

On the Witten Rigidity Theorem for String^c Manifolds

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Abstract

We establish the family rigidity and vanishing theorems on the equivariant K -theory level for the Witten type operators on String^c manifolds introduced by Chen-Han-Zhang [3].

1 Introduction

In [18], Witten derived a series of elliptic operators on the free loop space \mathcal{LM} of a spin manifold M . In particular, the index of the formal signature operator on loop space turns out to be exactly the elliptic genus constructed by Landweber-Stong [7] and Ochanine [16] in a topological way. Motivated by physics, Witten proposed that these elliptic operators should be rigid with respect to the circle action.

This claim of Witten was first proved by Taubes [17] and Bott-Taubes [2]. See also [5] and [6] for other interesting cases. By the modular invariance property, Liu ([10, 11]) presented a simple and unified proof of the above result as well as various further generalizations. In particular, several new vanishing theorems were established in [10, 11].

In a recent paper [3], Chen, Han and Zhang introduced a topological condition which they called String^c condition for even dimensional Spin^c manifolds. Under this String^c condition, they constructed a Witten type genus which is the index of a Witten type operator, a linear combination of twisted Spin^c Dirac operators. Furthermore, by applying Liu's method in [10, 11], Chen-Han-Zhang established the rigidity and vanishing theorems for this Witten type operator under relevant anomaly cancelation condition (cf. [3, Theorem 3.2]).

In many situations in geometry, it is rather natural and necessary to generalize the rigidity and vanishing theorems to the family case. On the equivariant Chern character level, Liu and Ma ([12, 13]) established several family rigidity and vanishing theorems. In [14, 15], inspired by [17], Liu, Ma and Zhang established the corresponding family rigidity and vanishing theorems on the equivariant K -theory level. As explained in [14, 15], taking the Chern character might kill some torsion elements involved in the index bundle. Therefore, the rigidity and vanishing properties on the K -theory level are more subtle than those on the Chern character level.

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The purpose of this paper is to establish the family rigidity and vanishing theorems on the equivariant K -theory level for the Witten type operators introduced by Chen-Han-Zhang [3]. In fact, our main results in Theorem 2.2 may be regarded as an analogue of [14, Theorem 2.1] and [15, Theorems 2.1, 2.2]. In particular, if the base manifold is a point, from our family rigidity theorem, one deduces Chen-Han-Zhang's theorem [3, Theorem 3.2(i)]. Both the statement and the proof of Theorem 2.2 are inspired by those of [14, Theorem 2.1] and [15, Theorems 2.1, 2.2], which essentially depend on the techniques developed by Taubes [17] and Bismut-Lebeau [1].

This paper is organized as follows. In Section 2, we state and prove our main results, Theorem 2.2, the rigidity and vanishing theorems for the family Witten type operators introduced by Chen-Han-Zhang [3]. Section 3 is devoted to the proofs of two intermediate results, Theorems 2.8 and 2.9, which are used in the proof of Theorem 2.2.

2 Rigidity and vanishing theorems in K -theory

In this section, we establish the main results of this paper, the rigidity and vanishing theorems on the equivariant K -theory level for a family of Spin^c manifolds. Such theorems hold under some anomaly cancelation assumption which is inspired by Chen-Han-Zhang's String^c condition [3]. For the particular case when the base manifold is a point, our results imply Chen-Han-Zhang's theorem [3, Theorem 3.2(i)].

This section is organized as follows. In Section 2.1, we reformulate a K -theory version of the equivariant family index theorem which is proved in [15, Theorem 1.2] and [14, Theorem 1.1]. In Section 2.2, we state our main results, the rigidity and vanishing theorems on the equivariant K -theory level for a family of Spin^c manifolds. In Section 2.3, we state two intermediate results on the relations between the family indices on the fixed point set, which will be used to prove our main results stated in Section 2.1. In Section 2.4, we prove the family rigidity and vanishing theorems.

2.1 A K -theory version of the equivariant family index theorem

Let M, B be two compact manifolds, and $\pi : M \rightarrow B$ a smooth fibration with compact fiber X such that $\dim X = 2l$. Let TX denote the relative tangent bundle carrying a Riemannian metric g^{TX} . We assume that TX is oriented. Let (W, h^W) be a complex Hermitian vector bundle over M .

Let (V, g^V) (resp. $(V', g^{V'})$) be a $2p$ (resp. $2p'$) dimensional oriented real Euclidean vector bundle over M . Let (L, h^L) be a complex Hermitian line bundle over M with the property that the vector bundle $U = TX \oplus V \oplus V'$ satisfies $\omega_2(U) = c_1(L) \pmod{2}$, where ω_2 denotes the second Stiefel-Whitney class, and c_1 denotes the first Chern class. Then the vector bundle U has a Spin^c -structure. Let $S(U, L)$ be the fundamental complex spinor bundle for (U, L) (cf. [8, Appendix D]).

Assume that there is a fiberwise S^1 action on M which lifts to V, V', L and W , and assume the metrics $g^{TX}, g^V, g^{V'}, h^L$ and h^W are S^1 -invariant. Also

assume that the S^1 actions on TX , V , V' , L lift to $S(U, L)$.

Let ∇^{TX} be the Levi-Civita connection on (TX, g^{TX}) along the fiber X . Let ∇^V (resp. $\nabla^{V'}$) be an S^1 -invariant Euclidean connection on (V, g^V) (resp. $(V', g^{V'})$). Let ∇^L (resp. ∇^W) be an S^1 -invariant Hermitian connection on (L, h^L) (resp. (W, h^W)).

The Clifford algebra bundle $C(TX)$ is the bundle of Clifford algebras over X whose fibre at $x \in X$ is the Clifford algebra $C(T_x X)$ (cf. [8]). Let $C(V)$ (resp. $C(V')$) be the Clifford algebra bundle of (V, g^V) (resp. $(V', g^{V'})$).

Let $\{e_i\}_{i=1}^{2l}$ (resp. $\{f_j\}_{j=1}^{2p}$) be an oriented orthonormal basis of (TX, g^{TX}) (resp. (V, g^V)). We denote by $c(\cdot)$ the Clifford action of $C(TX)$, $C(V)$ and $C(V')$ on $S(U, L)$. Let τ be the involution of $S(U, L)$ given by

$$\tau = (\sqrt{-1})^{l+p} c(e_1) \cdots c(e_{2l}) c(f_1) \cdots c(f_{2p}) . \quad (2.1)$$

In the rest of the paper, we will say τ the involution determined by $TX \oplus V$. We decompose $S(U, L) = S_+(U, L) \oplus S_-(U, L)$ corresponding to τ such that $\tau|_{S_{\pm}(U, L)} = \pm 1$. Let $\nabla^{S(U, L)}$ be the Hermitian connection on $S(U, L)$ induced by ∇^{TX} , ∇^V , $\nabla^{V'}$ and ∇^L (cf. [8, Appendix D]). Then $\nabla^{S(U, L)}$ preserves the \mathbb{Z}_2 -grading of $S(U, L)$ induced by (2.1). Let $\nabla^{S(U, L) \otimes W}$ be the Hermitian connection on $S(U, L) \otimes W$ obtained from the tensor product of $\nabla^{S(U, L)}$ and ∇^W . Let $D^X \otimes W$ be the family twisted Spin^c -Dirac operator on the fiber X defined by

$$D^X \otimes W = \sum_{i=1}^{2l} c(e_i) \nabla_{e_i}^{S(U, L) \otimes W} . \quad (2.2)$$

By [12, Proposition 1.1], the index bundle $\text{Ind}_{\tau}(D^X \otimes W)$ over B is well-defined in the equivariant K -group $K_{S^1}(B)$. Using the same notations as in [15, (1.4)-(1.7)], we write, as an identification of virtual S^1 -bundles,

$$\text{Ind}_{\tau}(D^X \otimes W) = \bigoplus_{n \in \mathbb{Z}} \text{Ind}_{\tau}(D^X \otimes W, n) \otimes [n], \quad (2.3)$$

where by $[n]$ ($n \in \mathbb{Z}$) we mean the one dimensional complex vector space on which S^1 acts as multiplication by g^n for a generator $g \in S^1$.

Let $F = \{F_{\alpha}\}$ be the fixed point set of the circle action on M . Then $\pi : F_{\alpha} \rightarrow B$ (resp. $\pi : F \rightarrow B$) is a smooth fibration with fiber Y_{α} (resp. Y). Let $\tilde{\pi} : N \rightarrow F$ denote the normal bundle to F in M . Then $N = TX/TY$. We identify N as the orthogonal complement of TY in $TX|_F$. Then $TX|_F$ admits a S^1 -equivariant decomposition (cf. [15, (1.8)])

$$TX|_F = \bigoplus_{v \neq 0} N_v \oplus TY, \quad (2.4)$$

where N_v is a complex vector bundle such that $g \in S^1$ acts on it by g^v with $v \in \mathbb{Z} \setminus \{0\}$. Clearly, $N = \bigoplus_{v \neq 0} N_v$. We will regard N as a complex vector bundle and write $N_{\mathbb{R}}$ for the underlying real vector bundle of N . For $v \neq 0$, let $N_{v, \mathbb{R}}$ denote the underlying real vector bundle of N_v .

Similarly, let (cf. [15, (1.9) and (1.46)])

$$V|_F = \bigoplus_{v \neq 0} V_v \oplus V_0^{\mathbb{R}}, \quad V'|_F = \bigoplus_{v \neq 0} V'_v \oplus V_0'^{\mathbb{R}}, \quad W|_F = \bigoplus_v W_v, \quad (2.5)$$

be the S^1 -equivariant decompositions of the restrictions of V , V' and W over F respectively, where V_v , V'_v and W_v ($v \in \mathbb{Z}$) are complex vector bundles over F on which $g \in S^1$ acts by g^v , and $V_0^{\mathbb{R}}$ (resp. $V_0'^{\mathbb{R}}$) is the real subbundle of V (resp. V') such that S^1 acts as identity. For $v \neq 0$, let $V_{v,\mathbb{R}}$ (resp. $V'_{v,\mathbb{R}}$) denote the underlying real vector bundle of V_v (resp. V'_v). Denote by $2p_0 = \dim V_0^{\mathbb{R}}$ and $2l_0 = \dim Y$.

Let us write (compare with [15, (1.47)])

$$L_F = L \otimes \left(\bigotimes_{v \neq 0} \det N_v \otimes \bigotimes_{v \neq 0} \det V_v \otimes \bigotimes_{v \neq 0} \det V'_v \right)^{-1}. \quad (2.6)$$

Then $TY \oplus V_0^{\mathbb{R}} \oplus V_0'^{\mathbb{R}}$ has a Spin^c -structure. Let $S(TY \oplus V_0^{\mathbb{R}} \oplus V_0'^{\mathbb{R}}, L_F)$ be the fundamental spinor bundle for $(TY \oplus V_0^{\mathbb{R}} \oplus V_0'^{\mathbb{R}}, L_F)$. Let R be a Hermitian complex vector bundle equipped with a Hermitian connection over F . We will denote by $D^Y \otimes R$ the family (twisted) Spin^c Dirac operator on $S(TY \oplus V_0^{\mathbb{R}} \oplus V_0'^{\mathbb{R}}, L_F) \otimes R$ defined as in (2.2) and by $D^{Y_\alpha} \otimes R$ its restriction to Y_α .

Recall that $N_{v,\mathbb{R}}$ and $V_{v,\mathbb{R}}$ are canonically oriented by their complex structures. The decompositions (2.4), (2.5) induce the orientations of TY and $V_0^{\mathbb{R}}$ respectively. Let $\{e_i\}_{i=1}^{2l_0}$, $\{f_j\}_{j=1}^{2p_0}$ be the corresponding oriented orthonormal basis of (TY, g^{TY}) and $(V_0^{\mathbb{R}}, g^{V_0^{\mathbb{R}}})$. The involution of $S(TY \oplus V_0^{\mathbb{R}} \oplus V_0'^{\mathbb{R}}, L_F)$ is canonically associated to that of $S(U, L)$, which we still denote by τ , is given by

$$\tau = (\sqrt{-1})^{l_0+p_0} c(e_1) \cdots c(e_{2l_0}) c(f_1) \cdots c(f_{2p_0}). \quad (2.7)$$

Let $S(TY \oplus V_0^{\mathbb{R}} \oplus V_0'^{\mathbb{R}}, L_F) = S_+(TY \oplus V_0^{\mathbb{R}} \oplus V_0'^{\mathbb{R}}, L_F) \oplus S_-(TY \oplus V_0^{\mathbb{R}} \oplus V_0'^{\mathbb{R}}, L_F)$ be the \mathbb{Z}_2 -grading of $S(TY \oplus V_0^{\mathbb{R}} \oplus V_0'^{\mathbb{R}}, L_F)$ induced by τ .

Let $C(N_{\mathbb{R}})$ (resp. $C(V_{v,\mathbb{R}})$) be the Clifford algebra bundle of $(N_{\mathbb{R}}, g^{TX}|_{N_{\mathbb{R}}})$ (resp. $(V_{v,\mathbb{R}}, g^V|_{V_{v,\mathbb{R}}})$). By [15, (1.10)], $\Lambda(\overline{N}^*)$ is a $C(N_{\mathbb{R}})$ -Clifford module with the involution $\tau^N|_{\Lambda^{\text{even/odd}}(\overline{N}^*)} = \pm 1$. Similarly as in [15, (1.10)], we can define the Clifford action of $C(V_{v,\mathbb{R}})$ on $\Lambda(\overline{V}_v^*)$. Then $\Lambda(\overline{V}_v^*)$ is a $C(V_{v,\mathbb{R}})$ -Clifford module with the involution $\tau_v^V|_{\Lambda^{\text{even/odd}}(\overline{V}_v^*)} = \pm 1$.

By restricting to F , one has the isomorphism of \mathbb{Z}_2 -graded $C(TX)$ -Clifford modules over F as follows (compare with [15, (1.49)]),

$$\begin{aligned} (S(U, L), \tau)|_F &\simeq \left(S(TY \oplus V_0^{\mathbb{R}} \oplus V_0'^{\mathbb{R}}, L_F), \tau \right) \widehat{\otimes} (\Lambda \overline{N}^*, \tau^N) \\ &\quad \widehat{\otimes} \bigotimes_{v \neq 0} (\Lambda \overline{V}_v^*, \tau_v^V) \widehat{\otimes} \bigotimes_{v \neq 0} (\Lambda \overline{V}_v^*, \text{id}), \end{aligned} \quad (2.8)$$

where id denotes the trivial involution and $\widehat{\otimes}$ denotes the \mathbb{Z}_2 -graded tensor product (cf. [8, pp. 11]). Furthermore, the isomorphism (2.8) gives the identifications of the canonical connections on the bundles (compare with [15, (1.13)]).

Let S^1 act on $L|_F$ by sending $g \in S^1$ to g^{l_c} ($l_c \in \mathbb{Z}$) on F . Then l_c is locally constant on F . Following [15, (1.50)], we define the following elements in $K(F)[[q^{\frac{1}{2}}]]$,

$$\begin{aligned} R(q) &= q^{\frac{1}{2}(\sum_v |v| \dim N_v - \sum_v v \dim V_v - \sum_v v \dim V'_v + l_c)} \bigotimes_{v>0} (\text{Sym}_{q^v}(N_v) \otimes \det N_v) \\ &\quad \otimes \bigotimes_{v<0} \text{Sym}_{q^{-v}}(\overline{N}_v) \otimes \bigotimes_{v \neq 0} \Lambda_{-q^v}(V_v) \otimes \bigotimes_{v \neq 0} \Lambda_{q^v}(V'_v) \otimes \left(\sum_v q^v W_v \right) \\ &= \sum_n R_n q^n, \end{aligned} \quad (2.9)$$

$$\begin{aligned} R'(q) &= q^{\frac{1}{2}(-\sum_v |v| \dim N_v - \sum_v v \dim V_v - \sum_v v \dim V'_v + l_c)} \bigotimes_{v>0} \text{Sym}_{q^{-v}}(\overline{N}_v) \\ &\quad \otimes \bigotimes_{v<0} (\text{Sym}_{q^v}(N_v) \otimes \det N_v) \otimes \bigotimes_{v \neq 0} \Lambda_{-q^v}(V_v) \\ &\quad \otimes \bigotimes_{v \neq 0} \Lambda_{q^v}(V'_v) \otimes \left(\sum_v q^v W_v \right) = \sum_n R'_n q^n. \end{aligned} \quad (2.10)$$

As explained in [15, pp. 139], since $TX \oplus V \oplus V' \oplus L$ is spin, one gets

$$\sum_v v \dim N_v + \sum_v v \dim V_v + \sum_v v \dim V'_v + l_c \equiv 0 \pmod{2}. \quad (2.11)$$

Therefore, $R(q), R'(q) \in K(F)[[q]]$.

The following theorem was essentially proved in [15, Theorem 1.2].

Theorem 2.1 *For $n \in \mathbb{Z}$, the following identity holds in $K(B)$,*

$$\begin{aligned} \text{Ind}_\tau(D^X \otimes W, n) &= \sum_\alpha (-1)^{\sum_{0<v} \dim N_v} \text{Ind}_\tau(D^{Y_\alpha} \otimes R_n) \\ &= \sum_\alpha (-1)^{\sum_{v<0} \dim N_v} \text{Ind}_\tau(D^{Y_\alpha} \otimes R'_n). \end{aligned} \quad (2.12)$$

2.2 Family rigidity and vanishing theorems

Let $\pi : M \rightarrow B$ be a fibration of compact manifolds with compact fiber X and $\dim X = 2l$. We assume that S^1 acts fiberwise on M and TX has an S^1 -invariant Spin^c structure. Let K_X be the S^1 -equivariant complex line bundle over M which is induced by the S^1 -invariant Spin^c structure of TX . Let $S(TX, K_X)$ be the complex spinor bundle of (TX, K_X) (cf. [8, Appendix D]).

Let V be an even dimensional real vector bundle over M . We assume that V has an S^1 -invariant spin structure. Let $S(V) = S^+(V) \oplus S^-(V)$ be the spinor bundle of V . Let W be an S^1 -equivariant complex vector bundle over M . Let $K_W = \det(W)$ be the determinant line bundle of W .

We define the following elements in $K(M)[[q^{1/2}]]$,

$$\begin{aligned}
R_1(V) &= \left(S^+(V) + S^-(V) \right) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}(V) , \\
R_2(V) &= \left(S^+(V) - S^-(V) \right) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{-q^n}(V) , \\
R_3(V) &= \bigotimes_{n=1}^{\infty} \Lambda_{-q^{n-1/2}}(V) , \quad R_4(V) = \bigotimes_{n=1}^{\infty} \Lambda_{q^{n-1/2}}(V) , \\
Q_1(W) &= \bigotimes_{n=0}^{\infty} \Lambda_{q^n}(\overline{W}) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}(W) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{-q^{n-1/2}}(\overline{W}) \\
&\quad \otimes \bigotimes_{n=1}^{\infty} \Lambda_{-q^{n-1/2}}(W) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^{n-1/2}}(\overline{W}) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^{n-1/2}}(W) .
\end{aligned} \tag{2.13}$$

For $N \in \mathbb{Z}$, $N \geq 1$, let $y = e^{2\pi i/N} \in \mathbb{C}$. Let G_y be the multiplicative group generated by y . Following [18], as in [14, Section 2.1], we consider the fiberwise action G_y on W and \overline{W} by sending $y \in G_y$ to y on W and y^{-1} on \overline{W} . Then G_y acts naturally on $Q_1(W)$.

Let $H_{S^1}^*(M, \mathbb{Z}) = H^*(M \times_{S^1} ES^1, \mathbb{Z})$ denote the S^1 -equivariant cohomology group of M , where ES^1 is the universal S^1 -principal bundle over the classifying space BS^1 of S^1 . So $H_{S^1}^*(M, \mathbb{Z})$ is a module over $H^*(BS^1, \mathbb{Z})$ induced by the projection $\pi : M \times_{S^1} ES^1 \rightarrow BS^1$. Let $p_1(\cdot)_{S^1}$ and $\omega_2(\cdot)_{S^1}$ denote the first S^1 -equivariant pontrjagin class and the second S^1 -equivariant Stiefel-Whitney class, respectively. As $V \times_{S^1} ES^1$ is spin over $M \times_{S^1} ES^1$, one knows that $\frac{1}{2}p_1(V)_{S^1}$ is well defined in $H_{S^1}^*(M, \mathbb{Z})$ (cf. [17, pp. 456-457]). Recall that

$$H^*(BS^1, \mathbb{Z}) = \mathbb{Z}[[u]] \tag{2.14}$$

with u a generator of degree 2.

In the following, we denote by $D^X \otimes R$ the family twisted Spin^c Dirac operator acting fiberwise on $S(TX, K_X) \otimes R$. Recall that if $\text{Ind}(D^X \otimes R, n)$ vanishes for $n \neq 0$, we say that $D^X \otimes R$ is rigid on the equivariant K -theory level for the S^1 action.

Now we can state the main results of this paper as follows, which can be thought of as an analogue of [14, Theorem 2.1].

Theorem 2.2 *Assume $w_2(W)_{S^1} = w_2(TX)_{S^1}$, $\frac{1}{2}p_1(V + 3W - TX)_{S^1} = e \cdot \pi^* u^2$ ($e \in \mathbb{Z}$) in $H_{S^1}^*(M, \mathbb{Z})$, and $c_1(W) = 0 \pmod{N}$. For $i = 1, 2, 3, 4$, consider the family of $G_y \times S^1$ -equivariant twisted Spin^c Dirac operators*

$$D^X \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes \bigotimes_{n=1}^{\infty} \text{Sym}_{q^n}(TX) \otimes R_i(V) \otimes Q_1(W) . \tag{2.15}$$

- (i) *If $e = 0$, then these operators are rigid on the equivariant K -theory level for the S^1 action.*

- (ii) If $e < 0$, then the index bundles of these operators are zero in $K_{G_y \times S^1}(B)$. In particular, these index bundles are zero in $K_{G_y}(B)$.

Remark 2.3 As explained in [14, Remark 2.1], $w_2(W)_{S^1} = w_2(TX)_{S^1}$ means that $\frac{1}{2}p_1(3W - TX)_{S^1}$ is well defined and that $c_1(K_W \otimes K_X^{-1})_{S^1} = 0 \pmod{2}$. By [4, Corollary 1.2], the S^1 action on M can be lifted to $(K_W \otimes K_X^{-1})^{1/2}$ and is compatible with the S^1 action on $K_W \otimes K_X^{-1}$.

Take $N = 1$, i.e., we forget the G_y action on W and remove the corresponding assumption $c_1(W) = 0 \pmod{N}$. Furthermore, take $W = K_X$ and $V = 0$. Then an interesting consequence of Theorem 2.2 is the following family rigidity and vanishing property, which may be thought of as an extension of [15, Theorem 2.3] to the Spin^c case. When the base manifold is a point, it turns out exactly to be Chen-Han-Zhang's theorem [3, Theorem 3.2(i)].

Corollary 2.4 Assume $\frac{1}{2}p_1(3K_X - TX)_{S^1} = e \cdot \bar{\pi}^* u^2$ ($e \in \mathbb{Z}$) in $H_{S^1}^*(M, \mathbb{Z})$. Consider the family of S^1 -equivariant twisted Spin^c Dirac operators

$$D^X \otimes \bigotimes_{n=1}^{\infty} \text{Sym}_{q^n}(TX) \otimes Q_1(K_X). \quad (2.16)$$

- (i) If $e = 0$, then these operators are rigid on the equivariant K -theory level for the S^1 action.
- (ii) If $e < 0$, then the index bundles of these operators are zero in $K_{S^1}(B)$. In particular, these index bundles are zero in $K(B)$.

Remark 2.5 The operators in (2.16) are the Witten type operators introduced by Chen-Han-Zhang [3]. By taking $N = 1$, $W = K_X$, $V = 0$, and letting the base manifold B be a point in [14, Theorem 2.1], we get Chen-Han-Zhang's theorem [3, Theorem 3.2(ii)]. It is rather natural to establish an analogue of [14, Theorem 2.1], which corresponds to Chen-Han-Zhang's theorem [3, Theorem 3.2(i)]. That is one of the motivation of Theorem 2.2.

Actually, as in [14, 15], our proof of Theorem 2.2 works under the following slightly weaker hypothesis. Let us first explain some notations.

For each $n > 1$, consider $\mathbb{Z}_n \subset S^1$, the cyclic subgroup of order n . We have the \mathbb{Z}_n -equivariant cohomology of M defined by $H_{\mathbb{Z}_n}^*(M, \mathbb{Z}) = H^*(M \times_{\mathbb{Z}_n} ES^1, \mathbb{Z})$, and there is a natural “forgetful” map $\alpha(S^1, \mathbb{Z}_n) : M \times_{\mathbb{Z}_n} ES^1 \rightarrow M \times_{S^1} ES^1$ which induces a pullback $\alpha(S^1, \mathbb{Z}_n)^* : H_{S^1}^*(M, \mathbb{Z}) \rightarrow H_{\mathbb{Z}_n}^*(M, \mathbb{Z})$. We denote by $\alpha(S^1, 1)$ the arrow which forgets the S^1 action. Thus $\alpha(S^1, 1)^* : H_{S^1}^*(M, \mathbb{Z}) \rightarrow H^*(M, \mathbb{Z})$ is induced by the inclusion of M into $M \times_{S^1} ES^1$ as a fiber over BS^1 .

Finally, note that if \mathbb{Z}_n acts trivially on a space Y , then there is a new arrow $t^* : H^*(Y, \mathbb{Z}) \rightarrow H_{\mathbb{Z}_n}^*(Y, \mathbb{Z})$ induced by the projection $t : Y \times_{\mathbb{Z}_n} ES^1 = Y \times B\mathbb{Z}_n \rightarrow Y$.

Let $\mathbb{Z}_{\infty} = S^1$. For each $1 < n \leq +\infty$, let $i : M(n) \rightarrow M$ be the inclusion of the fixed point set of $\mathbb{Z}_n \subset S^1$ in M , and so i induces $i_{S^1} : M(n) \times_{S^1} ES^1 \rightarrow M \times_{S^1} ES^1$.

In the rest of this paper, we suppose that there exists some integer $e \in \mathbb{Z}$ such that for $1 < n \leq +\infty$,

$$\begin{aligned} \alpha(S^1, \mathbb{Z}_n)^* \circ i_{S^1}^* \left(\frac{1}{2} p_1(V + 3W - TX)_{S^1} - e \cdot \bar{\pi}^* u^2 \right) \\ = t^* \circ \alpha(S^1, 1)^* \circ i_{S^1}^* \left(\frac{1}{2} p_1(V + 3W - TX)_{S^1} \right). \end{aligned} \quad (2.17)$$

As indicated in [14, Remark 2.4], the relation (2.17) clearly follows from the hypothesis of Theorem 2.2 by pulling back and forgetting. Thus it is a weaker hypothesis.

We can now state a slightly more general version of Theorem 2.2.

Theorem 2.6 *Under the hypothesis (2.17), we have*

- (i) *If $e = 0$, then the index bundles of the twisted Spin^c Dirac operators in Theorem 2.2 are rigid on the equivariant K -theory level for the S^1 action.*
- (ii) *If $e < 0$, then the index bundles of the twisted Spin^c Dirac operators in Theorem 2.2 are zero as elements in $K_{G_y \times S^1}(B)$. In particular, these index bundles are zero in $K_{G_y}(B)$.*

The rest of this section is devoted to a proof of Theorem 2.6.

2.3 Two recursive formulas

Let $F = \{F_\alpha\}$ be the fixed point set of the circle action. Then $\pi : F \rightarrow B$ is a fibration with compact fibre denoted by $Y = \{Y_\alpha\}$.

As in [14, (2.5)], we may and we will assume that

$$\begin{aligned} TX|_F &= TY \oplus \bigoplus_{v>0} N_v, \\ TX|_F \otimes_{\mathbb{R}} \mathbb{C} &= TY \otimes_{\mathbb{R}} \mathbb{C} \oplus \bigoplus_{v>0} (N_v \oplus \bar{N}_v), \end{aligned} \quad (2.18)$$

where N_v are complex vector bundles on which S^1 acts by sending $g \in S^1$ to g^v . We also assume that (cf. [14, (2.6)])

$$V|_F = V_0^{\mathbb{R}} \oplus \bigoplus_{v>0} V_v, \quad W|_F = \bigoplus_v W_v, \quad (2.19)$$

where V_v, W_v are complex vector bundles on which S^1 acts by sending g to g^v , and $V_0^{\mathbb{R}}$ is a real vector bundle on which S^1 acts as identity.

By (2.18), as in [14, (2.7)], there is a natural isomorphism between the \mathbb{Z}_2 -graded $C(TX)$ -Clifford modules over F ,

$$S(TX, K_X)|_F \simeq S\left(TY, K_X \otimes_{v>0} (\det N_v)^{-1}\right) \widehat{\otimes} \widehat{\bigotimes}_{v>0} \Lambda N_v. \quad (2.20)$$

For a complex vector bundle R over F , let $D^Y \otimes R, D^{Y_\alpha} \otimes R$ be the twisted Spin^c Dirac operator on $S(TY, K_X \otimes_{v>0} (\det N_v)^{-1}) \otimes R$ over F, F_α respectively.

We introduce the following locally constant functions on F (cf. [14, (2.8)]),

$$\begin{aligned} e(N) &= \sum_{v>0} v^2 \dim N_v, & d'(N) &= \sum_{v>0} v \dim N_v, \\ e(V) &= \sum_{v>0} v^2 \dim V_v, & d'(V) &= \sum_{v>0} v \dim V_v, \\ e(W) &= \sum_v v^2 \dim W_v, & d'(W) &= \sum_v v \dim W_v. \end{aligned} \quad (2.21)$$

As in [14, (2.9)], we write

$$\begin{aligned} L(N) &= \otimes_{v>0} (\det N_v)^v, & L(V) &= \otimes_{v>0} (\det V_v)^v, \\ L(W) &= \otimes_{v \neq 0} (\det W_v)^v, & L &= L(N)^{-1} \otimes L(V) \otimes L(W)^3. \end{aligned} \quad (2.22)$$

By using (2.17) and computing as in [14, (2.10)-(2.11)], one knows

$$c_1(L) = 0, \quad e(V) + 3 \cdot e(W) - e(N) = 2e, \quad (2.23)$$

which means L is a trivial complex line bundle over each component F_α of F , and S^1 acts on L by sending g to g^{2e} , and G_y acts on L by sending y to $y^{3d'(W)}$. From [14, Lemma 2.1], we know that $d'(W) \bmod (N)$ is constant on each connected component of M . Thus we can extend L to a trivial complex line bundle over M , and we extend the S^1 action on it by sending $g \in S^1$ on the canonical section 1 of L to $g^{2e} \cdot 1$, and G_y acts on L by sending y to $y^{3d'(W)}$.

In what follows, if $R(q) = \sum_{m \in \frac{1}{2}\mathbb{Z}} q^m R_m \in K_{S^1}(M)[[q^{1/2}]]$, we will also denote $\text{Ind}(D^X \otimes R_m, h)$ by $\text{Ind}(D^X \otimes R(q), m, h)$. For $i = 1, 2, 3, 4$, set

$$R_{i1} = (K_W \otimes K_X^{-1})^{1/2} \otimes R_i(V) \otimes Q_1(W). \quad (2.24)$$

As in [14, Proposition 2.1], by using Theorem 2.1 we first express the global equivariant family index via the family indices on the fixed point set.

Proposition 2.7 *For $m \in \frac{1}{2}\mathbb{Z}$, $h \in \mathbb{Z}$, $1 \leq i \leq 4$, we have the following identity in $K_{G_y}(B)$,*

$$\begin{aligned} & \text{Ind} \left(D^X \otimes \otimes_{n=1}^\infty \text{Sym}_{q^n}(TX) \otimes R_{i1}, m, h \right) \\ &= \sum_{\alpha} (-1)^{\sum_{v>0} \dim N_v} \text{Ind} \left(D^{Y_\alpha} \otimes \otimes_{n=1}^\infty \text{Sym}_{q^n}(TX|_F) \otimes R_{i1} \right. \\ & \quad \left. \otimes \text{Sym}(\oplus_{v>0} N_v) \otimes_{v>0} \det N_v, m, h \right). \end{aligned} \quad (2.25)$$

To simplify the notations, we use the same convention as in [14, pp. 945]. For $n_0 \in \mathbb{N}^*$, we define a number operator P on $K_{S^1}(M)[[q^{\frac{1}{n_0}}]]$ in the following way: if $R(q) = \oplus_{n \in \frac{1}{n_0}\mathbb{Z}} R_n q^n \in K_{S^1}(M)[[q^{\frac{1}{n_0}}]]$, then P acts on $R(q)$ by multiplication by n on R_n . From now on, we simply denote $\text{Sym}_{q^n}(TX)$, $\Lambda_{q^n}(V)$ and $\Lambda_{q^n}(W)$ by $\text{Sym}(TX_n)$, $\Lambda(V_n)$ and $\Lambda(W_n)$, respectively. In this way, P acts on TX_n , V_n and W_n by multiplication by n , and the action of P on $\text{Sym}(TX_n)$,

$\Lambda(V_n)$ and $\Lambda(W_n)$ is naturally induced by the corresponding action of P on TX_n , V_n and W_n . So the eigenspace of $P = n$ is just given by the coefficient of q^n of the corresponding element $R(q)$. For $R(q) = \oplus_{n \in \frac{1}{n_0}\mathbb{Z}} R_n q^n \in K_{S^1}(M)[[q^{\frac{1}{n_0}}]]$, we will also denote $\text{Ind}(D^X \otimes R_m, h)$ by $\text{Ind}(D^X \otimes R(q), m, h)$.

For $p \in \mathbb{N}$, we introduce the following elements in $K_{S^1}(F)[[q]]$ (cf. [14, (3.6)]),

$$\begin{aligned}\mathcal{F}_p(X) &= \bigotimes_{n=1}^{\infty} \text{Sym}(TY_n) \otimes \bigotimes_{v>0} \left(\bigotimes_{n=1}^{\infty} \text{Sym}(N_{v,n}) \bigotimes_{n>pv} \text{Sym}(\overline{N}_{v,n}) \right), \\ \mathcal{F}'_p(X) &= \bigotimes_{v>0} \bigotimes_{0 \leq n \leq pv} \left(\text{Sym}(N_{v,-n}) \otimes \det N_v \right), \\ \mathcal{F}^{-p}(X) &= \mathcal{F}_p(X) \otimes \mathcal{F}'_p(X).\end{aligned}\tag{2.26}$$

Then from (2.18), over F , we have

$$\mathcal{F}^0(X) = \bigotimes_{n=1}^{\infty} \text{Sym}_{q^n}(TX|_F) \otimes \text{Sym}(\oplus_{v>0} N_v) \otimes_{v>0} \det N_v.\tag{2.27}$$

We now state two intermediate results on the relations between the family indices on the fixed point set. These two recursive formulas will be used in the next subsection to prove Theorem 2.6.

Theorem 2.8 (Compare with [14, Theorem 2.3]) *For $1 \leq i \leq 4$, $h, p \in \mathbb{Z}$, $p > 0$, $m \in \frac{1}{2}\mathbb{Z}$, the following identity holds in $K_{G_y}(B)$,*

$$\begin{aligned}& \sum_{\alpha} (-1)^{\sum_{v>0} \dim N_v} \text{Ind}(D^{Y_{\alpha}} \otimes \mathcal{F}^0(X) \otimes R_{i1}, m, h) \\ &= \sum_{\alpha} (-1)^{pd'(N) + \sum_{v>0} \dim N_v} \text{Ind}(D^{Y_{\alpha}} \otimes \mathcal{F}^{-p}(X) \otimes R_{i1}, \\ & \quad m + \frac{1}{2}p^2e(N) + \frac{p}{2}d'(N), h).\end{aligned}\tag{2.28}$$

The proof of Theorem 2.8 will be given in Sections 3.2-3.4.

Theorem 2.9 (Compare with [14, Theorem 2.4]) *For each α , $1 \leq i \leq 4$, $h, p \in \mathbb{Z}$, $p > 0$, $m \in \frac{1}{2}\mathbb{Z}$, the following identity holds in $K_{G_y}(B)$,*

$$\begin{aligned}& \text{Ind}\left(D^{Y_{\alpha}} \otimes \mathcal{F}^{-p}(X) \otimes R_{i1}, m + \frac{1}{2}p^2e(N) + \frac{p}{2}d'(N), h\right) \\ &= (-1)^{pd'(W)} \text{Ind}\left(D^{Y_{\alpha}} \otimes \mathcal{F}^0(X) \otimes R_{i1} \otimes L^{-p}, m + ph + p^2e, h\right).\end{aligned}\tag{2.29}$$

The proof of Theorem 2.9 will be given in Section 3.1.

2.4 A proof of Theorem 2.6

As $\frac{1}{2}p_1(3W - TX)_{S^1} \in H_{S^1}^*(X, \mathbb{Z})$ is well defined, one has the same identity as in [14, (2.27)],

$$d'(N) + d'(W) = 0 \pmod{2}. \quad (2.30)$$

From Proposition 2.7, Theorems 2.8, 2.9 and (2.30), for $1 \leq i \leq 4$, $h, p \in \mathbb{Z}$, $p > 0$, $m \in \frac{1}{2}\mathbb{Z}$, we get the following identity (compare with [14, (2.28)]),

$$\begin{aligned} & \text{Ind}\left(D^X \otimes \bigotimes_{n=1}^{\infty} \text{Sym}_{q^n}(TX) \otimes R_{i1}, m, h\right) \\ &= \text{Ind}\left(D^X \otimes \bigotimes_{n=1}^{\infty} \text{Sym}_{q^n}(TX) \otimes R_{i1} \otimes L^{-p}, m', h\right), \end{aligned} \quad (2.31)$$

with

$$m' = m + ph + p^2e. \quad (2.32)$$

By (2.13) and (2.24), if $m < 0$ or $m' < 0$, then either side of (2.31) is identically zero, which completes the proof of Theorem 2.6. In fact,

- (i) Assume that $e = 0$. Let $h \in \mathbb{Z}$, $m_0 \in \frac{1}{2}\mathbb{Z}$, $h \neq 0$ be fixed. If $h > 0$, we take $m' = m_0$, then for p large enough, we get $m < 0$ in (2.31). If $h < 0$, we take $m = m_0$, then for p large enough, we get $m' < 0$ in (2.31).
- (ii) Assume that $e < 0$. For $h \in \mathbb{Z}$, $m_0 \in \frac{1}{2}\mathbb{Z}$, we take $m = m_0$, then for p large enough, we get $m' < 0$ in (2.31).

The proof of Theorem 2.6 is completed.

Remark 2.10 We point out here that there is a \mathbb{Z}/k version of Theorem 2.6, which is an analogue of [9, Theorem 4.4]. In fact, by using the mod k localization formula for \mathbb{Z}/k circle actions on \mathbb{Z}/k Spin^c manifolds established in [9, Theorem 2.7] (see also [19, Theorem 2.1] for the spin case), our proof of Theorem 2.6 can be applied to the case of \mathbb{Z}/k manifolds with little modification.

Remark 2.11 (Compare with [14, Remark 2.5]) If M is connected, by (2.31), for $1 \leq i \leq 4$, in $K_{G_y}(B)$, we get

$$\begin{aligned} & \text{Ind}\left(D^X \otimes \bigotimes_{n=1}^{\infty} \text{Sym}_{q^n}(TX) \otimes R_{i1}\right) \\ &= \text{Ind}\left(D^X \otimes \bigotimes_{n=1}^{\infty} \text{Sym}_{q^n}(TX) \otimes R_{i1}\right) \otimes [3d'(W)], \end{aligned} \quad (2.33)$$

where by $[3d'(W)]$ we mean the one dimensional complex vector space on which $y \in G_y$ acts by multiplication by $y^{3d'(W)}$. In particular, if B is a point, and $3d'(W) \neq 0 \pmod{N}$, we get the vanishing theorem for String^c manifolds analogue to the result of [5, § 10].

3 Proofs of Theorems 2.8 and 2.9

In this section, we prove those two intermediate results which are stated in Section 2.3 and used in Section 2.4 to prove our main results.

This section is organized as follows. In Section 3.1, following [14, Section 3.2], we prove Theorem 2.9. In Section 3.2, we introduce the same refined shift operators as in [14, Section 4.2]. In Section 3.3, we construct the twisted Spin^c Dirac operator on $M(n_j)$, the fixed point set of the naturally induced \mathbb{Z}_{n_j} -action on M . In Section 3.4, by applying the S^1 -equivariant index theorem in Section 2.1, we finally prove Theorem 2.8.

3.1 A proof of Theorem 2.9

Let H be the canonical basis of $\text{Lie}(S^1) = \mathbb{R}$, i.e., $\exp(tH) = \exp(2\sqrt{-1}\pi t)$, for $t \in \mathbb{R}$. On the fixed point F , let \mathbf{J}_H denote the operator which computes the weight of the S^1 action on $\Gamma(F, E|_F)$ for any S^1 -equivariant vector bundle E over M . Then \mathbf{J}_H can be explicitly given by (cf. [15, (3.2)])

$$\mathbf{J}_H = \frac{1}{2\pi\sqrt{-1}} \mathcal{L}_H|_{\Gamma(F, E|_F)}, \quad (3.1)$$

where \mathcal{L}_H denotes the infinitesimal action of H acting on $\Gamma(M, E)$.

Recall that the \mathbb{Z}_2 -grading on $S(TX, K_X) \otimes_{n=1}^{\infty} \text{Sym}(TX_n)$ (resp. $S(TY, K_X \otimes_{v>0} (\det N_v)^{-1}) \otimes \mathcal{F}^{-p}(X)$) is induced by the \mathbb{Z}_2 -grading on $S(TX, K_X)$ (resp. $S(TY, K_X \otimes_{v>0} (\det N_v)^{-1})$). Write

$$\begin{aligned} Q_W^1 &= \bigotimes_{n=0}^{\infty} \Lambda(\overline{W}_n) \otimes \bigotimes_{n=1}^{\infty} \Lambda(W_n), & Q_W^2 &= \bigotimes_{n \in \mathbb{N} + \frac{1}{2}} \Lambda(\overline{W}_n) \otimes \bigotimes_{n \in \mathbb{N} + \frac{1}{2}} \Lambda(W_n), \\ F_V^1 &= S(V) \otimes \bigotimes_{n=1}^{\infty} \Lambda(V_n), & F_V^2 &= \bigotimes_{n \in \mathbb{N} + \frac{1}{2}} \Lambda(V_n). \end{aligned} \quad (3.2)$$

There are two natural \mathbb{Z}_2 -gradings on F_V^1, F_V^2 (resp. Q_W^1, Q_W^2). The first grading is induced by the \mathbb{Z}_2 -grading of $S(V)$ and the forms of homogeneous degrees in $\bigotimes_{n=1}^{\infty} \Lambda(V_n), \bigotimes_{n \in \mathbb{N} + \frac{1}{2}} \Lambda(V_n)$ (resp. Q_W^2). We define $\tau_e|_{F_V^{i\pm}} = \pm 1$ ($i = 1, 2$) (resp. $\tau_e|_{Q_W^{2\pm}} = \pm 1$) to be the involution defined by this \mathbb{Z}_2 -grading. The second grading is the one for which F_V^i and Q_W^i ($i = 1, 2$) are purely even, i.e., $F_V^{i+} = F_V^i, Q_W^{i+} = Q_W^i$. We denote by $\tau_s = \text{id}$ the involution defined by this \mathbb{Z}_2 -grading. Set $Q(W) = Q_W^1 \otimes Q_W^2 \otimes Q_W^2$. We will denote by τ_1 the \mathbb{Z}_2 -grading on $Q(W)$ defined by

$$(Q(W), \tau_1) = (Q_W^1, \tau_s) \widehat{\otimes} (Q_W^2, \tau_e) \widehat{\otimes} (Q_W^2, \tau_s). \quad (3.3)$$

Then the coefficient of q^n ($n \in \frac{1}{2}\mathbb{Z}$) in (2.13) of $R_1(V)$ (resp. $R_2(V), R_3(V), R_4(V), Q_1(W)$) is exactly the \mathbb{Z}_2 -graded vector subbundle of (F_V^1, τ_s) (resp. $(F_V^1, \tau_e), (F_V^2, \tau_e), (F_V^2, \tau_s), (Q(W), \tau_1)$), on which P acts by multiplication by n .

Furthermore, we denote by τ_e (resp. τ_s) the \mathbb{Z}_2 -grading on $S(TX, K_X) \otimes \otimes_{n=1}^{\infty} \text{Sym}(TX_n) \otimes F_V^i$ ($i = 1, 2$) induced by the above \mathbb{Z}_2 -gradings. We will denote by τ_{e1} (resp. τ_{s1}) the \mathbb{Z}_2 -grading on $S(TX, K_X) \otimes \otimes_{n=1}^{\infty} \text{Sym}(TX_n) \otimes F_V^i \otimes Q(W)$ ($i = 1, 2$) defined by

$$\tau_{e1} = \tau_e \widehat{\otimes} \tau_1, \quad \tau_{s1} = \tau_s \widehat{\otimes} \tau_1. \quad (3.4)$$

We still denote by τ_{e1} (resp. τ_{s1}) the \mathbb{Z}_2 -grading on $S(TY, K_X \otimes_{v>0} (\det N_v)^{-1}) \otimes \mathcal{F}^{-p}(X) \otimes F_V^i \otimes Q(W)$ ($i = 1, 2$) which is induced as in (3.4).

By (2.19), as in (2.20), there is a natural isomorphism between the \mathbb{Z}_2 -graded $C(V)$ -Clifford modules over F ,

$$S(V)|_F \simeq S(V_0^{\mathbb{R}}, \otimes_{v>0} (\det V_v)^{-1}) \otimes \widehat{\bigotimes}_{v>0} \Lambda V_v. \quad (3.5)$$

Let $V_0 = V_0^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Using (2.19) and (3.5), we rewrite (3.2) on the fixed point set F as follows,

$$\begin{aligned} Q_W^1 &= \bigotimes_{n=0}^{\infty} \Lambda(\oplus_v \overline{W}_{v,n}) \otimes \bigotimes_{n=1}^{\infty} \Lambda(\oplus_v W_{v,n}), \\ Q_W^2 &= \bigotimes_{n \in \mathbb{N} + \frac{1}{2}} \Lambda(\oplus_v \overline{W}_{v,n}) \otimes \bigotimes_{n \in \mathbb{N} + \frac{1}{2}} \Lambda(\oplus_v W_{v,n}), \\ F_V^1 &= \bigotimes_{n=1}^{\infty} \Lambda(V_{0,n} \oplus \oplus_{v>0} (V_{v,n} \oplus \overline{V}_{v,n})) \\ &\quad \otimes S(V_0^{\mathbb{R}}, \otimes_{v>0} (\det V_v)^{-1}) \otimes_{v>0} \Lambda V_{v,0}, \\ F_V^2 &= \bigotimes_{n \in \mathbb{N} + \frac{1}{2}} \Lambda(V_{0,n} \oplus \oplus_{v>0} (V_{v,n} \oplus \overline{V}_{v,n})). \end{aligned} \quad (3.6)$$

We can reformulate Theorem 2.9 as follows.

Theorem 3.1 *For each $\alpha, h, p \in \mathbb{Z}$, $p > 0$, $m \in \frac{1}{2}\mathbb{Z}$, for $i = 1, 2$, $\tau = \tau_{e1}$ or τ_{s1} , the following identity holds in $K_{G_y}(B)$,*

$$\begin{aligned} &\text{Ind}_{\tau} \left(D^{Y_{\alpha}} \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes \mathcal{F}^{-p}(X) \otimes F_V^i \otimes Q(W), \right. \\ &\quad \left. m + \frac{1}{2}p^2e(N) + \frac{1}{2}pd'(N), h \right) \\ &= (-1)^{pd'(W)} \text{Ind}_{\tau} \left(D^{Y_{\alpha}} \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes \mathcal{F}^0(X) \otimes F_V^i \right. \\ &\quad \left. \otimes Q(W) \otimes L^{-p}, m + ph + p^2e, h \right). \end{aligned} \quad (3.7)$$

Following [17] in spirit, we introduce the same shift operators as in [14, (3.9)]. For $p \in \mathbb{N}$, we set

$$\begin{aligned} r_* : N_{v,n} &\rightarrow N_{v,n+pv}, & r_* : \overline{N}_{v,n} &\rightarrow \overline{N}_{v,n-pv}, \\ r_* : V_{v,n} &\rightarrow V_{v,n+pv}, & r_* : \overline{V}_{v,n} &\rightarrow \overline{V}_{v,n-pv}, \\ r_* : W_{v,n} &\rightarrow W_{v,n+pv}, & r_* : \overline{W}_{v,n} &\rightarrow \overline{W}_{v,n-pv}. \end{aligned} \quad (3.8)$$

Proposition 3.2 *For $p \in \mathbb{Z}$, $p > 0$, $i = 1, 2$, there are natural isomorphisms of vector bundles over F ,*

$$r_*(\mathcal{F}^{-p}(X)) \simeq \mathcal{F}^0(X) \otimes L(N)^p, \quad r_*(F_V^i) \simeq F_V^i \otimes L(V)^{-p}. \quad (3.9)$$

For any $p \in \mathbb{Z}$, $p > 0$, $i = 1, 2$, there are natural $G_y \times S^1$ -equivariant isomorphisms of vector bundles over F ,

$$r_*(Q_W^i) \simeq Q_W^i \otimes L(W)^{-p}. \quad (3.10)$$

In particular, one gets the $G_y \times S^1$ -equivariant bundle isomorphism

$$r_*(Q(W)) \simeq Q(W) \otimes L(W)^{-3p}. \quad (3.11)$$

Proof By [14, Proposition 3.1], only the $i = 2$ case in (3.10) needs to be proved.

Using [14, (3.14)-(3.16)], we have the natural $G_y \times S^1$ -equivariant isomorphisms of vector bundles over F ,

$$\begin{aligned} \bigotimes_{\substack{n \in \mathbb{N} + \frac{1}{2}, v > 0, \\ 0 < n < pv}} \Lambda^{i_n}(\overline{W}_{v, n-pv}) &\simeq \bigotimes_{\substack{n \in \mathbb{N} + \frac{1}{2}, v > 0, \\ 0 < n < pv}} \Lambda^{\dim W_v - i_n}(W_{v, -n+pv}) \\ &\quad \otimes \bigotimes_{v > 0} (\det \overline{W}_v)^{pv}, \\ \bigotimes_{\substack{n \in \mathbb{N} + \frac{1}{2}, v < 0, \\ 0 < n < -pv}} \Lambda^{i'_n}(W_{v, n+pv}) &\simeq \bigotimes_{\substack{n \in \mathbb{N} + \frac{1}{2}, v < 0, \\ 0 < n < -pv}} \Lambda^{\dim W_v - i'_n}(\overline{W}_{v, -n-pv}) \\ &\quad \otimes \bigotimes_{v > 0} (\det W_v)^{-pv}. \end{aligned} \quad (3.12)$$

From (2.22) and (3.12), we get (3.10) for the $i = 2$ case.

The proof of Proposition 3.2 is completed.

The following proposition, which is an analogue of [14, Proposition 3.2], is deduced from Proposition 3.2.

Proposition 3.3 *For $p \in \mathbb{Z}$, $p > 0$, $i = 1, 2$, the G_y -equivariant isomorphism of vector bundles over F induced by (3.9), (3.11),*

$$\begin{aligned} r_* : S(TY, K_X \otimes_{v>0} (\det N_v)^{-1}) \otimes (K_W \otimes K_X^{-1})^{1/2} \\ \otimes \mathcal{F}^{-p}(X) \otimes F_V^i \otimes Q(W) \\ \longrightarrow S(TY, K_X \otimes_{v>0} (\det N_v)^{-1}) \otimes (K_W \otimes K_X^{-1})^{1/2} \\ \otimes \mathcal{F}^0(X) \otimes F_V^i \otimes Q(W) \otimes L^{-p}, \end{aligned} \quad (3.13)$$

verifies the following identities,

$$\begin{aligned} r_*^{-1} \cdot \mathbf{J}_H \cdot r_* &= \mathbf{J}_H, \\ r_*^{-1} \cdot P \cdot r_* &= P + p\mathbf{J}_H + p^2e - \frac{1}{2}p^2e(N) - \frac{p}{2}d'(N). \end{aligned} \quad (3.14)$$

For the \mathbb{Z}_2 -gradings, we have

$$r_*^{-1} \tau_e r_* = \tau_e, \quad r_*^{-1} \tau_s r_* = \tau_s, \quad r_*^{-1} \tau_1 r_* = (-1)^{pd'(W)} \tau_1. \quad (3.15)$$

Proof By the proof of [14, Proposition 3.2], we need to compute the action of $r_*^{-1} \cdot P \cdot r_*$ on $\bigotimes_{\substack{n \in \mathbb{N} + \frac{1}{2}, v > 0, \\ 0 < n < pv}} \Lambda^{i_n}(\overline{W}_{v,n}) \bigotimes_{\substack{n \in \mathbb{N} + \frac{1}{2}, v < 0, \\ 0 < n < -pv}} \Lambda^{i'_n}(W_{v,n})$. In fact, by (3.12),

$$\begin{aligned} r_*^{-1} \cdot P \cdot r_* &= \sum_{\substack{n \in \mathbb{N} + \frac{1}{2}, v > 0, \\ 0 < n < pv}} (\dim W_v - i_n)(-n + pv) \\ &\quad + \sum_{\substack{n \in \mathbb{N} + \frac{1}{2}, v < 0, \\ 0 < n < -pv}} (\dim W_v - i'_n)(-n - pv) \\ &= P + p\mathbf{J}_H + \frac{1}{2}p^2e(W). \end{aligned} \quad (3.16)$$

By [14, (3.21)-(3.23)], (2.21)-(2.23) and (3.16), we deduce the second line of (3.14). The first line of (3.14) is obvious.

Consider the \mathbb{Z}_2 -gradings. The first two identities of (3.15) were proved in [15, (3.18)]. τ_1 changes only on $\bigotimes_{\substack{n \in \mathbb{N} + \frac{1}{2}, v > 0, \\ 0 < n < pv}} \Lambda^{i_n}(\overline{W}_{v,n}) \bigotimes_{\substack{n \in \mathbb{N} + \frac{1}{2}, v < 0, \\ 0 < n < -pv}} \Lambda^{i'_n}(W_{v,n})$. From (2.21) and (3.12), we get the third identity of (3.15).

The proof of Proposition 3.3 is completed.

Theorem 3.1 is a direct consequence of Proposition 3.3. The proof of Theorem 2.9 is completed.

The rest of this section is devoted to a proof of Theorem 2.8.

3.2 The refined shift operators

We first introduce a partition of $[0, 1]$ as in [14, pp. 942–943]. Set $J = \{v \in \mathbb{N} \mid \text{there exists } \alpha \text{ such that } N_v \neq 0 \text{ on } F_\alpha\}$ and

$$\Phi = \{\beta \in (0, 1] \mid \text{there exists } v \in J \text{ such that } \beta v \in \mathbb{Z}\}. \quad (3.17)$$

We order the elements in Φ so that $\Phi = \{\beta_i \mid 1 \leq i \leq J_0, J_0 \in \mathbb{N} \text{ and } \beta_i < \beta_{i+1}\}$. Then for any integer $1 \leq i \leq J_0$, there exist $p_i, n_i \in \mathbb{N}$, $0 < p_i \leq n_i$, with $(p_i, n_i) = 1$ such that

$$\beta_i = p_i/n_i. \quad (3.18)$$

Clearly, $\beta_{J_0} = 1$. We also set $p_0 = 0$ and $\beta_0 = 0$.

For $0 \leq j \leq J_0$, $p \in \mathbb{N}^*$, we write

$$\begin{aligned} I_j^p &= \left\{ (v, n) \in \mathbb{N} \times \mathbb{N} \mid v \in J, (p-1)v < n \leq pv, \frac{n}{v} = p-1 + \frac{p_j}{n_j} \right\}, \\ \overline{I}_j^p &= \left\{ (v, n) \in \mathbb{N} \times \mathbb{N} \mid v \in J, (p-1)v < n \leq pv, \frac{n}{v} > p-1 + \frac{p_j}{n_j} \right\}. \end{aligned} \quad (3.19)$$

Clearly, $I_0^p = \emptyset$, the empty set. We define $\mathcal{F}_{p,j}(X)$ as in [14, (2.21)], which are

analogous with (2.26). More specifically, we set

$$\begin{aligned}
\mathcal{F}_{p,j}(X) &= \bigotimes_{n=1}^{\infty} \text{Sym}(TY_n) \otimes \bigotimes_{v>0} \left(\bigotimes_{n=1}^{\infty} \text{Sym}(N_{v,n}) \otimes \bigotimes_{n>(p-1)v+\frac{p_j}{n_j}v} \text{Sym}(\overline{N}_{v,n}) \right) \\
&\quad \otimes \bigotimes_{\substack{v>0, \\ 0 \leq n \leq (p-1)v + \lceil \frac{p_j}{n_j} v \rceil}} \left(\text{Sym}(N_{v,-n}) \otimes \det N_v \right) \\
&= \mathcal{F}_p(X) \otimes \mathcal{F}'_{p-1}(X) \otimes \bigotimes_{(v,n) \in \overline{I}_j^p} \text{Sym}(\overline{N}_{v,n}) \\
&\quad \otimes \bigotimes_{(v,n) \in \cup_{i=0}^j I_i^p} \left(\text{Sym}(N_{v,-n}) \otimes \det N_v \right),
\end{aligned} \tag{3.20}$$

where we use the notation that for $s \in \mathbb{R}$, $[s]$ denotes the greatest integer which is less than or equal to s . Then

$$\mathcal{F}_{p,0}(X) = \mathcal{F}^{-p+1}(X), \quad \mathcal{F}_{p,J_0}(X) = \mathcal{F}^{-p}(X). \tag{3.21}$$

From the construction of β_i , we know that for $v \in J$, there is no integer in $(\frac{p_{j-1}}{n_{j-1}}v, \frac{p_j}{n_j}v)$. Furthermore (cf. [14, (4.24)]),

$$\begin{aligned}
\left\lfloor \frac{p_{j-1}}{n_{j-1}}v \right\rfloor &= \left\lfloor \frac{p_j}{n_j}v \right\rfloor - 1 \quad \text{if } v \equiv 0 \pmod{n_j}, \\
\left\lfloor \frac{p_{j-1}}{n_{j-1}}v \right\rfloor &= \left\lfloor \frac{p_j}{n_j}v \right\rfloor \quad \text{if } v \not\equiv 0 \pmod{n_j}.
\end{aligned} \tag{3.22}$$

We use the same shift operators r_{j*} , $1 \leq j \leq J_0$ as in [14, (4.21)], which refine the shift operator r_* defined in (3.8). For $p \in \mathbb{N} \setminus \{0\}$, set

$$\begin{aligned}
r_{j*} : N_{v,n} &\rightarrow N_{v,n+(p-1)v+p_jv/n_j}, \quad r_{j*} : \overline{N}_{v,n} \rightarrow \overline{N}_{v,n-(p-1)v-p_jv/n_j}, \\
r_{j*} : V_{v,n} &\rightarrow V_{v,n+(p-1)v+p_jv/n_j}, \quad r_{j*} : \overline{V}_{v,n} \rightarrow \overline{V}_{v,n-(p-1)v-p_jv/n_j}, \\
r_{j*} : W_{v,n} &\rightarrow W_{v,n+(p-1)v+p_jv/n_j}, \quad r_{j*} : \overline{W}_{v,n} \rightarrow \overline{W}_{v,n-(p-1)v-p_jv/n_j}.
\end{aligned} \tag{3.23}$$

For $1 \leq j \leq J_0$, we define $\mathcal{F}(\beta_j)$, $F_V^1(\beta_j)$, $F_V^2(\beta_j)$, $Q_W^1(\beta_j)$ and $Q_W^2(\beta_j)$ over F as follows (compare with [14, (4.13)]).

$$\begin{aligned}
\mathcal{F}(\beta_j) &= \bigotimes_{0 < n \in \mathbb{Z}} \text{Sym}(TY_n) \otimes \bigotimes_{\substack{v>0, \\ v \equiv 0, \frac{n_j}{2} \pmod{n_j}}} \bigotimes_{0 < n \in \mathbb{Z} + \frac{p_j}{n_j}v} \text{Sym}(N_{v,n} \oplus \overline{N}_{v,n}) \\
&\quad \otimes \bigotimes_{0 < v' < n_j/2} \text{Sym} \left(\bigoplus_{v \equiv v', -v' \pmod{n_j}} \left(\bigoplus_{0 < n \in \mathbb{Z} + \frac{p_j}{n_j}v} N_{v,n} \oplus \bigoplus_{0 < n \in \mathbb{Z} - \frac{p_j}{n_j}v} \overline{N}_{v,n} \right) \right), \\
F_V^1(\beta_j) &= \Lambda \left(\bigoplus_{0 < n \in \mathbb{Z}} V_{0,n} \oplus \bigoplus_{\substack{v>0, \\ v \equiv 0, \frac{n_j}{2} \pmod{n_j}}} \left(\bigoplus_{0 < n \in \mathbb{Z} + \frac{p_j}{n_j}v} V_{v,n} \oplus \bigoplus_{0 < n \in \mathbb{Z} - \frac{p_j}{n_j}v} \overline{V}_{v,n} \right) \right. \\
&\quad \left. \bigoplus_{0 < v' < n_j/2} \left(\bigoplus_{v \equiv v', -v' \pmod{n_j}} \left(\bigoplus_{0 < n \in \mathbb{Z} + \frac{p_j}{n_j}v} V_{v,n} \oplus \bigoplus_{0 < n \in \mathbb{Z} - \frac{p_j}{n_j}v} \overline{V}_{v,n} \right) \right) \right),
\end{aligned}$$

$$\begin{aligned}
F_V^2(\beta_j) &= \Lambda \left(\bigoplus_{0 < n \in \mathbb{Z} + \frac{1}{2}} V_{0,n} \bigoplus_{\substack{v > 0, \\ \frac{n_j}{2} \bmod(n_j)}} \left(\bigoplus_{0 < n \in \mathbb{Z} + \frac{p_j}{n_j} v + \frac{1}{2}} V_{v,n} \bigoplus_{0 < n \in \mathbb{Z} - \frac{p_j}{n_j} v + \frac{1}{2}} \overline{V}_{v,n} \right) \right. \\
&\quad \left. \bigoplus_{0 < v' < n_j/2} \left(\bigoplus_{v \equiv v', -v' \bmod(n_j)} \left(\bigoplus_{0 < n \in \mathbb{Z} + \frac{p_j}{n_j} v + \frac{1}{2}} V_{v,n} \bigoplus_{0 < n \in \mathbb{Z} - \frac{p_j}{n_j} v + \frac{1}{2}} \overline{V}_{v,n} \right) \right) \right), \\
Q_W^1(\beta_j) &= \Lambda \left(\bigoplus_v \left(\bigoplus_{0 < n \in \mathbb{Z} + \frac{p_j}{n_j} v} W_{v,n} \bigoplus_{0 \leq n \in \mathbb{Z} - \frac{p_j}{n_j} v} \overline{W}_{v,n} \right) \right), \\
Q_W^2(\beta_j) &= \Lambda \left(\bigoplus_v \left(\bigoplus_{0 < n \in \mathbb{Z} + \frac{p_j}{n_j} v + \frac{1}{2}} W_{v,n} \bigoplus_{0 < n \in \mathbb{Z} - \frac{p_j}{n_j} v + \frac{1}{2}} \overline{W}_{v,n} \right) \right). \quad (3.24)
\end{aligned}$$

Using (3.22), (3.24) and computing directly, we get an analogue of [14, Proposition 4.1] which refines Proposition 3.2.

Proposition 3.4 *For $p \in \mathbb{Z}$, $p > 0$, $1 \leq j \leq J_0$, there are natural isomorphisms of vector bundles over F ,*

$$\begin{aligned}
r_{j*}(\mathcal{F}_{p,j-1}(X)) &\simeq \mathcal{F}(\beta_j) \otimes \bigotimes_{v > 0, v \equiv 0 \bmod(n_j)} \text{Sym}(\overline{N}_{v,0}) \\
&\quad \otimes \bigotimes_{v > 0} (\det N_v)^{\left[\frac{p_j}{n_j} v\right] + (p-1)v+1} \otimes \bigotimes_{v > 0, v \equiv 0 \bmod(n_j)} (\det N_v)^{-1}, \\
r_{j*}(\mathcal{F}_{p,j}(X)) &\simeq \mathcal{F}(\beta_j) \otimes \bigotimes_{v > 0, v \equiv 0 \bmod(n_j)} \text{Sym}(N_{v,0}) \\
&\quad \otimes \bigotimes_{v > 0} (\det N_v)^{\left[\frac{p_j}{n_j} v\right] + (p-1)v+1}, \\
r_{j*}(F_V^1) &\simeq S\left(V_0^{\mathbb{R}}, \otimes_{v > 0} (\det V_v)^{-1}\right) \otimes F_V^1(\beta_j) \\
&\quad \otimes \bigotimes_{\substack{v > 0, \\ v \equiv 0 \bmod(n_j)}} \Lambda(V_{v,0}) \otimes \bigotimes_{v > 0} (\det \overline{V}_v)^{\left[\frac{p_j}{n_j} v\right] + (p-1)v}, \\
r_{j*}(F_V^2) &\simeq F_V^2(\beta_j) \otimes \bigotimes_{\substack{v > 0, \\ \frac{n_j}{2} \bmod(n_j)}} \Lambda(V_{v,0}) \otimes \bigotimes_{v > 0} (\det \overline{V}_v)^{\left[\frac{p_j}{n_j} v + \frac{1}{2}\right] + (p-1)v}.
\end{aligned}$$

For $p \in \mathbb{Z}$, $p > 0$, $1 \leq j \leq J_0$, there are natural $G_y \times S^1$ -equivariant isomor-

phisms of vector bundles over F ,

$$\begin{aligned}
r_{j*}(Q_W^1) &\simeq Q_W^1(\beta_j) \otimes \bigotimes_{\substack{v>0, \\ v \equiv 0 \pmod{n_j}}} \det W_v \otimes \\
&\quad \bigotimes_{v>0} (\det \overline{W}_v)^{\left[\frac{p_j}{n_j}v\right] + (p-1)v+1} \bigotimes_{v<0} (\det W_v)^{\left[-\frac{p_j}{n_j}v\right] - (p-1)v}, \\
r_{j*}(Q_W^2) &\simeq Q_W^2(\beta_j) \otimes \bigotimes_{\substack{v>0, \\ v \equiv \frac{n_j}{2} \pmod{n_j}}} \Lambda(W_{v,0}) \otimes \bigotimes_{\substack{v<0, \\ v \equiv -\frac{n_j}{2} \pmod{n_j}}} \Lambda(\overline{W}_{v,0}) \\
&\quad \bigotimes_{v>0} (\det \overline{W}_v)^{\left[\frac{p_j}{n_j}v + \frac{1}{2}\right] + (p-1)v} \bigotimes_{v<0} (\det W_v)^{\left[-\frac{p_j}{n_j}v + \frac{1}{2}\right] - (p-1)v}.
\end{aligned} \tag{3.25}$$

Proof By [14, Proposition 4.1], we need only to prove the second isomorphism in (3.25). In fact, using [14, (3.14)], we have the natural $G_y \times S^1$ -equivariant isomorphisms of vector bundles over F ,

$$\begin{aligned}
&\bigotimes_{\substack{0 < n \in \mathbb{Z} + \frac{1}{2}, v > 0, \\ n - (p-1)v - \frac{p_j}{n_j}v \leq 0}} \Lambda^{i_n} \left(\overline{W}_{v, n - (p-1)v - \frac{p_j}{n_j}v} \right) \simeq \bigotimes_{v>0} (\det \overline{W}_v)^{\left[\frac{p_j}{n_j}v + \frac{1}{2}\right] + (p-1)v} \\
&\quad \otimes \bigotimes_{\substack{0 < n \in \mathbb{Z} + \frac{1}{2}, v > 0, \\ n - (p-1)v - \frac{p_j}{n_j}v \leq 0}} \Lambda^{\dim W_v - i_n} \left(W_{v, -n + (p-1)v + \frac{p_j}{n_j}v} \right), \\
&\bigotimes_{\substack{0 < n \in \mathbb{Z} + \frac{1}{2}, v < 0, \\ n + (p-1)v + \frac{p_j}{n_j}v \leq 0}} \Lambda^{i'_n} \left(W_{v, n + (p-1)v + \frac{p_j}{n_j}v} \right) \simeq \bigotimes_{v<0} (\det W_v)^{\left[-\frac{p_j}{n_j}v + \frac{1}{2}\right] - (p-1)v} \\
&\quad \otimes \bigotimes_{\substack{0 < n \in \mathbb{Z} + \frac{1}{2}, v < 0, \\ n + (p-1)v + \frac{p_j}{n_j}v \leq 0}} \Lambda^{\dim W_v - i'_n} \left(\overline{W}_{v, -n - (p-1)v - \frac{p_j}{n_j}v} \right).
\end{aligned} \tag{3.26}$$

From (3.24) and (3.26), we get the second isomorphism in (3.25).

The proof of Proposition 3.4 is completed.

3.3 The Spin^c Dirac operators on $M(n_j)$

Recall that there is a nontrivial circle action on M which can be lifted to the circle actions on V and W .

For $n \in \mathbb{N} \setminus \{0\}$, let $\mathbb{Z}_n \subset S^1$ denote the cyclic subgroup of order n . Let $M(n_j)$ be the fixed point set of the induced \mathbb{Z}_{n_j} action on M . Then $\pi : M(n_j) \rightarrow B$ is a fibration with compact fiber $X(n_j)$. Let $N(n_j) \rightarrow M(n_j)$ be the normal bundle to $M(n_j)$ in M . As in [2, pp. 151] (see also [14, Section 4.1], [15, Section 4.1] or [17]), we see that $N(n_j)$ and V can be decomposed, as real vector bundles

over $M(n_j)$, into

$$\begin{aligned} N(n_j) &= \bigoplus_{0 < v < n_j/2} N(n_j)_v \oplus N(n_j)_{n_j/2}^{\mathbb{R}}, \\ V|_{M(n_j)} &= V(n_j)_0^{\mathbb{R}} \oplus \bigoplus_{0 < v < n_j/2} V(n_j)_v \oplus V(n_j)_{n_j/2}^{\mathbb{R}}, \end{aligned} \quad (3.27)$$

where $V(n_j)_0^{\mathbb{R}}$ is the real vector bundle on which \mathbb{Z}_{n_j} acts by identity, and $N(n_j)_{n_j/2}^{\mathbb{R}}$ (resp. $V(n_j)_{n_j/2}^{\mathbb{R}}$) is defined to be zero if n_j is odd. Moreover, for $0 < v < n_j/2$, $N(n_j)_v$ (resp. $V(n_j)_v$) admits a unique complex structure such that $N(n_j)_v$ (resp. $V(n_j)_v$) becomes a complex vector bundle on which $g \in \mathbb{Z}_{n_j}$ acts by g^v . We also denote by $V(n_j)_0$, $V(n_j)_{n_j/2}$ and $N(n_j)_{n_j/2}$ the corresponding complexification of $V(n_j)_0^{\mathbb{R}}$, $V(n_j)_{n_j/2}^{\mathbb{R}}$ and $N(n_j)_{n_j/2}^{\mathbb{R}}$.

Similarly, we also have the following \mathbb{Z}_{n_j} -equivariant decomposition of W over $M(n_j)$ into complex vector bundles,

$$W|_{M(n_j)} = \bigoplus_{0 \leq v < n_j} W(n_j)_v, \quad (3.28)$$

where for $0 \leq v < n_j$, $g \in \mathbb{Z}_{n_j}$ acts on $W(n_j)_v$ by sending g to g^v .

By [14, Lemma 4.1] (which generalizes [2, Lemmas 9.4 and 10.1] and [17, Lemma 5.1]), we know that the vector bundles $TX(n_j)$ and $V(n_j)_0^{\mathbb{R}}$ are orientable and even dimensional. Thus $N(n_j)$ is orientable over $M(n_j)$. By (3.27), $V(n_j)_{n_j/2}^{\mathbb{R}}$ and $N(n_j)_{n_j/2}^{\mathbb{R}}$ are also orientable and even dimensional. In what follows, we fix the orientations of $N(n_j)_{n_j/2}^{\mathbb{R}}$ and $V(n_j)_{n_j/2}^{\mathbb{R}}$. Then $TX(n_j)$ and $V(n_j)_0^{\mathbb{R}}$ are naturally oriented by (3.27) and the orientations of TX , V , $N(n_j)_{n_j/2}^{\mathbb{R}}$ and $V(n_j)_{n_j/2}^{\mathbb{R}}$. Let $W(n_j)_{n_j/2}^{\mathbb{R}}$ be the underlying real vector bundle of $W(n_j)_{n_j/2}$, which are canonically oriented by its complex structure.

By (2.18), (2.19), (3.27) and (3.28), we get the following identifications of complex vector bundles over F (cf. [14, (4.9) and (4.12)]), for $0 < v \leq n_j/2$,

$$\begin{aligned} N(n_j)_v|_F &= \bigoplus_{v' > 0, v' \equiv v \pmod{n_j}} N_{v'} \oplus \bigoplus_{v' > 0, v' \equiv -v \pmod{n_j}} \overline{N}_{v'}, \\ V(n_j)_v|_F &= \bigoplus_{v' > 0, v' \equiv v \pmod{n_j}} V_{v'} \oplus \bigoplus_{v' > 0, v' \equiv -v \pmod{n_j}} \overline{V}_{v'}, \end{aligned} \quad (3.29)$$

for $0 \leq v < n_j$,

$$W(n_j)_v|_F = \bigoplus_{v' \equiv v \pmod{n_j}} W_{v'}. \quad (3.30)$$

Also we get the following identifications of real vector bundles over F (cf. [14, (4.11)]),

$$\begin{aligned} TX(n_j)|_F &= TY \oplus \bigoplus_{\substack{v > 0, \\ v \equiv 0 \pmod{n_j}}} N_v, \quad N(n_j)_{\frac{n_j}{2}}^{\mathbb{R}}|_F = \bigoplus_{\substack{v > 0, \\ v \equiv \frac{n_j}{2} \pmod{n_j}}} N_v, \\ V(n_j)_0^{\mathbb{R}}|_F &= V_0^{\mathbb{R}} \oplus \bigoplus_{\substack{v > 0, \\ v \equiv 0 \pmod{n_j}}} V_v, \quad V(n_j)_{\frac{n_j}{2}}^{\mathbb{R}}|_F = \bigoplus_{\substack{v > 0, \\ v \equiv \frac{n_j}{2} \pmod{n_j}}} V_v. \end{aligned} \quad (3.31)$$

Moreover, we have the identifications of complex vector bundles over F ,

$$\begin{aligned} TX(n_j)|_F \otimes_{\mathbb{R}} \mathbb{C} &= TY \otimes_{\mathbb{R}} \mathbb{C} \oplus \bigoplus_{v>0, v \equiv 0 \pmod{n_j}} (N_v \oplus \overline{N}_v), \\ V(n_j)_0|_F &= V_0^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \oplus \bigoplus_{v>0, v \equiv 0 \pmod{n_j}} (V_v \oplus \overline{V}_v). \end{aligned} \quad (3.32)$$

As $(p_j, n_j) = 1$, we know that, for $v \in \mathbb{Z}$, $\frac{p_j}{n_j}v \in \mathbb{Z}$ if and only if $\frac{v}{n_j} \in \mathbb{Z}$. Also, $\frac{p_j}{n_j}v \in \mathbb{Z} + \frac{1}{2}$ if and only if $\frac{v}{n_j} \in \mathbb{Z} + \frac{1}{2}$. Remark if $v \equiv -v' \pmod{n_j}$, then $\{n \mid 0 < n \in \mathbb{Z} + \frac{p_j}{n_j}v\} = \{n \mid 0 < n \in \mathbb{Z} - \frac{p_j}{n_j}v'\}$. Using the identifications (3.29), (3.30) and (3.32), we can rewrite $\mathcal{F}(\beta_j)$, $F_V^1(\beta_j)$, $F_V^2(\beta_j)$, $Q_W^1(\beta_j)$ and $Q_W^2(\beta_j)$ over F defined over in (3.24) as follows (compare with [14, (4.7)]),

$$\begin{aligned} \mathcal{F}(\beta_j) &= \bigotimes_{0 < n \in \mathbb{Z}} \text{Sym}(TX(n_j)_n) \otimes \bigotimes_{0 < v < n_j/2} \text{Sym}\left(\bigoplus_{0 < n \in \mathbb{Z} + \frac{p_j}{n_j}v} N(n_j)_{v,n}\right. \\ &\quad \left. \oplus \bigoplus_{0 < n \in \mathbb{Z} - \frac{p_j}{n_j}v} \overline{N(n_j)_{v,n}}\right) \oplus \bigoplus_{0 < n \in \mathbb{Z} + \frac{1}{2}} \text{Sym}(N(n_j)_{n_j/2,n}), \end{aligned} \quad (3.33)$$

$$\begin{aligned} F_V^1(\beta_j) &= \Lambda\left(\bigoplus_{0 < n \in \mathbb{Z}} V(n_j)_{0,n} \oplus \bigoplus_{0 < v < n_j/2} \left(\bigoplus_{0 < n \in \mathbb{Z} + \frac{p_j}{n_j}v} V(n_j)_{v,n}\right.\right. \\ &\quad \left.\left. \oplus \bigoplus_{0 < n \in \mathbb{Z} - \frac{p_j}{n_j}v} \overline{V(n_j)_{v,n}}\right) \oplus \bigoplus_{0 < n \in \mathbb{Z} + \frac{1}{2}} V(n_j)_{n_j/2,n}\right), \end{aligned} \quad (3.34)$$

$$\begin{aligned} F_V^2(\beta_j) &= \Lambda\left(\bigoplus_{0 < n \in \mathbb{Z}} V(n_j)_{n_j/2,n} \oplus \bigoplus_{0 < v < n_j/2} \left(\bigoplus_{0 < n \in \mathbb{Z} + \frac{p_j}{n_j}v + \frac{1}{2}} V(n_j)_{v,n}\right.\right. \\ &\quad \left.\left. \oplus \bigoplus_{0 < n \in \mathbb{Z} - \frac{p_j}{n_j}v + \frac{1}{2}} \overline{V(n_j)_{v,n}}\right) \oplus \bigoplus_{0 < n \in \mathbb{Z} + \frac{1}{2}} V(n_j)_{0,n}\right), \end{aligned} \quad (3.35)$$

$$Q_W^1(\beta_j) = \Lambda\left(\bigoplus_{0 \leq v < n_j} \left(\bigoplus_{0 < n \in \mathbb{Z} + \frac{p_j}{n_j}v} W(n_j)_{v,n} \oplus \bigoplus_{0 \leq n \in \mathbb{Z} - \frac{p_j}{n_j}v} \overline{W(n_j)_{v,n}}\right)\right), \quad (3.36)$$

$$\begin{aligned} Q_W^2(\beta_j) &= \Lambda\left(\bigoplus_{0 \leq v < n_j} \left(\bigoplus_{0 < n \in \mathbb{Z} + \frac{p_j}{n_j}v + \frac{1}{2}} W(n_j)_{v,n}\right.\right. \\ &\quad \left.\left. \oplus \bigoplus_{0 < n \in \mathbb{Z} - \frac{p_j}{n_j}v + \frac{1}{2}} \overline{W(n_j)_{v,n}}\right)\right). \end{aligned} \quad (3.37)$$

We indicate here that $\mathcal{F}(\beta_j)$, $F_V^1(\beta_j)$, $F_V^2(\beta_j)$, $Q_W^1(\beta_j)$ and $Q_W^2(\beta_j)$ in (3.24) are the restrictions of the corresponding vector bundles in the right hand side of (3.33)-(3.37) over $M(n_j)$, which will still be denoted as $\mathcal{F}(\beta_j)$, $F_V^1(\beta_j)$, $F_V^2(\beta_j)$, $Q_W^1(\beta_j)$ and $Q_W^2(\beta_j)$. Write

$$Q_W(\beta_j) = Q_W^1(\beta_j) \otimes Q_W^2(\beta_j) \otimes Q_W^2(\beta_j), \quad (3.38)$$

which we now think of as a vector bundle over $M(n_j)$.

We now define the Spin^c Dirac operators on $M(n_j)$. The following Lemma follows from the proof of [2, Lemmas 11.3 and 11.4].

Lemma 3.5 (Compare with [14, Lemma 4.2]) *Assume that (2.17) holds. Let*

$$L(n_j) = \bigotimes_{0 < v < n_j/2} \left(\det(N(n_j)_v) \otimes \det(\overline{V(n_j)_v}) \right. \\ \left. \otimes \left(\det(\overline{W(n_j)_v}) \otimes \det(W(n_j)_{n_j-v}) \right)^3 \right)^{(r(n_j)+1)v} \quad (3.39)$$

be the complex line bundle over $M(n_j)$. Then we have

- (i) $L(n_j)$ has an n_j^{th} root over $M(n_j)$.
- (ii) Let $U_1 = TX(n_j) \oplus V(n_j)_0^{\mathbb{R}} \oplus W(n_j)_{n_j/2}^{\mathbb{R}} \oplus W(n_j)_{n_j/2}^{\mathbb{R}}$,

$$L_1 = K_X \otimes \bigotimes_{0 < v < n_j/2} \left(\det(N(n_j)_v) \otimes \det(\overline{V(n_j)_v}) \right) \\ \otimes \left(\det(W(n_j)_{n_j/2}) \right)^3 \otimes L(n_j)^{\frac{r(n_j)}{n_j}}.$$

$$\text{Let } U_2 = TX(n_j) \oplus V(n_j)_{n_j/2}^{\mathbb{R}} \oplus W(n_j)_{n_j/2}^{\mathbb{R}} \oplus W(n_j)_{n_j/2}^{\mathbb{R}},$$

$$L_2 = K_X \otimes \bigotimes_{0 < v < n_j/2} \left(\det(N(n_j)_v) \right) \otimes \left(\det(W(n_j)_{n_j/2}) \right)^3 \otimes L(n_j)^{\frac{r(n_j)}{n_j}}.$$

Then U_1 (resp. U_2) has a Spin^c structure defined by L_1 (resp. L_2).

Remark that in order to define an S^1 (resp. G_y) action on $L(n_j)^{r(n_j)/n_j}$, we must replace the S^1 (resp. G_y) action by its n_j -fold action. Here by abusing notation, we still say an S^1 (resp. G_y) action without causing any confusion.

Let $S(U_1, L_1)$ (resp. $S(U_2, L_2)$) be the fundamental complex spinor bundle for (U_1, L_1) (resp. (U_2, L_2)) (cf. [8, Appendix D]). There are two \mathbb{Z}_2 -gradings on $S(U_1, L_1)$ (resp. $S(U_2, L_2)$). The first grading, which we denote by τ_s , is induced by the involutions on $S(U_1, L_1)$ and $S(U_2, L_2)$ determined by $TX(n_j) \oplus W(n_j)_{n_j/2}^{\mathbb{R}}$ as in (2.1). The second grading, which we denote by τ_e , is induced by the involution on $S(U_1, L_1)$ (resp. $S(U_2, L_2)$) determined by $TX(n_j) \oplus V(n_j)_0^{\mathbb{R}} \oplus W(n_j)_{n_j/2}^{\mathbb{R}}$ (resp. $U_2 = TX(n_j) \oplus V(n_j)_{n_j/2}^{\mathbb{R}} \oplus W(n_j)_{n_j/2}^{\mathbb{R}}$) as in (2.1).

In what follows, by $D^{X(n_j)}$ we mean the S^1 -equivariant Spin^c Dirac operator on $S(U_1, L_1)$ or $S(U_2, L_2)$ over $M(n_j)$.

Corresponding to (2.8), by (3.29) and (3.30), we define $S(U_1, L_1)'$ (resp. $S(U_2, L_2)'$) equipped with its involutions τ'_s and τ'_e as follows (compare with

[14, (4.16)]),

$$\begin{aligned}
\left(S(U_1, L_1)', \tau'_s / \tau'_e \right) &= \left(S \left(TY \oplus V_0^{\mathbb{R}}, L_1 \otimes \bigotimes_{\substack{v>0, \\ v \equiv 0 \pmod{n_j}}} (\det N_v \otimes \det V_v)^{-1} \right. \right. \\
&\quad \left. \otimes \bigotimes_{v \equiv \frac{n_j}{2} \pmod{n_j}} (\det W_v)^{-2} \right), \tau'_s / \tau'_e \Big) \otimes \bigotimes_{\substack{v>0, \\ v \equiv 0 \pmod{n_j}}} \Lambda_{\pm 1}(V_v) \\
&\otimes \bigotimes_{v \equiv \frac{n_j}{2} \pmod{n_j}} \Lambda_{-1}(W_v) \otimes \bigotimes_{v \equiv \frac{n_j}{2} \pmod{n_j}} \Lambda(W_v) , \tag{3.40}
\end{aligned}$$

$$\begin{aligned}
\left(S(U_2, L_2)', \tau'_s / \tau'_e \right) &= S \left(TY, L_2 \otimes \bigotimes_{\substack{v>0, \\ v \equiv 0 \pmod{n_j}}} (\det N_v)^{-1} \right. \\
&\quad \left. \otimes \bigotimes_{\substack{v>0, \\ v \equiv \frac{n_j}{2} \pmod{n_j}}} (\det V_v)^{-1} \otimes \bigotimes_{v \equiv \frac{n_j}{2} \pmod{n_j}} (\det W_v)^{-2} \right) \tag{3.41} \\
&\otimes \bigotimes_{\substack{v>0, \\ v \equiv \frac{n_j}{2} \pmod{n_j}}} \Lambda_{\pm 1}(V_v) \otimes \bigotimes_{v \equiv \frac{n_j}{2} \pmod{n_j}} \Lambda_{-1}(W_v) \otimes \bigotimes_{v \equiv \frac{n_j}{2} \pmod{n_j}} \Lambda(W_v) .
\end{aligned}$$

Then by (2.8), for $i = 1, 2$, we have the following isomorphisms of Clifford modules over F preserving the \mathbb{Z}_2 -gradings (compare with [14, (4.17)]),

$$\left(S(U_i, L_i), \tau_s / \tau_e \right) \Big|_F \simeq \left(S(U_i, L_i)', \tau'_s / \tau'_e \right) \otimes \bigotimes_{v>0, v \equiv 0 \pmod{n_j}} \Lambda_{-1}(N_v). \tag{3.42}$$

As in [14, pp. 952], we introduce formally the following complex line bundles over F ,

$$\begin{aligned}
L'_1 &= \left(L_1^{-1} \otimes \bigotimes_{\substack{v>0, \\ v \equiv 0 \pmod{n_j}}} (\det N_v \otimes \det V_v) \otimes \bigotimes_{v \equiv \frac{n_j}{2} \pmod{n_j}} (\det W_v)^2 \right. \\
&\quad \left. \otimes \bigotimes_{v>0} (\det N_v \otimes \det V_v)^{-1} \otimes K_X \right)^{\frac{1}{2}}, \tag{3.43}
\end{aligned}$$

$$\begin{aligned}
L'_2 &= \left(L_2^{-1} \otimes \bigotimes_{\substack{v>0, \\ v \equiv 0 \pmod{n_j}}} \det N_v \otimes \bigotimes_{\substack{v>0, \\ v \equiv \frac{n_j}{2} \pmod{n_j}}} \det V_v \right. \\
&\quad \left. \otimes \bigotimes_{v \equiv \frac{n_j}{2} \pmod{n_j}} (\det W_v)^2 \otimes \bigotimes_{v>0} (\det N_v)^{-1} \otimes K_X \right)^{\frac{1}{2}}. \tag{3.44}
\end{aligned}$$

In fact, from (2.8), Lemma 3.5 and the assumption that V is spin, one verifies easily that $c_1(L_i'^2) = 0 \pmod{2}$ for $i = 1, 2$, which implies that L'_1 and L'_2 are well-defined complex line bundles over F .

Then by [14, (3.14)], (3.40)-(3.44) and the definitions of L_1, L_2 , we get the following identifications of Clifford modules over F (compare with [14, (4.19)]),

$$\begin{aligned}
& \left(S(U_1, L_1)' \otimes L_1', (\tau'_s/\tau'_e) \otimes \text{id} \right) = S(TY, K_X \otimes_{v>0} (\det N_v)^{-1}) \\
& \quad \otimes \left(S(V_0^{\mathbb{R}}, \otimes_{v>0} (\det V_v)^{-1}), \text{id}/\tau \right) \otimes \bigotimes_{\substack{v>0, \\ v \equiv 0 \pmod{n_j}}} \Lambda_{\pm 1}(V_v) \\
& \quad \otimes \bigotimes_{\substack{v>0, \\ v \equiv \frac{n_j}{2} \pmod{n_j}}} \Lambda_{-1}(W_v) \otimes \bigotimes_{\substack{v<0, \\ v \equiv \frac{n_j}{2} \pmod{n_j}}} \Lambda_{-1}(\overline{W}_v) \quad (3.45) \\
& \quad \otimes \bigotimes_{\substack{v>0, \\ v \equiv \frac{n_j}{2} \pmod{n_j}}} \Lambda(W_v) \otimes \bigotimes_{\substack{v<0, \\ v \equiv \frac{n_j}{2} \pmod{n_j}}} \Lambda(\overline{W}_v) \otimes \bigotimes_{\substack{v<0, \\ v \equiv \frac{n_j}{2} \pmod{n_j}}} (\det W_v)^2,
\end{aligned}$$

$$\begin{aligned}
& \left(S(U_2, L_2)' \otimes L_2', (\tau'_s/\tau'_e) \otimes \text{id} \right) = S(TY, K_X \otimes_{v>0} (\det N_v)^{-1}) \\
& \quad \otimes \bigotimes_{\substack{v>0, \\ v \equiv \frac{n_j}{2} \pmod{n_j}}} \Lambda_{\pm 1}(V_v) \otimes \bigotimes_{\substack{v>0, \\ v \equiv \frac{n_j}{2} \pmod{n_j}}} \Lambda_{-1}(W_v) \otimes \bigotimes_{\substack{v<0, \\ v \equiv \frac{n_j}{2} \pmod{n_j}}} \Lambda_{-1}(\overline{W}_v) \\
& \quad \otimes \bigotimes_{\substack{v>0, \\ v \equiv \frac{n_j}{2} \pmod{n_j}}} \Lambda(W_v) \otimes \bigotimes_{\substack{v<0, \\ v \equiv \frac{n_j}{2} \pmod{n_j}}} \Lambda(\overline{W}_v) \otimes \bigotimes_{\substack{v<0, \\ v \equiv \frac{n_j}{2} \pmod{n_j}}} (\det W_v)^2. \quad (3.46)
\end{aligned}$$

Now we compare the \mathbb{Z}_2 -gradings in (3.45) and (3.46). Set (compare with [14, (4.20)])

$$\begin{aligned}
\Delta(n_j, N) &= \sum_{\frac{n_j}{2} < v' < n_j} \sum_{0 < v, v \equiv v' \pmod{n_j}} \dim N_v + o\left(N(n_j) \frac{\mathbb{R}}{2}\right), \\
\Delta(n_j, V) &= \sum_{\frac{n_j}{2} < v' < n_j} \sum_{0 < v, v \equiv v' \pmod{n_j}} \dim V_v + o\left(V(n_j) \frac{\mathbb{R}}{2}\right), \quad (3.47) \\
\Delta(n_j, W) &= \sum_{v < 0, v \equiv \frac{n_j}{2} \pmod{n_j}} \dim W_v,
\end{aligned}$$

with $o(N(n_j) \frac{\mathbb{R}}{2})$ (resp. $o(V(n_j) \frac{\mathbb{R}}{2})$) equals 0 or 1, depending on whether the given orientation on $N(n_j) \frac{\mathbb{R}}{2}$ (resp. $V(n_j) \frac{\mathbb{R}}{2}$) agrees or disagrees with the complex orientation of $\oplus_{v>0, v \equiv \frac{n_j}{2} \pmod{n_j}} N_v$ (resp. $\oplus_{v>0, v \equiv \frac{n_j}{2} \pmod{n_j}} V_v$).

As explained in [15, pp. 166], for the \mathbb{Z}_2 -gradings induced by τ_s , the differences of the \mathbb{Z}_2 -gradings of (3.45) and (3.46) are both $(-1)^{\Delta(n_j, N) + \Delta(n_j, W)}$; for the \mathbb{Z}_2 -gradings induced by τ_e , the difference of the \mathbb{Z}_2 -gradings of (3.45) (resp. (3.46)) is $(-1)^{\Delta(n_j, N) + \Delta(n_j, V) + \Delta(n_j, W)}$ (resp. $(-1)^{\Delta(n_j, N) + o(V(n_j) \frac{\mathbb{R}}{n_j/2}) + \Delta(n_j, W)}$).

Lemma 3.6 (Compare with [14, Lemma 4.3]) *Let us write*

$$\begin{aligned}
L(\beta_j)_1 &= L'_1 \otimes \bigotimes_{v>0} (\det N_v)^{\left[\frac{p_j}{n_j}v\right] + (p-1)v+1} \otimes \bigotimes_{v>0} (\det \overline{V}_v)^{\left[\frac{p_j}{n_j}v\right] + (p-1)v} \\
&\otimes \bigotimes_{\substack{v>0, \\ v \equiv 0 \pmod{n_j}}} (\det N_v)^{-1} \otimes \bigotimes_{v<0} (\det W_v)^{\left[-\frac{p_j}{n_j}v\right] + 2\left[-\frac{p_j}{n_j}v + \frac{1}{2}\right] - 3(p-1)v} \\
&\otimes \bigotimes_{v>0} (\det \overline{W}_v)^{\left[\frac{p_j}{n_j}v\right] + 2\left[\frac{p_j}{n_j}v + \frac{1}{2}\right] + 3(p-1)v+1} \\
&\otimes \bigotimes_{\substack{v>0, \\ v \equiv 0 \pmod{n_j}}} \det W_v \otimes \bigotimes_{\substack{v<0, \\ v \equiv \frac{n_j}{2} \pmod{n_j}}} (\det \overline{W}_v)^2, \\
L(\beta_j)_2 &= L'_2 \otimes \bigotimes_{v>0} (\det N_v)^{\left[\frac{p_j}{n_j}v\right] + (p-1)v+1} \otimes \bigotimes_{v>0} (\det \overline{V}_v)^{\left[\frac{p_j}{n_j}v + \frac{1}{2}\right] + (p-1)v} \\
&\otimes \bigotimes_{\substack{v>0, \\ v \equiv 0 \pmod{n_j}}} (\det N_v)^{-1} \otimes \bigotimes_{v<0} (\det W_v)^{\left[-\frac{p_j}{n_j}v\right] + 2\left[-\frac{p_j}{n_j}v + \frac{1}{2}\right] - 3(p-1)v} \\
&\otimes \bigotimes_{v>0} (\det \overline{W}_v)^{\left[\frac{p_j}{n_j}v\right] + 2\left[\frac{p_j}{n_j}v + \frac{1}{2}\right] + 3(p-1)v+1} \\
&\otimes \bigotimes_{\substack{v>0, \\ v \equiv 0 \pmod{n_j}}} \det W_v \otimes \bigotimes_{\substack{v<0, \\ v \equiv \frac{n_j}{2} \pmod{n_j}}} (\det \overline{W}_v)^2.
\end{aligned}$$

Then $L(\beta_j)_1$ and $L(\beta_j)_2$ can be extended naturally to $G_y \times S^1$ -equivariant complex line bundles over $M(n_j)$ which we will still denote by $L(\beta_j)_1$ and $L(\beta_j)_2$ respectively.

Proof We introduce the following line bundle over $M(n_j)$,

$$\begin{aligned}
L^\omega(\beta_j) &= \bigotimes_{0 < v < n_j/2} \left(\det(N(n_j)_v) \otimes \det(\overline{V}(n_j)_v) \right. \\
&\quad \left. \otimes (\det(\overline{W}(n_j)_v) \otimes \det(W(n_j)_{n_j-v}))^3 \right)^{-\omega(v) - r(n_j)v}.
\end{aligned} \tag{3.48}$$

where as in [15, (4.35)] we write $\left[\frac{p_j}{n_j}v\right] = \frac{p_j}{n_j}v - \frac{\omega(v)}{n_j}$.

As in [15, (4.38)] and [14, (4.28)], Lemma 3.5 implies that $L^\omega(\beta_j)^{1/n_j}$ is well-defined over $M(n_j)$. Direct calculation shows that

$$\begin{aligned}
L(\beta_j)_1 &= L^{-(p-1) - \frac{p_j}{n_j}} \otimes L^\omega(\beta_j)^{\frac{1}{n_j}} \otimes \bigotimes_{0 < v < \frac{n_j}{2}} \det(\overline{W}(n_j)_v) \otimes (\det(\overline{W}(n_j)_{\frac{n_j}{2}}))^2 \\
&\otimes \bigotimes_{1 \leq m \leq \frac{p_j}{2}} \bigotimes_{m - \frac{1}{2} < \frac{p_j}{n_j}v < m} (\det(\overline{W}(n_j)_v) \otimes \det(W(n_j)_{n_j-v}))^2,
\end{aligned}$$

$$\begin{aligned}
L(\beta_j)_2 &= L^{-(p-1)-\frac{p_j}{n_j}} \otimes L^\omega(\beta_j)^{\frac{1}{n_j}} \otimes \bigotimes_{0 < v < \frac{n_j}{2}} \det(\overline{W(n_j)}_v) \otimes (\det(\overline{W(n_j)}_{\frac{n_j}{2}}))^2 \\
&\otimes \bigotimes_{1 \leq m \leq \frac{p_j}{2}} \bigotimes_{m-\frac{1}{2} < \frac{p_j}{n_j} v < m} \left((\det(\overline{W(n_j)}_v) \otimes \det(W(n_j)_{n_j-v}))^2 \otimes \det(\overline{V(n_j)}_v) \right).
\end{aligned}$$

The proof of Lemma 3.6 is completed.

To simplify the notations, we introduce the following locally constant functions on F (compare with [15, (4.45)], [14, (4.30)]),

$$\begin{aligned}
\varepsilon_W^1 &= -\frac{1}{2} \sum_{v>0} (\dim W_v) \cdot \left(([\frac{p_j}{n_j}v] + (p-1)v)([\frac{p_j}{n_j}v] + (p-1)v + 1) \right. \\
&\quad \left. - (\frac{p_j}{n_j}v + (p-1)v)(2([\frac{p_j}{n_j}v] + (p-1)v + 1)) \right) \\
&\quad - \frac{1}{2} \sum_{v<0} (\dim W_v) \cdot \left(([-\frac{p_j}{n_j}v] - (p-1)v)([-\frac{p_j}{n_j}v] - (p-1)v + 1) \right. \\
&\quad \left. + (\frac{p_j}{n_j}v + (p-1)v)(2([-\frac{p_j}{n_j}v] - (p-1)v + 1)) \right), \tag{3.49}
\end{aligned}$$

$$\begin{aligned}
\varepsilon_W^2 &= -\frac{1}{2} \sum_{v>0} (\dim W_v) \cdot \left(([\frac{p_j}{n_j}v + \frac{1}{2}] + (p-1)v)^2 \right. \\
&\quad \left. - 2(\frac{p_j}{n_j}v + (p-1)v)([\frac{p_j}{n_j}v + \frac{1}{2}] + (p-1)v) \right) \\
&\quad - \frac{1}{2} \sum_{v<0} (\dim W_v) \cdot \left(([-\frac{p_j}{n_j}v + \frac{1}{2}] - (p-1)v)^2 \right. \\
&\quad \left. + 2(\frac{p_j}{n_j}v + (p-1)v)([-\frac{p_j}{n_j}v + \frac{1}{2}] - (p-1)v) \right), \tag{3.50}
\end{aligned}$$

$$\begin{aligned}
\varepsilon_1 &= \frac{1}{2} \sum_{v>0} (\dim N_v - \dim V_v) \left(([\frac{p_j}{n_j}v] + (p-1)v)([\frac{p_j}{n_j}v] + (p-1)v + 1) \right. \\
&\quad \left. - (\frac{p_j}{n_j}v + (p-1)v)(2([\frac{p_j}{n_j}v] + (p-1)v + 1)) \right), \tag{3.51}
\end{aligned}$$

$$\begin{aligned}
\varepsilon_2 &= \frac{1}{2} \sum_{v>0} (\dim N_v) \cdot \left(([\frac{p_j}{n_j}v] + (p-1)v)([\frac{p_j}{n_j}v] + (p-1)v + 1) \right. \\
&\quad \left. - (\frac{p_j}{n_j}v + (p-1)v)(2([\frac{p_j}{n_j}v] + (p-1)v + 1)) \right) \\
&\quad - \frac{1}{2} \sum_{v>0} (\dim V_v) \cdot \left(([\frac{p_j}{n_j} + \frac{1}{2}] + (p-1)v)^2 \right. \\
&\quad \left. - 2(\frac{p_j}{n_j}v + (p-1)v)([\frac{p_j}{n_j} + \frac{1}{2}] + (p-1)v) \right). \tag{3.52}
\end{aligned}$$

As in [14, (2.23)], for $0 \leq j \leq J_0$, we set

$$\begin{aligned}
e(p, \beta_j, N) &= \frac{1}{2} \sum_{v>0} (\dim N_v) \cdot ([\frac{p_j}{n_j}v] + (p-1)v)([\frac{p_j}{n_j}v] + (p-1)v + 1), \\
d'(p, \beta_j, N) &= \sum_{v>0} (\dim N_v) \cdot ([\frac{p_j}{n_j}v] + (p-1)v). \tag{3.53}
\end{aligned}$$

Then $e(p, \beta_j, N)$ and $d'(p, \beta_j, N)$ are locally constant functions on F . In particular, we have

$$\begin{aligned} e(p, \beta_0, N) &= \frac{1}{2}(p-1)^2 e(N) + \frac{1}{2}(p-1) d'(N) , \\ e(p, \beta_{J_0}, N) &= \frac{1}{2} p^2 e(N) + \frac{1}{2} p d'(N) , \\ d'(p, \beta_{J_0}, N) &= d'(p+1, \beta_0, N) = p d'(N) . \end{aligned} \tag{3.54}$$

Proposition 3.7 (Compare with [14, Proposition 4.2]) *For $i = 1, 2$, the G_y -equivariant isomorphisms of complex vector bundles over F induced by Proposition 3.4 and (3.45)-(3.46),*

$$\begin{aligned} r_{i1} : S(TY, K_X \otimes_{v>0} (\det N_v)^{-1}) \otimes (K_W \otimes K_X^{-1})^{1/2} \\ \otimes \mathcal{F}_{p,j-1}(X) \otimes F_V^i \otimes Q(W) \\ \longrightarrow S(U_i, L_i)' \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes \mathcal{F}(\beta_j) \otimes F_V^i(\beta_j) \\ \otimes Q_W(\beta_j) \otimes L(\beta_j)_i \otimes \bigotimes_{\substack{v>0, \\ v \equiv 0 \pmod{n_j}}} \text{Sym}(\bar{N}_{v,0}) , \\ \\ r_{i2} : S(TY, K_X \otimes_{v>0} (\det N_v)^{-1}) \otimes (K_W \otimes K_X^{-1})^{1/2} \\ \otimes \mathcal{F}_{p,j}(X) \otimes F_V^i \otimes Q(W) \\ \longrightarrow S(U_i, L_i)' \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes \mathcal{F}(\beta_j) \otimes F_V^i(\beta_j) \\ \otimes Q_W(\beta_j) \otimes L(\beta_j)_i \otimes \bigotimes_{\substack{v>0, \\ v \equiv 0 \pmod{n_j}}} \left(\text{Sym}(N_{v,0}) \otimes \det N_v \right) , \end{aligned}$$

have the following properties:

(i) for $i = 1, 2$, $\gamma = 1, 2$,

$$\begin{aligned} r_{i\gamma}^{-1} \cdot \mathbf{J}_H \cdot r_{i\gamma} &= \mathbf{J}_H , \\ r_{i\gamma}^{-1} \cdot P \cdot r_{i\gamma} &= P + \left(\frac{pj}{n_j} + (p-1) \right) \mathbf{J}_H + \varepsilon_{i\gamma} , \end{aligned} \tag{3.55}$$

where $\varepsilon_{i\gamma}$ are given by

$$\begin{aligned} \varepsilon_{i1} &= \varepsilon_i + \varepsilon_W^1 + 2\varepsilon_W^2 - e(p, \beta_{j-1}, N) , \\ \varepsilon_{i2} &= \varepsilon_i + \varepsilon_W^1 + 2\varepsilon_W^2 - e(p, \beta_j, N) . \end{aligned} \tag{3.56}$$

(ii) for $i = 1, 2$, $\gamma = 1, 2$,

$$\begin{aligned} r_{i\gamma}^{-1} \tau_e r_{i\gamma} &= (-1)^{\mu_i} \tau_e , \quad r_{i\gamma}^{-1} \tau_s r_{i\gamma} = (-1)^{\mu_3} \tau_s , \\ r_{i\gamma}^{-1} \tau_1 r_{i\gamma} &= (-1)^{\mu_4} \tau_1 , \end{aligned} \tag{3.57}$$

where μ_i are given by

$$\begin{aligned}
\mu_1 &= - \sum_{v>0} (\dim V_v) \left[\frac{p_j}{n_j} v \right] + \Delta(n_j, N) + \Delta(n_j, V) + \Delta(n_j, W) \pmod{2}, \\
\mu_2 &= - \sum_{v>0} (\dim V_v) \cdot \left[\frac{p_j}{n_j} v + \frac{1}{2} \right] + \Delta(n_j, N) \\
&\quad + o(V(n_j) \frac{\mathbb{R}}{n_j}) + \Delta(n_j, W) \pmod{2}, \\
\mu_3 &= \Delta(n_j, N) + \Delta(n_j, W) \pmod{2}, \\
\mu_4 &= \sum_{v>0} (\dim W_v) \cdot \left(\left[\frac{p_j}{n_j} v + \frac{1}{2} \right] + (p-1)v \right) \\
&\quad + \sum_{v<0} (\dim W_v) \cdot \left(\left[-\frac{p_j}{n_j} v + \frac{1}{2} \right] - (p-1)v \right) \pmod{2}.
\end{aligned}$$

Proof By the proof of [14, Proposition 4.2], we need to compute the action of $r_*^{-1} \cdot P \cdot r_*$ on $\bigotimes_{\substack{0 < n \in \mathbb{Z} + \frac{1}{2}, v > 0, \\ n - (p-1)v - \frac{p_j}{n_j} v \leq 0}} \Lambda^{i_n}(\overline{W}_{v,n}) \otimes \bigotimes_{\substack{0 < n \in \mathbb{Z} + \frac{1}{2}, v < 0, \\ n + (p-1)v + \frac{p_j}{n_j} v \leq 0}} \Lambda^{i'_n}(W_{v,n})$. In fact, by (3.26), as in (3.16), we get

$$\begin{aligned}
r_*^{-1} \cdot P \cdot r_* &= \sum_{\substack{0 < n \in \mathbb{Z} + \frac{1}{2}, v > 0, \\ n - (p-1)v - \frac{p_j}{n_j} v \leq 0}} (\dim W_v - i_n)(-n + (p-1)v + \frac{p_j}{n_j} v) \\
&\quad + \sum_{\substack{0 < n \in \mathbb{Z} + \frac{1}{2}, v < 0, \\ n + (p-1)v + \frac{p_j}{n_j} v \leq 0}} (\dim W_v - i'_n)(-n - (p-1)v - \frac{p_j}{n_j} v) \quad (3.58) \\
&= P + (p-1 + \frac{p_j}{n_j}) \mathbf{J}_H + \varepsilon_W^2.
\end{aligned}$$

By [14, (4.36)-(4.38)] and (3.58), we deduce the second line of (3.55). The first line of (3.55) is obvious.

Consider the \mathbb{Z}_2 -gradings. By [15, (4.49)-(4.50)] and the discussion following (3.47), we get the identities in the first line of (3.57). Observe that τ_1 changes only on $\bigotimes_{\substack{0 < n \in \mathbb{Z} + \frac{1}{2}, v > 0, \\ n - (p-1)v - \frac{p_j}{n_j} v \leq 0}} \Lambda^{i_n}(\overline{W}_{v,n}) \otimes \bigotimes_{\substack{0 < n \in \mathbb{Z} + \frac{1}{2}, v < 0, \\ n + (p-1)v + \frac{p_j}{n_j} v \leq 0}} \Lambda^{i'_n}(W_{v,n})$. From (3.26),

we get the identity in the second line of (3.57).

The proof of Proposition 3.7 is completed.

3.4 A proof of Theorem 2.8

Lemma 3.8 (Compare with [14, Lemmas 4.4 and 4.6]) *For each connected component M' of $M(n_j)$, the following functions are independent on the connected components of F in M' ,*

$$\begin{aligned}
\varepsilon_i + \varepsilon_W^1 + 2\varepsilon_W^2, \quad i &= 1, 2, \\
d'(p, \beta_j, N) + \mu_i + \mu_4 \pmod{2}, \quad i &= 1, 2, 3, \\
d'(p, \beta_{j-1}, N) + \sum_{0 < v} \dim N_v + \mu_i + \mu_4 \pmod{2}, \quad i &= 1, 2, 3.
\end{aligned} \quad (3.59)$$

Proof Recall that $[\frac{p_j}{n_j}v] = \frac{p_j}{n_j}v - \frac{\omega(v)}{n_j}$. By using (3.30), we express ε_W^1 and ε_W^2 defined in (3.49)-(3.50) explicitly as follows,

$$\begin{aligned} \varepsilon_W^1 &= \frac{1}{2}(p-1 + \frac{p_j}{n_j})^2 e(W) + \frac{1}{8} \dim W(n_j)_{\frac{n_j}{2}} \\ &\quad + \frac{1}{2} \sum_{0 < v < \frac{n_j}{2}} \frac{\omega(v)\omega(-v)}{n_j^2} (\dim W(n_j)_v + \dim W(n_j)_{n_j-v}), \end{aligned} \quad (3.60)$$

$$\begin{aligned} \varepsilon_W^2 &= \frac{1}{2}(p-1 + \frac{p_j}{n_j})^2 e(W) - \frac{1}{8} \dim W(n_j)_{\frac{n_j}{2}} \\ &\quad - \frac{1}{2} \sum_{0 \leq m \leq \frac{p_j-1}{2}} \sum_{m < \frac{p_j}{n_j}v < m + \frac{1}{2}} \left(\frac{\omega(v)}{n_j} \right)^2 (\dim W(n_j)_v + \dim W(n_j)_{n_j-v}) \\ &\quad - \frac{1}{2} \sum_{0 \leq m \leq \frac{p_j}{2}} \sum_{m - \frac{1}{2} < \frac{p_j}{n_j}v < m} \left(\frac{\omega(-v)}{n_j} \right)^2 (\dim W(n_j)_v + \dim W(n_j)_{n_j-v}). \end{aligned} \quad (3.61)$$

By using (2.23), (3.60), (3.61) and the explicit expressions of ε_i given in [15, (4.56)-(4.57)], we know the functions in the first line of (3.59) are independent on the connected components of F in M' .

Now consider the functions in the rest lines of (3.59). By (2.30), (3.29), (3.31), (3.47) and [14, Lemma 4.5], we get

$$\begin{aligned} d'(p, \beta_j, N) + \mu_i + \mu_4 &\equiv \sum_{0 < m \leq \frac{p_j}{2}} \sum_{\substack{0 < v < \frac{n_j}{2} \\ m - \frac{1}{2} < \frac{p_j}{n_j}v < m}} \dim N(n_j)_v + \frac{1}{2} \dim_{\mathbb{R}} N(n_j)_{\frac{n_j}{2}} \\ &\quad + \sum_{v > 0} (\dim N_v) \left[\frac{p_j}{n_j}v + \frac{1}{2} \right] + \sum_{v > 0} (\dim W_v) \left[\frac{p_j}{n_j}v + \frac{1}{2} \right] \\ &\quad + \sum_{v < 0} (\dim W_v) \left[-\frac{p_j}{n_j}v + \frac{1}{2} \right] + o\left(N(n_j)_{\frac{n_j}{2}}^{\mathbb{R}}\right) + \Delta(n_j, W) \pmod{2}. \end{aligned} \quad (3.62)$$

But by [14, Lemma 4.5], as $w_2(W \oplus TX)_{S^1} = 0$, we know that, mod(2),

$$\begin{aligned} &\sum_{v > 0} (\dim N_v) \left[\frac{p_j}{n_j}v + \frac{1}{2} \right] + \sum_{v > 0} (\dim W_v) \left[\frac{p_j}{n_j}v + \frac{1}{2} \right] \\ &\quad + \sum_{v < 0} (\dim W_v) \left[-\frac{p_j}{n_j}v + \frac{1}{2} \right] + o\left(N(n_j)_{\frac{n_j}{2}}^{\mathbb{R}}\right) + \Delta(n_j, W) \end{aligned} \quad (3.63)$$

is independent on the connected components of F in M' . Thus, the independence on the connected components of F in M' of the functions in the second line of (3.59) is proved, which, combining with [14, (4.42)], implies the the same independent property of the functions in the third line of (3.59).

The proof of Lemma 3.8 is completed.

By (3.33)-(3.38) and Lemma 3.6, we know that the Dirac operator $D^{X(n_j)} \otimes \mathcal{F}(\beta_j) \otimes F_V^i(\beta_j) \otimes Q_W(\beta_j) \otimes L(\beta_j)_i$ ($i = 1, 2$) is well-defined on $M(n_j)$. Observe that (2.12) in Theorem 2.1 is compatible with the G_y action. Thus, by using

Proposition 3.7, Lemma 3.8 and applying Theorem 2.1 to each connected component of $M(n_j)$ separately, we deduce that for $i = 1, 2$, $1 \leq j \leq J_0$, $m \in \frac{1}{2}\mathbb{Z}$, $h \in \mathbb{Z}$, $\tau = \tau_{e1}$ or τ_{s1} ,

$$\begin{aligned}
& \sum_{\alpha} (-1)^{d'(p, \beta_{j-1}, N) + \sum_{v>0} \dim N_v} \text{Ind}_{\tau} \left(D^{Y_{\alpha}} \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes \mathcal{F}_{p, j-1}(X) \right. \\
& \quad \left. \otimes F_V^i \otimes Q(W), m + e(p, \beta_{j-1}, N), h \right) \\
&= \sum_{\beta} (-1)^{d'(p, \beta_{j-1}, N) + \sum_{v>0} \dim N_v + \mu} \text{Ind}_{\tau} \left(D^{X(n_j)} \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes \mathcal{F}(\beta_j) \right. \\
& \quad \left. \otimes F_V^i(\beta_j) \otimes Q_W(\beta_j) \otimes L(\beta_j)_i, m + \varepsilon_i + \varepsilon_W^1 + 2\varepsilon_W^2 + \left(\frac{p_j}{n_j} + (p-1)\right)h, h \right) \\
&= \sum_{\alpha} (-1)^{d'(p, \beta_j, N) + \sum_{v>0} \dim N_v} \text{Ind}_{\tau} \left(D^{Y_{\alpha}} \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes \mathcal{F}_{p, j}(X) \right. \\
& \quad \left. \otimes F_V^i \otimes Q(W), m + e(p, \beta_j, N), h \right), \tag{3.64}
\end{aligned}$$

where \sum_{β} means the sum over all the connected components of $M(n_j)$. In (3.64), if $\tau = \tau_{s1}$, then $\mu = \mu_3 + \mu_4$; if $\tau = \tau_{e1}$, then $\mu = \mu_i + \mu_4$. Combining (3.54) with (3.64), we get (2.28).

The proof of Theorem 2.8 is completed.

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