\widetilde{Z} := \mathrm{Pr}'^*TZ$, $\widetilde{L}_Z := \mathrm{Pr}'^*L_Z$ and $\widetilde{E} := \mathrm{Pr}'^*E$ are naturally equipped with metrics $g^{T\widetilde{Z}}$, $h^{\widetilde{L}_Z}$, $h^{\widetilde{E}}$ and connections $\widetilde{\nabla}$, $\nabla^{\widetilde{L}_Z}$, $\nabla^{\widetilde{E}}$. Then the fiberwise G action can be naturally extended to $\widetilde{\pi} : \widetilde{W} \to \widetilde{S}$ such that G acts as identity on $\widetilde{S}$ and $g^{T\widetilde{Z}}, h^{\widetilde{L}_Z}$, $h^{\widetilde{E}}$, $\widetilde{\nabla}$, $\nabla^{\widetilde{L}_Z}$, $\nabla^{\widetilde{E}}$ are G-invariant. Let $D^{\widetilde{Z}}$ be the fiberwise Dirac operator associated to $(T^H\widetilde{W}, g^{T\widetilde{Z}}, \nabla^{\widetilde{L}_Z}, \nabla^{\widetilde{E}})$.

Assumption 1.6. We assume that there exists such a smooth path such that $\ker D^{\widetilde{Z}}$ is locally constant.

Under Assumption 1.6, from (1.73), we can define the connection $\nabla^{\ker D^{\widetilde{Z}}}$ on $\ker D^{\widetilde{Z}}$. From [28, Theorem B.5.4], modulo exact forms, the Chern-Simons forms

$$\begin{aligned}
&\widetilde{\widehat{\mathrm{A}}}_g(TZ, \nabla, \nabla') := -\int_0^1 [\widehat{\mathrm{A}}_g(TZ, \widetilde{\nabla})]^{ds} ds, \\
&\widetilde{\mathrm{ch}}_g(L_Z^{1/2}, \nabla^{L_Z^{1/2}}, \nabla'^{L_Z^{1/2}}) := -\int_0^1 [\mathrm{ch}_g(\widetilde{L}_Z^{1/2}, \nabla^{\widetilde{L}_Z^{1/2}})]^{ds} ds, \\
&\widetilde{\mathrm{ch}}_g(E, \nabla^E, \nabla'^E) := -\int_0^1 [\mathrm{ch}_g(\widetilde{E}, \nabla^{\widetilde{E}})]^{ds} ds, \\
&\widetilde{\mathrm{ch}}_g(\ker D^Z, \nabla^{\ker D^Z}, \nabla'^{\ker D^Z}) := -\int_0^1 [\mathrm{ch}_g(\ker D^{\widetilde{Z}}, \nabla^{\ker D^{\widetilde{Z}}})]^{ds} ds
\end{aligned} \tag{1.86}$$

do not depend on the choices of the objects with $\widetilde{}$. Moreover,

$$
\begin{aligned}
&d\widetilde{\widehat{\mathrm{A}}}_g(TZ,\nabla,\nabla')=\widehat{\mathrm{A}}_g(TZ,\nabla')-\widehat{\mathrm{A}}_g(TZ,\nabla),\\
&d\widetilde{\mathrm{ch}}_g(L_Z^{1/2},\nabla^{L_Z^{1/2}},\nabla'^{L_Z^{1/2}})=\mathrm{ch}_g(L_Z^{1/2},\nabla'^{L_Z^{1/2}})-\mathrm{ch}_g(L_Z^{1/2},\nabla^{L_Z^{1/2}}),\\
&d\widetilde{\mathrm{ch}}_g(E,\nabla^E,\nabla'^E)=\mathrm{ch}_g(E,\nabla'^E)-\mathrm{ch}_g(E,\nabla^E),\\
&d\widetilde{\mathrm{ch}}_g(\ker D^Z,\nabla^{\ker D^Z},\nabla'^{\ker D^Z})=\mathrm{ch}_g(\ker D^Z,\nabla'^{\ker D^Z})-\mathrm{ch}_g(\ker D^Z,\nabla^{\ker D^Z}).
\end{aligned}
\tag{1.87}
$$

Now we can obtain the anomaly formula for the equivariant eta forms.

Theorem 1.7. *Assume that Assumption 1.6 holds.*

i) When n is odd, modulo exact forms on S, we have

$$
\begin{aligned}
&\tilde{\eta}_g(T'^HW,g'^{TZ},h'^{L_Z},h'^E,\nabla'^{L_Z},\nabla'^E)-\tilde{\eta}_g(T^HW,g^{TZ},h^{L_Z},h^E,\nabla^{L_Z},\nabla^E)\\
&=\int_{Z^g}\widetilde{\widehat{\mathrm{A}}}_g(TZ,\nabla^{TZ},\nabla'^{TZ})\wedge\mathrm{ch}_g(L_Z^{1/2},\nabla^{L_Z^{1/2}})\wedge\mathrm{ch}_g(E,\nabla^E)\\
&+\int_{Z^g}\widehat{\mathrm{A}}_g(TZ,\nabla'^{TZ})\wedge\widetilde{\mathrm{ch}}_g(L_Z^{1/2},\nabla^{L_Z^{1/2}},\nabla'^{L_Z^{1/2}})\wedge\mathrm{ch}_g(E,\nabla^E)\\
&\qquad+\int_{Z^g}\widehat{\mathrm{A}}_g(TZ,\nabla'^{TZ})\wedge\mathrm{ch}_g(L_Z^{1/2},\nabla'^{L_Z^{1/2}})\wedge\widetilde{\mathrm{ch}}_g(E,\nabla^E,\nabla'^E).
\end{aligned}
\tag{1.88}
$$

ii) When n is even, modulo exact forms on S, we have

$$
\begin{aligned}
&\tilde{\eta}_g(T'^HW,g'^{TZ},h'^{L_Z},h'^E,\nabla'^{L_Z},\nabla'^E)-\tilde{\eta}_g(T^HW,g^{TZ},h^{L_Z},h^E,\nabla^{L_Z},\nabla^E)\\
&=\int_{Z^g}\widetilde{\widehat{\mathrm{A}}}_g(TZ,\nabla^{TZ},\nabla'^{TZ})\wedge\mathrm{ch}_g(L_Z^{1/2},\nabla^{L_Z^{1/2}})\wedge\mathrm{ch}_g(E,\nabla^E)\\
&+\int_{Z^g}\widehat{\mathrm{A}}_g(TZ,\nabla'^{TZ})\wedge\widetilde{\mathrm{ch}}_g(L_Z^{1/2},\nabla^{L_Z^{1/2}},\nabla'^{L_Z^{1/2}})\wedge\mathrm{ch}_g(E,\nabla^E)\\
&+\int_{Z^g}\widehat{\mathrm{A}}_g(TZ,\nabla'^{TZ})\wedge\mathrm{ch}_g(L_Z^{1/2},\nabla'^{L_Z^{1/2}})\wedge\widetilde{\mathrm{ch}}_g(E,\nabla^E,\nabla'^E)\\
&\qquad-\widetilde{\mathrm{ch}}_g(\ker D^Z,\nabla^{\ker D^Z},\nabla'^{\ker D^Z}).
\end{aligned}
\tag{1.89}
$$

Proof. Let $\widetilde{B}$ be the Bismut superconnection associated to $(T^H\widetilde{W},\widetilde{g}^{TZ},h^{\widetilde{L_Z}},\nabla^{\widetilde{L_Z}},\nabla^{\widetilde{E}})$. From (1.59),

$$
\widehat{\widetilde{B}}=\widetilde{B}_t+dt\wedge\frac{\partial}{\partial t}-\frac{n}{4t}dt
\tag{1.90}
$$

is the Bismut superconnection associated to the fibration $(0,+\infty)\times[0,1]\times W\to(0,+\infty)\times[0,1]\times S$. We decompose

$$
\psi_S\widetilde{\mathrm{Tr}}[g\exp(-\widehat{\widetilde{B}}^2)]=dt\wedge\gamma+ds\wedge r'+dt\wedge ds\wedge r''+r''',
\tag{1.91}
$$

where γ,r',r'',r''' do not contain dt neither ds and by (1.63),

$$
r'(t,s)=\psi_S\widetilde{\mathrm{Tr}}[g\exp(-\widetilde{B}_t^2)]^{ds}|_{(t,s)}.
\tag{1.92}
$$

From (1.65), (1.91) and Definition 1.3, we have

$$\tilde{\eta}_g(T_s^H W, g_s^{TZ}, h_s^{L_Z}, h_s^E, \nabla_s^L, \nabla_s^E) := -\int_0^\infty \gamma(t,s)dt. \tag{1.93}$$

Since $(dt \wedge \frac{\partial}{\partial t} + ds \wedge \frac{\partial}{\partial s} + d^S)\psi_S \widetilde{\mathrm{Tr}}[g\exp(-\widehat{\widetilde{B}}^{\,2})] = 0$, we have

$$\frac{\partial \gamma}{\partial s} = \frac{\partial r'}{\partial t} + d^S r''. \tag{1.94}$$

From (1.93), we have

$$\begin{aligned}
&\tilde{\eta}_g(T'^H W, g'^{TZ}, h'^{L_Z}, h'^E, \nabla'^{L_Z}, \nabla'^E) - \tilde{\eta}_g(T^H W, g^{TZ}, h^{L_Z}, h^E, \nabla^{L_Z}, \nabla^E)\\
&= \int_0^{+\infty} (\gamma(t,1) - \gamma(t,0))dt = \int_0^{+\infty}\int_0^1 \frac{\partial}{\partial s}\gamma(t,s)dtds\\
&= \int_0^{+\infty}\int_0^1 \frac{\partial}{\partial t} r'(t,s)dtds + d^S \int_0^{+\infty}\int_0^1 r''(t,s)dtds\\
&= -\int_0^1 (r'(0,s) - r'(\infty,s))ds + d^S \int_0^{+\infty}\int_0^1 r''(t,s)dtds.
\end{aligned} \tag{1.95}$$

The commutative property among derivative and integrals in the above formula is granted by (1.72) and (1.77).

Let $\nabla^{T\widetilde{Z}}$ be the Euclidean connection associated to $(T^H\widetilde{W}, g^{T\widetilde{Z}})$ as in (1.15). By (1.70), (1.75) and (1.92), we have

$$r'(0,s) = \left\{\int_{Z^g} \widehat{A}_g(TZ, \nabla^{T\widetilde{Z}}) \wedge \mathrm{ch}_g(\widetilde{L}_Z^{1/2}, \nabla^{\widetilde{L}_Z^{1/2}}) \wedge \mathrm{ch}_g(\widetilde{E}, \nabla^{\widetilde{E}})\right\}^{ds}\Bigg|_{\{s\}\times S} \tag{1.96}$$

and

$$r'(\infty,s) = \begin{cases} \{\mathrm{ch}_g(\ker D^{\widetilde{Z}}, \nabla^{\ker D^{\widetilde{Z}}})\}^{ds}|_{\{s\}\times S}, & \text{if } n \text{ is even};\\ 0, & \text{if } n \text{ is odd}. \end{cases} \tag{1.97}$$

Then Theorem 1.7 follows from (1.86), (1.95), (1.96) and (1.97).

The proof of Theorem 1.7 is complete. □

2. Functoriality of equivariant eta forms

In this section, we state our main result.

2.1. **Functoriality of equivariant eta forms.** Let W, V, S be smooth manifolds. Let $\pi_1 : W \to V$, $\pi_2 : V \to S$ be smooth fibrations with closed oriented fibers X, Y, with $\dim X = n - m$, $\dim Y = m$. Then $\pi_3 = \pi_2 \circ \pi_1 : W \to S$ is a smooth fibration with closed oriented fiber Z with $\dim Z = n$. Then we have the diagram of smooth fibrations:

$$\begin{tikzcd}
X \arrow[r] & Z \arrow[r] \arrow[d] & W \arrow[d, "\pi_1"'] \arrow[dr, "\pi_3"] & \\
 & Y \arrow[r] & V \arrow[r, "\pi_2"] & S.
\end{tikzcd}$$

Let TX, TY, TZ be the relative tangent bundles. We assume that TX and TY have the Spin^c structures with complex line bundles L_X and L_Y respectively. Let

$$L_Z = \pi_1^*(L_Y) \otimes L_X. \tag{2.1}$$

Then TZ have a Spin^c structure with complex line bundle L_Z. Recall the notations in Section 1, we take quadruples $(T_1^H W, g^{TX}, h^{L_X}, \nabla^{L_X})$, $(T_2^H V, g^{TY}, h^{L_Y}, \nabla^{L_Y})$ and $(T_3^H W$, g^{TZ}, h^{L_Z}, $\nabla^{L_Z})$ with respect to fibrations π_1, π_2 and π_3 respectively. Then we can define connections ∇^{TX}, ∇^{TY}, ∇^{TZ}, fundamental spinors $\mathcal{S}(TX, L_X)$, $\mathcal{S}(TY, L_Y)$, $\mathcal{S}(TZ, L_Z)$, metrics $h^{\mathcal{S}_X}$, $h^{\mathcal{S}_Y}$, $h^{\mathcal{S}_Z}$ and connections $\nabla^{\mathcal{S}_X}$, $\nabla^{\mathcal{S}_Y}$, $\nabla^{\mathcal{S}_Z}$ as in Section 1.2. If $U \in TS$, $U' \in TV$, let $U_1'^H \in T_1^H W$, $U_2^H \in T_2^H V$, $U_3^H \in T_3^H W$ be the horizontal lifts of U', U, U, so that $\pi_{1,*}(U_1'^H) = U'$, $\pi_{2,*}(U_2^H) = U$, $\pi_{3,*}(U_3^H) = U$.

Set $T^H Z := T_1^H W \cap TZ$. Then we have the splitting of smooth vector bundles over W,

$$TZ = T^H Z \oplus TX, \tag{2.2}$$

and

$$T^H Z \cong \pi_1^* TY. \tag{2.3}$$

Let ${}^0\nabla^{TZ}$ be the connection on $TZ = T^H Z \oplus TX$ defined by

$${}^0\nabla^{TZ} = \pi^* \nabla^{TY} \oplus \nabla^{TX} \tag{2.4}$$

as in (1.16). Set

$${}^0\nabla^{L_Z} = \pi_1^* \nabla^{L_Y} \otimes 1 + 1 \otimes \nabla^{L_X}. \tag{2.5}$$

Let D^X and D^Z be the Dirac operators associated to $(T_1^H W, g^{TX}, \nabla^{L_X})$ and $(T_3^H W$, g^{TZ}, $\nabla^{L_Z})$. For $v \in V$, let $\mathscr{E}_{X,v}$ be the set of smooth sections over X_v of $\mathcal{S}(TX, L_X)$. We still regard $\mathscr{E}_X$ as an infinite dimensional fiber bundle over V. For any $v \in V$, $s_1, s_2 \in \mathscr{E}_{X,v}$, as in (1.22), we define the scalar product

$$\langle s_1, s_2 \rangle_{\mathscr{E}_{X,v}} = \int_{X_v} \langle s_1(x), s_2(x) \rangle_X \, dv_X, \tag{2.6}$$

where $\langle \cdot, \cdot \rangle_X = h^{\mathcal{S}_X}(\cdot, \cdot)$. Let $\{e_i\}$ be a local orthonormal frame of (TX, g^{TX}). As in (1.23) and (1.24), for $U \in TV$, we set

$$\nabla_U^{\mathscr{E}_X, u} := \nabla_{U_1^H}^{\mathcal{S}_X} - \frac{1}{2} \langle S_1(e_i) e_i, U_1^H \rangle. \tag{2.7}$$

Then $\nabla^{\mathscr{E}_X, u}$ preserves the scalar product $\langle \cdot, \cdot \rangle_{\mathscr{E}_X}$.

We assume that $\ker D^X$ is locally constant. Then $\ker D^X$ forms a vector bundle over V. Let $P^{\ker D^X} : \mathscr{E}_X \to \ker D^X$ be the orthonomal projection with respect to the scalar product (2.6). Let $h^{\ker D^X}$ be the L^2 metric induced by $h^{\mathcal{S}_X}$ and

$$\nabla^{\ker D^X} := P^{\ker D^X} \nabla^{\mathscr{E}_X, u} P^{\ker D^X}. \tag{2.8}$$

Then $\nabla^{\ker D^X}$ preserves the metric $h^{\ker D^X}$. Let D^Y be the Dirac operator associated to $(T_2^H V, g^{TY}, \nabla^{\mathcal{S}_Y \otimes \ker D^X})$.

Assumption 2.1. We assume that the quadruples $(T_1^H W, g^{TX}, h^{L_X}, \nabla^{L_X})$ and $(T_2^H V$, g^{TY}, h^{L_Y}, $\nabla^{L_Y})$ satisfy the conditions that $\ker D^X$ is locally constant and $\ker D^Y = 0$.

Let G be a compact Lie group which acts on W such that for any $g \in G$, $g \cdot \pi_1 = \pi_1 \cdot g$ and $\pi_3 \cdot g = \pi_3$. Then we know that G acts as identity on S. We assume that the action of G preserves the Spin^c structures of TX, TY, TZ and the quadruples $(T_1^H W, g^{TX}, h^{L_X}, \nabla^{L_X})$, $(T_2^H V, g^{TY}, h^{L_Y}, \nabla^{L_Y})$, $(T_3^H W,\, g^{TZ},\, h^{L_Z},\, \nabla^{L_Z})$ are G-invariant.

On the other hand, we take another equivariant horizontal subbundle $T_3'^H W \subset TW$, which is complement of TZ, such that

$$T_3'^H W \subset T_1^H W. \tag{2.9}$$

Let g'^{TZ} be another metric on TZ such that

$$g'^{TZ} = \pi_1^* g^{TY} \oplus g^{TX}. \tag{2.10}$$

Let ∇'^{TZ} be the connection associated to $(T_3'^H W, g'^{TZ})$ as in (1.15).

Let $\mathcal{S}'(TZ, L_Z)$ be the fundamental spinor associated to (g'^{TZ}, L_Z). Then

$$\mathcal{S}'(TZ, L_Z) := \pi_1^* \mathcal{S}(TY, L_Y) \otimes \mathcal{S}(TX, L_X). \tag{2.11}$$

Set

$$h'^{L_Z} := \pi_1^* h^{L_Y} \otimes h^{L_X}. \tag{2.12}$$

Let

$$g_T'^{TZ} = \pi_1^* g^{TY} \oplus T^{-2} g^{TX}. \tag{2.13}$$

We denote the Clifford algebra bundle of TZ with respect to $g_T'^{TZ}$ by $C_T(TZ)$. Let $\{f_p\}$ be a local orthonormal frame of (TY, g^{TY}). Then $\{Te_i\} \cup \{f_{p,1}^H\}$ is a local orthonormal frame of $(TZ, g_T'^{TZ})$. We define a Clifford algebra isomorphism

$$\mathcal{G}_T : C_T(TZ) \to C(TZ) \tag{2.14}$$

by

$$\mathcal{G}_T(c(f_{p,1}^H)) = c(f_{p,1}^H), \quad \mathcal{G}_T(c_T(Te_i)) = c(e_i). \tag{2.15}$$

Under this isomorphism, we can consider $\mathcal{S}'(TZ, L_Z)$ in (2.11) as a spinor associated to $(TZ, g_T'^{TZ})$. Let D_T^Z be the fiberwise Dirac operator associated to $(T_3'^H W, g_T'^{TZ},\, {}^0\nabla^{L_Z})$.

Comparing with [20, Theorem 1.5], we can get the following lemma.

Lemma 2.2. *If Assumption 2.1 holds, there exists $T_0 \geq 1$, such that when $T \geq T_0$, $\ker D_T^Z = 0$.*

We will give another proof of this lemma in Section 4.3.

Now we state an analogue of Assumption 1.6 as follows.

Assumption 2.3. We assume that there exist an equivariant horizontal subbundle $T_3'^H W \subset TW$ satisfying (2.9) and a smooth path constructed as the argument before Assumption 1.6, connecting the quadruples $(T_3^H W,\, g^{TZ},\, h^{L_Z},\, \nabla^{L_Z})$ and $(T_3'^H W,\, g_{T_0}'^{TZ},\, h'^{L_Z},\, {}^0\nabla^{L_Z})$, such that $\ker(D^{\widetilde{Z}}) = 0$.

For any $g \in G$, let $T_1^H(W|_{V^g}) = T_1^H W|_{V^g} \cap T(W|_{V^g})$ be the equivariant horizontal subbundle of $T(W|_{V^g})$. We state our main result as follows.

Theorem 2.4. *If Assumption 2.1 and 2.3 hold, for any $g \in G$, we have the following identity in $\Omega^*(S)/d^S\Omega^*(S)$,*

$$
\begin{aligned}
(2.16)\quad \tilde{\eta}_g(T_3^H W, g^{TZ}, h^{L_Z}, \nabla^{L_Z}) &= \tilde{\eta}_g(T_2^H V, g^{TY}, h^{L_Y}, h^{\ker D^X}, \nabla^{L_Y}, \nabla^{\ker D^X}) \\
&+ \int_{Y^g} \widehat{\mathrm{A}}_g(TY, \nabla^{TY}) \wedge \mathrm{ch}_g(L_Y^{1/2}, \nabla^{L_Y^{1/2}}) \wedge \tilde{\eta}_g(T_1^H(W|_{V^g}), g^{TX}, h^{L_X}, \nabla^{L_X}) \\
&+ \int_{Z^g} \widetilde{\widehat{\mathrm{A}}}_g(TZ, \nabla^{TZ}, {}^0\nabla^{TZ}) \wedge \mathrm{ch}_g(L_Z^{1/2}, \nabla^{L_Z^{1/2}}) \\
&+ \int_{Z^g} \widehat{\mathrm{A}}_g(TZ, {}^0\nabla^{TZ}) \wedge \widetilde{\mathrm{ch}}_g(L_Z^{1/2}, \nabla^{L_Z^{1/2}}, {}^0\nabla^{L_Z^{1/2}}).
\end{aligned}
$$

2.2. **Simplifying assumptions.** By anomaly formula Theorem 1.7, we only need to prove Theorem 2.4 when $(T_3^H W,\ g^{TZ},\ h^{L_Z},\ \nabla^{L_Z})$=$(T_3^{H'} W, g_{T_0}^{'TZ}, h'^{L_Z}, \nabla'^{L_Z})$. Therefore, in the following sections, we assume that

$$
(2.17)\quad \begin{gathered} T_3^H W \subset T_1^H W, \quad g^{TZ} = g^{TX} \oplus \pi_1^* g^{TY}, \quad h^{L_Z} = \pi_1^* h^{L_Y} \otimes h^{L_X}, \\ \nabla^{L_Z} = \pi_1^* \nabla^{L_Y} \otimes 1 + 1 \otimes \nabla^{L_X}. \end{gathered}
$$

Let

$$
(2.18)\quad g_T^{TZ} = \pi_1^* g^{TY} \oplus \frac{1}{T^2} g^{TX}
$$

and D_T^Z be the fiberwise Dirac operator associated to $(T_3^H W, g_T^{TZ}, \nabla^{L_Z})$. We assume that $\ker D^X$ is locally constant, $\ker D^Y = 0$ and for any $T \geq 1$, $\ker D_T^Z = 0$.

3. Proof of Theorem 2.4

In this section, we use the assumptions and the notations in Section 2.2.

This Section is organized as follows. In Section 3.1, we introduce a 1-form on $\mathbb{R}_+ \times \mathbb{R}_+$. In Section 3.2, we state some intermediate results which we need for the proof of Theorem 2.4, whose proofs are delayed to Section 4-8. In Section 3.3, we prove Theorem 2.4.

3.1. **A fundamental 1-form.** Let ∇_T^{TZ} be the connection associated to $(T_3^H W, g_T^{TZ})$ as in (1.15). Let $S_{1,T}$ be the tensor associated to $(T_1^H W, T^{-2} g^{TX})$ as in (1.17). Comparing with [6, (3.10)] and [27, Theorem 5.1], we have

$$
(3.1)\quad \nabla_T^{TZ} = {}^0\nabla^{TZ} + P^{TZ} S_{1,T} P^{TZ} = {}^0\nabla^{TZ} + P^{TX} S_1 P^{T^H Z} + \frac{1}{T^2} P^{T^H Z} S_1 P^{TZ}.
$$

Let $\nabla^{\mathcal{S}_Z, T}$ be the connection on $\mathcal{S}(TZ, L_Z)$ induced by ∇_T^{TZ} and ∇^{L_Z}. Set

$$
(3.2)\quad {}^0\nabla^{\mathcal{S}_Z} := \pi_1^* \nabla^{\mathcal{S}_Y} \otimes 1 + 1 \otimes \nabla^{\mathcal{S}_X}.
$$

Then by (3.1),

$$
(3.3)\quad \nabla^{\mathcal{S}_Z, T} = {}^0\nabla^{\mathcal{S}_Z} + \frac{1}{2T} \langle S_1(\cdot) e_i, f_{p,1}^H \rangle c(e_i) c(f_{p,1}^H) + \frac{1}{4T^2} \langle S_1(\cdot) f_{p,1}^H, f_{q,1}^H \rangle c(f_{p,1}^H) c(f_{q,1}^H).
$$

As the construction in Section 1.4, We consider the space $\widehat{S} := \mathbb{R}_{+,T} \times \mathbb{R}_{+,u} \times S$. Let $\mathrm{pr}_S : \widehat{S} \to S$ denote the projection and consider the fibration $\widehat{\pi}_3 : \widehat{W} := \mathbb{R}_{+,T} \times \mathbb{R}_{+,u} \times W \to \widehat{S}$. Let $\mathrm{Pr}_W : \widehat{W} \to W$ be the canonical projection. Set $T^H \widehat{W} = T(\mathbb{R}_+ \times \mathbb{R}_+) \oplus \mathrm{Pr}_W^*(T_1^H W)$. Then $T^H \widehat{W}$ is a horizontal subbundle of $T\widehat{W}$ as in (1.11). We define the metric $\widehat{g}^{TZ}$ such that it restricts to $u^{-2}g_T^{TZ}$ over $(T,u)\times W$. Let $h^{\widehat{L}z}=\mathrm{Pr}_W^* h^{Lz}$ and $\nabla^{\widehat{L}z}=\mathrm{Pr}_W^*\nabla^{Lz}$. We naturally extend the G-actions to this case such that the G-action is identity on $\widehat{S}$.

We denote by $B_{3,u^2,T}$ the Bismut superconnection associated to $(T_3^HW, u^{-2}g_T^{TZ}, h^{Lz}, \nabla^{Lz})$. We know that the G-action commutes with this Bismut superconnection.

Let $\widehat{B}$ be the Bismut superconnection for the fibration $\widehat{W}\to\widehat{S}$, by the arguments above (1.59), we can get

$$\widehat{B}_{(T,u,b)}=B_{3,u^2,T}+dT\wedge\frac{\partial}{\partial T}+du\wedge\frac{\partial}{\partial u}+\frac{n}{4u^2}du+\frac{n-m}{4T^2}dT. \tag{3.4}$$

Definition 3.1. We define $\beta_g=du\wedge\beta_g^u+dT\wedge\beta_g^T$ to be the part of $\psi_S\widetilde{\mathrm{Tr}}[g\exp(-\widehat{B}^2)]$ of degree one with respect to the coordinates (T,u), with functions $\beta_g^u,\ \beta_g^T:\mathbb{R}_{+,T}\times\mathbb{R}_{+,u}\to\Omega^*(S)$.

From (1.62) and (3.4), we have

$$\begin{aligned}
&\beta_g^u(T,u)=\begin{cases}-\dfrac{1}{2\sqrt{-1}\sqrt{\pi}}\psi_S\,\mathrm{Tr}_s\left[g\dfrac{\partial B_{3,u^2,T}}{\partial u}\exp(-B^2_{3,u^2,T})\right], & \text{if } n \text{ is even;}\\[2ex] -\dfrac{1}{\sqrt{\pi}}\psi_S\,\mathrm{Tr}^{\mathrm{even}}\left[g\dfrac{\partial B_{3,u^2,T}}{\partial u}\exp(-B^2_{3,u^2,T})\right], & \text{if } n \text{ is odd,}\end{cases}\\
&\beta_g^T(T,u)=\begin{cases}-\dfrac{1}{2\sqrt{-1}\sqrt{\pi}}\psi_S\,\mathrm{Tr}_s\left[g\dfrac{\partial B_{3,u^2,T}}{\partial T}\exp(-B^2_{3,u^2,T})\right], & \text{if } n \text{ is even;}\\[2ex] -\dfrac{1}{\sqrt{\pi}}\psi_S\,\mathrm{Tr}^{\mathrm{even}}\left[g\dfrac{\partial B_{3,u^2,T}}{\partial T}\exp(-B^2_{3,u^2,T})\right], & \text{if } n \text{ is odd.}\end{cases}
\end{aligned} \tag{3.5}$$

By Definition 1.3 and Remark 1.4, we know that

$$\widetilde{\eta}_g(T_1^HW, g_T^{TZ}, h^{Lz}, \nabla^{Lz})=-\int_0^{+\infty}\beta_g^u(T,u)du. \tag{3.6}$$

Proposition 3.2. *There exists a smooth family $\alpha_g:\mathbb{R}_{+,T}\times\mathbb{R}_{+,u}\to\Omega^*(S)$ such that*

$$\left(du\wedge\frac{\partial}{\partial u}+dT\wedge\frac{\partial}{\partial T}\right)\beta_g=dT\wedge du\ d^S\alpha_g. \tag{3.7}$$

Proof. We denote by α_g the coefficient of $du\wedge dT$ component of $\psi_S\widetilde{\mathrm{Tr}}[g\exp(-\widehat{B}^2)]$. Then

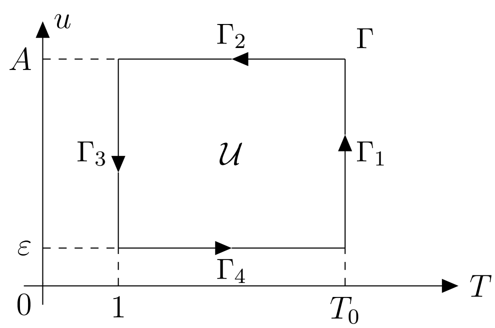
$$\psi_S\widetilde{\mathrm{Tr}}[g\exp(-\widehat{B}^2)]=\psi_S\widetilde{\mathrm{Tr}}[g\exp(-B^2_{3,u^2,T})]+\beta_g+du\wedge dT\ \alpha_g. \tag{3.8}$$

Since $\psi_S\widetilde{\mathrm{Tr}}[g\exp(-\widehat{B}^2)]$ and $\psi_S\widetilde{\mathrm{Tr}}[g\exp(-B^2_{3,u^2,T})]$ are closed forms, we have

$$\begin{aligned}\left(du\wedge\frac{\partial}{\partial u}+dT\wedge\frac{\partial}{\partial T}\right)\psi_S\widetilde{\mathrm{Tr}}[g\exp(-B^2_{3,u^2,T})]-dT\wedge du\ d^S\alpha_g+d^S\beta_g&\\ +\left(du\wedge\frac{\partial}{\partial u}+dT\wedge\frac{\partial}{\partial T}\right)\beta_g=0.&\end{aligned} \tag{3.9}$$

Then Proposition 3.2 follows from comparing the coefficient of $dT\wedge du$ in (3.9). $\square$

Take ε, A, T_0, $0 < \varepsilon \leq 1 \leq A < \infty$, $1 \leq T_0 < \infty$. Let $\Gamma = \Gamma_{\varepsilon,A,T_0}$ be the oriented contour in $\mathbb{R}_{+,T} \times \mathbb{R}_{+,u}$.



The contour Γ is made of four oriented pieces $\Gamma_1, \cdots, \Gamma_4$ indicated in the above picture. For $1 \leq k \leq 4$, set $I_k^0 = \int_{\Gamma_k} \beta_g$. Then by Stocks' formula and Proposition 3.2,

$$\sum_{k=1}^{4} I_k^0 = \int_{\partial \mathcal{U}} \beta_g = \int_{\mathcal{U}} \left(du \wedge \frac{\partial}{\partial u} + dT \wedge \frac{\partial}{\partial T} \right) \beta_g = d^S \left(\int_{\mathcal{U}} \alpha_g dT \wedge du \right). \tag{3.10}$$

3.2. **Intermediate results.** Now we state without proof some intermediate results, which will play an essential role in the proof of Theorem 2.4. The proofs of these results are delayed to Section 4-8.

In the sequence, we will assume for simplicity that S is compact. If S is non-compact, the various constants $C > 0$ depend explicitly on the compact subset of S on which the given estimate is valid.

Let $\mathrm{Pr}_V : \widehat{V} = \mathbb{R}_+ \times V \to V$ be the projection. For the fibration $\widehat{V} \to \widehat{S} = \mathbb{R}_+ \times S$, let $(T_2^H \widehat{V}, \widehat{g}^{TY}, h^{\widehat{L}_Y}, \nabla^{\widehat{L}_Y})$ be the quadruple such that $T_2^H \widehat{V} = T(\mathbb{R}_+) \oplus \mathrm{Pr}_V^*(T_2^H V)$, $\widehat{g}^{TY}_{(t,v)} = t^{-2} g^{TY}_v$ for $t \in \mathbb{R}_+$, $v \in V$, $\widehat{L}_Y = \mathrm{Pr}_V^* L_Y$, $h^{\widehat{L}_Y} = \mathrm{Pr}_V^* h^{L_Y}$ and $\nabla^{\widehat{L}_Y} = \mathrm{Pr}_V^* \nabla^{L_Y}$. Let $h^{\ker D^{\widehat{X}}}$ and $\nabla^{\ker D^{\widehat{X}}}$ be the induced metric and connection on the vector bundle $\ker D^{\widehat{X}}$. Let $h^{\widehat{\mathcal{S}}_Y}$ and $\nabla^{\widehat{\mathcal{S}}_Y}$ be the induced metric and connection on $\mathrm{Pr}_V^* \mathcal{S}(TY, L_Y)$. We naturally extend the G-action to this case such that the G-action is identity on $\mathbb{R}_+ \times S$.

Let B_2, $\widehat{B}_2$ and B_{2,u^2} be the Bismut superconnections associated to $(T_2^H V, g^{TY}, h^{L_Y}, h^{\ker D^X}, \nabla^{L_Y}, \nabla^{\ker D^X})$, $(T_2^H \widehat{V}, \widehat{g}^{TY}, h^{\widehat{L}_Y}, h^{\ker D^{\widehat{X}}}, \nabla^{\widehat{L}_Y}, \nabla^{\ker D^{\widehat{X}}})$ and $(T_2^H V, u^{-2} g^{TY}, h^{L_Y}, h^{\ker D^X}, \nabla^{L_Y}, \nabla^{\ker D^X})$ respectively. For any $g \in G$, let us decompose

$$\psi_S \widetilde{\mathrm{Tr}}[g \exp(-\widehat{B}_2^2)] = dt \wedge \gamma_2(t) + r_2(t), \tag{3.11}$$

where $\gamma_2(t), r_2(t) \in \Omega^*(S)$. By Definition 1.3 and Remark 1.4,

$$\int_0^{+\infty} \gamma_2(t) dt = -\widetilde{\eta}_g(T_2^H V, g^{TY}, h^{L_Y}, h^{\ker D^X}, \nabla^{L_Y}, \nabla^{\ker D^X}). \tag{3.12}$$

Theorem 3.3. *i) For any $u > 0$, we have*

$$\lim_{T \to \infty} \beta_g^u(T, u) = \gamma_2(u). \tag{3.13}$$

ii) For $0 < u_1 < u_2$ fixed, there exists $C > 0$ such that, for $u \in [u_1, u_2]$, $T \geq 1$, we have

$$|\beta_g^u(T,u)| \leq C. \tag{3.14}$$

iii) We have the following identity:

$$\lim_{T\to+\infty} \int_1^\infty \beta_g^u(T,u)du = \int_1^\infty \gamma_2(u)du. \tag{3.15}$$

Theorem 3.4. *We have the following identity:*

$$\lim_{u\to+\infty} \int_1^\infty \beta_g^T(T,u)dT = 0. \tag{3.16}$$

Let $\mathrm{Pr}_W|_{V^g} : \widehat{W}|_{V^g} = \mathbb{R}_+ \times W|_{V^g} \to W|_{V^g}$ be the projection. For the fibration $\widehat{W}|_{V^g} \to \widehat{V}^g = \mathbb{R}_+ \times V^g$, let $(T_1^H(\widehat{W}|_{V^g}), \widehat{g}^{TX}, h^{\widehat{L}_X}, \nabla^{\widehat{L}_X})$ be the quadruple such that $T_1^H(\widehat{W}|_{V^g}) = T(\mathbb{R}_+) \oplus (\mathrm{Pr}_W|_{V^g})^* T_1^H(W|_{V^g})$, $\widehat{g}^{TX}_{(t,w)} = t^{-2} g^{TX}_w$ for $t \in \mathbb{R}_+$, $w \in W|_{V^g}$, $\widehat{L}_X = (\mathrm{Pr}_W|_{V^g})^* L_X$, $h^{\widehat{L}_X} = (\mathrm{Pr}_W|_{V^g})^* h^{L_X}$ and $\nabla^{\widehat{L}_X} = (\mathrm{Pr}_W|_{V^g})^* \nabla^{L_X}$. We naturally extend the G-actions to this case such that g acts trivially on $\widehat{V}^g$.

Let $\widehat{B}_1$ be the Bismut superconnection associated to $(T_1^H(\widehat{W}|_{V^g}), \widehat{g}^{TX}, \nabla^{\widehat{L}_X})$. For any $g \in G$, let us decompose

$$\psi_{V^g} \widetilde{\mathrm{Tr}}[g \exp(-\widehat{B}_1^2)] = dt \wedge \gamma_1(t) + r_1(t), \tag{3.17}$$

where $\gamma_1(t), r_1(t) \in \Omega^*(S)$. By Definition 1.3 and Remark 1.4,

$$\int_0^{+\infty} \gamma_1(t)dt = -\widetilde{\eta}_g(T_1^H(W|_{V^g}), g^{TX}, h^{L_X}, \nabla^{L_X}). \tag{3.18}$$

By (1.44), $\widehat{\mathrm{A}}_g(TZ, \nabla^{TZ})$ only depends on $g \in G$ and R^{TZ}. So we can denote it by $\widehat{\mathrm{A}}_g(R^{TZ})$. Let R_T^{TZ} be the curvature of ∇_T^{TZ}. Set

$$\gamma_{\mathcal{A}}(T) = -\left.\frac{\partial}{\partial b}\right|_{b=0} \widehat{\mathrm{A}}_g\left(R_T^{TZ} + b\frac{\partial \nabla_T^{TZ}}{\partial T}\right). \tag{3.19}$$

By a standard argument in Chern-Weil theory, we know that

$$\frac{\partial}{\partial T}\widetilde{\widehat{\mathrm{A}}}_g(TZ, \nabla^{TZ}, \nabla_T^{TZ}) = -\gamma_{\mathcal{A}}(T). \tag{3.20}$$

Proposition 3.5. *When $T \to +\infty$, we have $\gamma_{\mathcal{A}}(T) = O(T^{-2})$. Moreover, modulo exact forms on W^g, we have*

$$\widetilde{\widehat{\mathrm{A}}}_g(TZ, \nabla^{TZ}, {}^0\nabla^{TZ}) = -\int_1^{+\infty} \gamma_{\mathcal{A}}(T)dT. \tag{3.21}$$

Theorem 3.6. *i) For any $u > 0$, there exist $C > 0$ and $\delta > 0$ such that, for $T \geq 1$, we have*

$$|\beta_g^T(T,u)| \leq \frac{C}{T^{1+\delta}}. \tag{3.22}$$

ii) For any $T > 0$, we have

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}\beta_g^T(T\varepsilon^{-1}, \varepsilon) = \int_{Y^g} \widehat{\mathrm{A}}_g(TY, \nabla^{TY}) \wedge \mathrm{ch}_g(L_Y^{1/2}, \nabla^{L_Y^{1/2}}) \wedge \gamma_1(T). \tag{3.23}$$

iii) There exists $C>0$ such that for $\varepsilon\in(0,1]$, $\varepsilon\leq T\leq 1$,

$$\varepsilon^{-1}\left|\beta_g^T(T\varepsilon^{-1},\varepsilon)-\int_{Z^g}\gamma_{\mathcal{A}}(T\varepsilon^{-1})\wedge \mathrm{ch}_g(L_Z^{1/2},\nabla^{L_Z^{1/2}})\right|\leq C. \tag{3.24}$$

iv) There exist $\delta\in(0,1]$, $C>0$ such that, for $\varepsilon\in(0,1]$, $T\geq 1$,

$$\varepsilon^{-1}|\beta_g^T(T\varepsilon^{-1},\varepsilon)|\leq \frac{C}{T^{1+\delta}}. \tag{3.25}$$

3.3. **Proof of Theorem 2.4.** We now finish the proof of Theorem 2.4 under the simplifying assumptions in Section 2.2. By (3.10), we know that

$$\begin{aligned}\int_\varepsilon^A \beta_g^u(T_0,u)du-\int_1^{T_0}\beta_g^T(T,A)dT-\int_\varepsilon^A\beta_g^u(1,u)du+\int_1^{T_0}\beta_g^T(T,\varepsilon)dT\\ =I_1+I_2+I_3+I_4\end{aligned} \tag{3.26}$$

is an exact form. We take the limits $A\to\infty$, $T\to\infty$ and then $\varepsilon\to 0$ in the indicated order. Let I_j^k, $j=1,2,3,4$, $k=1,2,3$ denote the value of the part I_j after the kth limit. By [21, §22, Theorem 17], $d\Omega(S)$ is closed under uniformly convergence on compact sets of S. Thus,

$$\sum_{j=1}^4 I_j^3\equiv 0 \bmod d\Omega^*(S). \tag{3.27}$$

From (3.6), we obtain that

$$I_3^3=\tilde{\eta}_g(T_3^HW,,g^{TZ},h^{L_Z},\nabla^{L_Z}). \tag{3.28}$$

Furthermore, by Theorem 3.4, we get

$$I_2^2=I_2^3=0. \tag{3.29}$$

From (3.12) and Theorem 3.3, we conclude that

$$I_1^3=-\tilde{\eta}_g(T_2^HV,g^{TY},h^{L_Y},h^{\ker D^X},\nabla^{L_Y},\nabla^{\ker D^X}). \tag{3.30}$$

Finally, using Theorem 3.6, we get

$$\begin{aligned}I_4^3=-\int_{Y^g}\widehat{\mathrm{A}}_g(TY,\nabla^{TY})\wedge\mathrm{ch}_g(L_Y^{1/2},\nabla^{L_Y^{1/2}})\wedge\tilde{\eta}_g(T_1^H(W|_{V^g}),g^{TX},h^{L_X},\nabla^{L_X})\\ -\int_{Z^g}\widetilde{\widehat{\mathrm{A}}}_g(TZ,\nabla^{TZ},{}^0\nabla^{TZ})\wedge\mathrm{ch}_g(L_Z^{1/2},\nabla^{L_Z^{1/2}})\end{aligned} \tag{3.31}$$

as follows: We write

$$\int_1^{+\infty}\beta_g^T(T,\varepsilon)dT=\int_\varepsilon^{+\infty}\varepsilon^{-1}\beta_g^T(T\varepsilon^{-1},\varepsilon)dT. \tag{3.32}$$

Convergence of the integrals above is granted by (3.22). Using (3.23), (3.25) and Proposition 3.5, we get

(3.33)

$$\lim_{\varepsilon\to 0}\int_1^{+\infty}\varepsilon^{-1}\beta_g^T(T\varepsilon^{-1},\varepsilon)dT=\int_{Y^g}\widehat{\mathrm{A}}_g(TY,\nabla^{TY})\wedge\mathrm{ch}_g(L_Y^{1/2},\nabla^{L_Y^{1/2}})\wedge\int_1^{+\infty}\gamma_1(T)dT$$

and

$$\lim_{\varepsilon\to 0}\int_\varepsilon^1 \varepsilon^{-1}\left[\beta_g^T(T\varepsilon^{-1},\varepsilon)dT - \int_{Z^g}\gamma_{\mathcal{A}}(T\varepsilon^{-1})\wedge \mathrm{ch}_g(L_Z^{1/2},\nabla^{L_Z^{1/2}})\right]dT \\ = \int_{Y^g}\widehat{\mathrm{A}}_g(TY,\nabla^{TY})\wedge \mathrm{ch}_g(L_Y^{1/2},\nabla^{L_Y^{1/2}})\wedge\int_0^1\gamma_1(T)dT. \tag{3.34}$$

The remaining part of the integral yields by (3.24)

$$\int_\varepsilon^1 \varepsilon^{-1}\int_{Z^g}\gamma_{\mathcal{A}}(T\varepsilon^{-1})\wedge \mathrm{ch}_g(L_Z^{1/2},\nabla^{L_Z^{1/2}})dT = \int_{Z^g}\int_1^{+\infty}\gamma_{\mathcal{A}}(T)\wedge \mathrm{ch}_g(L_Z^{1/2},\nabla^{L_Z^{1/2}})dT \\ = -\int_{Z^g}\widetilde{\widehat{\mathrm{A}}}_g(TZ,\nabla^{TZ},{}^0\nabla^{TZ})\wedge \mathrm{ch}_g(L_Z^{1/2},\nabla^{L_Z^{1/2}}). \tag{3.35}$$

These four equations for I_k^3, $k=1,2,3,4$, imply Theorem 2.4.

4. Proof of Theorem 3.3

In this section, we use the assumptions and the notations of Section 2.2 except D_T^Z is invertible for any $T\geq 1$.

This Section is organized as follows. In Section 4.1, we make some estimates of the fibrewise Dirac operator D_T^Z. In Section 4.2, we write the operator $\mathcal{B}_T$ in a matrix form. In Section 4.3, we state two intermediate results, from which Theorem 3.3 follows easily. We prove one of them in Section 4.3 and leave the proof of the other one to Section 4.4. In Section 4.5, we prove Proposition 3.5.

4.1. Estimates of $D_T^{Z,2}$.

Definition 4.1. For $v\in V$, $b\in S$, let $\mathbb{E}_v$, $\mathbb{E}_{0,b}$ (resp. $\mathbb{E}_{1,b}$) be the vector spaces of the smooth sections of $\pi_3^*\Lambda(T^*S)\widehat{\otimes}\mathcal{S}(TZ,L_Z)$ on X_v, Z_b (resp. $\pi_2^*\Lambda(T^*S)\widehat{\otimes}\mathcal{S}(TY,L_Y)\otimes\ker D^X$ on Y_b). For $\mu\in\mathbb{R}$, let $\mathbb{E}_v^\mu$, $\mathbb{E}_{0,b}^\mu$, $\mathbb{E}_{1,b}^\mu$ be the Sobolev spaces of the order μ of sections of $\pi_3^*\Lambda(T^*S)\widehat{\otimes}\mathcal{S}(TZ,L_Z)$, $\pi_3^*\Lambda(T^*S)\widehat{\otimes}\mathcal{S}(TZ,L_Z)$, $\pi_2^*\Lambda(T^*S)\widehat{\otimes}\mathcal{S}(TY,L_Y)\otimes\ker D^X$ on X_v, Z_b, Y_b with Sobolev norms $\|\cdot\|_{X,\mu}$, $\|\cdot\|_\mu$, $\|\cdot\|_{Y,\mu}$.

For $v\in V$, in this section, we simply denote by P_b the projection from $\mathbb{E}_{0,b}^0$ to $\mathbb{E}_{1,b}^0$ and let $P^\perp=1-P$. Let $\mathbb{E}_1^{0,\perp}$ be the orthogonal bundle to $\mathbb{E}_1^0$ in $\mathbb{E}_0^0$. Let $\mathbb{E}_1^{\mu,\perp}=\mathbb{E}_1^{0,\perp}\cap\mathbb{E}_0^\mu$.

Let $\{e_i\}$, $\{f_p\}$, $\{g_\alpha\}$ be the local orthonormal frames of TX, TY, TS respectively and $\{e^i\}$, $\{f^p\}$, $\{g^\alpha\}$ be their dual. Recall that $\nabla^{\mathscr{E}_X,u}$ is the connection in (2.7). Set

$$\nabla^{\mathcal{S}_Y\otimes\mathscr{E}_X,u}=\nabla^{\mathcal{S}_Y}\otimes 1+1\otimes\nabla^{\mathscr{E}_X,u}. \tag{4.1}$$

Let

$$D^H=c(f_{p,1}^H)\nabla^{\mathcal{S}_Y\otimes\mathscr{E}_X,u}_{f_{p,1}^H}. \tag{4.2}$$

By (2.8), we have

$$PD^HP=D^Y. \tag{4.3}$$

Let S_2 and S_3 be the tensor associated to $(T_2^H V, g^{TY})$ and $(T_3^H W, g^{TZ})$ as in (1.17). Let T_1 T_2, T_3, be the torsion tensors defined before (1.25) associated to $(T_1^H W, g^{TX})$, $(T_2^H V, g^{TY})$, $(T_3^H W, g^{TZ})$. By (1.25), we have

$$\langle T_3(g^H_{\alpha,3}, g^H_{\beta,3}), f^H_{p,1}\rangle = \langle T_2(g^H_{\alpha,2}, g^H_{\beta,2}), f_p\rangle. \tag{4.4}$$

From (1.28) and [4, Theorem 10.19], the Dirac operator D^Z associated to $(T_3^H W, g^{TZ}, \nabla^{TZ})$ can be written by

$$D^Z = D^X + D^H - \frac{1}{8}\langle T_1(f^H_{p,1}, f^H_{q,1}), e_i\rangle c(e_i)c(f^H_{p,1})c(f^H_{q,1}). \tag{4.5}$$

If we replace the metric g^{TZ} by g^{TZ}_T, by (1.25), we have

$$D^Z_T = TD^X + D^H + \frac{1}{8T}\langle [f^H_{p,1}, f^H_{q,1}], e_i\rangle c(e_i)c(f^H_{p,1})c(f^H_{q,1}). \tag{4.6}$$

Definition 4.2. For $s, s' \in \mathbb{E}_0$, we set

$$|s|^2_{T,0} := \|s\|^2_0, \tag{4.7}$$

$$|s|^2_{T,1} := \|Ps\|^2_0 + T^2\|P^\perp s\|^2_0 + \sum_p \|{}^0\nabla^{\mathcal{S}_Z}_{f^H_{p,1}} s\|^2_0 + T^2 \sum_i \|{}^0\nabla^{\mathcal{S}_Z}_{e_i} P^\perp s\|^2_0. \tag{4.8}$$

Set

$$|s|_{T,-1} = \sup_{0\neq s'\in\mathbb{E}^1_0} \frac{|\langle s, s'\rangle_0|}{|s'|_{T,1}}. \tag{4.9}$$

Lemma 4.3. *There exist $C_1, C_2, C_3 > 0$, $T_0 \geq 1$, such that for any $T \geq T_0$, $s, s' \in \mathbb{E}_0$,*

$$\begin{aligned} \langle D^{Z,2}_T s, s\rangle_0 &\geq C_1|s|^2_{T,1} - C_2|s|^2_{T,0}, \\ |\langle D^{Z,2}_T s, s'\rangle_0| &\leq C_3|s|_{T,1}|s'|_{T,1}. \end{aligned} \tag{4.10}$$

Proof. The proof of Lemma 4.3 is almost the same as that of [5, theorem 5.19]. For the completeness of this paper, we state the proof here.

Easy to check that D^Z_T is a fiberwisely self-adjoint operator associated to $\langle\cdot,\cdot\rangle_0$ in (1.22). Set

$$D^H_T = D^H + \frac{1}{8T}\langle [f^H_{p,1}, f^H_{q,1}], e_i\rangle c(e_i)c(f^H_{p,1})c(f^H_{q,1}). \tag{4.11}$$

Then by (4.6),

$$D^{Z,2}_T = T^2 D^{X,2} + D^{H,2}_T + T[D^X, D^H_T]. \tag{4.12}$$

The family of operators (D^X, D^H_T) is uniformly elliptic. So there exists $C'_1, C'_2 > 0$, such that for $T \in [1, +\infty]$, $s \in \mathbb{E}_0$,

$$\|D^X s\|^2_0 + \|D^H_T s\|^2_0 \geq C'_1\|s\|^2_1 - C'_2\|s\|^2_0. \tag{4.13}$$

Since $\ker D^X$ is a vector bundle, there exists $C'_3 > 0$,

$$\|D^X P^\perp s\|^2_0 \geq C'_3\|P^\perp s\|^2_0. \tag{4.14}$$

Using (4.13) and (4.14), we get for $T \in [1, +\infty)$,

$$\begin{aligned} T^2\|D^X P^\perp s\|_0^2 + \|D_T^H P^\perp s\|_0^2 \geq C_1'\|P^\perp s\|_1^2 + \frac{T^2-1}{2}\|D^X P^\perp s\|_0^2 \\ + \left(\frac{C_3'(T^2-1)}{2} - C_2'\right)\|P^\perp s\|_0^2. \end{aligned} \tag{4.15}$$

By elliptic estimate associated to the norm $\|\cdot\|_{X,\mu}$ and (4.14), there exists $C_4' > 0$, such that

$$\|D^X P^\perp s\|_0^2 \geq C_4' \sum_i \|{}^0\nabla_{e_i}^{\mathcal{S}_Z} P^\perp s\|_0^2. \tag{4.16}$$

Let ${}^0R$ be the curvature of ${}^0\nabla^{\mathcal{S}_Z} - \frac{1}{2}\langle S_1(e_i)e_i, \cdot\rangle$. Then from a easy computation given by [6, Theorem 2.5], we have

$$[D^X, D^H] = c(e_i)c(f_{p,1}^H)\left({}^0R(e_i, f_{p,1}^H) - {}^0\nabla^{\mathcal{S}_Z}_{T_1(e_i, f_{p,1}^H)}\right). \tag{4.17}$$

Since $T_1(e_i, f_{p,1}^H) \in TX$, $[D^X, D^H]$ is a fiberwise first order elliptic operator along the fibers X. By (4.14), (4.16) and (4.17), there exists $C_5', C_6' > 0$, such that for $T \geq 1$, $s \in \mathbb{E}_0$,

$$|\langle T[D^X, D_T^H]s, s\rangle_0| \leq T|\langle [D^X, D^H]P^\perp s, P^\perp s\rangle_0| + C_5'\|P^\perp s\|_0^2 \leq C_6' T\|D^X P^\perp s\|_0^2. \tag{4.18}$$

From (4.8), (4.12), (4.16) and (4.18), there exist $C_1'', C_2'' > 0$, $T_0 \geq 1$ such that for any $T \geq T_0$, $s \in \mathbb{E}_0$

$$\langle D_T^{Z,2} P^\perp s, P^\perp s\rangle_0 \geq C_1''|P^\perp s|_{T,1}^2 + C_1'\|P^\perp s\|_1^2 - C_2''\|s\|_0^2. \tag{4.19}$$

From (4.12) and (4.13), we have

$$\langle D_T^{Z,2} Ps, Ps\rangle_0 \geq C_1'\|Ps\|_1^2 - C_2'\|s\|_0^2. \tag{4.20}$$

Since

$$\begin{aligned} &\langle D_T^{H,2} P^\perp s, Ps\rangle_0 = \langle P^\perp s, D_T^{H,2} Ps\rangle_0 \\ =&2\langle P^\perp s, [D_T^H, P]D_T^H s\rangle_0 + \langle P^\perp s, [D_T^H, [D_T^H, P]]s\rangle_0 \end{aligned} \tag{4.21}$$

and $[D_T^H, P]$, $[D_T^H, [D_T^H, P]]$ are operators with smooth kernels along the fiber X, there exists $C_3'' > 0$, such that

$$|\langle D_T^{H,2} P^\perp s, Ps\rangle_0| \leq C_3''|s|_{T,1}\|P^\perp s\|_0. \tag{4.22}$$

As in (4.18), there exists $C_4'' > 0$, such that

$$|\langle T[D^X, D_T^H]P^\perp s, Ps\rangle_0| \leq C_4''|P^\perp s|_{T,1}\|Ps\|_0. \tag{4.23}$$

So by (4.12),

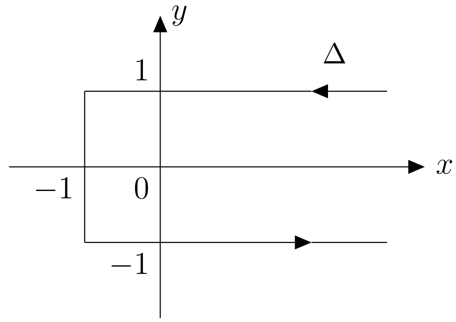
$$|\langle D_T^{Z,2} P^\perp s, Ps\rangle_0| \leq (C_3'' + C_4'')|s|_{T,1}|s|_{T,0}. \tag{4.24}$$

Since $[{}^0\nabla^{\mathcal{S}_Z}, P]$ and $[{}^0\nabla^{\mathcal{S}_Z}, P^\perp]$ are bounded operators, there exists $C > 0$, such that

$$\|P^\perp s\|_1 + \|Ps\|_1 \geq \sum_p \|{}^0\nabla^{\mathcal{S}_Z}_{f_{p,1}^H} s\|_0^2 + \sum_i \|{}^0\nabla^{\mathcal{S}_Z}_{e_i} P^\perp s\|_0^2 - C\|s\|_0^2. \tag{4.25}$$

So from (4.19), (4.20), (4.24) and (4.25), we get the first inequality of (4.10). The second inequality follows directly from (4.12) and (4.18).

The proof of Lemma 4.3 is complete. $\square$



Let Δ be the oriented contour in the above picture.

If $A \in \mathscr{L}(\mathbb{E}_0^0)$ (resp. $\mathscr{L}(\mathbb{E}_0^{-1}, \mathbb{E}_0^1)$), we note $\|A\|$ (resp. $|A|_T^{-1,1}$) the norm of A with respect to the norm $\|\cdot\|_0$ (resp. the norms $|\cdot|_{T,-1}$ and $|\cdot|_{T,1}$). Comparing with [14, Theorem 11.27], we have the following lemma.

Lemma 4.4. *There exist $T_0 \geq 1, C > 0$, such that for $T \geq T_0$, $\lambda \in \Delta$, the resolvent $(\lambda - D_T^{Z,2})^{-1}$ exists, extends to a continuous linear operator from $\mathbb{E}_0^{-1}$ into $\mathbb{E}_0^1$, and moreover*

$$\begin{aligned} &\|(\lambda - D_T^{Z,2})^{-1}\| \leq C, \\ &|(\lambda - D_T^{Z,2})^{-1}|_T^{-1.1} \leq C(1 + |\lambda|)^2. \end{aligned} \tag{4.26}$$

Proof. Since D_T^Z is fiberwisely self-adjoint, for $\lambda \in \mathbb{C} \backslash \mathbb{R}^+$, $(\lambda - D_T^{Z,2})^{-1}$ exists.

For $\lambda = a \pm i \in \mathbb{C}$, $s \in \mathbb{E}_0^2$,

$$|\langle (D_T^{Z,2} - \lambda)s, s\rangle_0| \geq \|s\|_0^2. \tag{4.27}$$

So there exists $C > 0$, such that for any $\lambda \in \Delta$,

$$\|(\lambda - D_T^{Z,2})^{-1} s\|_0 \leq C\|s\|_0. \tag{4.28}$$

So we get the first inequality of (4.26).

For $\lambda_0 \in \mathbb{R}$, $\lambda_0 \leq -2C_2$, by (4.10), we have

$$|\langle (\lambda_0 - D_T^{Z,2})s, s\rangle_0| \geq C_1 |s|_{T,1}^2. \tag{4.29}$$

Then by (4.9) and (4.29),

$$|(\lambda_0 - D_T^{Z,2})s|_{T,-1} = \sup_{0 \neq s' \in \mathbb{E}_0^1} \frac{|\langle (\lambda_0 - D_T^{Z,2})s, s'\rangle_0|}{|s'|_{T,1}} \geq C_1 |s|_{T,1}. \tag{4.30}$$

For $\lambda \in \Delta$,

$$(\lambda - D_T^{Z,2})^{-1} = (\lambda_0 - D_T^{Z,2})^{-1} + (\lambda - \lambda_0)(\lambda - D_T^{Z,2})^{-1}(\lambda_0 - D_T^{Z,2})^{-1}. \tag{4.31}$$

From (4.28), (4.30) and (4.31), we deduce that $(\lambda - D_T^{Z,2})^{-1}$ extends to a linear map from $\mathbb{E}_0^{-1}$ into $\mathbb{E}_0^0$ and

(4.32)
$$\begin{aligned} |(\lambda - D_T^{Z,2})^{-1}s|_{T,0} &\leq |(\lambda_0 - D_T^{Z,2})^{-1}s|_{T,0} + |\lambda_0 - \lambda||(\lambda - D_T^{Z,2})^{-1}(\lambda_0 - D_T^{Z,2})^{-1}s|_{T,0} \\ &\leq C_1^{-1}|s|_{T,-1} + C|\lambda_0 - \lambda||(\lambda_0 - D_T^{Z,2})^{-1}s|_{T,0} \\ &\leq (C_1^{-1} + CC_1^{-1}|\lambda_0 - \lambda|)|s|_{T,-1}. \end{aligned}$$

On the other hand,

$$(\lambda - D_T^{Z,2})^{-1} = (\lambda_0 - D_T^{Z,2})^{-1} + (\lambda - \lambda_0)(\lambda_0 - D_T^{Z,2})^{-1}(\lambda - D_T^{Z,2})^{-1}. \tag{4.33}$$

So from (4.30), (4.32) and (4.33), we deduce that $(\lambda - D_T^{Z,2})^{-1}$ extends to a linear map from $\mathbb{E}_0^{-1}$ into $\mathbb{E}_0^1$ and

$$\begin{aligned}|(\lambda - D_T^{Z,2})^{-1}s|_{T,1} &\leq |(\lambda_0 - D_T^{Z,2})^{-1}s|_{T,1} + |\lambda_0 - \lambda||(\lambda_0 - D_T^{Z,2})^{-1}(\lambda - D_T^{Z,2})^{-1}s|_{T,1} \\ &\leq C_1^{-1}|s|_{T,-1} + C_1^{-1}|\lambda_0 - \lambda||(\lambda - D_T^{Z,2})^{-1}s|_{T,0} \\ &\leq (C_1^{-1} + C_1^{-1}|\lambda_0 - \lambda|(C_1^{-1} + CC_1^{-1}|\lambda_0 - \lambda|))|s|_{T,-1}.\end{aligned} \tag{4.34}$$

Then we get the second inequality of (4.26). □

4.2. **The matrix structure.** In the sequence, if α_T ($T \in [1, +\infty]$) is a family of tensors (resp. differential operators), we write that as $T \to +\infty$,

$$\alpha_T = \alpha_\infty + O\left(\frac{1}{T^k}\right), \tag{4.35}$$

if for any $p \in \mathbb{N}$, there exists $C > 0$, such that for $T \geq 1$, the sup of the norms of the coefficients of $\alpha_T - \alpha_\infty$ and their derivatives of order $\leq p$ is dominated by C/T^k.

Recall that $\mathscr{E}_Z$ is the infinite dimensional fiber bundle over S, whose fibers are the set of smooth sections over Z of $\mathcal{S}(TZ, L_Z)$. Comparing with (1.24), for $U \in TS$, we define the connections on $\mathscr{E}_Z$

$$\begin{aligned}{}^0\nabla_U^{\mathscr{E}_Z,u} &= {}^0\nabla_{U_3^H}^{\mathcal{S}_Z} - \frac{1}{2}\langle S_3(e_i)e_i, U_3^H\rangle - \frac{1}{2}\langle S_3(f_{p,1}^H, f_{p,1}^H), U_3^H\rangle, \\ \nabla_U^{\mathscr{E}_Z,T,u} &= \nabla_{U_3^H}^{\mathcal{S}_Z,T} - \frac{1}{2}\langle S_3(e_i)e_i, U_3^H\rangle - \frac{1}{2}\langle S_3(f_{p,1}^H, f_{p,1}^H), U_3^H\rangle.\end{aligned} \tag{4.36}$$

By (3.3) and (4.36), we have

$$\nabla_U^{\mathscr{E}_Z,T,u} = {}^0\nabla_U^{\mathscr{E}_Z,u} + \frac{1}{2T}\langle S_1(U_3^H)e_i, f_{p,1}^H\rangle c(e_i)c(f_{p,1}^H). \tag{4.37}$$

Recall that $B_{3,u^2,T}$ is the Bismut superconnection associated to $(T_3^HW, u^{-2}g_T^{TZ}, h^{L_Z}, \nabla^{L_Z})$. Denote by $B_{3,T} = B_{3,1,T}$. From (1.28), (1.33), (2.14), (4.6), (4.4), (4.36) and (4.37), we can calculate $B_{3,T}$ and $B_{3,u^2,T}$ exactly.

Proposition 4.5. *For $T > 0$ and $u > 0$,*

$$\begin{aligned}B_{3,T} = TD^X + {}^0\nabla^{\mathscr{E}_Z,u} + D^H - \frac{c(T_2)}{4} - \frac{1}{8T}\langle T_1(f_{p,1}^H, f_{q,1}^H), e_i\rangle c(e_i)c(f_{p,1}^H)c(f_{q,1}^H) \\ + \frac{1}{2T}\langle S_1(g_\alpha^H)e_i, f_{p,1}^H\rangle c(e_i)c(f_{p,1}^H)g^\alpha \wedge - \frac{1}{8T}\langle T_3(g_{\alpha,3}^H, g_{\beta,3}^H), e_i\rangle c(e_i)g^\alpha \wedge g^\beta \wedge,\end{aligned} \tag{4.38}$$

and

$$B_{3,u^2,T} = uTD^X + uD^H - \frac{u}{8T}\langle T_1(f_{p,1}^H, f_{q,1}^H), e_i\rangle c(e_i)c(f_{p,1}^H)c(f_{q,1}^H) + {}^0\nabla^{\mathscr{E}_Z,u} + \frac{1}{2T}\langle S_1(g_\alpha^H)e_i, f_{p,1}^H\rangle c(e_i)c(f_{p,1}^H)g^\alpha\wedge - \frac{c(T_2)}{4u} - \frac{1}{8uT}\langle T_3(g_{\alpha,3}^H, g_{\beta,3}^H), e_i\rangle c(e_i)g^\alpha\wedge g^\beta\wedge . \tag{4.39}$$

Let $\mathscr{E}_Y$ be the infinite dimensional fiber bundle over S, whose fibers are the set of smooth sections over Y of $\mathcal{S}(TY, L_Y)\otimes\ker D^X$. By (1.24), for $U\in TS$, we define the connections on $\mathscr{E}_Y$

$$\nabla_U^{\mathscr{E}_Y,u} = \nabla_{U_2^H}^{\mathcal{S}_Y\otimes\ker D^X} - \frac{1}{2}\langle S_2(f_p)f_p, U_2^H\rangle. \tag{4.40}$$

From [27, Theorem 5.2], we have

$$\langle S_3(f_{p,1}^H, f_{q,1}^H), U_3^H\rangle = \langle S_2(f_p)f_p, U_2^H\rangle. \tag{4.41}$$

So by (2.7), (2.8), (4.36), (4.40) and (4.41), we have

$$\nabla^{\mathscr{E}_Y,u} = P\,{}^0\nabla^{\mathscr{E}_Z,u}P. \tag{4.42}$$

Recall that B_2 is the Bismut superconnection associated to $(T_2^HV, g^{TY}, h^{L_Y}, h^{\ker D^X}, \nabla^{L_Y}, \nabla^{\ker D^X})$ and $B_{2,u^2} = u^2\delta_{u^2}B_2\delta_{u^2}^{-1}$. Then by (1.28),

$$B_2 = D^Y + \nabla^{\mathscr{E}_Y,u} - c(T_2)/4. \tag{4.43}$$

Lemma 4.6. *For any $T\in[1,+\infty]$, the operator $PB_{3,T}P$ is a superconnection on $\mathbb{E}_1$. When $T\to+\infty$,*

$$PB_{3,T}P = B_2 + O\left(\frac{1}{T}\right). \tag{4.44}$$

Proof. Set

$$\mathcal{C} = {}^0\nabla^{\mathscr{E}_Z,u} + D^H - \frac{c(T_2)}{4}. \tag{4.45}$$

By (4.39), we have

$$PB_{3,T}P = P\mathcal{C}P + O\left(\frac{1}{T}\right). \tag{4.46}$$

From (4.3), (4.42) and (4.43), we get

$$P\mathcal{C}P = B_2. \tag{4.47}$$

So Lemma 4.6 follows from (4.46) and (4.47). ∎

Set

$$\mathcal{B}_T = B_{3,T}^2 + u^{-2}du\wedge\,\delta_{u^2}^{-1}\frac{\partial B_{3,u^2,T}}{\partial u}\delta_{u^2}. \tag{4.48}$$

Then $\mathcal{B}_T$ is a differential operator along the fiber Z with values in $\Lambda(T^*(\mathbb{R}_+\times S))$. Set

$$\mathcal{B}_{u,T} = u^2\delta_{u^2}\mathcal{B}_T\delta_{u^2}^{-1} = B_{3,u^2,T}^2 + du\wedge\frac{\partial B_{3,u^2,T}}{\partial u}. \tag{4.49}$$

Then by (3.5), we have

$$\beta_g^u = \psi_S \widetilde{\mathrm{Tr}}[g\exp(-\mathcal{B}_{u,T})]^{du} = \psi_S \delta_{u^2} \widetilde{\mathrm{Tr}}[g\exp(-u^2\mathcal{B}_T)]^{du}. \tag{4.50}$$

From Proposition 4.5,

$$\delta_{u^2}^{-1}\frac{\partial B_{3,u^2,T}}{\partial u}\delta_{u^2} = TD^X + D^H + \frac{c(T_2)}{4} + O\left(\frac{1}{T}\right). \tag{4.51}$$

Set

$$\mathcal{B}_2 = B_2^2 + u^{-2}du\wedge \delta_{u^2}^{-1}\frac{\partial B_{2,u^2}}{\partial u}\delta_{u^2}. \tag{4.52}$$

By (3.11), we have

$$\gamma_2(u) = \psi_S \delta_{u^2}\widetilde{\mathrm{Tr}}[g\exp(-u^2\mathcal{B}_2)]^{du}. \tag{4.53}$$

From (1.33), (4.43), (4.51) and Lemma 4.6, we have

$$P\mathcal{B}_T P = \mathcal{B}_2 + O\left(\frac{1}{T}\right). \tag{4.54}$$

Put

$$\begin{aligned} E_T &= P\mathcal{B}_T P, \quad & F_T &= P\mathcal{B}_T P^{\perp}, \\ G_T &= P^{\perp}\mathcal{B}_T P, \quad & H_T &= P^{\perp}\mathcal{B}_T P^{\perp}. \end{aligned} \tag{4.55}$$

Then we write $\mathcal{B}_T$ in matrix form with respect to the splitting $\mathbb{E}_0 = \mathbb{E}_1^0 \oplus \mathbb{E}_1^{0,\perp}$,

$$\mathcal{B}_T = \begin{pmatrix} E_T & F_T \\ G_T & H_T \end{pmatrix}. \tag{4.56}$$

Similarly as in [27, Theorem 5.5], we have

Proposition 4.7. *There exist operators E, F, G, H such that, as $T \to +\infty$,*

$$\begin{aligned} E_T &= E + O(1/T), \qquad & F_T &= TF + O(1), \\ G_T &= TG + O(1), \qquad & H_T &= T^2H + O(T). \end{aligned} \tag{4.57}$$

Let

$$Q = [D^X, \mathcal{C}]. \tag{4.58}$$

Then $Q(\mathbb{E}_1^0) \subset \mathbb{E}_1^{0,\perp}$, and Q is a smooth family of first order elliptic operators acting along the fibers X. Moreover,

$$\begin{aligned} E &= P(\mathcal{C}^2 + u^{-2}du\wedge(D^Y - c(T_2)/4))P, \quad & F &= PQP^{\perp}, \\ G &= P^{\perp}QP, & H &= P^{\perp}D^{X,2}P^{\perp}, \end{aligned} \tag{4.59}$$

and

$$\mathcal{B}_2 = E - FH^{-1}G. \tag{4.60}$$

Proof. By (4.38) and (4.45), we have

$$B_{3,T} = TD^X + \mathcal{C} + O\left(\frac{1}{T}\right). \tag{4.61}$$

From (4.48) and (4.55), we get (4.59).

Let ${}^0R_Z$ be the curvature of ${}^0\nabla^{\mathcal{S}_Z} - \frac{1}{2}\langle S_3(e_i)e_i,\cdot\rangle - \frac{1}{2}\langle S_3(f^H_{p,1})f^H_{p,1},\cdot\rangle$. As in (4.17), we have

$$\begin{aligned}[]
&[D^X,\, {}^0\nabla^{\mathscr{E}_Z,u}] = c(e_i)g_3^{\alpha,H}\wedge\left({}^0R_Z(e_i, g^H_{\alpha,3}) - {}^0\nabla^{\mathcal{S}_Z}_{T_3(e_i,g^H_{\alpha,3})}\right),\\
&({}^0\nabla^{\mathscr{E}_Z,u})^2 = g_3^{\alpha,H}\wedge g_3^{\beta,H}\wedge\left({}^0R_Z(g^H_{\alpha,3}, g^H_{\alpha,3}) - {}^0\nabla^{\mathcal{S}_Z}_{T_3(g^H_{\alpha,3},g^H_{\alpha,3})}\right)
\end{aligned} \tag{4.62}$$

and $T_3(e_i, g^H_{\alpha,3}) \in TX$, $T_3(g^H_{\alpha,3}, g^H_{\alpha,3}) \in TZ$. By (4.17), (4.45) and (4.62), we know that $Q = [D^X, \mathcal{C}]$ is a smooth family of first order elliptic operators acting along the fibers X and $Q(\mathbb{E}^0_1) \subset \mathbb{E}^{0,\perp}_1$.

By (4.43), (4.52) and (4.59), we know that

$$\begin{aligned}
E - FH^{-1}G &= P(\mathcal{C}^2 + u^{-2}du\wedge(D^Y - c(T_2)/4))P\\
&\quad - P\mathcal{C}D^XP^\perp(D^{X,2})^{-2}P^\perp D^X\mathcal{C}P = (P\mathcal{C}P)^2 + u^{-2}du\wedge(D^Y - c(T_2)/4) = \mathcal{B}_2
\end{aligned} \tag{4.63}$$

The proof of Proposition 4.7 is complete. □

4.3. **Proof of Theorem 3.3.** If C is an operator, let $\mathrm{Sp}(C)$ be the spectrum of C. The following lemma is an analogue of [8, Proposition 9.2].

Lemma 4.8. *For any $u > 0$, $T \geq 1$,*

$$\begin{aligned}
\mathrm{Sp}(\mathcal{B}_2) &= \mathrm{Sp}(D^{Y,2}),\\
\mathrm{Sp}(\mathcal{B}_{u,T}) &= \mathrm{Sp}(u^2D_T^{Z,2}) = \mathrm{Sp}(u^2\mathcal{B}_T).
\end{aligned} \tag{4.64}$$

Proof. We only prove the first formula. The proof of the second one is the same.

By (4.43), set

$$\begin{aligned}
\mathcal{R} := \mathcal{B}_2 - D^{Y,2} = \left(\nabla^{\mathscr{E}_Y,u} - \frac{1}{4}c(T_2)\right)^2 + \left[D^Y, \nabla^{\mathscr{E}_Y,u} - \frac{1}{4}c(T_2)\right]\\
+ \frac{1}{u^2}du\wedge\left(D^Y - \frac{c(T_2)}{4}\right).
\end{aligned} \tag{4.65}$$

Take $\lambda \notin \mathrm{Sp}(D^{Y,2})$. Then

$$(\lambda - \mathcal{B}_2)^{-1} - (\lambda - D^{Y,2})^{-1} = (\lambda - D^{Y,2})^{-1}\mathcal{R}(\lambda - \mathcal{B}_2)^{-1}. \tag{4.66}$$

Inductively,

$$\begin{aligned}
(\lambda - \mathcal{B}_2)^{-1} &= (\lambda - D^{Y,2})^{-1} + (\lambda - D^{Y,2})^{-1}\mathcal{R}(\lambda - D^{Y,2})^{-1}\\
&\quad + (\lambda - D^{Y,2})^{-1}\mathcal{R}(\lambda - D^{Y,2})^{-1}\mathcal{R}(\lambda - D^{Y,2})^{-1} + \cdots.
\end{aligned} \tag{4.67}$$

Since $\mathcal{R}$ has positive degree in $\Lambda(T^*S)$, the expansion above has finite terms.

By elliptic estimate, there exist $c_1, c_2 > 0$, such that for any $s \in \mathbb{E}_1$,

$$\|(\lambda - D^{Y,2})s\|_{Y,0} \geq c_1\|s\|_{Y,2} - c_2\|s\|_{Y,0}. \tag{4.68}$$

Then there exists $c > 0$ such that

$$\|(\lambda - D^{Y,2})^{-1}s\|_{Y,2} \leq \frac{1}{c_1}\|s\|_{Y,0} + \frac{c_2}{c_1}\|(\lambda - D^{Y,2})^{-1}s\|_{Y,0} \leq c\|s\|_{Y,0}. \tag{4.69}$$

From (4.62) and (4.65), there exists $c > 0$ such that

$$\|\mathcal{R}s\|_{Y,0} \leq c\|s\|_{Y,1}. \tag{4.70}$$

By (4.67), (4.69) and (4.70), there exists $c > 0$, such that

$$\|(\lambda - \mathcal{B}_2)^{-1}s\|_{Y,0} \leq c\|s\|_{Y,0}. \tag{4.71}$$

So $\lambda \notin \mathrm{Sp}(\mathcal{B}_2)$.

Exchange $\mathcal{B}_2$ and $D^{Y,2}$, we get the first formula of (4.64). □

By Lemma 4.8, we have

$$\begin{aligned} \exp(-u^2\mathcal{B}_T) &= \frac{1}{2\pi\sqrt{-1}}\int_\Delta \frac{\exp(-u^2\lambda)}{\lambda - \mathcal{B}_T}d\lambda, \\ \exp(-u^2\mathcal{B}_2) &= \frac{1}{2\pi\sqrt{-1}}\int_\Delta \frac{\exp(-u^2\lambda)}{\lambda - \mathcal{B}_2}d\lambda. \end{aligned} \tag{4.72}$$

Lemma 4.9. *There exist $T_0 \geq 1, C > 0, k \in \mathbb{N}$, such that for $T \geq T_0$, $\lambda \in \Delta$, the resolvent $(\lambda - \mathcal{B}_T)^{-1}$ exists, extends to a continuous linear operator from $\mathbb{E}_0^{-1}$ into $\mathbb{E}_0^1$, and moreover*

$$\begin{aligned} \|(\lambda - \mathcal{B}_T)^{-1}\| &\leq C(1 + |\lambda|)^k, \\ |(\lambda - \mathcal{B}_T)^{-1}|_T^{-1,1} &\leq C(1 + |\lambda|)^k. \end{aligned} \tag{4.73}$$

Proof. Set

$$\mathcal{R}_T := \mathcal{B}_T - D_T^{Z,2}. \tag{4.74}$$

By (4.17), (4.38), (4.48) and (4.62), we know that $\mathcal{R}_T$ is a first order fiberwise differential operator along the fiber Z. Moreover, from (4.8), for $i = -1, 0$, there exists $C_i > 0$, such that for any $s \in \mathbb{E}_0^i$,

$$|\mathcal{R}_T s|_{T,i} \leq C_i|s|_{T,i+1}. \tag{4.75}$$

Take $\lambda \in \Delta$. Then

$$\begin{aligned} (\lambda - \mathcal{B}_T)^{-1} &= (\lambda - D_T^{Z,2})^{-1} + (\lambda - D_T^{Z,2})^{-1}\mathcal{R}_T(\lambda - D_T^{Z,2})^{-1} \\ &\quad + (\lambda - D_T^{Z,2})^{-1}\mathcal{R}_T(\lambda - D_T^{Z,2})^{-1}\mathcal{R}_T(\lambda - D_T^{Z,2})^{-1} + \cdots. \end{aligned} \tag{4.76}$$

Since $\mathcal{R}_T$ has positive degree in $\Lambda(T^*(\mathbb{R} \times S))$, the expansion above has finite terms.

From (4.75), and (4.76) and Lemma 4.4, there exist $T_0 \geq 1, C > 0, k \in \mathbb{N}$, such that for $T \geq T_0$, $\lambda \in \Delta$, the resolvent $(\lambda - \mathcal{B}_T)^{-1}$ exists, extends to a continuous linear operator from $\mathbb{E}_0^{-1}$ into $\mathbb{E}_0^1$, and moreover

$$\begin{aligned} \|(\lambda - \mathcal{B}_T)^{-1}\| &\leq C(1 + |\lambda|)^k, \\ |(\lambda - \mathcal{B}_T)^{-1}|_T^{-1,1} &\leq C(1 + |\lambda|)^k. \end{aligned} \tag{4.77}$$

The proof of Lemma 4.9 is complete. □

Similarly, there exist $C > 0, k \in \mathbb{N}$, such that for $\lambda \in \Delta$, the resolvent $(\lambda - \mathcal{B}_2)^{-1}$ exists, and for any $s \in \mathbb{E}_1^0$, $s' \in \mathbb{E}_1^{-1}$, we have

$$\begin{aligned} \|(\lambda - \mathcal{B}_2)^{-1} s\|_{Y,0} &\leq C(1+|\lambda|)^k \|s\|_{Y,0}, \\ \|(\lambda - \mathcal{B}_2)^{-1} s'\|_{Y,1} &\leq C(1+|\lambda|)^k \|s'\|_{Y,-1}. \end{aligned} \tag{4.78}$$

Replacing $\mathcal{B}_T$ by H_T and $D_T^{Z,2}$ by $P^\perp D_T^{Z,2} P^\perp$ in the proof of Lemma 4.9, we can get the following lemma.

Lemma 4.10. *There exist $T_0 \geq 1, C > 0, k \in \mathbb{N}$, such that for $T \geq T_0, \lambda \in \Delta$, the resolvent $(\lambda - H_T)^{-1}$ exists, and for any $s \in \mathbb{E}_0^{2,\perp}$, we have*

$$\begin{aligned} \|(\lambda - H_T)^{-1} s\|_0 &\leq C(1+|\lambda|)^k \|s\|_0, \\ |(\lambda - H_T)^{-1} s|_{T,1} &\leq C(1+|\lambda|)^k |s|_{T,-1}. \end{aligned} \tag{4.79}$$

Choose $s, s' \in \mathbb{E}_0$ such that $s = (\lambda - \mathcal{B}_T)^{-1} s'$, $\lambda \in \Delta$. Then by (4.55), we have

$$\begin{aligned} P s' &= (\lambda - E_T) P s - F_T P^\perp s, \\ P^\perp s' &= -G_T P s + (\lambda - H_T) P^\perp s. \end{aligned} \tag{4.80}$$

Let

$$\mathcal{E}_T(\lambda) = \lambda - E_T - F_T (\lambda - H_T)^{-1} G_T. \tag{4.81}$$

Then

$$P(\lambda - \mathcal{B}_T)^{-1} P = \mathcal{E}_T(\lambda)^{-1}. \tag{4.82}$$

By (4.82) and Lemma 4.9, there exist $T_0 \geq 1, C > 0, k \in \mathbb{N}$, such that for $T \geq T_0$, $\lambda \in \Delta$, $s \in \mathbb{E}_0$,

$$\begin{aligned} \|\mathcal{E}_T(\lambda)^{-1} s\|_0 &\leq C(1+|\lambda|)^k \|s\|_0, \\ |\mathcal{E}_T(\lambda)^{-1} s|_{T,1} &\leq C(1+|\lambda|)^k |s|_{T,-1}. \end{aligned} \tag{4.83}$$

Lemma 4.11. *There exist $C > 0$, $T_0 \geq 1$, $k \in \mathbb{N}$, such that for $T \geq T_0$, $\lambda \in \Delta$, $s \in \mathbb{E}_0$,*

$$\|(\mathcal{E}_T(\lambda)^{-1} - P(\lambda - \mathcal{B}_2)^{-1} P) s\|_0 \leq \frac{C(1+|\lambda|)^k}{T} \|s\|_0. \tag{4.84}$$

Proof. We know that

$$\mathcal{E}_T(\lambda)^{-1} - P(\lambda - \mathcal{B}_2)^{-1} P = P \mathcal{E}_T(\lambda)^{-1} (\lambda - \mathcal{B}_2 - \mathcal{E}_T(\lambda))(\lambda - \mathcal{B}_2)^{-1} P. \tag{4.85}$$

By (4.60) and (4.81),

$$\begin{aligned} \lambda - \mathcal{B}_2 - \mathcal{E}_T(\lambda) &= E_T + F_T(\lambda - H_T)^{-1} G_T - E + F H^{-1} G \\ &= E_T - E + (F_T - TF)(\lambda - H_T)^{-1} G_T P s + TF((\lambda - H_T)^{-1} + T^{-2} H^{-1}) G_T P s \\ &\qquad + T^{-1} F H^{-1} (G_T - TG) P s. \end{aligned} \tag{4.86}$$

From Proposition 4.7, there exists $C > 0$ such that for $s, s' \in \mathbb{E}_0$,

$$\|(E_T - E) P s\|_0 \leq \frac{C}{T} \|P s\|_1, \tag{4.87}$$

and

$$|\langle G_T Ps, s'\rangle_0| \leq C\|Ps\|_1 |s'|_{T,1}. \tag{4.88}$$

By (4.88),

$$|G_T Ps|_{T,-1} \leq C\|Ps\|_1. \tag{4.89}$$

Similarly, we have

$$|(G_T - TG)Ps|_{T,-1} \leq \frac{C}{T}\|Ps\|_1. \tag{4.90}$$

Since $[P, {}^0\nabla^{\mathcal{S}_Z}_{f^H_{p,1}}]$ is a bounded operator, by Proposition 4.7, there exists $C > 0$, such that for $s \in \mathbb{E}_0$, we have

$$\begin{aligned}\|F_T s\|_0 &\leq CT \sum_i \|{}^0\nabla^{\mathcal{S}_Z}_{e_i} P^\perp s\|_0 + C\sum_p \|P\, {}^0\nabla^{\mathcal{S}_Z}_{f^H_{p,1}} P^\perp s\|_0 \\ &\leq CT \sum_i \|{}^0\nabla^{\mathcal{S}_Z}_{e_i} P^\perp s\|_0 + C\sum_p \|[P, {}^0\nabla^{\mathcal{S}_Z}_{f^H_{p,1}}]P^\perp s\|_0 \leq C|s|_{T,1}.\end{aligned} \tag{4.91}$$

From (4.91), we have

$$\|(F_T - TF)s\|_0 \leq \frac{C}{T}\|s\|_{T,1}. \tag{4.92}$$

So from (4.8), (4.89), (4.92), Proposition 4.7 and Lemma 4.10, we have

$$\begin{aligned}&\|(F_T - TF)(\lambda - H_T)^{-1} G_T Ps\|_0 \leq \frac{C}{T}|(\lambda - H_T)^{-1} G_T Ps|_{T,1} \\ \leq &\frac{C(1+|\lambda|)^k}{T}|G_T Ps|_{T,-1} \leq \frac{C(1+|\lambda|)^k}{T}\|Ps\|_1.\end{aligned} \tag{4.93}$$

By (4.75), there exists $C > 0$, such that for any $s \in \mathbb{E}_0^{2,\perp}$,

$$|(H_T - T^2 H)s|_{T,-1} \leq C_i |s|_{T,0}. \tag{4.94}$$

From (4.8), (4.14), (4.89), (4.94), Proposition 4.7 and Lemma 4.10, we have

$$\begin{aligned}&\|TF((\lambda - H_T)^{-1} + T^{-2}H^{-1})G_T Ps\|_0 \\ \leq &C|(\lambda - H_T)^{-1}(\lambda - (H_T - T^2 H))T^{-2}H^{-1} G_T Ps|_{T,1} \\ \leq &C|\lambda||(\lambda - H_T)^{-1}(T^2H)^{-1} G_T Ps|_{T,1} \\ &+ C|(\lambda - H_T)^{-1}(H_T - T^2H)(T^2H)^{-1} G_T Ps|_{T,1} \\ \leq &C(1+|\lambda|)^{k+1}|(T^2H)^{-1} G_T Ps|_{T,-1} \\ &+ C(1+|\lambda|)^k|(H_T - T^2H)(T^2H)^{-1} G_T Ps|_{T,-1} \\ \leq &\frac{C(1+|\lambda|)^{k+1}}{T^2}|G_T Ps|_{T,-1} + \frac{C(1+|\lambda|)^k}{T^2}\|G_T Ps\|_0 \\ \leq &\frac{C(1+|\lambda|)^{k+1}}{T}\|Ps\|_1.\end{aligned} \tag{4.95}$$

From (4.14), (4.16) and (4.59), there exists $C > 0$, such that for any $s \in \mathbb{E}_0^{2,\perp}$, we have

$$\langle T^2 Hs, s\rangle \geq C|s|_{T,1}. \tag{4.96}$$

By (4.96),

$$|(T^2H)^{-1}s|_{T,1} \leq C|s|_{T,-1}. \tag{4.97}$$

From (4.90), (4.97) and Proposition 4.7, we have

$$\begin{aligned} \|T^{-1}FH^{-1}G(T)Ps\|_0 &\leq C|(T^2H)^{-1}(G_T - TG)Ps|_{T,1} \\ &\leq C|(G_T - TG)Ps|_{T,-1} \leq \frac{C}{T}\|Ps\|_1. \end{aligned} \tag{4.98}$$

Therefore, from (4.86), (4.87), (4.93), (4.95) and (4.98), there exists $C > 0$, such that

$$\|(\lambda - \mathcal{B}_2 - \mathcal{E}_T(\lambda))Ps\|_0 \leq \frac{C(1+|\lambda|)^{k+1}}{T}\|Ps\|_1. \tag{4.99}$$

Then Lemma 4.11 follows from (4.78), (4.83), (4.85) and (4.99). □

Lemma 4.12. *There exist $C > 0$, $T_0 \geq 1$, $k \in \mathbb{N}$, such that for $T \geq T_0$, $\lambda \in \Delta$,*

$$\begin{aligned} &\|P(\lambda - \mathcal{B}_T)^{-1}P - P(\lambda - \mathcal{B}_2)^{-1}P\| \leq \frac{C(1+|\lambda|)^k}{T}, \\ &\|P(\lambda - \mathcal{B}_T)^{-1}P^\perp\| \leq \frac{C(1+|\lambda|)^k}{T}, \\ &\|P^\perp(\lambda - \mathcal{B}_T)^{-1}P\| \leq \frac{C(1+|\lambda|)^k}{T}, \\ &\|P^\perp(\lambda - \mathcal{B}_T)^{-1}P^\perp\| \leq \frac{C(1+|\lambda|)^k}{T^2}. \end{aligned} \tag{4.100}$$

Proof. The first inequality follows from (4.82) and Lemma 4.11.

By (4.80), we find that

$$\begin{aligned} &P(\lambda - \mathcal{B}_T)^{-1}P^\perp = \mathcal{E}_T(\lambda)^{-1}F_T(\lambda - H_T)^{-1}, \\ &P^\perp(\lambda - \mathcal{B}_T)^{-1}P = (\lambda - H_T)^{-1}G_T\mathcal{E}_T(\lambda)^{-1}, \\ &P^\perp(\lambda - \mathcal{B}_T)^{-1}P^\perp = (\lambda - H_T)^{-1}(1 + G_TP(\lambda - \mathcal{B}_T)^{-1}P^\perp). \end{aligned} \tag{4.101}$$

From (4.83), (4.91) and Lemma 4.10, there exists $C > 0$, such that for $s \in \mathbb{E}_0$,

$$\begin{aligned} &\|\mathcal{E}_T(\lambda)^{-1}F_T(\lambda - H_T)^{-1}P^\perp s\|_0 \leq C\|F_T(\lambda - H_T)^{-1}P^\perp s\|_0 \\ \leq &C|(\lambda - H_T)^{-1}P^\perp s|_{T,1} \leq C(1+|\lambda|)^k|P^\perp s|_{T,-1} \leq \frac{C(1+|\lambda|)^k}{T}\|s\|_0. \end{aligned} \tag{4.102}$$

From (4.8), (4.83), (4.89) and Lemma 4.10, there exists $C > 0$, such that for $s \in \mathbb{E}_0$,

$$\begin{aligned} &\|(\lambda - H_T)^{-1}G_T\mathcal{E}_T(\lambda)^{-1}Ps\|_0 \leq \frac{1}{T}|(\lambda - H_T)^{-1}G_T\mathcal{E}_T(\lambda)^{-1}Ps|_{T,1} \\ \leq &\frac{C(1+|\lambda|)^k}{T}|G_T\mathcal{E}_T(\lambda)^{-1}Ps|_{T,-1} \leq \frac{C(1+|\lambda|)^k}{T}\|\mathcal{E}_T(\lambda)^{-1}Ps\|_1 \\ \leq &\frac{C(1+|\lambda|)^k}{T}|\mathcal{E}_T(\lambda)^{-1}Ps|_{T,1} \leq \frac{C(1+|\lambda|)^{2k}}{T}\|s\|_0. \end{aligned} \tag{4.103}$$

From (4.102) and (4.103), there exists $C>0$, such that for $s\in\mathbb{E}_0$,

$$\begin{aligned}&\|(\lambda-H_T)^{-1}G_T\mathcal{E}_T(\lambda)^{-1}F_T(\lambda-H_T)^{-1}P^{\perp}s\|_0\\ &\leq\frac{C(1+|\lambda|)^{2k}}{T}\|F_T(\lambda-H_T)^{-1}P^{\perp}s\|_0\leq\frac{C(1+|\lambda|)^{3k}}{T^2}\|s\|_0.\end{aligned}\tag{4.104}$$

From (4.8) and Lemma 4.10, we have

$$\begin{aligned}\|(\lambda-H_T)^{-1}s\|_0\leq\frac{1}{T}|(\lambda-H_T)^{-1}s|_{T,1}\leq\frac{C(1+|\lambda|)^k}{T}|P^{\perp}s|_{T,-1}\\ \leq\frac{C(1+|\lambda|)^k}{T^2}\|s\|_0.\end{aligned}\tag{4.105}$$

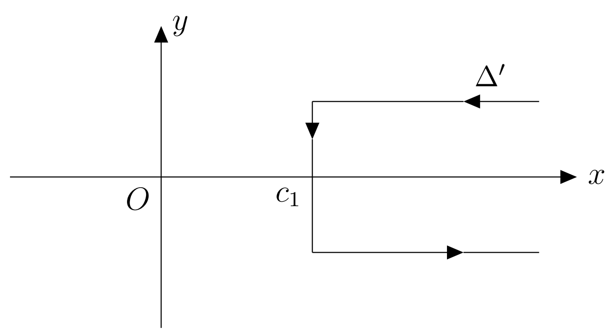
By (4.104) and (4.105), we get the last estimate in (4.100)

Then the proof of Lemma 4.12 is complete. □

We assume that $\ker D^Y=0$. There exists $c_1>0$, such that $\mathrm{Sp}(\mathcal{B}_2)=\mathrm{Sp}(D^{Y,2})\subset[2c_1,+\infty)$. By Lemma 4.8 and Proposition 4.12, we know that when T is sufficiently large,

$$\mathrm{Sp}(D_T^{Z,2})=\mathrm{Sp}(\mathcal{B}_T)\subset[c_1,+\infty).\tag{4.106}$$

Note that in this section, we need not assume that $\ker D_T^Z=0$. Therefore, we get another proof of Lemma 2.2.



Let Δ' be the oriented contour in the above picture. Then all the estimates in this Section hold for any $\lambda\in\Delta'$. From (4.106), there exists $T_0\geq 1$, for $u>0$, $T\geq T_0$,

$$\exp(-u^2\mathcal{B}_T)=\frac{1}{2\pi\sqrt{-1}}\int_{\Delta'}\frac{e^{-u^2\lambda}}{\lambda-\mathcal{B}_T}d\lambda.\tag{4.107}$$

From (4.72) and Lemma 4.12, we get the following theorem.

Theorem 4.13. *For $u_0>0$ fixed, there exist $C,C'>0$ and $T_0\geq 1$ such that for $T\geq T_0$, $u\geq u_0$,*

$$\|\exp(-u^2\mathcal{B}_T)-P\exp(-u^2\mathcal{B}_2)P\|\leq\frac{C}{T}\exp(-C'u^2).\tag{4.108}$$

Let $\exp(-u^2\mathcal{B}_T)(z,z')$, $P\exp(-u^2\mathcal{B}_2)P(z,z')$ $(z,z'\in Z_b, b\in S)$ be the smooth kernels of the operators $\exp(-u^2\mathcal{B}_T)$, $P\exp(-u^2\mathcal{B}_2)P$ calculated with respect to $dv_Z(z')$.

By using the proof of [25, Theorems 5.22] and the fact that $\ker D^Y=0$, we have

Proposition 4.14. *(i) For $u_0>0$ fixed, for $m\in\mathbb{N}$, $b\in S$, there exist $C,C'>0$, $T_0\geq 1$, such that for $z,z'\in Z_b$, $u\geq u_0$, $T\geq T_0$,*

$$\sup_{|\alpha|,|\alpha'|\leq m}\left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial z^{\alpha}\partial z'^{\alpha'}}\exp(-u^2\mathcal{B}_T)(z,z')\right|\leq C\exp(-C'u^2). \tag{4.109}$$

(ii) For $u_0>0$ fixed, for $m\in\mathbb{N}$, $b\in S$, there exist $C,C'>0$, $T_0\geq 1$, such that for $z,z'\in Z_b$, $u\geq u_0$, $T\geq T_0$,

$$\sup_{|\alpha|,|\alpha'|\leq m}\left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial z^{\alpha}\partial z'^{\alpha'}}P\exp(-u^2\mathcal{B}_2)P(z,z')\right|\leq C\exp(-C'u^2). \tag{4.110}$$

The complete proof of Proposition 4.14 is left to the next subsection.

From Proposition 4.14 i), we get Theorem 3.3 ii).

Let inj^Z be the injectivity radius of (Z_b,g^{TZ_b}). For $z\in Z_b$, we will identify $B^{T_zZ_b}(0,\varepsilon)$ with $B^{Z_b}(z,\varepsilon)$ by the canonical exponential map when $\varepsilon<\mathrm{inj}^Z$.

Let $\phi:\mathbb{R}^n\to[0,1]$ be a smooth function with compact support in $B(0,\mathrm{inj}^Z/2)$, equal 1 near 0 such that $\int_{\mathbb{R}^n}\phi(W)dv(W)=1$. Take $v\in(0,1]$. By Taylor expansion and Proposition 4.14, there exists $c>0$, such that

$$\begin{aligned}|(\exp(-u^2\mathcal{B}_T)-P\exp(-u^2\mathcal{B}_2)P)(vW,vW')&\\ -(\exp(-u^2\mathcal{B}_T)-P\exp(-u^2\mathcal{B}_2)P)(0,0)|&\leq cv\exp(-C'u^2)\end{aligned} \tag{4.111}$$

for $|W|,|W'|$ are sufficiently small. Then for $U,U'\in\mathbb{E}_0$,

$$\begin{aligned}&|\langle(\exp(-u^2\mathcal{B}_T)-P\exp(-u^2\mathcal{B}_2)P)(0,0)U,U'\rangle_0\\ &-\int_{\mathbb{R}^n\times\mathbb{R}^n}\langle(\exp(-u^2\mathcal{B}_T)-P\exp(-u^2\mathcal{B}_2)P)(vW,vW')U,U'\rangle_0\\ &\quad\times\phi(W)\phi(W')dv(W)dv(W')|\leq cv|U||U'|\exp(-C'u^2).\end{aligned} \tag{4.112}$$

On the other hand, By Theorem 4.13,

$$\begin{aligned}&\left|\int_{\mathbb{R}^n\times\mathbb{R}^n}\langle(\exp(-u^2\mathcal{B}_T)-P\exp(-u^2\mathcal{B}_2)P)(vW,vW')U,U'\rangle_0\right.\\ &\qquad\times\phi(W)\phi(W')dv(W)dv(W')|\\ &\leq\frac{c}{Tv^n}|U||U'|\exp(-C'u^2).\end{aligned} \tag{4.113}$$

Take $v=T^{-\frac{1}{n+1}}$. From (4.112) and (4.113), we get

$$|(\exp(-u^2\mathcal{B}_T)-P\exp(-u^2\mathcal{B}_2)P)(0,0)|\leq cT^{-\frac{1}{n+1}}\exp(-C'u^2). \tag{4.114}$$

Therefore, we can get the following theorem.

Theorem 4.15. *For $u_0>0$ fixed, there exist $C,C'>0$, $T_0\geq 1$, $\delta>0$, such that for $u\geq u_0$, $T\geq T_0$,*

$$\left|\psi_S\delta_{u^2}\widetilde{\mathrm{Tr}}[g\exp(-u^2\mathcal{B}_T)]-\psi_S\delta_{u^2}\widetilde{\mathrm{Tr}}[g\exp(-u^2\mathcal{B}_2)]\right|\leq\frac{C}{T^{\delta}}\exp(-C'u^2). \tag{4.115}$$

By (4.50) and (4.53), we can get Theorem 3.3 by taking the coefficients of du in (5.5).

The proof of Theorem 3.3 is complete.

4.4. **Proof of Theorem 4.14.** Recall that we assume that S is compact for simplicity in Section 3.2. There exist a family of $\mathscr{C}^\infty$ sections of TY (resp. TX), $U_1, \cdots, U_r$ (resp. $U_1', \cdots, U_{r'}'$), such that for any $y \in V$ (resp. $x \in W$), $U_1(y), \cdots, U_r(y)$ (resp. $U_1'(x), \cdots, U_{r'}(x)$) span T_yY (resp. T_xX).

Definition 4.16. Let $\mathcal{D}$ be a family of operators on $\mathbb{E}_0$,

$$\mathcal{D} = \left\{ P\,{}^0\nabla^{\mathcal{S}_Z}_{U^H_{p,1}} P + \ P^\perp\,{}^0\nabla^{\mathcal{S}_Z}_{U^H_{p,1}} P^\perp,\ P^\perp\,{}^0\nabla^{\mathcal{S}_Z}_{U_i'} P^\perp \right\}. \tag{4.116}$$

Note that in [25, (5.60)], the corresponding set of operators is stated as $\{p_T\,{}^0\nabla^{\Lambda(T^{*(0,1)}Z)\otimes\xi}_{U^H_{l,1}} p_T$, $p_T^\perp\,{}^0\nabla^{\Lambda(T^{*(0,1)}Z)\otimes\xi}_{U^H_{l,1}} p_T^\perp,\ p_T^\perp\,{}^0\nabla^{\Lambda(T^{*(0,1)}Z)\otimes\xi}_{U_i'} p_T^\perp\}$. We need to read [25, (5.60)] as $\mathcal{D}_T = \{p_T\,{}^0\nabla^{\Lambda(T^{*(0,1)}Z)\otimes\xi}_{U^H_{l,1}} p_T + \ p_T^\perp\,{}^0\nabla^{\Lambda(T^{*(0,1)}Z)\otimes\xi}_{U^H_{l,1}} p_T^\perp,\ p_T^\perp\,{}^0\nabla^{\Lambda(T^{*(0,1)}Z)\otimes\xi}_{U_i'} p_T^\perp\}$. In this way, the corresponding commutator $[Q_1, [Q_2, \cdots [Q_k, A_T^2], \cdots]]$ has the same structure as A_T^2 (see the following proof of Lemma 4.17).

Lemma 4.17. *For any $k \in \mathbb{N}$ fixed, there exists $C_k > 0$, $T_0 \geq 1$ such that for $T \geq T_0$, $Q_1, \cdots, Q_k \in \mathcal{D}$ and $s, s' \in \mathbb{E}_0^2$, we have*

$$|\langle [Q_1, [Q_2, \cdots [Q_k, \mathcal{B}_T], \cdots]]s, s'\rangle_0| \leq C_k |s|_{T,1} |s'|_{T,1}. \tag{4.117}$$

Proof. Set $\mathscr{S}$ be the set of uniformly bounded operators along the fiber X with smooth kernel. Set

$$\begin{aligned}
\Theta_1 &= \left\{ a_{ij}\,{}^0\nabla^{\mathcal{S}_Z}_{U_i'}\,{}^0\nabla^{\mathcal{S}_Z}_{U_j'} + b \,:\, a_{ij} \in \mathscr{C}^\infty(W, C(TZ)), b \in \mathscr{S} \right\},\\
\Theta_2 &= \left\{ a_i\,{}^0\nabla^{\mathcal{S}_Z}_{U_i'} + b \,:\, a_i \in \mathscr{C}^\infty(W, C(TZ)), b \in \mathscr{S} \right\},\\
\Theta_3 &= \Big\{ b_{pq}\,{}^0\nabla^{\mathcal{S}_Z}_{U_p}\,{}^0\nabla^{\mathcal{S}_Z}_{U_q} + b_p\,{}^0\nabla^{\mathcal{S}_Z}_{U_p} + a_i\,{}^0\nabla^{\mathcal{S}_Z}_{U_i'} + b \,:\, a_i \in \mathscr{C}^\infty(W, C(TZ)),\\
&\qquad\qquad b_{pq}, b_p, b \in \mathscr{S} \Big\}.
\end{aligned} \tag{4.118}$$

By (4.17), (4.38), (4.48), (4.51) and (4.62), we can split the operator $\mathcal{B}_T$ such that

$$\mathcal{B}_T = T^2 P^\perp A_1 P^\perp + T(P^\perp A_2 P^\perp + P A_2' P^\perp + P^\perp A_2' P) + A_3, \tag{4.119}$$

where $A_1 \in \Theta_1$, $A_2, A_2' \in \Theta_2$, $A_3 \in \Theta_3$.

First, we consider the case when $k = 1$.

a) The case where $Q = P\,{}^0\nabla^{\mathcal{S}_Z}_{U^H_{p,1}} P + \ P^\perp\,{}^0\nabla^{\mathcal{S}_Z}_{U^H_{p,1}} P^\perp$.

We observe that if $b \in \mathscr{S}$, so are $\left[{}^0\nabla^{\mathcal{S}_Z}_{U^H_{p,1}}, b\right]$, ${}^0\nabla^{\mathcal{S}_Z}_{U_i'}\, b$ and $b\,{}^0\nabla^{\mathcal{S}_Z}_{U_i'}$.

Then we have

$$\begin{aligned}
[Q, P^\perp A_1 P^\perp] &= P^\perp \left(\left[{}^0\nabla^{\mathcal{S}_Z}_{U^H_{p,1}}, A_1\right] + \left[{}^0\nabla^{\mathcal{S}_Z}_{U^H_{p,1}}, P\right] A_1 + A_1 \left[{}^0\nabla^{\mathcal{S}_Z}_{U^H_{p,1}}, P\right] \right) P^\perp,\\
[Q, P^\perp A_2 P^\perp] &= P^\perp \left(\left[{}^0\nabla^{\mathcal{S}_Z}_{U^H_{p,1}}, A_2\right] + \left[{}^0\nabla^{\mathcal{S}_Z}_{U^H_{p,1}}, P\right] A_2 + A_2 \left[{}^0\nabla^{\mathcal{S}_Z}_{U^H_{p,1}}, P\right] \right) P^\perp,\\
[Q, P A_2' P^\perp] &= P \left(\left[{}^0\nabla^{\mathcal{S}_Z}_{U^H_{p,1}}, A_2'\right] + \left[{}^0\nabla^{\mathcal{S}_Z}_{U^H_{p,1}}, P\right] A_2' - A_2' \left[{}^0\nabla^{\mathcal{S}_Z}_{U^H_{p,1}}, P\right] \right) P^\perp,\\
[Q, P^\perp A_2' P] &= P^\perp \left(\left[{}^0\nabla^{\mathcal{S}_Z}_{U^H_{p,1}}, A_2'\right] - \left[{}^0\nabla^{\mathcal{S}_Z}_{U^H_{p,1}}, P\right] A_2' + A_2' \left[{}^0\nabla^{\mathcal{S}_Z}_{U^H_{p,1}}, P\right] \right) P,
\end{aligned} \tag{4.120}$$

and $\left[{}^0\nabla^{\mathcal{S}_Z}_{U^H_{p,1}}, A_i\right] \in \Theta_i$, $A_i\left[{}^0\nabla^{\mathcal{S}_Z}_{U^H_{p,1}}, P\right] \in \Theta_i$, $\left[{}^0\nabla^{\mathcal{S}_Z}_{U^H_{p,1}}, A'_2\right] \in \Theta_2$, $A'_2\left[{}^0\nabla^{\mathcal{S}_Z}_{U^H_{p,1}}, P\right] \in \Theta_2$. For the element in Θ_3, since the principal symbol of Q is identity, we have $[Q, A_3] \in \Theta_3$.

So $[Q, \mathcal{B}_T]$ has the same structure as $\mathcal{B}_T$ in (4.119).

b) The case where $Q = P^\perp\, {}^0\nabla^{\mathcal{S}_Z}_{U'_i} P^\perp$.

As in (4.120), we have

$$\begin{aligned}
[Q, P^\perp A_1 P^\perp] &= P^\perp \left(\left[{}^0\nabla^{\mathcal{S}_Z}_{U'_i}, A_1\right] + {}^0\nabla^{\mathcal{S}_Z}_{U'_i} P A_1 + A_1 P\, {}^0\nabla^{\mathcal{S}_Z}_{U'_i} \right) P^\perp, \\
[Q, P^\perp A_2 P^\perp] &= P^\perp \left(\left[{}^0\nabla^{\mathcal{S}_Z}_{U'_i}, A_2\right] + {}^0\nabla^{\mathcal{S}_Z}_{U'_i} P A_2 + A_2 P\, {}^0\nabla^{\mathcal{S}_Z}_{U'_i} \right) P^\perp, \\
[Q, P A'_2 P^\perp] &= P \left(- \left[{}^0\nabla^{\mathcal{S}_Z}_{U'_i}, A'_2\right] + A'_2 P\, {}^0\nabla^{\mathcal{S}_Z}_{U'_i} \right) P^\perp, \\
[Q, P^\perp A'_2 P] &= P^\perp \left(\left[{}^0\nabla^{\mathcal{S}_Z}_{U'_i}, A'_2\right] - {}^0\nabla^{\mathcal{S}_Z}_{U'_i} P A'_2 \right) P.
\end{aligned} \tag{4.121}$$

Since $[Q, A_3] \in \Theta_3$, we know that $[Q, \mathcal{B}_T]$ has the same structure as $\mathcal{B}_T$ in (4.119).

c) Higher order commutators

The estimate of higher order commutators are obtained inductively from a) and b).

The proof of Lemma 4.17 is complete. □

For $k \in \mathbb{N}$, let $\mathcal{D}^k$ be the family of operators Q which can be written in the form

$$Q = Q_1 \cdots Q_k, \quad Q_i \in \mathcal{D}. \tag{4.122}$$

If $k \in \mathbb{N}$, we define the Hilbert norm $\|\cdot\|'_k$ by

$$\|s\|'^2_k = \sum_{\ell=0}^{k} \sum_{Q \in \mathcal{D}^\ell} \|Qs\|_0^2. \tag{4.123}$$

Since $[{}^0\nabla^{\mathcal{S}_Z}_{f^H_{p,1}}, P]$, $P\, {}^0\nabla^{\mathcal{S}_Z}_{e_i}$ and ${}^0\nabla^{\mathcal{S}_Z}_{e_i} P$ are operators along the fiber X with smooth kernels, for any $T \geq 1$, the sobolev norm $\|\cdot\|'_k$ is equivalent to the canonical sobolev norm $\|\cdot\|_k$.

Thus, we also denote the Sobolev space with respect to $\|\cdot\|'_k$ by $\mathbb{E}^k_0$.

Lemma 4.18. *For any $m \in \mathbb{N}$, there exist $p_m \in \mathbb{N}$, $C_m > 0$ and $T_0 \geq 1$ such that for $T \geq T_0$, $\lambda \in \Delta'$, $s \in \mathbb{E}^m_0$,*

$$\|(\lambda - \mathcal{B}_T)^{-1} s\|'_{m+1} \leq C_m (1 + |\lambda|)^{p_m} \|s\|'_m. \tag{4.124}$$

Proof. Clearly for $T \geq 1$,

$$\|s\|'_1 \leq C |s|_{T,1}. \tag{4.125}$$

When $m = 0$, we obtain the lemma from (4.125) and Lemma 4.9.

For the general case, let $\mathscr{R}_T$ be the family of operators

$$\mathscr{R}_T = \{[Q_{i_1}, [Q_{i_2}, \cdots [Q_{i_p}, \mathcal{B}_T], \cdots]]\} \tag{4.126}$$

where $Q_{i_1}, \cdots Q_{i_p} \in \mathcal{D}$. We can express

$$Q_1 \cdots Q_{k+1} (\lambda - \mathcal{B}_T)^{-1} \tag{4.127}$$

as a linear combination of operators of the type

$$(\lambda-\mathcal{B}_T)^{-1}\mathscr{R}_1(\lambda-\mathcal{B}_T)^{-1}\mathscr{R}_2\cdots\mathscr{R}_{k'}(\lambda-\mathcal{B}_T)^{-1}Q_{k'+1}\cdots Q_{k+1},\quad k'\leq k, \tag{4.128}$$

with $\mathscr{R}_1,\cdots,\mathscr{R}_{k'}\in\mathscr{R}_T$. By Lemma 4.17, we have

$$|\mathscr{R}_i s|_{T,-1}\leq |s|_{T,1}. \tag{4.129}$$

From (4.125), (4.129) and Lemma 4.9, we have

$$\begin{aligned}
&\|(\lambda-\mathcal{B}_T)^{-1}s\|'_{k+1}\leq C\sum\|Q_2\cdots Q_{k+1}(\lambda-\mathcal{B}_T)^{-1}s\|'_1\\
\leq& C\sum\|(\lambda-\mathcal{B}_T)^{-1}\mathscr{R}_2(\lambda-\mathcal{B}_T)^{-1}\mathscr{R}_3\cdots\mathscr{R}_{k'}(\lambda-\mathcal{B}_T)^{-1}Q_{k'+1}\cdots Q_{k+1}s\|'_1\\
\leq& C_k(1+|\lambda|)^{p_k}\sum\|Q_{k'+1}\cdots Q_{k+1}s\|_0\\
\leq& C_k(1+|\lambda|)^{p_k}\|s\|'_k.
\end{aligned} \tag{4.130}$$

The proof of Lemma 4.18 is complete. □

Now we can complete the proof of Theorem 4.14.

From (4.107), for any $k\in\mathbb{N}^*$,

$$\exp(-u^2\mathcal{B}_T)=\frac{1}{2\pi i}\int_{\Delta'}\frac{e^{-u^2\lambda}}{(\lambda-\mathcal{B}_T)}d\lambda=\frac{(-1)^{k-1}(k-1)!}{2\pi i u^{k-1}}\int_{\Delta'}\frac{e^{-u^2\lambda}}{(\lambda-\mathcal{B}_T)^k}d\lambda. \tag{4.131}$$

By Lemma 4.18, there exist $C>0$, $r\in\mathbb{N}^*$, such that for any m'-order (resp. m'') fiberwise differential operator R (resp. R') along Z, $m',m''\geq n/2$, choosing $k\geq m'+m''$,

$$\|R(\lambda-\mathcal{B}_T)^{-k}R's\|_0\leq C\|(\lambda-\mathcal{B}_T)^{-k}R's\|'_{m'}\leq C(1+|\lambda|)^r\|s\|_0. \tag{4.132}$$

From (4.131) and (4.132), there exist $C,C'>0$, such that

$$\|R\exp(-u^2\mathcal{B}_T)R's\|_0\leq C\exp(-C'u^2)\|s\|_0. \tag{4.133}$$

Now applying Sobolev embedding theorem, for R'' a fiberwise differential operator of order $m'-n/2$ along Z, there exists $C>0$, such that for any $s\in\mathbb{E}_0$,

$$|R''\exp(-u^2\mathcal{B}_T)R's|_{\mathscr{C}^0}\leq C\exp(-C'u^2)\|s\|_0, \tag{4.134}$$

and

$$(R''\exp(-u^2\mathcal{B}_T)R's)(z)=\int_Z(R'_{z'}R''_z\exp(-u^2\mathcal{B}_T)(z,z'))s(z')dv_Z(z'), \tag{4.135}$$

here $R'_{z'}$ acts on $\mathcal{S}(TZ,L_Z)^*$ by identifying $\mathcal{S}(TZ,L_Z)^*$ to $\mathcal{S}(TZ,L_Z)$ by $h^{\mathcal{S}_Z}$. Thus, we have

$$\|R'_{\cdot}R''_z\exp(-u^2\mathcal{B}_T)(z,\cdot)\|_0\leq C\exp(-C'u^2). \tag{4.136}$$

Applying the Sobolev embedding theorem to the z'-variable, from (4.136), we can get (4.109).

From (4.78), for any $m\in\mathbb{N}$, there exist $p_m\in\mathbb{N}$, $C_m>0$ and $T_0\geq 1$ such that for $T\geq T_0$, $\lambda\in\Delta'$, $s\in\mathbb{E}_0^m$,

$$\|P(\lambda-\mathcal{B}_2)^{-1}Ps\|'_{m+1}\leq C_m(1+|\lambda|)^{p_m}\|Ps\|'_m. \tag{4.137}$$

Following the same process, we get (4.110).

4.5. **Proof of Proposition 3.5.** Let N_X be the number operator acting on TZ such that for $s \in TZ$,

$$N_X P^{TX} s = P^{TX} s, \quad N_X P^{T^H Z} s = 0. \tag{4.138}$$

Let

$$'\nabla_T^{TZ} = T^{-N_X} \nabla_T^{TZ} T^{N_X}. \tag{4.139}$$

Let $'R_T^{TZ}$ be the curvature of $'\nabla_T^{TZ}$. By (3.1), we have

$$'\nabla_T^{TZ} = {}^0\nabla^{TZ} + \frac{1}{T}(P^{TX} S_1 P^{T^H Z} + P^{T^H Z} S_1 P^{TX}) + \frac{1}{T^2} P^{T^H Z} S_1 P^{T^H Z}. \tag{4.140}$$

Then by 3.19, we have

$$\gamma_{\mathcal{A}}(T) = \frac{\partial}{\partial b}\Big|_{b=0} \widehat{\mathrm{A}}_g \left('R_T^{TZ} + b \frac{\partial' \nabla_T^{TZ}}{\partial T} \right). \tag{4.141}$$

From (4.140), we have

$$\frac{\partial' \nabla_T^{TZ}}{\partial T} = O\left(\frac{1}{T^2}\right) \quad \text{and} \quad 'R_T^{TZ} = O(1). \tag{4.142}$$

Then Proposition 3.5 follows from $'\nabla_\infty^{TZ} = {}^0\nabla^{TZ}$.

5. Proof of Theorem 3.4 and Theorem 3.6 i)

In this Section, we use the notations in Section 4.

Set

$$\mathcal{B}'_T = B_{3,T}^2 + dT \wedge \frac{\partial B_{3,T}}{\partial T}. \tag{5.1}$$

By (4.38), we have

$$\begin{aligned} \frac{\partial B_{3,T}}{\partial T} = D^X - \frac{1}{8T^2} \big(&\langle [f_{p,1}^H, f_{q,1}^H], e_i \rangle c(e_i) c(f_{p,1}^H) c(f_{q,1}^H) \\ &+ \langle [g_{\alpha,3}^H, g_{\beta,3}^H], e_i \rangle c(e_i) g^\alpha \wedge g^\beta \wedge + 4 \langle S_1(g_{\alpha,3}^H) e_i, f_{p,1}^H \rangle c(e_i) c(f_{p,1}^H) g^\alpha \wedge \big). \end{aligned} \tag{5.2}$$

By Definition 3.1, we have

$$\beta_g^T(T,u) = \psi_S \delta_{u^2} \widetilde{\mathrm{Tr}}[g \exp(-u^2 \mathcal{B}'_T)]^{dT}. \tag{5.3}$$

Recall that B_2 is the Bismut superconnection in (4.43). Comparing with (4.54), by Lemma 4.6, we have

$$P \mathcal{B}'_T P = B_2 + O\left(\frac{1}{T}\right). \tag{5.4}$$

By (5.4), if we replace $\mathcal{B}_T$ to $\mathcal{B}'_T$ and $\mathcal{B}_2$ to B_2, then everything in Section 4 works well. As an analogue of Theorem 4.15, we can get the following theorem.

Theorem 5.1. *For $u_0 > 0$ fixed, there exist $C, C' > 0$, $T_0 \geq 1$, $\delta > 0$, such that for $u \geq u_0$, $T \geq T_0$,*

$$\left| \psi_S \delta_{u^2} \widetilde{\mathrm{Tr}}[g \exp(-u^2 \mathcal{B}'_T)] - \psi_S \delta_{u^2} \widetilde{\mathrm{Tr}}[g \exp(-u^2 B_2)] \right| \leq \frac{C}{T^\delta} \exp(-C' u^2). \tag{5.5}$$

Take $s > 0$. By replacing T to sT in Theorem 5.1 and taking the coefficient of ds, for$sT \geq T_0$, we have

$$\left|\psi_S \delta_{u^2} \widetilde{\mathrm{Tr}}[g\exp(-u^2\mathcal{B}'_{sT})]^{ds}\right| \leq \frac{C}{(sT)^\delta}\exp(-C'u^2). \tag{5.6}$$

By (5.3), for $T \geq T_0$, we have

$$\begin{aligned}\beta_g^T(T,u) = \psi_S \delta_{u^2} \widetilde{\mathrm{Tr}}[g\exp(-u^2\mathcal{B}'_{sT})]^{d(sT)}\Big|_{s=1} \\ = T^{-1}\cdot \psi_S \delta_{u^2} \widetilde{\mathrm{Tr}}[g\exp(-u^2\mathcal{B}'_{sT})]^{ds}\Big|_{s=1}.\end{aligned} \tag{5.7}$$

From (5.6) and (5.7), for $u_0 > 0$ fixed, there exist $C, C' > 0$, $T_0 \geq 1$, $\delta > 0$, such that for $u \geq u_0$, $T \geq T_0$, we have

$$\left|\beta_g^T(T,u)\right| \leq \frac{C}{T^{1+\delta}}\exp(-C'u^2). \tag{5.8}$$

Then we get Theorem 3.4 and Theorem 3.6 i).

6. Proof of Theorem 3.6 ii)

In this section, we use the notations in Section 2.2, 4, 5 and assumptions in Section 2.2.

In the first three subsections, we prove Theorem 3.6 ii) when $\dim Y$ and $\dim Z$ are all even. In Section 6.4, we discuss the other cases. In Section 6.5, we prove the technical result Theorem 6.5.

6.1. **The proof is local on $\pi_1^{-1}(V^g)$.** Recall that $\mathcal{B}'_T$ is the operator defined in (5.1).

As in (4.49), we set

$$\mathcal{B}'_{\varepsilon,T/\varepsilon} = \varepsilon^2\delta_{\varepsilon^2}\mathcal{B}'_{T/\varepsilon}\delta_{\varepsilon^2}^{-1} = B^2_{3,\varepsilon^2,T/\varepsilon} + \varepsilon^{-1}dT\wedge \frac{\partial B_{3,\varepsilon^2,T'}}{\partial T'}\Big|_{T'=T\varepsilon^{-1}}. \tag{6.1}$$

By Definition 3.1, we have

$$\varepsilon^{-1}\beta_g^T(T/\varepsilon,\varepsilon) = \psi_S\widetilde{\mathrm{Tr}}[g\exp(-\mathcal{B}'_{\varepsilon,T/\varepsilon})]^{dT}. \tag{6.2}$$

Precisely, by (4.39), we have

$$\begin{aligned}B_{3,\varepsilon^2,T/\varepsilon} = TD^X + \varepsilon D^H + \frac{\varepsilon^2}{8T}\langle [f^H_{p,1}, f^H_{q,1}], e_i\rangle c(e_i)c(f^H_{p,1})c(f^H_{q,1}) \\ + {}^0\nabla^{\mathscr{E}_Z,u} - \frac{c(T_2)}{4\varepsilon} + \frac{\varepsilon}{2T}\langle S_1(g^H_{\alpha,3})e_i, f^H_{p,1}\rangle c(e_i)c(f^H_{p,1})g_3^\alpha\wedge \\ + \frac{1}{8T}\langle [g^H_{\alpha,3}, g^H_{\beta,3}], e_i\rangle c(e_i)g^\alpha\wedge g^\beta\wedge,\end{aligned} \tag{6.3}$$

and

$$\begin{aligned}\varepsilon^{-1}\frac{\partial B_{3,\varepsilon^2,T'}}{\partial T'}\Big|_{T'=T\varepsilon^{-1}} = D^X - \frac{1}{8T^2}(\langle \varepsilon^2[f^H_{p,1}, f^H_{q,1}], e_i\rangle c(e_i)c(f^H_{p,1})c(f^H_{q,1}) \\ + 4\varepsilon\langle S_1(g^H_{\alpha,3})e_i, f^H_{p,1}\rangle c(e_i)c(f^H_{p,1})g_3^\alpha\wedge + \langle [g^H_{\alpha,3}, g^H_{\beta,3}], e_i\rangle c(e_i)g^\alpha\wedge g^\beta\wedge).\end{aligned} \tag{6.4}$$

Set $B_1|_{V^g}$ be the Bismut superconnection associated to $(T_1^H(W|_{V^g}), g^{TX}, h^{L_X}, \nabla^{L_X})$. For $t>0$, we denote δ_t^V the operator on $\Lambda^i(T^*V^g)$ by multiplying by $t^{-i/2}$. As in (1.32), set

$$B_{1,T^2}|_{V^g} = T\delta_{T^2}^V \circ B_1|_{V^g} \circ (\delta_{T^2}^V)^{-1}. \tag{6.5}$$

As in (4.49), we set

$$\mathcal{B}''_{T^2}|_{V^g} = (B_{1,T^2}|_{V^g})^2 + dT \wedge \left.\frac{\partial B_{1,T^2}}{\partial T}\right|_{V^g}. \tag{6.6}$$

Then by (3.17), we have

$$\gamma_1(T) = \psi_{V^g}\widetilde{\mathrm{Tr}}[g\exp(-\mathcal{B}''_{T^2}|_{V^g})]^{dT}. \tag{6.7}$$

In the first three subsections we assume that $\dim Y = m$ and $\dim Z = n$ are all even.

Let d^V, d^W be the distance functions on V, W associated to g^{TV}, g^{TW}. Let Inj^V, Inj^W be the injective radius of V, W. In the sequel, we assume that given $0<\alpha<\alpha_0<\inf\{\mathrm{Inj}^V,\mathrm{Inj}^W\}$ are chosen small enough so that if $y\in V$, $d^V(g^{-1}y,y)\le\alpha$, then $d^V(y,V^g)\le \frac14\alpha_0$, and if $z\in W$, $d^W(g^{-1}z,z)\le\alpha$, then $d^W(z,W^g)\le\frac14\alpha_0$.

Let f be a smooth even function defined on $\mathbb{R}$ with values in $[0,1]$, such that

$$f(t) = \begin{cases} 1, & |t|\le \alpha/2;\\ 0, & |t|\ge\alpha.\end{cases} \tag{6.8}$$

For $t\in(0,1]$, $a\in\mathbb{C}$, set

$$\begin{cases} \mathbf{F}_t(a) = \displaystyle\int_{-\infty}^{+\infty}\cos(\sqrt{2}va)e^{-\frac{v^2}{2}}f(\sqrt{t}v)\frac{dv}{\sqrt{2\pi}},\\ \mathbf{G}_t(a) = \displaystyle\int_{-\infty}^{+\infty}\cos(\sqrt{2}va)e^{-\frac{v^2}{2}}(1-f(\sqrt{t}v))\frac{dv}{\sqrt{2\pi}}.\end{cases} \tag{6.9}$$

Clearly,

$$\mathbf{F}_t(a)+\mathbf{G}_t(a) = \exp(-a^2). \tag{6.10}$$

The functions $\mathbf{F}_t(a)$ and $\mathbf{G}_t(a)$ are even holomorphic functions and the restrictions of $\mathbf{F}_t(a)$, $\mathbf{G}_t(a)$ to $\mathbb{R}$ lie in the Schwartz space. So there exist holomorphic functions $\widetilde{\mathbf{F}}_t(a)$ and $\widetilde{\mathbf{G}}_t(a)$ on $\mathbb{C}$ such that

$$\mathbf{F}_t(a) = \widetilde{\mathbf{F}}_t(a^2), \quad \mathbf{G}_t(a) = \widetilde{\mathbf{G}}_t(a^2). \tag{6.11}$$

From (6.10), we deduce that

$$\exp(-\mathcal{B}'_{\varepsilon,T/\varepsilon}) = \widetilde{\mathbf{F}}_{\varepsilon^2}(\mathcal{B}'_{\varepsilon,T/\varepsilon}) + \widetilde{\mathbf{G}}_{\varepsilon^2}(\mathcal{B}'_{\varepsilon,T/\varepsilon}). \tag{6.12}$$

Fix $b\in S$. For $z,z'\in Z_b$, let $\widetilde{\mathbf{F}}_{\varepsilon^2}(\mathcal{B}'_{\varepsilon,T/\varepsilon})(z,z')$ and $\widetilde{\mathbf{G}}_{\varepsilon^2}(\mathcal{B}'_{\varepsilon,T/\varepsilon})(z,z')$ be the smooth kernels associated to $\widetilde{\mathbf{F}}_{\varepsilon^2}(\mathcal{B}'_{\varepsilon,T/\varepsilon})$ and $\widetilde{\mathbf{G}}_{\varepsilon^2}(\mathcal{B}'_{\varepsilon,T/\varepsilon})$ with respect to the volume form $dv_Z(z')$.

Lemma 6.1. *For $\delta>0$ fixed , there exist $C_1,C_2>0$, such that for any $z,z'\in Z_b$, $0<\varepsilon\le\delta$, $T\ge1$,*

$$\left|\widetilde{\mathbf{G}}_{\frac{\varepsilon^2}{T^2}}(\mathcal{B}'_{\frac{\varepsilon}{T},T})(z,z')\right| \le C_1\exp\left(-\frac{C_2T^2}{\varepsilon^2}\right). \tag{6.13}$$

In particular,

$$\left|\psi_S \operatorname{Tr}_s\left[g\widetilde{\mathbf{G}}_{\frac{\varepsilon^2}{T^2}}(\mathcal{B}'_{\frac{\varepsilon}{T},T})\right]\right| \leq C_1 \exp\left(-\frac{C_2T^2}{\varepsilon^2}\right). \tag{6.14}$$

Proof. By (4.38), (5.1) and the elliptic estimate, there exists $C > 0$ such that for any $T \geq 1$,

$$\|s\|_2 \leq C\|\mathcal{B}'_T s\|_0 + CT^2\|s\|_0. \tag{6.15}$$

Then for a m-order fiberwise differential operator Q along Z with scalar principal symbol, by (6.15), we have

$$\|Qs\|_2 \leq C\|\mathcal{B}'_T Qs\|_0 + CT^2\|Qs\|_0 \leq C\|Q\mathcal{B}'_T s\|_0 + CT^2\|Qs\|_0 + C\|[\mathcal{B}'_T, Q]s\|_0. \tag{6.16}$$

By (4.38) and (5.1), we have

$$\|[\mathcal{B}'_T, Q]s\|_0 \leq CT^2\|s\|_{m+1}. \tag{6.17}$$

Thus we get the estimate

$$\|s\|_{m+2} \leq C\|\mathcal{B}'_T s\|_m + CT^2\|s\|_{m+1} \leq CT^2(\|\mathcal{B}'_T s\|_{m+1} + \|s\|_{m+1}). \tag{6.18}$$

By induction, there exist $c_k > 0$ for $0 \leq k \leq m$, such that

$$\|s\|_m \leq T^{2m}\sum_{k=0}^{m} c_k\|(\mathcal{B}'_T)^k s\|_0. \tag{6.19}$$

Let $\mathcal{B}'^*_T$ be the adjoint of $\mathcal{B}'_T$. Similarly, we have

$$\|s\|_m \leq T^{2m}\sum_{k=0}^{m} c_k\|(\mathcal{B}'^*_T)^k s\|_0. \tag{6.20}$$

For m-order fiberwise differential operator Q, for $m' \in \mathbb{N}$, by (6.19) and (6.20), we have

$$\begin{aligned}
&\left|\left\langle (\mathcal{B}'_T)^{m'}\widetilde{\mathbf{G}}_{\frac{\varepsilon^2}{T^2}}\left(\frac{\varepsilon^2}{T^2}\mathcal{B}'_T\right)Qs, s'\right\rangle\right| \\
&= \left|\left\langle s, Q^*\widetilde{\mathbf{G}}_{\frac{\varepsilon^2}{T^2}}\left(\frac{\varepsilon^2}{T^2}\mathcal{B}'^*_T\right)(\mathcal{B}'^*_T)^{m'}s'\right\rangle\right| \leq \left\|\widetilde{\mathbf{G}}_{\frac{\varepsilon^2}{T^2}}\left(\frac{\varepsilon^2}{T^2}\mathcal{B}'^*_T\right)(\mathcal{B}'^*_T)^{m'}s'\right\|_m \|s\|_0 \\
&\leq \left(T^m\sum_{k=0}^{m} c_k\left\|(\mathcal{B}'^*_T)^k\widetilde{\mathbf{G}}_{\frac{\varepsilon^2}{T^2}}\left(\frac{\varepsilon^2}{T^2}\mathcal{B}'^*_T\right)(\mathcal{B}'^*_T)^{m'}s'\right\|_0\right)\|s\|_0
\end{aligned} \tag{6.21}$$

By [8, (11.18)], for $m \in \mathbb{N}$, there exist $c'_m > 0$ and $c > 0$, such that for any $0 < \varepsilon \leq \delta$, $T \geq 1$,

$$\sup_{\lambda\in\Delta}|\lambda|^m\left|\widetilde{\mathbf{G}}_{\frac{\varepsilon^2}{T^2}}\left(\frac{\varepsilon^2}{T^2}\lambda\right)\right| \leq c'_m \exp\left(-\frac{cT^2}{\varepsilon^2}\right). \tag{6.22}$$

From (6.21) and (6.22), there exists $c_{m,m'} > 0$, such that

$$\left\|(\mathcal{B}'_T)^{m'}\widetilde{\mathbf{G}}_{\frac{\varepsilon^2}{T^2}}\left(\frac{\varepsilon^2}{T^2}\mathcal{B}'_T\right)Q\right\|_0 \leq c_{m,m'}\exp\left(-\frac{cT^2}{2\varepsilon^2}\right). \tag{6.23}$$

Let P be a fiberwise differential operators along Z of order m'. Then by (6.19) and (6.23), there exists $c'_{m,m'} > 0$, such that for any $0 < \varepsilon \leq \delta$, $T \geq 1$,

$$\left\| P \widetilde{\mathbf{G}}_{\frac{\varepsilon^2}{T^2}} \left(\frac{\varepsilon^2}{T^2} \mathcal{B}'_T \right) Q \right\|_0 \leq \left\| \widetilde{\mathbf{G}}_{\frac{\varepsilon^2}{T^2}} \left(\frac{\varepsilon^2}{T^2} \mathcal{B}'_T \right) Q \right\|_{m'} \leq c'_{m,m'} \exp\left(-\frac{cT^2}{2\varepsilon^2} \right). \tag{6.24}$$

Following the same process in (4.134)-(4.136), there exist $C_1, C_2 > 0$, such that for any $z, z' \in Z_b$, $0 < \varepsilon \leq \delta$, $T \geq 1$,

$$\left| \widetilde{\mathbf{G}}_{\frac{\varepsilon^2}{T^2}} \left(\frac{\varepsilon^2}{T^2} \mathcal{B}'_T \right)(z, z') \right| \leq C_1 \exp\left(-\frac{C_2 T^2}{\varepsilon^2} \right). \tag{6.25}$$

Since $\mathcal{B}'_{\frac{\varepsilon}{T}, T} = \frac{\varepsilon^2}{T^2} \delta_{\frac{\varepsilon^2}{T^2}} \mathcal{B}'_T \delta^{-1}_{\frac{\varepsilon^2}{T^2}}$, we get the proof of Lemma 6.1. □

Using Lemma 6.1 with $\varepsilon = T$ and T replace by T/ε, for $T \geq 1$ fixed, we find

$$\begin{aligned} &\left| \widetilde{\mathbf{G}}_{\varepsilon^2}(\mathcal{B}'_{\varepsilon, T/\varepsilon})(z, z') \right| \leq C_1 \exp\left(-\frac{C_2}{\varepsilon^2} \right), \\ &\left| \psi_S \operatorname{Tr}_s \left[g \widetilde{\mathbf{G}}_{\varepsilon^2}(\mathcal{B}'_{\varepsilon, T/\varepsilon}) \right] \right| \leq C_1 \exp\left(-\frac{C_2}{\varepsilon^2} \right). \end{aligned} \tag{6.26}$$

From (6.12) and (6.26), by the finite propagation speed for the solution of the hyperbolic equations for $\cos\left(s\sqrt{\mathcal{B}'_{\varepsilon, T/\varepsilon}} \right)$ (cf. [19, §7.8] and [32, §4.4]), it is clear that for $0 < \varepsilon \leq 1$, $T \geq 1$, $z, z' \in Z_b$, if $d^V(\pi_1 z, \pi_1 z') \geq \alpha$, then

$$\widetilde{\mathbf{F}}_{\varepsilon^2}(\mathcal{B}'_{\varepsilon, T/\varepsilon})(z, z') = 0, \tag{6.27}$$

and moreover, given $z \in Z_b$, $\widetilde{\mathbf{F}}_{\varepsilon^2}(\mathcal{B}'_{\varepsilon, T/\varepsilon})(z, \cdot)$ only depends on the restriction of $\mathcal{B}'_{\varepsilon, T/\varepsilon}$ to $\pi_1^{-1}(B^Y(\pi_1 z, \alpha))$.

Let $\mathcal{U}_{\alpha_0}(Y_b^g)$ be the set of $y \in Y_b$ such that $d^Y(y, Y_b^g) < \alpha_0/4$. We identify $\mathcal{U}_{\alpha_0}(Y_b^g)$ to $\{(y, U) : y \in Y_b^g, U \in N_{Y^g/Y}, |U| < \alpha_0/4\}$ by using geodesic coordinates normal to Y^g in Y, where $N_{Y^g/Y}$ is the real normal bundle associated to $g \in G$ in Y. Let dv_{Y^g} and dv_{N_Y} be the corresponding volume forms on TY^g and N_Y induced by g^{TY}. Then there exists the function k_Y on $\mathcal{U}_{\alpha_0}(Y_b^g)$, such that

$$dv_Z(z) = k_Y(y, U) dv_{Y^g}(y) dv_{N_Y}(U) dv_X(x). \tag{6.28}$$

Thus, from (6.27),

$$\begin{aligned} &\operatorname{Tr}_s \left[g \widetilde{\mathbf{F}}_{\varepsilon^2}(\mathcal{B}'_{\varepsilon, T/\varepsilon}) \right] \\ &= \int_Z \operatorname{Tr}_s \left[g \widetilde{\mathbf{F}}_{\varepsilon^2}(\mathcal{B}'_{\varepsilon, T/\varepsilon})(g^{-1} z, z) \right] dv_Z(z) \\ &= \int_{Y^g} \int_{U \in N, |U| < \alpha_0/4} \int_X \operatorname{Tr}_s \left[g \widetilde{\mathbf{F}}_{\varepsilon^2}(\mathcal{B}'_{\varepsilon, T/\varepsilon})(g^{-1}(y, U, x), (y, U, x)) \right] \\ &\quad \cdot k_Y(y, U) dv_{Y^g}(y) dv_{N_Y}(U) dv_X(x). \end{aligned} \tag{6.29}$$

Therefore, from (6.2), (6.29) and Lemma 6.1, we see that the proof of Theorem 3.6 ii) is local near $\pi_1^{-1}(V^g)$.

6.2. **Rescaling of the variable U and of the Clifford variables.** By (1.34), (3.3), (4.41) and (6.1), we have

$$
\begin{aligned}
\mathcal{B}'_{\varepsilon,T/\varepsilon} = &-\Big(T\,{}^0\nabla^{\mathcal{S}_Z}_{e_i} + \frac{\varepsilon}{2}\langle S_1(e_i)e_j, f^H_{p,1}\rangle c(e_j)c(f^H_{p,1}) \\
&+ \frac{\varepsilon^2}{4T}\langle S_1(e_i)f^H_{p,1}, f^H_{q,1}\rangle c(f^H_{p,1})c(f^H_{q,1}) + \frac{1}{2}\langle S_3(e_i)e_j, g^H_{\alpha,3}\rangle c(e_j)g^\alpha\wedge \\
&+ \frac{\varepsilon}{2T}\langle S_3(e_i)f^H_{p,1}, g^H_{\alpha,3}\rangle c(f^H_{p,1})g^\alpha\wedge + \frac{1}{4T}\langle S_3(e_i)g^H_{\alpha,3}, g^H_{\beta,3}\rangle g^\alpha\wedge g^\beta\wedge\Big)^2 \\
&+ dT\wedge\Big(c(e_i)\,{}^0\nabla^{\mathcal{E}}_{e_i} - \frac{1}{8T^2}\big(\varepsilon^2\langle[f^H_{p,1}, f^H_{q,1}], e_i\rangle c(e_i)c(f^H_{p,1})c(f^H_{q,1}) \\
&\quad +4\varepsilon\langle S_1(g^H_{\alpha,3})e_i, f^H_{p,1}\rangle c(e_i)c(f^H_{p,1})g^\alpha_3\wedge + \langle[g^H_{\alpha,3}, g^H_{\beta,3}], e_i\rangle c(e_i)g^\alpha\wedge g^\beta\wedge\big)\Big) \\
&-\varepsilon^2\Big({}^0\nabla^{\mathcal{S}_Z}_{f^H_{p,1}} + \frac{\varepsilon}{2T}\langle S_1(f^H_{p,1})e_i, f^H_{q,1}\rangle c(e_i)c(f^H_{q,1}) + \frac{1}{2T}\langle S_3(f^H_{p,1})e_i, g^H_{\alpha,3}\rangle c(e_i)g^\alpha\wedge \\
&+\frac{1}{2\varepsilon}\langle S_2(f_p)f_q, g^H_{\alpha,2}\rangle c(f^H_{q,1})g^\alpha\wedge + \frac{1}{4\varepsilon^2}\langle S_2(f_p)g^H_{\alpha,2}, g^H_{\beta,2}\rangle g^\alpha\wedge g^\beta\wedge\Big)^2 \\
&+\frac{\varepsilon^2}{4}K^Z_{T/\varepsilon} + \frac{T^2}{4}R^{L_Z}(e_i,e_j)c(e_i)c(e_j) + \frac{T\varepsilon}{2}R^{L_Z}(e_i, f^H_{p,1})c(e_i)c(f^H_{p,1}) \\
&+\frac{\varepsilon^2}{4}R^{L_Z}(f^H_{p,1}, f^H_{q,1})c(f^H_{p,1})c(f^H_{q,1}) + \frac{1}{4}R^{L_Z}(g^H_{\alpha,3}, g^H_{\beta,3})g^\alpha\wedge g^\beta\wedge \\
&+\frac{\varepsilon}{2}R^{L_Z}(f^H_{p,1}, g^H_{\alpha,3})c(f^H_{p,1})g^\alpha\wedge + \frac{T}{2}R^{L_Z}(e_i, g^H_{\alpha,3})c(e_i)g^{\alpha,H}_3\wedge.
\end{aligned}
\tag{6.30}
$$

Set

$$
\begin{aligned}
\nabla'_{f^H_{p,1}} = {}^0\nabla^{\mathcal{S}_Z}_{f^H_{p,1}} - \frac{1}{2}\langle S_1(e_i)e_i, f^H_{p,1}\rangle + \frac{1}{2\varepsilon}\langle S_2(f_p)f_q, g^H_{\alpha,2}\rangle c(f^H_{q,1})g^\alpha\wedge \\
+ \frac{1}{4\varepsilon^2}\langle S_2(f_p)g^H_{\alpha,2}, g^H_{\beta,2}\rangle g^\alpha\wedge g^\beta\wedge.
\end{aligned}
\tag{6.31}
$$

Recall that $\mathscr{E}_{X,y_0} = \mathscr{C}^\infty(X_{y_0}, \mathcal{S}(TX, L_X))$, which is naturally equipped with a Hermitian product attached to g^{TX} and $h^{\mathcal{S}_X}$ as in (1.22). By (1.24), the connection ∇' preserves the scalar product on $\mathscr{E}_X$.

Take $y_0 \in V^g$ and $\pi_2(y_0) = b$. We identify $B^{Y_b}(y_0, \alpha_0)$ with $B(0, \alpha_0) \subset T_{y_0}Y = \mathbb{R}^m$ by using normal coordinates. Take a vector $U \in \mathbb{R}^m$. We identify $TY|_U$ to $TY|_{\{0\}}$ by parallel transport along the curve $t \mapsto tU$ with respect to the connection ∇^{TY}. We lift horizontally the paths $t \in \mathbb{R}^*_+ \mapsto tU$ into paths $t \in \mathbb{R}^*_+ \mapsto x_t \in Z_b$ with $x_t \in X_{tU}$, $dx_t/dt \in T^HZ_b$. If $x_0 \in X_{y_0}$, we identify $T_{x_t}X$, $\mathcal{S}(TZ, L_Z)_{x_t}$ to $T_{x_0}X$, $\mathcal{S}(TZ, L_Z)_{x_0}$ by parallel transport along the curve $t \mapsto x_t$ with respect to the connection ∇^{TX}, ∇'. Then we can define the operator $\mathcal{B}'_{\varepsilon,T/\varepsilon}$ to a neighborhood of $\{0\}\times X_{y_0}$ in $T_{y_0}Y \times X_{y_0}$.

Let $\rho : T_{y_0}Y \to [0,1]$ be a smooth function such that

$$
\rho(U) = \begin{cases} 1, & |U| \le \alpha_0/4; \\ 0, & |U| \ge \alpha_0/2. \end{cases}
\tag{6.32}
$$

Let Δ^{TY} be the ordinary Laplacian operator on $T_{y_0}Y$.

Recall that $\ker D^X|_{B^Y(y_0,\alpha_0/2)}$ is a smooth vector subbundle of $\mathscr{E}_{X,y_0}$ on $B^Y(y_0,\alpha_0/2)$. If $\alpha_0>0$ is small enough, there is a vector bundle $K\subset\mathscr{E}_{X,y_0}$ over $T_{y_0}Y$, which coincides with $\ker D^X$ on $B(0,\alpha_0/2)$, with $\ker D^X_{y_0}$ on $T_{y_0}Y\backslash B(0,\alpha_0)$, such that if $K^\perp$ is the orthogonal bundle to K in $\mathscr{E}_{X,y_0}$, then

$$K^\perp\cap\ker D^X_{y_0}=\{0\}. \tag{6.33}$$

For $U\in T_{y_0}Y$, in the following sections, let P^K_U be the orthogonal projection operator from $\mathscr{E}_{X,y_0}$ to K_U. Set $P^{K,\perp}_U=1-P^K_U$.

Set

$$L^1_{\varepsilon,T}=(1-\rho^2(U))(-\varepsilon^2\Delta^{TY}+T^2P^{K,\perp}_UD^{X,2}_{y_0}P^{K,\perp}_U)+\rho^2(U)(\mathcal{B}'_{\varepsilon,T/\varepsilon}). \tag{6.34}$$

Comparing with (6.26), for any $m\in\mathbb{N}$ and $T\geq 1$ fixed, there exist $C_1,C_2>0$, such that for $|U|$, $|U'|<\alpha_0/4$, $0<\varepsilon\leq 1$,

$$|\widetilde{\mathbf{G}}_{\varepsilon^2}(L^1_{\varepsilon,T})((U,x),(U',x'))|\leq C_1\exp\left(-\frac{C_2}{\varepsilon^2}\right). \tag{6.35}$$

For $(U,x)\in N_{Y^g/Y,y_0}\times X_{y_0}$, $|U|<\alpha_0/4$, $\varepsilon>0$, set

$$(S_\varepsilon s)(U,x)=s\left(U/\varepsilon,x\right). \tag{6.36}$$

Put

$$\begin{aligned}L^2_{\varepsilon,T}:=S_\varepsilon^{-1}L^1_{\varepsilon,T}S_\varepsilon=(1-\rho^2(\varepsilon U))&(-S_\varepsilon^{-1}\varepsilon^2\Delta^{TY}S_\varepsilon+T^2P^{K,\perp}_{\varepsilon U}D^{X,2}_{y_0}P^{K,\perp}_{\varepsilon U})\\&+\rho^2(\varepsilon U)S_\varepsilon^{-1}\mathcal{B}'_{\varepsilon,T/\varepsilon}S_\varepsilon.\end{aligned} \tag{6.37}$$

Let $\dim T_{y_0}Y^g=l'$ and $\dim N_{Y^g/Y,y_0}=2l''$. Then $l'+2l''=m$. Let $\{f_1,\cdots,f_{l'}\}$ be an orthonormal basis of $T_{y_0}Y^g$ and let $\{f_{l'+1},\cdots,f_{l'+2l''}\}$ be an orthonormal basis of $N_{Y^g/Y,y_0}$. For $\alpha\in\mathbb{C}(f^p\wedge i_{f_p})_{1\leq p\leq l'}$, let $[\alpha]^{max}\in\mathbb{C}$ be the coefficient of $f^1\wedge\cdots\wedge f^{l'}$ in the expansion of α. Let R_ε be a rescaling such that

$$\begin{aligned}&R_\varepsilon(c(e_i))=c(e_i),\\&R_\varepsilon(c(f^H_{p,1}))=\frac{f^{p,H}_1\wedge}{\varepsilon}-\varepsilon\, i_{f^H_{p,1}},\quad\text{for } 1\leq p\leq l',\\&R_\varepsilon(c(f^H_{p,1}))=c(f^H_{p,1}),\quad\text{for } l'+1\leq p\leq l'+2l''.\end{aligned} \tag{6.38}$$

Then R_ε is a Clifford algebra homomorphism. Set

$$L^3_{\varepsilon,T}=R_\varepsilon(L^2_{\varepsilon,T}). \tag{6.39}$$

Let $\exp(-L^i_{\varepsilon,T})((U,x),(U',x'))$, $\widetilde{\mathbf{F}}_{\varepsilon^2}(L^i_{\varepsilon,T})((U,x),(U',x'))$ ($(U,x),(U',x')\in T_{y_0}Y\times X_{y_0}$, $i=1,2,3$) be the smooth kernels of $\exp(-L^i_{\varepsilon,T})$, $\widetilde{\mathbf{F}}_{\varepsilon^2}(L^i_{\varepsilon,T})$ with respect to the volume form $dv_{T_{y_0}Y}(U')dv_{X_{y_0}}(x')$. Using finite propagation speed as in (6.27), we see that if $(U,x)\in N_{Y^g/Y,y_0}\times X_{y_0}$, $|U|<\alpha_0/4$, then

$$\widetilde{\mathbf{F}}_{\varepsilon^2}(\mathcal{B}'_{\varepsilon,T/\varepsilon})(g^{-1}(y_0,U,x),(y_0,U,x))k_Y(y_0,U)=\widetilde{\mathbf{F}}_{\varepsilon^2}(L^1_{\varepsilon,T})(g^{-1}(U,x),(U,x)). \tag{6.40}$$

By (6.12), (6.26), (6.35) and (6.40), there exist $C_1, C_2 > 0$, such that for $|U| < \alpha_0/4$, $x \in X_{y_0}$,

$$\begin{aligned}(6.41)\quad &|\exp(-\mathcal{B}'_{\varepsilon,T/\varepsilon})(g^{-1}(y_0,U,x),(y_0,U,x))k_Y(y_0,U)\\ &\qquad - \exp(-L^1_{\varepsilon,T})(g^{-1}(U,x),(U,x))| \leq C_1 \exp\left(-\frac{C_2}{\varepsilon^2}\right).\end{aligned}$$

Since $T_{y_0}Y_b$ is an Euclidean space, on $T_{y_0}Y_b$,

$$(6.42)\qquad \mathcal{S}(TY, L_Y)_{y_0} = \mathcal{S}(TY^g)\widehat{\otimes}\mathcal{S}(N_{Y^g/Y}) \otimes L_Y^{1/2},$$

where $\mathcal{S}(\cdot)$ is the spinor space. From (6.38), we know that $L^3_{\varepsilon,T}((U,x),(U',x'))$ lies in

$$(6.43)\qquad \pi_2^*\Lambda(T_b^*S)\widehat{\otimes}(\mathrm{End}(\Lambda(T^*Y^g))\widehat{\otimes}C(N_{Y^g/Y}) \otimes \mathrm{End}(L_Y^{1/2}))_{y_0}\widehat{\otimes}\,\mathrm{End}(\mathcal{S}(TX, L_X))$$

and acts on

$$(6.44)\qquad \pi_2^*\Lambda(T_b^*S)\widehat{\otimes}(\Lambda(T^*Y^g)\widehat{\otimes}\mathcal{S}(N_{Y^g/Y}) \otimes L_Y^{1/2})_{y_0}\widehat{\otimes}\mathcal{S}(TX, L_X).$$

Recall that $\widetilde{c}_{TY^g}$ is the trace element defined in (1.8).

Lemma 6.2. *For $t > 0$, $(U,x) \in N_{Y^g/Y,y_0} \times X_{y_0}$ and $g \in G$, we have*

$$\begin{aligned}(6.45)\quad &\int_{Y^g}\int_{\substack{U\in N_{Y^g/Y},\\ |U|\leq\alpha_0/4}}\int_X \mathrm{Tr}_s[g\exp(-L^1_{\varepsilon,T})(g^{-1}(U,x),(U,x))]dv_{Y^g}(y)dv_{N_Y}(U)dv_X(x)\\ &= \int_{Y^g}\int_{\substack{U\in N_{Y^g/Y},\\ |U|\leq\alpha_0/4\varepsilon}}\int_X \widetilde{c}_{TY^g}\,\mathrm{Tr}_s\left[g\exp(-L^3_{\varepsilon,T})\left(g^{-1}\left(U,x\right),\left(U,x\right)\right)\right]^{max}\\ &\qquad\qquad\qquad \cdot dv_{Y^g}(y)dv_{N_Y}(U)dv_X(x).\end{aligned}$$

Proof. From (6.37) and the uniqueness of the heat kernel, we have

$$(6.46)\qquad \exp(-L^2_{\varepsilon,T}) = S_\varepsilon^{-1}\exp(-L^1_{\varepsilon,T})S_\varepsilon.$$

For $U \in T_{y_0}Y$, $x \in X_{y_0}$, $\mathrm{supp}\,\phi \subset B(0,\alpha_0/2) \times X_{y_0}$, we have

$$\begin{aligned}(6.47)\quad &\int_{T_{y_0}Y}\int_X \exp(-L^2_{\varepsilon,T})((U,x),(U',x'))\phi(U',x')dv_{TY}(U')dv_X(x')\\ &=(\exp(-L^2_{\varepsilon,T})\phi)(U,x) = (S_\varepsilon^{-1}\exp(-L^1_{\varepsilon,T})S_\varepsilon\phi)(U,x) = (\exp(-L^1_{\varepsilon,T})S_\varepsilon\phi)(\varepsilon U,x)\\ &=\int_{T_{y_0}Y}\int_X \exp(-L^1_{\varepsilon,T})((\varepsilon U,x),(U',x'))(S_\varepsilon\phi)(U',x')dv_{TY}(U')dv_X(x')\\ &=\varepsilon^{\dim Y}\cdot\int_{T_{y_0}Y}\int_X \exp(-L^1_{\varepsilon,T})((\varepsilon U,x),(\varepsilon U',x'))\phi(U',x')dv_{TY}(U')dv_X(x').\end{aligned}$$

Thus,

$$(6.48)\qquad \exp(-L^1_{\varepsilon,T})(g^{-1}(U,x),(U,x)) = \varepsilon^{-\dim Y}\exp(-L^2_{\varepsilon,T})\left(g^{-1}(U/\varepsilon,x),(U/\varepsilon,x)\right).$$

By (1.8), (1.10), (6.43), (6.48) and the definition of $L^3_{\varepsilon,T}$, we have

$$\begin{aligned} &\operatorname{Tr}_s\left[g\exp(-L^3_{\varepsilon,T})\left(g^{-1}(U/\varepsilon,x),(U/\varepsilon,x)\right)\right]^{max} \\ =&\sum_j \widetilde{c}^{\,-1}_{TY^g}\,\varepsilon^{-\dim Y^g}\operatorname{Tr}_s\left[g\exp(-L^2_{\varepsilon,T})\left(g^{-1}(U/\varepsilon,x),(U/\varepsilon,x)\right)\right] \\ =&\widetilde{c}^{\,-1}_{TY^g}\varepsilon^{\dim_{\mathbb{R}} N}\operatorname{Tr}_s\left[g\exp(-L^1_{\varepsilon,T})(g^{-1}(U,x),(U,x))\right]. \end{aligned} \tag{6.49}$$

The proof of Lemma 6.2 is complete. □

6.3. **Proof of Theorem 3.6 ii).** Let K^X be the scalar curvature of the fibers (TX, g^{TX}). Comparing with [6, (3.15)-(3.17)], for $T\geq 1$, we can compute that

$$\lim_{\varepsilon\to 0}\varepsilon^2 K^Z_{T/\varepsilon} = T^2K^X. \tag{6.50}$$

Let Γ' be the connection form of ∇', which is defined in (6.31). By using [1, Proposition 3.7], we see that for $U\in TY=\mathbb{R}^m$,

$$\Gamma'_U = \frac{1}{2}(\nabla')^2(U,\cdot) + O\big(|U|^2\big). \tag{6.51}$$

Lemma 6.3. *For $U,V\in TY$, the following identity holds.*

$$\begin{aligned} (\nabla')^2(U_1^H,V_1^H) =& \frac{1}{4}\langle R^{TX}(U_1^H,V_1^H)e_i,e_j\rangle c(e_i)c(e_j) + \frac{1}{2}R^{L_Z}(U_1^H,V_1^H) \\ &+\frac{1}{4}\langle R^{TY}(f_p,f_q)U,V\rangle c(f^H_{p,1})c(f^H_{q,1}) + \frac{1}{4\varepsilon^2}\langle R^{TY}(g^H_{\alpha,2},g^H_{\beta,2})U,V\rangle g^\alpha\wedge g^\beta\wedge \\ &-\frac{1}{2}d(\langle S_1(e_i)e_i,\cdot\rangle)(U_1^H,V_1^H) + \frac{1}{2\varepsilon}\langle R^{TY}(f_p,g^H_{\alpha,2})U,V\rangle c(f^H_{p,1})g^\alpha\wedge. \end{aligned} \tag{6.52}$$

Proof. By the fundamental identity of [6, Theorem 4.14] (see also [27, (7.15)]), for $Z,W\in TV$,

$$\begin{aligned} \langle R^{TY}(U,V)P^{TY}Z,P^{TY}W\rangle &+ \langle (S_2P^{TY}S_2)(U,V)Z,W\rangle \\ &+\langle(\nabla^{TY}S_2)(U,V)Z,W\rangle = \langle R^{TY}(Z,W)U,V\rangle. \end{aligned} \tag{6.53}$$

Since S_2 maps TY to T_2^HV, we have

$$(S_2P^{TY}S_2)(U,V)f_p=0,\quad \langle(\nabla^{TY}S_2)(U,V)f_p,f_q\rangle=0. \tag{6.54}$$

Then Lemma 6.3 follows from (6.31), (6.53) and (6.54). □

Lemma 6.4. *When $\varepsilon\to 0$, the limit $L^3_{0,T}=\lim_{\varepsilon\to 0}L^3_{\varepsilon,T}$ exists and*

$$L^3_{0,T}|_{V^g} = -\left(\partial_p + \frac{1}{4}\langle R^{TY}|_{V^g}U, f^H_{p,1}\rangle\right)^2 + \frac{1}{2}R^{L_Y}|_{V^g} + \mathcal{B}_{1,T^2}|_{V^g}. \tag{6.55}$$

Proof. By (6.51) and Lemma 6.3, we have

$$\begin{aligned}(6.56)\quad \lim_{\varepsilon\to 0} R_{\varepsilon^2}[\varepsilon S_{\varepsilon^2}^{-1}\nabla'_{f_p}|_U S_{\varepsilon^2}] &= \partial_p + \lim_{\varepsilon\to 0} R_{\varepsilon^2}[\varepsilon^2 (S_\varepsilon^{-1}(\nabla')^2 S_\varepsilon)(U,f_p)] \\ = \partial_p + \frac{1}{4}\sum_{1\leq q,r\leq l'} \langle R^{TY}(f_q,f_r)U,f_p\rangle f^q\wedge f^r\wedge &+ \frac{1}{4}\langle R^{TY}(g_{\alpha,2}^H, g_{\beta,2}^H)U,f_p\rangle g^\alpha\wedge g^\beta\wedge \\ &+ \frac{1}{2}\sum_{1\leq q\leq l'}\langle R^{TY}(f_q, g_{\alpha,2}^H)U,f_p\rangle f^q\wedge g^\alpha\wedge .\end{aligned}$$

Then by (6.30), (6.50) and the definition of $L^3_{\varepsilon,T}$, we have

(6.57)

$$\begin{aligned}\lim_{\varepsilon\to 0} L^3_{\varepsilon,T} = &-\Bigg(T\,{}^0\nabla^{\mathcal{S}_Z}_{e_i} + \frac{1}{2}\sum_{1\leq p\leq l'}\langle S_1(e_i)e_j, f_{p,1}^H\rangle c(e_j) f^p\wedge \\ &+ \frac{1}{4T}\sum_{1\leq p,q\leq l'}\langle S_1(e_i)f_{p,1}^H, f_{q,1}^H\rangle f^p\wedge f^q\wedge + \frac{1}{2}\langle S_3(e_i)e_j, g_{\alpha,3}^H\rangle c(e_j) g^\alpha\wedge \\ &+ \frac{1}{2T}\sum_{1\leq p\leq l'}\langle S_3(e_i) f_{p,1}^H, g_{\alpha,3}^H\rangle f^p\wedge g^\alpha\wedge + \frac{1}{4T}\langle S_3(e_i)g_{\alpha,3}^H, g_{\beta,3}^H\rangle g^\alpha\wedge g^\beta\wedge\Bigg)^2 \\ &+ dT\wedge\Bigg(D^X - \frac{1}{8T^2}\Bigg(\sum_{1\leq p,q\leq l'}\langle [f_{p,1}^H, f_{q,1}^H], e_i\rangle c(e_i) f^p\wedge f^q\wedge \\ &\quad +4\sum_{1\leq p\leq l'}\langle S_1(g_{\alpha,3}^H)e_i, f_{p,1}^H\rangle c(e_i) f^p\wedge g^\alpha\wedge + \langle [g_{\alpha,3}^H, g_{\beta,3}^H], e_i\rangle c(e_i) g^\alpha\wedge g^\beta\wedge\Bigg)\Bigg) \\ &-\Bigg(\partial_p + \frac{1}{4}\sum_{1\leq q,r\leq l'}\langle R^{TY}(U,f_p)f_q, f_r\rangle f^q\wedge f^r\wedge \\ &+ \frac{1}{4}\langle R^{TY}(U,f_p)g_{\alpha,2}^H, g_{\beta,2}^H\rangle g^\alpha\wedge g^\beta\wedge + \frac{1}{2}\sum_{1\leq q\leq l'}\langle R^{TY}(U,f_p)f_q, g_{\alpha,2}^H\rangle f^q\wedge g^\alpha\wedge\Bigg)^2 \\ &+ \frac{T^2}{4}K^X + \frac{T^2}{4}R^{L_Z}(e_i,e_j)c(e_i)c(e_j) + \frac{T}{2}\sum_{1\leq p\leq l'} R^{L_Z}(e_i, f_{p,1}^H)c(e_i)f^p\wedge \\ &+ \frac{1}{4}\sum_{1\leq p,q\leq l'} R^{L_Z}(f_{p,1}^H, f_{q,1}^H) f^p\wedge f^q\wedge + \frac{1}{4}R^{L_Z}(g_{\alpha,3}^H, g_{\beta,3}^H) g^\alpha\wedge g^\beta\wedge \\ &+ \frac{1}{2}\sum_{1\leq p\leq l'} R^{L_Z}(f_{p,1}^H, g_{\alpha,3}^H) f^p\wedge g^\alpha\wedge + \frac{T}{2}R^{L_Z}(e_i, g_{\alpha,3}^H)c(e_i) g_3^{\alpha,H}\wedge .\end{aligned}$$

By (1.34) and (6.5), we have

$$(6.58)\quad (\mathcal{B}''_{T^2}|_{V^g})^2 = -\Bigg(T\,{}^0\nabla^{\mathcal{S}_Z}_{e_i} + \frac{1}{2}\sum_{1\le p\le l'}\langle S_1(e_i)e_j, f^H_{p,1}\rangle c(e_j)f^p\wedge$$
$$+\frac{1}{4T}\sum_{1\le p,q\le l'}\langle S_1(e_i)f^H_{p,1}, f^H_{q,1}\rangle f^p\wedge f^q\wedge + \frac{1}{2}\langle S_3(e_i)e_j, g^H_{\alpha,3}\rangle c(e_j)g^\alpha\wedge$$
$$+\frac{1}{2T}\sum_{1\le p\le l'}\langle S_3(e_i)f^H_{p,1}, g^H_{\alpha,3}\rangle f^p\wedge g^\alpha\wedge + \frac{1}{4T}\langle S_3(e_i)g^H_{\alpha,3}, g^H_{\beta,3}\rangle g^\alpha\wedge g^\beta\wedge\Bigg)^2$$
$$+\frac{T^2}{4}K^X + \frac{T^2}{4}R^{L_X}(e_i,e_j)c(e_i)c(e_j) + \frac{T}{2}\sum_{1\le p\le l'}R^{L_X}(e_i, f^H_{p,1})c(e_i)f^p\wedge$$
$$+\frac{1}{4}\sum_{1\le p,q\le l'}R^{L_X}(f^H_{p,1}, f^H_{q,1})f^p\wedge f^q\wedge + \frac{1}{4}R^{L_X}(g^H_{\alpha,3}, g^H_{\beta,3})g^\alpha\wedge g^\beta\wedge$$
$$+\frac{1}{2}\sum_{1\le p\le l'}R^{L_X}(f^H_{p,1}, g^H_{\alpha,3})f^p\wedge g^\alpha\wedge + \frac{T}{2}R^{L_X}(e_i, g^H_{\alpha,3})c(e_i)g_3^{\alpha,H}\wedge.$$

So

$$(6.59)\qquad \lim_{\varepsilon\to 0}L^3_{\varepsilon,T} = -\left(\partial_p + \frac{1}{4}\langle R^{TY}|_{V^g}U, f^H_{p,1}\rangle\right)^2 + \frac{1}{2}R^{L_Y}|_{V^g} + \mathcal{B}''_{T^2}|_{V^g}.$$

The proof of Lemma 6.4 is complete. □

Theorem 6.5. *i) For $T\ge 1$ fixed and $k\in\mathbb{N}$, there exist $c>0, C>0, r\in\mathbb{N}$ such that for any $(U,x),(U',x')\in T_{y_0}Y\times X_{y_0}$, $\varepsilon\in(0,1]$,*

$$(6.60)\quad \sup_{|\alpha|,|\alpha'|\le k}\left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial U^\alpha\partial U'^{\alpha'}}\exp(-L^3_{\varepsilon,T})((U,x),(U',x'))\right| \le c(1+|U|+|U'|)^r\exp(-C|U-U'|^2).$$

ii) For $T\ge 1$ fixed, there exist $c>0, C>0, r\in\mathbb{N}$, $\gamma>0$, such that for any $(U,x),(U',x')\in T_{y_0}Y\times X_{y_0}$, $\varepsilon\in(0,1]$,

$$(6.61)\quad |(\exp(-L^3_{\varepsilon,T})-\exp(-L^3_{0,T}))((U,x),(U',x'))| \le c\varepsilon^\gamma(1+|U|+|U'|)^r\exp(-C|U-U'|^2).$$

The proof of Theorem 6.5 is left to the next subsection.

On the vector space $N_{Y^g/Y,y_0}$, there exists $c>0$, such that for any $U\in N_{Y^g/Y,y_0}$,

$$(6.62)\qquad |g^{-1}U-U|\ge c|U|.$$

Then by (6.41), Lemma 6.2, 6.4, Theorem 6.5 and the dominated convergence theorem, we have

$$(6.63)\quad \lim_{\varepsilon\to 0}\psi_S\operatorname{Tr}_s[g\exp(-\mathcal{B}'_{\varepsilon,T/\varepsilon})] = \int_{Y^g}\int_{N_{Y^g/Y}}\int_X \widetilde{c}_{TY^g}\,\psi_S\operatorname{Tr}_s\left[g\exp(-L^3_{0,T})(g^{-1}(U,x),(U,x))\right]dv_N(U)dv_X(x).$$

By Mehler's formula (cf. [24, (1.33)]) and (1.47),

$$\begin{aligned}
&\int_X \mathrm{Tr}_s \left[g\exp(-L^3_{0,T})(g^{-1}(U,x),(U,x))\right] dv_X(x)\\
=&(4\pi)^{-\frac{1}{2}\dim Y}\det{}^{\frac{1}{2}}\left(\frac{R^{TY}/2}{\sinh(R^{TY}/2)}\right)\exp\left\{-\frac{1}{4}\left\langle\frac{R^{TY}/2}{\tanh(R^{TY}/2)}U,U\right\rangle\right.\\
&\left.-\frac{1}{4}\left\langle\frac{R^{TY}/2}{\tanh(R^{TY}/2)}g^{-1}U,g^{-1}U\right\rangle \quad +\frac{1}{2}\left\langle\frac{R^{TY}/2}{\sinh(R^{TY}/2)}\exp(R^{TY}/2)U,g^{-1}U\right\rangle\right\}\\
&\cdot\mathrm{Tr}_s[g|_{\mathcal{S}(N)}]\wedge\mathrm{Tr}\left[g\exp\left(-\frac{1}{2}R^{L_Y}|_{V^g}\right)\right]\wedge\mathrm{Tr}_s[g\exp(\mathcal{B}_{1,T^2}|_{V^g})].
\end{aligned}\tag{6.64}$$

Following the same computations in [24, (1.33)-(1.38)], by (1.43), (1.44), (1.56) and (6.63), we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\psi_S\,\mathrm{Tr}_s[g\exp(-\mathcal{B}'_{\varepsilon,T/\varepsilon})]\\
&=\psi_S\int_{Y^g}\widetilde{c}_{TY^g}(4\pi)^{-\frac{\dim Y^g}{2}}\psi^{-1}_{V^g}\left(\widehat{\mathrm{A}}_g(TY,\nabla^{TY})\wedge\mathrm{ch}_g(L_Y^{1/2},\nabla^{L_Y^{1/2}})\wedge\psi_{V^g}\,\mathrm{Tr}_s[g\exp(\mathcal{B}''_{T^2})|_{V^g}]\right).
\end{aligned}\tag{6.65}$$

Using (1.55), (6.2) and (6.7), we get Theorem 3.6 ii) when $\dim Z$ and $\dim Y$ are all even.

6.4. **General case.** When $\dim Y$ is odd and $\dim Z$ is even, by (1.10), following the same process in this section, we can get an analogue of (6.65):

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\psi_S\,\mathrm{Tr}^{\mathrm{odd}}[g\exp(-\mathcal{B}'_{\varepsilon,T/\varepsilon})]\\
&=\psi_S\int_{Y^g}\widetilde{c}_{TY^g}(4\pi)^{-\frac{\dim Y^g}{2}}\psi^{-1}_{V^g}\left(\widehat{\mathrm{A}}_g(TY,\nabla^{TY})\wedge\mathrm{ch}_g(L_Y^{1/2},\nabla^{L_Y^{1/2}})\wedge\psi_{V^g}\,\mathrm{Tr}_s[g\exp(\mathcal{B}''_{T^2})|_{V^g}]\right).
\end{aligned}\tag{6.66}$$

Then Theorem 3.6 ii) in this case follows from (1.55), (6.2), (6.7) and (6.66).

When $\dim Y$ is even and $\dim Z$ is odd, it is the same as the case above.

When $\dim Y$ and $\dim Z$ are all odd, by (1.10), as in (6.66), we have

$$\begin{aligned}
\lim_{\varepsilon\to 0}\psi_S\,\mathrm{Tr}_s[g\exp(-\mathcal{B}'_{\varepsilon,T/\varepsilon})]&=2\sqrt{-1}\psi_S\int_{Y^g}\widetilde{c}_{TY^g}(4\pi)^{-\frac{\dim Y^g}{2}}\\
&\cdot\psi^{-1}_{V^g}\left(\widehat{\mathrm{A}}_g(TY,\nabla^{TY})\wedge\mathrm{ch}_g(L_Y^{1/2},\nabla^{L_Y^{1/2}})\wedge\psi_{V^g}\,\mathrm{Tr}[g\exp(\mathcal{B}''_{T^2}|_{V^g})]\right).
\end{aligned}\tag{6.67}$$

Since the left hand side of (6.67) takes value in even forms and $\dim Y^g$ is odd, by (1.7) and (1.55), we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\psi_S\,\mathrm{Tr}_s[g\exp(-\mathcal{B}'_{\varepsilon,T/\varepsilon})]\\
&\qquad=\int_{Y^g}\widehat{\mathrm{A}}_g(TY,\nabla^{TY})\wedge\mathrm{ch}_g(L_Y^{1/2},\nabla^{L_Y^{1/2}})\wedge\psi_{V^g}\,\mathrm{Tr}^{\mathrm{odd}}[g\exp(\mathcal{B}''_{T^2}|_{V^g})].
\end{aligned}\tag{6.68}$$

The proof of Theorem 3.6 ii) is complete.

6.5. **Proof of Theorem 6.5.** We prove Theorem 6.5 by following the process of [14, Section 11] and [7, Section 11].

Let I^0 be the vector space of square integrable sections of $\pi_2^*\Lambda(T^*S)\widehat{\otimes}\Lambda(TY^g)\widehat{\otimes}\mathcal{S}(N_{Y^g/Y})\otimes L_Y^{1/2}\widehat{\otimes}\mathcal{S}(TX,L_X)$ over $T_{y_0}Y_b\times X_{y_0}$. For $0\leq q\leq \dim Y^g$, let I_q^0 be the vector space of square integrable sections of $\pi_2^*\Lambda(T^*S)\widehat{\otimes}\Lambda^q(TY^g)\widehat{\otimes}\mathcal{S}(N_{Y^g/Y})\otimes L_Y^{1/2}\widehat{\otimes}\mathcal{S}(TX,L_X)$. Then $I^0=\oplus_{q=0}^{l'}I_q^0$. Similarly, if $p\in\mathbb{R}$, I^p and I_q^p denote the corresponding p-th Sobolev spaces.

For $U\in T_{y_0}Y^g$, set

$$g_\varepsilon(U)=1+(1+|U|^2)^{\frac{1}{2}}\rho\left(\frac{\varepsilon U}{2}\right). \tag{6.69}$$

If $s\in I_q^0$, set

$$|s|^2_{\varepsilon,0}=\int_{T_{y_0}Y_b\times X_{y_0}}|s(U,x)|^2g_\varepsilon(U)^{2(l'-q)}dv_{TY}(U)dv_X(x). \tag{6.70}$$

Let $\langle\cdot,\cdot\rangle_{\varepsilon,0}$ be the Hermitian product attached to $|\cdot|_{\varepsilon,0}$.

So, for $1\leq p\leq l'$, $s\in I_p$, we can get

$$\begin{aligned}&\left|1_{\varepsilon|U|\leq\alpha_0/2}|U|\ (f^p\wedge-\varepsilon^2i_{f_p})s\right|^2_{\varepsilon,0}=\left|1_{\varepsilon|U|\leq\alpha_0/2}|U|f^p\wedge s\right|^2_{\varepsilon,0}+\left|1_{\varepsilon|U|\leq\alpha_0/2}|U|\varepsilon^2i_{f_p}s\right|^2_{\varepsilon,0}\\&\qquad=\int_{|U|\leq\frac{\alpha_0}{2\varepsilon}}|s|^2|U|^2(1+(1+|U|^2)^{\frac{1}{2}})^{2(l-p-1)}dv_{TY}(U)\\&\qquad\quad+\int_{|U|\leq\frac{\alpha_0}{2\varepsilon}}\varepsilon^4|s|^2|U|^2(1+(1+|U|^2)^{\frac{1}{2}})^{2(l-p+1)}dv_{TY}(U).\end{aligned} \tag{6.71}$$

Since there exists $C>0$, such that

$$\frac{|U|}{1+(1+|U|^2)^{\frac{1}{2}}}\leq 1,\quad \varepsilon^4|U|^2(1+(1+|U|^2)^{\frac{1}{2}})^2\leq C, \tag{6.72}$$

we have the following lemma.

Lemma 6.6. *The operators $1_{\varepsilon|U|\leq\alpha_0/2}(f^p\wedge-\varepsilon^2i_{f_p})$ and $1_{\varepsilon|U|\leq\alpha_0/2}|U|(f^p\wedge-\varepsilon^2i_{f_p})$ are uniformly bounded with respect to the norm $|\cdot|_{\varepsilon,0}$.*

Lemma 6.7. *(cf.* [14, Theorem 11.26]*) For $T\geq 1$ fixed, there exist $c_1,c_2,c_3,c_4>0$, such that for any $\varepsilon\in(0,1]$, $s\in I^1$,*

$$\begin{aligned}&\mathrm{Re}\langle L^3_{\varepsilon,T}s,s\rangle_{\varepsilon,0}\geq c_1|s|^2_{\varepsilon,1}-c_2|s|^2_{\varepsilon,0},\\&|\mathrm{Im}\langle L^3_{\varepsilon,T}s,s\rangle_{\varepsilon,0}|\leq c_3|s|_{\varepsilon,1}|s|_{\varepsilon,0},\\&|\langle L^3_{\varepsilon,T}s,s'\rangle_{\varepsilon,0}|\leq c_4|s|_{\varepsilon,1}|s'|_{\varepsilon,1}.\end{aligned} \tag{6.73}$$

Proof. let ∇ denote the gradient in the variable U. Since ρ has compact support, there exists $C>0$, such that

$$|\nabla\left(g_\varepsilon(U)\right)|\leq C. \tag{6.74}$$

From Lemma 6.6 and the definition of $L^3_{\varepsilon,T}$, we can get Lemma 6.7. $\square$

As in (4.9), set

$$|s|_{\varepsilon,-1} := \sup_{0\neq s'\in I^1} \frac{\langle s, s'\rangle_{\varepsilon,0}}{|s'|_{\varepsilon,1}}. \tag{6.75}$$

Lemma 6.8. *There exist $c, C > 0$ such that if*

$$\lambda \in U = \left\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \leq \frac{\mathrm{Im}(\lambda)^2}{4c^2} - c^2\right\}, \tag{6.76}$$

the resolvent $(\lambda - L^3_{\varepsilon,T})^{-1}$ exists, and moreover for any $\varepsilon \in (0, 1]$, $s \in I^1$,

$$\begin{aligned} &|(\lambda - L^3_{\varepsilon,T})^{-1}s|_{\varepsilon,0} \leq C|s|_{\varepsilon,0}, \\ &|(\lambda - L^3_{\varepsilon,T})^{-1}s|_{\varepsilon,1} \leq C(1 + |\lambda|)^2|s|_{\varepsilon,-1}. \end{aligned} \tag{6.77}$$

Proof. Take c_2 in Lemma 6.7. If $\lambda \in \mathbb{R}$, $\lambda \leq -2c_2$, for $s \in I^1$, we have

$$\mathrm{Re}\langle(L^3_{\varepsilon,T} - \lambda)s, s\rangle_{\varepsilon,0} \geq c_1|s|^2_{\varepsilon,0}. \tag{6.78}$$

So

$$|s|_{\varepsilon,0} \leq c_1^{-1}|(L^3_{\varepsilon,T} - \lambda)s|_{\varepsilon,0}. \tag{6.79}$$

Since $|s|_{\varepsilon,0} \leq c(\varepsilon)|s|_0$ for $c(\varepsilon) > 0$,

$$|s|_0 \leq |s|_{\varepsilon,0} \leq c_1^{-1}|(L^3_{\varepsilon,T} - \lambda)s|_{\varepsilon,0} \leq c(\varepsilon)c_1^{-1}|(L^3_{\varepsilon,T} - \lambda)s|_0. \tag{6.80}$$

So $(L^3_{\varepsilon,T} - \lambda)^{-1}$ exists for $\lambda \in \mathbb{R}$, $\lambda \leq -2c_2$.

Set $\lambda = a + ib \in \mathbb{C}$. Then by Lemma 6.7,

$$\begin{aligned} |\langle(L^3_{\varepsilon,T} - \lambda)s, s\rangle_{\varepsilon,0}| &\geq \max\{\mathrm{Re}\langle L^3_{\varepsilon,T}s, s\rangle_{\varepsilon,0} - a|s|^2_{\varepsilon,0}, |\mathrm{Im}\langle L^3_{\varepsilon,T}s, s\rangle_{\varepsilon,0} - b|s|^2_{\varepsilon,0}|\} \\ &\geq \max\{c_1|s|^2_{\varepsilon,1} - (c_2 + a)|s|^2_{\varepsilon,0}, -c_3|s|_{\varepsilon,1}|s|_{\varepsilon,0} + |b||s|^2_{\varepsilon,0}\}. \end{aligned} \tag{6.81}$$

Set

$$C(\lambda) = \inf_{t\in\mathbb{R},t\geq 1} \max\{c_1t^2 - (c_2 + a), -c_3t + |b|\}. \tag{6.82}$$

If $c > 0$ is small enough, we can get

$$c_0 = \inf_{\lambda\in U} C(\lambda) > 0. \tag{6.83}$$

Since $|s|_{\varepsilon,0} \leq |s|_{\varepsilon,1}$, if the resolvent $(\lambda - L^3_{\varepsilon,T})^{-1}$ exists, then

$$|(\lambda - L^3_{\varepsilon,T})^{-1}s|_{\varepsilon,0} \leq c_0^{-1}|s|_{\varepsilon,0}. \tag{6.84}$$

From (6.84), if $\lambda' \in U$, $|\lambda' - \lambda| \leq c_0/2$, then the resolvent $(\lambda' - L^3_{\varepsilon,T})^{-1}$ exists. By (6.80), we get the first inequality of (6.77).

For $\lambda_0 \in \mathbb{R}$, $\lambda_0 \leq -2c_2$ and $s \in I^1$, by Lemma 6.7, we have

$$|\langle(\lambda_0 - L^3_{\varepsilon,T})s, s\rangle_{\varepsilon,0}| \geq c_1|s|^2_{\varepsilon,1}. \tag{6.85}$$

Following the same process in (4.30)-(4.34), we get the second estimate of (6.77).

The proof of Lemma 6.8 is complete. ∎

Set $\mathcal{D}_H = \{\partial_p, \nabla^{\mathcal{S}_X}_{e_i}\}$. Set

$$|s|^2_{\varepsilon,k} = \sum_{l=0}^{k} \sum_{Q_i \in \mathcal{D}_H} |Q_1 \cdots Q_l s|^2_{\varepsilon,0}. \tag{6.86}$$

As in Lemma 4.17, since $[Q, L^3_{\varepsilon,T}]$ has the same structure as $L^3_{\varepsilon,T}$ for $Q \in \mathcal{D}_H$, for any $k \in \mathbb{N}$ fixed, there exists $C_k > 0$ such that for $\varepsilon \in (0,1]$, $Q_1, \cdots, Q_k \in \mathcal{D}_H$ and $s, s' \in I^2$, we have

$$|\langle [Q_1, [Q_2, \cdots [Q_k, L^3_{\varepsilon,T}], \cdots]] s, s' \rangle_{\varepsilon,0}| \leq C_k |s|_{\varepsilon,1} |s'|_{\varepsilon,1}. \tag{6.87}$$

Then using the proof of Lemma 4.18, we can get the Lemma as follows.

Lemma 6.9. *For any $\varepsilon \in (0,1]$, λ satisfies (6.76) and $m \in \mathbb{N}$, there exist $C_m > 0$ and $p_m \in \mathbb{N}$, such that*

$$|(\lambda - L^3_{\varepsilon,T})^{-1} s|_{\varepsilon,m+1} \leq C_m (1 + |\lambda|)^{p_m} |s|_{\varepsilon,m}. \tag{6.88}$$

Set

$$\Gamma = \partial U = \left\{ \lambda \in \mathbb{C} : \mathrm{Re}(\lambda) = \frac{\mathrm{Im}(\lambda)^2}{4c^2} - c^2 \right\}, \tag{6.89}$$

and

$$\Gamma' = \{ \lambda \in \mathbb{C} : |\mathrm{Im} \lambda| \leq c \}. \tag{6.90}$$

Then the map $\lambda \mapsto \lambda^2$ sends Γ' to Γ. Let $\Delta = -\Delta^{TY} + D^{X,2}_{y_0}$. For $\lambda \in \Gamma$, $k, m \in \mathbb{N}$ and $k \leq m$, from Lemma 6.9, there exist $C_k > 0$ and $p'_m > 0$ such that

$$\begin{aligned} &|\Delta^k (\lambda - L^3_{\varepsilon,T})^{-m} s|_{\varepsilon,0} \leq |(\lambda - L^3_{\varepsilon,T})^{-m} s|_{\varepsilon,k} \\ \leq & C_k (1 + |\lambda|)^{p'_m} |(\lambda - L^3_{\varepsilon,T})^{-m+k} s|_{\varepsilon,0} \leq C_k (1 + |\lambda|)^{p'_m} |s|_{\varepsilon,0}. \end{aligned} \tag{6.91}$$

Denote by $L^{3,*}_{\varepsilon,T}$ the formal adjoint of $L^3_{\varepsilon,T}$ with respect to the usual Hermitian product in I^0. Then $L^{3,*}_{\varepsilon,T}$ has the same structure as $L^3_{\varepsilon,T}$ except that we replace the operators $f^p \wedge$, i_{f_p} by i_{f_p} and $f^p \wedge$. For $s \in I^0_q$, set

$$|s|'^2_{\varepsilon,0} = \int_{T_{y_0} Y_b \times X_{y_0}} |s(U,x)|^2 g_\varepsilon(U)^{2(q-l')} dv_{TY}(U) dv_X(x). \tag{6.92}$$

From the above analysis associated to $|\cdot|'_{\varepsilon,0}$, we obtain (6.91) for $L^{3,*}_{\varepsilon,T}$ and $|\cdot|'_{\varepsilon,0}$. Taking adjoint with respect to the usual Hermitian product in I^0, we have

$$|(\lambda - L^3_{\varepsilon,T})^{-m} \Delta^k s|_{\varepsilon,0} \leq C_k (1 + |\lambda|)^{p'_m} |s|_{\varepsilon,0}. \tag{6.93}$$

So for $k, k', m \in \mathbb{N}$ and $m \geq k + k'$, there exists $C_{k,k'} > 0$, such that

$$\begin{aligned} |\Delta^k \exp(-L^3_{\varepsilon,T}) \Delta^{k'} s|_{\varepsilon,0} &= \left| \frac{(-1)^{m-1}(m-1)!}{2\pi i} \int_\Gamma e^{-\lambda} \Delta^k (\lambda - L^3_{\varepsilon,T})^{-m} \Delta^{k'} s \right|_{\varepsilon,0} \\ &\leq C_{k,k'} \left(\int_\Gamma e^{-\lambda} (1 + |\lambda|)^{p'_m} d\lambda \right) |s|_{\varepsilon,0} \\ &= C_{k,k'} \left(\int_{\Gamma'} e^{-\lambda^2} (1 + |\lambda^2|)^{p'_m} d\lambda \right) |s|_{\varepsilon,0} \leq C |s|_{\varepsilon,0}. \end{aligned} \tag{6.94}$$

Take $p \in \mathbb{N}$. Let J^0_{p,y_0} be the set of square integrable sections of $\Lambda(TV^g)\widehat{\otimes}\mathcal{S}(N_{Y^g/Y})\otimes L_Y^{1/2}\widehat{\otimes}\mathcal{S}(TX, L_X)$ over

$$\left\{(U,x)\in T_{y_0}Y\times X_{y_0}; x\in X_{y_0}, |U|\leq p+\frac{1}{2}\right\}. \tag{6.95}$$

We equip J^0_{p,y_0} with the Hermitian product for $s\in J^0_{p,y_0}$,

$$|s|^2_{(p),0}=\int_{|U|\leq p+\frac{1}{2}}\int_{X_{y_0}}|s(U,x)|^2 dv_{T_{y_0}Y}dv_X. \tag{6.96}$$

Obviously, there exists $C>0$ such that for any $p\in\mathbb{N}$, $s\in J^0_{p,y_0}$,

$$|s|_{(p),0}\leq |s|_{\varepsilon,0}\leq C(1+p)^{l'}|s|_{(p),0}. \tag{6.97}$$

By (6.94) and (6.97), we find for any $k\leq m$, $k'\leq m'$, there exists $C'>0$ such that for $\varepsilon\in(0,1]$, $p\in\mathbb{N}$, $s\in J^0_{p,y_0}$,

$$|\Delta^k\exp(-L^3_{\varepsilon,T})\Delta^{k'}s|_{(p),0}\leq \left|\Delta^k\exp(-L^3_{\varepsilon,T})\Delta^{k'}s\right|_{\varepsilon,0}\leq C'(1+p)^{l'}|s|_{(p),0}. \tag{6.98}$$

Thus, following the same process in (4.134) - (4.136), for $k,k'\in\mathbb{N}$ there exists $C>0$, $r>0$ such that for $\varepsilon\in(0,1]$, $p\in\mathbb{N}$,

$$\sup_{|U|,|U'|\leq p+1/4}|\Delta^k_{(U,x)}\Delta^{k'}_{(U',x')}\exp(-L^3_{\varepsilon,T})((U,x),(U',x'))|\leq C(1+p)^r. \tag{6.99}$$

So we get the bounds in (6.60) with $C=0$.

By (6.9) and (6.11), we write

$$\begin{aligned}\widetilde{\mathbf{G}}_u(L^3_{\varepsilon,T})&((U,x),(U',x'))\\ &=\int_{-\infty}^{+\infty}\cos\left(\sqrt{2}v\sqrt{L^3_{\varepsilon,T}}\right)((U,x),(U',x'))e^{-\frac{v^2}{2}}(1-f(\sqrt{u}v))\frac{dv}{\sqrt{2\pi}}.\end{aligned} \tag{6.100}$$

Lemma 6.10. *There exist $C_1,C_2>0$, $r>0$, such that for $\varepsilon\in(0,1]$, $m,m'\in\mathbb{N}$,*

$$\sup_{|\beta|\leq m,|\beta'|\leq m'}|\partial^\beta_U\partial^{\beta'}_{U'}\widetilde{\mathbf{G}}_u(L^3_{\varepsilon,T})((U',x'),(U,x))|\leq C_1(1+|U|+|U'|)^r\exp\left(-\frac{C_2}{u}\right). \tag{6.101}$$

Proof. After replacing $\exp(-L^3_{\varepsilon,T})$ to $\widetilde{\mathbf{G}}_u(L^3_{\varepsilon,T})$ in (6.94)-(6.99) and using (6.22), we get Lemma 6.10. □

If $|\sqrt{u}v|\leq\alpha/2$, then $f(\sqrt{u}v)=0$. Using finite propagation speed of the hyperbolic equation for the solution of hyperbolic equations for $\cos(s\sqrt{L^3_{\varepsilon,T}})$ (cf. [19, §7.8], [32, §4.4]), there exists a constant $C_0'>0$, such that

$$\widetilde{\mathbf{G}}_u(L^3_{\varepsilon,T})((U,x),(U',x'))=\exp(L^3_{\varepsilon,T})((U,x),(U',x')), \tag{6.102}$$

if $|U-U'|\geq C_0'/\sqrt{u}$.

Then by (6.102) and Lemma 6.10, For $m, m' \in \mathbb{N}$, there exists $C_1, C_2 > 0$, $r > 0$, such that for $\varepsilon \in (0,1]$,

$$\begin{aligned}\sup_{|\beta|\leq m, |\beta'|\leq m'} &\left|\partial_U^{\beta}\partial_{U'}^{\beta'} \exp(-L^3_{\varepsilon,T})((U,x),(U',x'))\right| \\ &\leq C_1(1+|U|+|U'|)^r \exp\left(-\frac{C_2|U-U'|^2}{C_0'^2}\right).\end{aligned} \tag{6.103}$$

So we get the bounds in (6.60).

For $U \in T_{y_0}Y$, set $U = U_p f_p$. Let $|\cdot|_{0,k}$ be the limit norm of $|\cdot|_{\varepsilon,k}$ as $\varepsilon \to 0$ for $k \in \{-1,0,1\}$. Note that all the estimates in this subsection work for $\varepsilon = 0$. For $k \in \{-1,0,1\}$ and $k' \in \mathbb{N}$, set

$$I_0^{k,k'} = \{s \in I^k : U^\alpha s \in I^k \text{ for } |\alpha| \leq k'\}.$$

For $s \in I_0^{k,k'}$, set

$$|s|^2_{0,(k,k')} = \sum_{|\alpha|\leqslant k'} |U^\alpha s|^2_{0,k}. \tag{6.104}$$

Lemma 6.11. *There exist $C > 0$, $k, k' \in \mathbb{N}$ such that for $s \in I$,*

$$\left|\left[\left(\lambda - L^3_{\varepsilon,T}\right)^{-1} - \left(\lambda - L^3_{0,T}\right)^{-1}\right]s\right|_{\varepsilon,0} \leqslant C\varepsilon\left(1+|\lambda|\right)^k |s|_{0,(0,k')}. \tag{6.105}$$

Proof. Clearly,

$$\left(\lambda - L^3_{\varepsilon,T}\right)^{-1} - \left(\lambda - L^3_{0,T}\right)^{-1} = \left(\lambda - L^3_{\varepsilon,T}\right)^{-1}\left(L^3_{\varepsilon,T} - L^3_{0,T}\right)\left(\lambda - L^3_{0,T}\right)^{-1}. \tag{6.106}$$

Since $|\cdot|_{\varepsilon,0} \leqslant |\cdot|_{0,0}$, then by (6.51),

$$\left|\left\langle (L^3_{\varepsilon,T} - L^3_{0,T})s, s'\right\rangle_{\varepsilon,0}\right| \leqslant C\varepsilon |s|_{0,(1,4)} |s'|_{\varepsilon,1}, \tag{6.107}$$

which implies that

$$\left|(L^3_{\varepsilon,T} - L^3_{0,T})s\right|_{\varepsilon,-1} \leqslant C\varepsilon |s|_{0,(1,4)}. \tag{6.108}$$

On the other hand, we have

$$\left|\left\langle \left[U_{i_1}, [\cdots[U_{i_p},\ L^3_{0,T}]\cdots]s, s'\right\rangle\right]\right|_{0,0} \leqslant C_p |s|_{0,1}|s'|_{0,1}. \tag{6.109}$$

From (6.109) and the argument as in the proof of Theorem 4.18, we obtain

$$\left|(\lambda - L^3_{0,T})^{-1}s\right|_{0,(1,k)} \leqslant C\left(1+|\lambda|\right)^k |s|_{0,(0,k)}. \tag{6.110}$$

This completes the proof of Lemma 6.11. $\square$

By (6.97) and Lemma 6.11, there exists $r \in \mathbb{N}$ for $s \in J^0_{p,y_0}$,

$$|((\lambda - L^3_{\varepsilon,T})^{-1} - (\lambda - L^3_{0,T})^{-1})s|_{(p),0} \leq c\varepsilon(1+|\lambda|)^2(1+p)^r |s|_{(p),0}. \tag{6.111}$$

So there exists $C > 0$, $r \in \mathbb{N}$, such that for $\varepsilon \in (0,1]$, $p \in \mathbb{N}$,

$$|(\exp(-L^3_{\varepsilon,T}) - \exp(-L^3_{0,T}))s|_{(p),0} \leq C\varepsilon(1+p)^r |s|_{(p),0}. \tag{6.112}$$

By the same process in (4.111)-(4.114), there exist $c>0, C>0, r\in\mathbb{N}$, such that for any $(U,x),(U',x')\in T_{y_0}Y\times X_{y_0}$, $\varepsilon\in(0,1]$,

$$
\begin{aligned}
&|(\exp(-L^3_{\varepsilon,T})-\exp(-L^3_{0,T}))((U,x),(U',x'))|\\
&\qquad\leq c\varepsilon^{(\dim Y+1)^{-1}}(1+|U|+|U'|)^r\exp(-C|U-U'|^2).
\end{aligned}
\tag{6.113}
$$

Then the proof of Theorem 6.5 is complete.

7. Proof of Theorem 3.6 iii)

In this Section, we use the notations and assumptions in Section 2.2 and 6.

7.1. **Localization of the problem near $\pi_1^{-1}(V^g)$.** We replace T by u and T/ε by T'.

By Lemma 6.1, there exist $C_1, C_2>0$, such that for any $z,z'\in Z_b$ and $u\in(0,1]$, $T'\geq 1$,

$$
\left|\widetilde{\mathbf{G}}_{u^2/T'^2}\left(\frac{u^2}{T'^2}\mathcal{B}'_{T'}\right)(z,z')\right|\leq C_1\exp\left(-\frac{C_2T'^2}{u^2}\right),
\tag{7.1}
$$

and

$$
\left|\psi_S\widetilde{\operatorname{Tr}}\left[g\widetilde{\mathbf{G}}_{u^2/T'^2}\left(\mathcal{B}'_{u/T',T'}\right)\right]\right|\leq C_1\exp\left(-\frac{C_2T'^2}{u^2}\right).
\tag{7.2}
$$

We trivialize the bundle $\pi_3^*\Lambda(T^*S)\widehat{\otimes}\mathcal{S}(TZ,L_Z)$ as in Section 6.2. By (6.34), we can get

$$
L^1_{u/T',u}=u^2\delta_{u^2}L^1_{1/T',1}\delta^{-1}_{u^2}.
\tag{7.3}
$$

Comparing with (6.41), there exists $C>0$, such that for $|U|<\alpha_0/4$,

$$
\begin{aligned}
&\big|\exp\left(-u^2\mathcal{B}'^2_{1/T'}\right)(g^{-1}(U,x),(U,x))k_Y(y_0,U)\\
&\qquad-\exp(-u^2L^1_{1/T',1})(g^{-1}(U,x),(U,x))\big|\leq C\exp\left(-\frac{C_2T'^2}{u^2}\right).
\end{aligned}
\tag{7.4}
$$

Then we can replace the fiber Z by $T_{y_0}Y\times X_{y_0}$ for $y_0\in V^g$.

7.2. **Proof of Theorem 3.6 iii).** We will use the notation of Section 6.2 with ε replaced by $1/T'$, and T by 1. By Lemma 6.4, we see that as $T'\to+\infty$

$$
L^3_{1/T',1}\to L^3_{0,1}.
\tag{7.5}
$$

Let $\exp(-u^2L^i_{\varepsilon,T})((U,x),(U',x'))$ $((U,x),(U'x')\in T_{y_0}Y\times X_{y_0})$ $(i=1,2,3)$ be the smooth kernel associated to the operator $\exp(-u^2L^i_{\varepsilon,T})$ with respect to $dv_{T_{y_0}Y}(U')dv_{X_{y_0}}(x')$. Then by (6.45),

$$
\begin{aligned}
&\psi_S\int_{Y^g}\int_{\substack{U\in N,\\|U|\leq\alpha_0/4}}\int_X\delta_{u^2}\widetilde{\operatorname{Tr}}\left[g\exp\left(-u^2L^1_{1/T',1}\right)\left(g^{-1}(U,x),(U,x)\right)\right]dv_{Y^g}dv_N(U)dv_X(x)\\
&\quad=\psi_S\int_{Y^g}\int_{\substack{U\in N,\\|U|\leq T'\alpha_0/4}}\int_X\tilde{c}_{TY^g}\delta_{u^2}\widetilde{\operatorname{Tr}}\left[g\exp(-u^2L^3_{1/T',1})\left(g^{-1}(U,x),(U,x)\right)\right]^{max}\\
&\qquad\qquad\cdot dv_{Y^g}dv_N(U)dv_X(x).
\end{aligned}
\tag{7.6}
$$

By (7.6) and the argument of Section 6.2, to calculate the asymptotic of the left hand side of (7.6) as $u \to 0$ uniformly in $T \geq 1$, we have to find the asymptotic as $u \to 0$ of

$$\psi_S \int_{U \in N} \int_X \tilde{c}_{TY^g} \delta_{u^2} \widetilde{\mathrm{Tr}} \left[g \exp(-u^2 L^3_{1/T,1}) \left(g^{-1}(U,x),(U,x) \right) \right]^{max} dv_N(U) dv_X(x). \tag{7.7}$$

The following lemma is a modification of Lemma 6.5.

Lemma 7.1. *There exist $C_1, C_2 > 0, p, r \in \mathbb{N}$ such that for any $(U,x),(U',x') \in T_{y_0}Y \times X_{y_0}$, $\varepsilon \in [0,1]$, $u \in (0,1]$,*

$$\begin{aligned} &|u^p \exp(-u^2 L^3_{\varepsilon,1})((U,x),(U',x'))| \\ &\qquad \leq C_1(1+|U|+|U'|)^r \cdot \exp\left(-C_2 \frac{|U-U'|^2 + d^X(x,x')^2}{u^2} \right). \end{aligned} \tag{7.8}$$

Proof. By (6.94),

$$\begin{aligned} |\Delta^k \exp(-u^2 L^3_{\varepsilon,1}) \Delta^{k'} s|_{\varepsilon,0} &\leq C \left(\int_\Gamma e^{-u^2\lambda}(1+|\lambda|)^{p_m} d\lambda \right) |s|_{\varepsilon,0} \\ \leq C u^{-2p_m-2} &\left(\int_{u^2\Gamma} e^{-\lambda}(1+|\lambda|)^{p'_m} d\lambda \right) |s|_{\varepsilon,0} \leq C u^{-2p_m-2} |s|_{\varepsilon,0}. \end{aligned} \tag{7.9}$$

So, there exists $p \in \mathbb{N}$, such that

$$|u^p \Delta^k \exp(-u^2 L^3_{\varepsilon,1}) \Delta^{k'} s|_{\varepsilon,0} \leq C |s|_{\varepsilon,0}. \tag{7.10}$$

Following the process in (6.95)-(6.99), we have

$$|u^p \exp(-u^2 L^3_{\varepsilon,1})((U,x),(U',x'))| \leq C(1+|U|+|U'|)^r. \tag{7.11}$$

Following the process in (6.100)-(6.103), We get Lemma 7.1. □

Let $N_{X^g/X}$ be the normal bundle to X^g in X. We identify $N_{X^g/X}$ to the orthogonal bundle to TX^g in TX. Let g^{N_X} be the metric on $N_{X^g/X}$ induced by g^{TX}. Let dv_{N_X} be the Riemannian volume form on $(N_{X^g/X}, g^{N_X})$.

For $U \in T_{y_0}Y$, $x \in X^g$, $V \in N_{X^g/X}$, $|U|, |V| \leq \alpha_0/4$, let $k_X(U,x,V)$ be defined by

$$dv_X(U,x,V) = k_X(U,x,V) dv_{N_{X^g/X}}(V) dv_{X^g}(x). \tag{7.12}$$

Set $n' = \dim Z^g$. By standard results on heat kernel (cf. [4, Theorem 6.11]), there exist smooth functions $a'_{T,-n'}(x), \cdots, a'_{T,0}(x)$ $(x \in W^g)$ such that as $u \to 0$, for $x \in X^g_{y_0}$,

$$\begin{aligned} \int_{\substack{V \in N_X, U \in N_Y, \\ |U|,|V| \leq \alpha_0/4}} &\delta_{u^2} \widetilde{\mathrm{Tr}} \left[g \exp(-u^2 L^3_{1/T',1}) \left(g^{-1}(U,x,V),(U,x,V) \right) \right]^{max} \\ &\cdot k_X(U,x,V) dv_{N_X} dv_{N_Y} = \sum_{j=-n'}^{0} a'_{T',j}(x) u^j + O(u), \end{aligned} \tag{7.13}$$

where the $a'_{T',j}(x)$ only depend on the operator $L^3_{1/T',1}$ and its higher derivatives on x. By (7.5), $a'_{T',j}(x)$ is continuous on $T' \in [1, +\infty]$.

By (6.29), (7.5)-(7.8) and (7.13), there exist $a_{T',j}$ depending continuously on $T' \in [1,+\infty]$ such that for any $u \in (0,1]$, $T' \in [1,+\infty]$,

$$\left| \psi_S \widetilde{\mathrm{Tr}} \left[g \exp\left(-\mathcal{B}'_{u/T',T'}\right)\right] - \sum_{j=-n'}^{0} a_{T',j} u^j \right| \leq Cu. \tag{7.14}$$

Since $\varepsilon = u/T'$, (7.14) is reformulated by

$$\left| \psi_S \widetilde{\mathrm{Tr}} \left[g \exp\left(-\mathcal{B}'_{\varepsilon,T'}\right)\right] - \sum_{j=-n'}^{0} a_{T',j} (\varepsilon T')^j \right| \leq C\varepsilon T'. \tag{7.15}$$

Following the process in (5.6)-(5.8), we have

$$\left| \psi_S \widetilde{\mathrm{Tr}} \left[g \exp\left(-\mathcal{B}'_{\varepsilon,T'}\right)\right]^{dT'} - \sum_{j=-n'}^{0} [a_{T',j}]^{dT'} (\varepsilon T')^j \right| \leq C\varepsilon. \tag{7.16}$$

For $T' \geq 1$ fixed, by Theorem 1.2 and (3.20), we have

$$\lim_{\varepsilon \to 0} \psi_S \widetilde{\mathrm{Tr}} \left[g \exp\left(-\mathcal{B}'_{\varepsilon,T'}\right)\right]^{dT'} = \int_{Z^g} \gamma_{\mathcal{A}}(T') \wedge \mathrm{ch}_g(L_Z^{1/2}, \nabla^{L_Z^{1/2}}). \tag{7.17}$$

From (7.15) and (7.17),

$$[a_{T',j}]^{dT'} = 0 \quad \text{if} \quad j < -1, \quad [a_{T',0}]^{dT'} = \int_{Z^g} \gamma_{\mathcal{A}}(T') \wedge \mathrm{ch}_g(L_Z^{1/2}, \nabla^{L_Z^{1/2}}). \tag{7.18}$$

Since $T' = \varepsilon T$,

$$[a_{T',j}]^{dT} = \varepsilon^{-1} [a_{T',j}]^{dT'}. \tag{7.19}$$

From (7.18) and (7.19), comparing the coefficients of dT in (7.15), we have

$$\left| \psi_S \widetilde{\mathrm{Tr}} \left[g \exp\left(-\mathcal{B}'_{\varepsilon,T/\varepsilon}\right)\right]^{dT} - \varepsilon^{-1} \int_{Z^g} \gamma_{\mathcal{A}}(T/\varepsilon) \wedge \mathrm{ch}_g(L_Z^{1/2}, \nabla^{L_Z^{1/2}}) \right| \leq C. \tag{7.20}$$

By (6.2) and (7.20), we get Theorem 3.6 iii).

8. Proof of Theorem 3.6 iv)

In this section, we prove Theorem 3.6 iv) by following the process of [5, Section IX] and [26, Section 9]. In Section 8.1, as in Section 6.1, we reduce the problem to a local problem near $\pi_1^{-1}(V^g)$. In Section 8.2, we study the matrix structure of $L^3_{\varepsilon,T}$ as in Section 4.2. In Section 8.3, we prove Theorem 3.6 iv).

We use the same notation as in Section 4, 6 and the assumptions in Section 2.2.

8.1. Finite propagation speed and localization.

Proposition 8.1. *There exist $C > 0$, $C' > 0$, $\delta > 0$, $T_0 \geq 1$, such that for $0 < \varepsilon \leq 1$, $T \geq T_0$,*

$$\left| \psi_S \widetilde{\mathrm{Tr}} \left[g \widetilde{\boldsymbol{G}}_{\varepsilon^2}(\mathcal{B}'_{\varepsilon,T/\varepsilon})\right]^{dT} \right| \leq \frac{C}{T^{1+\delta}}. \tag{8.1}$$

Proof. As we noted in Section 5, if we replace $\mathcal{B}_T$ by $\mathcal{B}'_{T/\varepsilon}$ and $\mathcal{B}_2$ to B_2, everything in Section 4 works well. So there exist $C>0$, $\delta>0$, $T_0\geq 1$, such that for $0<\varepsilon\leq 1$, $T\geq T_0$,

$$\left|\psi_S\widetilde{\mathrm{Tr}}\left[g\widetilde{\mathbf{G}}_{\varepsilon^2}(\varepsilon^2\mathcal{B}'_{T/\varepsilon})\right]-\psi_S\widetilde{\mathrm{Tr}}\left[g\widetilde{\mathbf{G}}_{\varepsilon^2}(\varepsilon^2 B_2)\right]\right|\leq\frac{C}{T^\delta}. \tag{8.2}$$

Since the second term above does not involve dT part, by (6.1) and following the argument in (5.5)-(5.8), we get Proposition 8.1. □

By Proposition 8.1, to establish Theorem 3.6 iv), we only need to prove the following result.

Theorem 8.2. *There exist $C>0$, $C'>0$, $\delta>0$, and $T_0\geq 1$ such that for $0<\varepsilon\leq 1$, $T\geq T_0$*

$$\left|\psi_S\widetilde{\mathrm{Tr}}\left[g\widetilde{\boldsymbol{F}}_{\varepsilon^2}(\mathcal{B}'_{\varepsilon,T/\varepsilon})\right]^{dT}\right|\leq\frac{C}{T^{1+\delta}}. \tag{8.3}$$

By the finite propagation speed as in (6.27), if $x\in W$, $\widetilde{\mathbf{F}}_{\varepsilon^2}(\mathcal{B}'_{\varepsilon,T/\varepsilon})(x,\cdot)$ only depends on the restriction of $\mathcal{B}'_{\varepsilon,T/\varepsilon}$ to $\pi^{-1}(B^Y(\pi_1 x,\alpha))$.

Now we can use the same argument as discussed in (6.27)-(6.29) to know the proof of Theorem 8.2 is local near $\pi_1^{-1}(V^g)$.

8.2. **The matrix structure of the operator $L^3_{\varepsilon,T}$ as $T\to+\infty$.** We use the same trivialization and notations as in Section 6.1.

By (6.45),

$$\begin{aligned}&\int_{Y^g}\int_{\substack{U\in N_Y,\\|U|\leq\alpha_0/4}}\widetilde{\mathrm{Tr}}[g\widetilde{\mathbf{F}}_{\varepsilon^2}(L^1_{\varepsilon,T})(g^{-1}(U,x),(U,x))]dv_{N_Y}dv_{Y^g}\\ &=\int_{Y^g}\int_{\substack{U\in N_Y,\\|U|\leq\alpha_0/4\varepsilon}}\tilde{c}_{TY^g}\widetilde{\mathrm{Tr}}\left[g\widetilde{\mathbf{F}}_{\varepsilon^2}(L^3_{\varepsilon,T})\left(g^{-1}\left(U,x\right),\left(U,x\right)\right)\right]dv_{N_Y}.\end{aligned} \tag{8.4}$$

Recall that the vector bundle K was defined in the argument before (6.33) and the operator S_ε was defined in (6.36). Let $\mathbb{F}^0_\varepsilon$ be the vector space of square integrable sections of $\Lambda(T^*V^g)\widehat{\otimes}\mathcal{S}(N_{Y^g/Y})\widehat{\otimes}S_\varepsilon^{-1*}K\otimes L_Y^{1/2}$ over $T_{y_0}Y$. Then $\mathbb{F}^0_\varepsilon$ is a Hilbert subspace of I^0. Let $\mathbb{F}^{0,\perp}_\varepsilon$ be its orthogonal complement in I^0. Let p_ε be the orthogonal projection operator from I^0 on $\mathbb{F}^0_\varepsilon$. Set $p_\varepsilon^\perp=1-p_\varepsilon$. Then if $s\in I^0$,

$$p_\varepsilon s(U)=P^K_{\varepsilon U}s(U,\cdot)\quad U\in T_{y_0}Y. \tag{8.5}$$

Put

$$\begin{aligned}E_{\varepsilon,T}&=p_\varepsilon L^3_{\varepsilon,T}p_\varepsilon,\quad F_{\varepsilon,T}=p_\varepsilon L^3_{\varepsilon,T}p_\varepsilon^\perp,\\ G_{\varepsilon,T}&=p_\varepsilon^\perp L^3_{\varepsilon,T}p_\varepsilon,\quad H_{\varepsilon,T}=p_\varepsilon^\perp L^3_{\varepsilon,T}p_\varepsilon^\perp.\end{aligned} \tag{8.6}$$

Then we write $L^3_{\varepsilon,T}$ in matrix form with respect to the splitting $I^0=\mathbb{F}^0_\varepsilon\oplus\mathbb{F}^{0,\perp}_\varepsilon$,

$$L^3_{\varepsilon,T}=\begin{pmatrix}E_{\varepsilon,T}&F_{\varepsilon,T}\\G_{\varepsilon,T}&H_{\varepsilon,T}\end{pmatrix}. \tag{8.7}$$

The following lemma is an analogue of Proposition 4.7.

Lemma 8.3. *There exist operators E_ε, F_ε, G_ε, H_ε such that as $T\to\infty$,*

$$\begin{aligned} E_{\varepsilon,T} &= E_\varepsilon + O(1/T), \qquad & F_{\varepsilon,T} &= TF_\varepsilon + O(1),\\ G_{\varepsilon,T} &= TG_\varepsilon + O(1), \qquad & H_{\varepsilon,T} &= T^2H_\varepsilon + O(T). \end{aligned} \tag{8.8}$$

Set

$$Q_\varepsilon := \rho^2(\varepsilon U)R_\varepsilon S_\varepsilon^{-1}\left[D^X, \varepsilon D^H + {}^0\nabla^{\mathscr{E}_Z,u}\right] S_\varepsilon. \tag{8.9}$$

Then Q_ε maps $\mathbb{F}^0_\varepsilon$ into $\mathbb{F}^{0,\perp}_\varepsilon$. Moreover,

$$\begin{aligned} F_\varepsilon &= p_\varepsilon Q_\varepsilon p_\varepsilon^\perp,\\ G_\varepsilon &= p_\varepsilon^\perp Q_\varepsilon p_\varepsilon,\\ H_\varepsilon &= p_\varepsilon^\perp(\rho^2(\varepsilon|U|)D^{X,2}_{\varepsilon U} + (1-\rho^2(\varepsilon U))D^{X,2}_{y_0})p_\varepsilon^\perp. \end{aligned} \tag{8.10}$$

Proof. From (6.1), (6.3), (6.37) and (6.39), we find the coefficient of T^2 in the expansion of $L^3_{\varepsilon,T}$ is given by

$$H_\varepsilon = (1-\rho^2(\varepsilon|U|))P^{K,\perp}_{\varepsilon U}D^{X,2}_{y_0}P^{K,\perp}_{\varepsilon U} + \rho^2(\varepsilon|U|)D^{X,2}_{\varepsilon U}. \tag{8.11}$$

When $\rho(\varepsilon|U|)\neq 0$, $K_{\varepsilon U} = \ker D^{X,2}_{\varepsilon U}$. So

$$H_\varepsilon = P^{K,\perp}_{\varepsilon U}\left((1-\rho^2(\varepsilon|U|))D^{X,2}_{y_0} + \rho^2(\varepsilon|U|)D^{X,2}_{\varepsilon U}\right)P^{K,\perp}_{\varepsilon U}. \tag{8.12}$$

Using (8.5), we see that (8.12) fits with the last formula in (8.10).

By (6.1), (6.3), (6.37) and (6.39), we find that the coefficient of T in the expansion of $L^3_{\varepsilon,T}$ is the operator Q_ε.

Using (8.9), it is clear that Q_ε maps $\mathbb{F}^0_\varepsilon$ into $\mathbb{F}^{0,\perp}_\varepsilon$. Also (8.8) and the remaining equations in (8.10) follow.

The proof of Theorem 8.3 is complete. □

Clearly, for $U\in T_{y_0}Y$, $H_{\varepsilon U}$, the operator H_ε at U, is an elliptic operator acting along X_{y_0}.

Proposition 8.4. *For any $\varepsilon>0$,*

$$\ker H_{\varepsilon U} = \Lambda(T^*V^g)\widehat{\otimes}\mathcal{S}(N_{Y^g/Y})\widehat{\otimes}K_{\varepsilon U}\otimes L_Y^{1/2}. \tag{8.13}$$

Proof. By (8.10), if $s\in\Lambda(T^*V^g)\widehat{\otimes}\mathcal{S}(N_{Y^g/Y})\widehat{\otimes}K_{\varepsilon U}\otimes L_Y^{1/2}$, then

$$H_\varepsilon s = 0. \tag{8.14}$$

The operator $H_{\varepsilon U}$ is self-adjoint and nonnegative. Therefore if $H_\varepsilon s=0$, then

$$\begin{aligned} &P^{K,\perp}_{\varepsilon U}\rho^2(\varepsilon|U|)D^{X,2}_{\varepsilon U}P^{K,\perp}_{\varepsilon U}s = 0,\\ &P^{K,\perp}_{\varepsilon U}(1-\rho^2(\varepsilon U))D^{X,2}_{y_0})P^{K,\perp}_{\varepsilon U}s = 0. \end{aligned} \tag{8.15}$$

If $\rho^2(\varepsilon|U|)\neq 0$, we deduce from the first identity in (8.15) that $P^{K,\perp}_{\varepsilon U}s = 0$, i.e. $s\in\Lambda(T^*V^g)\widehat{\otimes}\mathcal{S}(N_{Y^g/Y})\widehat{\otimes}K_{\varepsilon U}\otimes L_Y^{1/2}$. If $\rho^2(\varepsilon|U|)=0$, by the second identity in (8.15), $P^{K,\perp}_{\varepsilon U}s\in\ker D^X_{y_0}$. Using (6.33), we deduce that $P^{K,\perp}_{\varepsilon U}s=0$, i.e., $s\in\Lambda(T^*V^g)\widehat{\otimes}\mathcal{S}(N_{Y^g/Y})\widehat{\otimes}K_{\varepsilon U}\otimes L_Y^{1/2}$.

The proof of proposition 8.4 is complete. □

8.3. **Proof of Theorem 8.2.** For $s \in I$, put

$$|s|^2_{\varepsilon,T,1} := |P^K_{\varepsilon U} s|^2_{\varepsilon,0} + T^2 |P^{K,\perp}_{\varepsilon U} s|^2_{\varepsilon,0} + \sum_p |\nabla_{f_p} s|^2_{\varepsilon,0} + T^2 \sum_i |{}^0\nabla^{\mathcal{S}_Z}_{e_i} P^{K,\perp}_{\varepsilon U} s|^2_{\varepsilon,0}. \tag{8.16}$$

Lemma 8.5. *There exist $c_1, c_2, c_3, c_4 > 0$, $T_0 \geq 1$, such that for any $s, s' \in I$ with compact support, $\varepsilon \in (0,1]$, $T \geq T_0$, we have*

$$\begin{aligned} \mathrm{Re}\langle L^3_{\varepsilon,T} s, s\rangle_{\varepsilon,0} &\geq c_1 |s|^2_{\varepsilon,T,1} - c_2 |s|^2_{\varepsilon,0}, \\ |\mathrm{Im}\langle L^3_{\varepsilon,T} s, s\rangle_{\varepsilon,0}| &\leq c_3 |s|_{\varepsilon,T,1} |s|_{\varepsilon,0}, \\ |\langle L^3_{\varepsilon,T} s, s'\rangle_{\varepsilon,0}| &\leq c_4 |s|_{\varepsilon,T,1} |s'|_{\varepsilon,T,1}. \end{aligned} \tag{8.17}$$

Proof. By (6.1), (6.3), (6.37) and (6.39), the 2-order term of the differential operator $L^3_{\varepsilon,T}$ is a fiberwise elliptic operator

$$T^2 H_\varepsilon + \Delta^{TY}. \tag{8.18}$$

From (8.9), since K is a vector bundle over $T_{y_0}Y \times S$, for $s \in I$ with compact support, there exists $C_1 > 0$, such that

$$\langle H_\varepsilon P^{K,\perp}_{\varepsilon U} s, P^{K,\perp}_{\varepsilon U} s\rangle_{\varepsilon,0} \geq C_1 |P^{K,\perp}_{\varepsilon U} s|^2_{\varepsilon,0}. \tag{8.19}$$

Since H_ε is a fiberwise selfadjoint elliptic operator along the fibers X, from the elliptic estimates, there exist $C_2, C_3 > 0$, such that

$$\langle H_\varepsilon P^{K,\perp}_{\varepsilon U} s, P^{K,\perp}_{\varepsilon U} s\rangle_{\varepsilon,0} \geq C_2 \sum_i |{}^0\nabla^{\mathcal{S}_Z}_{e_i} P^{K,\perp}_{\varepsilon U} s|^2_{\varepsilon,0} - C_3 |P^{K,\perp}_{\varepsilon U} s|^2_{\varepsilon,0}. \tag{8.20}$$

From (8.19) and (8.20), there exists $C_4 > 0$, such that

$$\langle H_\varepsilon P^{K,\perp}_{\varepsilon U} s, P^{K,\perp}_{\varepsilon U} s\rangle_{\varepsilon,0} \geq C_4 \left(\sum_i |{}^0\nabla^{\mathcal{S}_Z}_{e_i} P^{K,\perp}_{\varepsilon U} s|^2_{\varepsilon,0} + |P^{K,\perp}_{\varepsilon U} s|^2_{\varepsilon,0} \right). \tag{8.21}$$

By (6.74), there exist $C_5, C_6 > 0$, such that

$$\langle \Delta^{TY} s, s\rangle_{\varepsilon,0} \geq C_5 \sum_p |\nabla_{f_p} s|^2_{\varepsilon,0} - C_6 |s|^2_{\varepsilon,0}. \tag{8.22}$$

Then there exist $C_1', C_2' > 0$, such that

$$\langle (T^2 H_\varepsilon + \Delta^{TY}) s, s\rangle_{\varepsilon,0} \geq C_1' |s|^2_{\varepsilon,T,1} - C_2' |s|^2_{\varepsilon,0}. \tag{8.23}$$

By Lemma 6.6 and (8.9), there exist $C > 0$, such that

$$|\langle T Q_\varepsilon s, s\rangle_{\varepsilon,0}| \leq C |s|_{\varepsilon,T,1} |s|_{\varepsilon,0}. \tag{8.24}$$

Then Lemma 8.5 follows from (6.74), (8.23) and (8.24). □

Set $\mathcal{D}_\varepsilon = \{P^K_{\varepsilon U} \partial_p P^K_{\varepsilon U} + P^{K,\perp}_{\varepsilon U} \partial_p P^{K,\perp}_{\varepsilon U},\ P^{K,\perp}_{\varepsilon U} \nabla^{\mathcal{S}_X}_{e_i} P^{K,\perp}_{\varepsilon U}\}$.
Let Ξ_ε be the operator from $\mathbb{F}_\varepsilon$ to itself,

$$\Xi_\varepsilon = E_\varepsilon - F_\varepsilon H_\varepsilon^{-1} G_\varepsilon. \tag{8.25}$$

Following the same argument in (4.72)-(4.137), we can get an analogue of Theorem 4.15.

Theorem 8.6. *There exist $C > 0$, $\delta > 0$, and $T_0 \geq 1$ such that for $0 < \varepsilon \leq 1$, $T \geq T_0$,*

$$\left|\psi_S \widetilde{\mathrm{Tr}}\left[g\widetilde{\boldsymbol{F}}_{\varepsilon^2}(L^3_{\varepsilon,T})\right] - \psi_S \widetilde{\mathrm{Tr}}\left[g\widetilde{\boldsymbol{F}}_{\varepsilon^2}(\Xi_\varepsilon)\right]\right| \leq \frac{C}{T^\delta}. \tag{8.26}$$

Since there is no dT part in the second term above, as in (5.5)-(5.8), we get Theorem 8.2.

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