# Rigidity and Vanishing Theorems on $\mathbb{Z}/k$ Spin$^c$ manifolds

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**Abstract**

In this paper, we first establish an $S^1$-equivariant index theorem for Dirac operators on $\mathbb{Z}/k$ Spin$^c$ manifolds, then combining with the methods developed by Taubes [24] and Liu-Ma-Zhang [19, 20], we extend Witten's rigidity theorem to the case of $\mathbb{Z}/k$ Spin$^c$ manifolds. Among others, our results resolve a conjecture of Devoto [6].

## 1 Introduction

In [25], Witten derived a series of elliptic operators on the free loop space $\mathcal{L}M$ of a spin manifold $M$. In particular, the index of the formal signature operator on loop space turns out to be exactly the elliptic genus constructed by Landweber-Stong [13] and Ochanine [23] in a topological way. Motivated by physics, Witten conjectured that these elliptic operators should be rigid with respect to the circle action.

This conjecture was first proved by Taubes [24] and Bott-Taubes [4]. See also [10] and [12] for other interesting cases. By the modular invariance property, Liu ([15, 16]) presented a simple and unified proof of the above conjecture as well as various further generalizations. In particular, several new vanishing theorems were established in [15, 16]. Furthermore, on the equivariant Chern character level, Liu and Ma ([17, 18]) generalized Witten's rigidity theorem to the family case, and also obtained several vanishing theorems for elliptic genera. In [19, 20], inspired by [24], Liu, Ma and Zhang established the corresponding family rigidity and vanishing theorems on the equivariant $K$-theory level.

In [27], Zhang established an equivariant index theorem for circle actions on $\mathbb{Z}/k$ spin manifolds and pointed out that by combining with the analytic arguments developed in [20], one can prove an extension of Witten's rigidity theorem to $\mathbb{Z}/k$ spin manifolds. The purpose of this paper is to extend the result of [27] to $\mathbb{Z}/k$ Spin$^c$ manifolds and then establish Witten's rigidity theorem for $\mathbb{Z}/k$ Spin$^c$ manifolds. Recall that a $\mathbb{Z}/k$-manifold $X$ is a smooth manifold with boundary $\partial X$ which consists of $k$ disjoint pieces, each of which is diffeomorphic to a given closed manifold $Y$ (cf. [22]). It is interesting that for a

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Dirac operator $D$ on a $\mathbb{Z}/k$-manifold, the APS-ind$(D)$ mod $k\mathbb{Z}$ determines a topological invariant in $\mathbb{Z}/k\mathbb{Z}$, where APS-ind$(D)$ is the index of $D$ which is imposed the boundary condition of Atiyah-Patodi-Singer type [1]. Freed and Melrose [7] proved a mod $k$ index theorem,

$$\text{APS-ind}(D) \bmod k\mathbb{Z} = \text{t-ind}(D)\,, \tag{1.1}$$

giving the APS-ind$(D)$ mod $k\mathbb{Z}$ a purely topological interpretation.

Assume that $X$ is a $\mathbb{Z}/k$ manifold which admits a $\mathbb{Z}/k$ circle action (cf. Section 2.2). Let $D$ be a Dirac operator on $X$ which commutes with the circle action. Let $R(S^1)$ denote the representation ring of $S^1$. The equivariant topological index of $D$ is defined by Freed and Melrose [7] as an element of $\mathbb{Z}/k\mathbb{Z} \otimes R(S^1)$, and we denote it by t-ind$_{S^1}(D)$. Then there exist $R_n \in \mathbb{Z}/k\mathbb{Z}$ such that

$$\text{t-ind}_{S^1}(D) = \sum_{n\in\mathbb{Z}} R_n \otimes [n]\,, \tag{1.2}$$

where by $[n]$ ($n \in \mathbb{Z}$) we mean the one dimensional complex vector space on which $S^1$ acts as multiplication by $g^n$ for a generator $g \in S^1$.

On the other hand, by applying the equivariant index theorem for $\mathbb{Z}/k$-manifolds established by Freed and Melrose in [7], one gets for $n \in \mathbb{Z}$,

$$R_n = \text{APS-ind}(D, n) \bmod k\mathbb{Z}\,. \tag{1.3}$$

See (2.9) for the definition of APS-ind$(D, n)$.

The Dirac operator $D$ on $X$ is said to be *rigid in $\mathbb{Z}/k$ category* for the circle action if its equivariant topological index t-ind$_{S^1}(D)$ verifies that for $n \in \mathbb{Z}$, $n \neq 0$, one has

$$R_n = 0 \quad \text{in } \mathbb{Z}/k\mathbb{Z}. \tag{1.4}$$

Furthermore, we say $D$ has *vanishing property in $\mathbb{Z}/k$ category* if its equivariant topological index t-ind$_{S^1}(D)$ is identically zero, i.e., (1.4) holds for any $n \in \mathbb{Z}$.

In [6], Devoto introduced what he called mod $k$ elliptic genus for $\mathbb{Z}/k$ spin manifolds as an $S^1$-equivariant topological index in the sense of [7] of some twisted Dirac operator and conjectured that this mod $k$ elliptic genus is *rigid in $\mathbb{Z}/k$ category.* In this paper, following the suggestion in [27, Remark 1], we present a proof of Devoto's conjecture. Moreover, we establish our results for $\mathbb{Z}/k$ Spin$^c$ manifolds, thus generalizing [16, Theorems A and B] to the case of $\mathbb{Z}/k$ Spin$^c$ manifolds.

Our proof of these rigidity results consists of two steps. In step 1 (Sections 2 and 3), we extend the $\mathbb{Z}/k$ equivariant index theorem of Zhang [27] to the Spin$^c$ case. In step 2 (Sections 4 and 5), using the mod $k$ localization index theorem established in step 1 and modifying the process in [19, 20], we prove the main results of this paper.

This paper is organized as follows. In Section 2, we state an $S^1$-equivariant index theorem for Dirac operators on $\mathbb{Z}/k$ Spin$^c$ manifolds (cf. Theorem 2.7). As an application, we extend Hattori's vanishing theorem [8] to the case of $\mathbb{Z}/k$ almost complex manifolds. In Section 3, we prove the $S^1$-equivariant index theorem stated in Section 2. In Section 4, we prove our main results (cf. Theorem 4.1), the rigidity and vanishing theorems for $\mathbb{Z}/k$ Spin$^c$ manifolds, which generalize [16, Theorems A and B]. When applied to $\mathbb{Z}/k$ spin manifolds, our results resolve a conjecture of Devoto [6]. Section 5 is devoted to a proof of the recursive formula which has been used in Section 4 in the proof of our main results.

# 2 Spin$^c$-Dirac operators and a mod $k$ localization formula

In this section, for a $\mathbb{Z}/k$ Spin$^c$ manifold which admits a nontrivial $\mathbb{Z}/k$ circle action, we state a mod $k$ localization formula, whose proof will be given in Section 3. As an application, we deduce the rigidity and vanishing property for several Dirac operators on a $\mathbb{Z}/k$ almost complex manifold. In particular, we extend Hattori's vanishing theorem [8] to the case of $\mathbb{Z}/k$ almost complex manifolds.

This section is organized as follows. In Section 2.1, we review the construction of Dirac operators on $\mathbb{Z}/k$ Spin$^c$ manifolds and the Atiyah-Patodi-Singer boundary problems. In Section 2.2, we recall the circle actions on $\mathbb{Z}/k$ manifolds and present a variation formula for the indices of these boundary problems. In Section 2.3, we state a mod $k$ localization formula for $\mathbb{Z}/k$ circle actions. As an application, in Section 2.4, we extend Hattori's vanishing theorem [8] to the case of $\mathbb{Z}/k$ almost complex manifolds.

## 2.1 Spin$^c$-Dirac operators on $\mathbb{Z}/k$-manifolds

We first recall the definition of $\mathbb{Z}/k$-manifolds introduced by Morgan and Sullivan (cf. [22]).

**Definition 2.1** (cf. [27, Definition 1.1]) *A compact $\mathbb{Z}/k$ manifold is a compact manifold $X$ with boundary $\partial X$, which admits a decomposition $\partial X = \cup_{i=1}^{k}(\partial X)_i$ into $k$ disjoint manifolds and $k$ diffeomorphisms $\pi_i : (\partial X)_i \to Y$ to a closed manifold $Y$.*

Let $\pi : \partial X \to Y$ be the induced map. In what follows, as in [27], we will call an object $\alpha$ (e.g., metrics, connections, etc.) of $X$ a $\mathbb{Z}/k$-object if there will be a corresponding object $\beta$ on $Y$ such that $\alpha|_{\partial X} = \pi^*\beta$.

We point out here that in this paper when consider the topological objects (e.g., cohomology, characteristic classes, $K$ group, etc.) on a $\mathbb{Z}/k$-manifold $X$, we always regard $X$ as a quotient space obtained by identifying each of the $k$ disjoint pieces of the boundary $\partial X$. Then $X$ has the homotopy type of a CW complex, which implies that the first Chern class $c_1$ induces a 1-to-1 correspondence between the equivalence classes of the complex line bundles over $X$ and the elements of $H^2(X;\mathbb{Z})$. As will be seen, this is essential in our proof.

We make the assumption that $X$ is $\mathbb{Z}/k$ oriented and of dimension $2l$.

Let $V$ be a $\mathbb{Z}/k$ real vector bundle over $X$ which is of dimension $2p$ and $\mathbb{Z}/k$ oriented. Let $L$ be a $\mathbb{Z}/k$ complex line bundle over $X$ with the property that the vector bundle $U = TX \oplus V$ satisfies $\omega_2(U) = c_1(L) \mod (2)$, where $\omega_2$

denotes the second Stiefel-Whitney class, and $c_1$ denotes the first Chern class. Then the $\mathbb{Z}/k$ vector bundle $U$ has a $\mathbb{Z}/k$ Spin$^c$-structure.

Let $g^{TX}$ be a $\mathbb{Z}/k$ Riemannian metric on $X$. Let $g^{T\partial X}$ be its restriction on $T\partial X$. Let $\epsilon_0 > 0$ be less than the injectivity radius of $g^{TX}$. We use the inward geodesic flow to identify a neighborhood of the boundary with the collar $[0, \epsilon_0) \times \partial X$. We assume that $g^{TX}$ is of product structure near $\partial X$. That is, there is an open neighborhood $\mathscr{U}_\epsilon = [0, \epsilon) \times \partial X$ of $\partial X$ in $X$ with $0 < \epsilon \leq \epsilon_0$ such that one has the orthogonal splitting on $\mathscr{U}_\epsilon$,

$$g^{TX}|_{\mathscr{U}_\epsilon} = dr^2 \oplus \pi_\epsilon^* g^{T\partial X}, \tag{2.1}$$

where $\pi_\epsilon : [0, \epsilon) \times \partial X \to \partial X$ is the obvious projection onto the second factor.

Let $\nabla^{TX}$ be the Levi-Civita connection on $(TX, g^{TX})$. Then $\nabla^{TX}$ is a $\mathbb{Z}/k$ connection.

Let $W$ be a $\mathbb{Z}/k$ complex vector bundle over $X$ with a $\mathbb{Z}/k$ Hermitian metric $g^W$. Let $\nabla^W$ be a $\mathbb{Z}/k$ Hermitian connection on $W$ with respect to $g^W$. We make the assumption that $g^W$ and $\nabla^W$ are both of product structure near $\partial X$. That is, over the open neighborhood $\mathscr{U}_\epsilon$ of $\partial X$, one has

$$g^W|_{\mathscr{U}_\epsilon} = \pi_\epsilon^*(g^W|_{\partial X}), \quad \text{and} \quad \nabla^W|_{\mathscr{U}_\epsilon} = \pi_\epsilon^*(\nabla^W|_{\partial X}). \tag{2.2}$$

Let $g^V$ (resp. $g^L$) be a $\mathbb{Z}/k$ Euclidean (resp. Hermitian) metric on $V$ (resp. $L$), and $\nabla^V$ (resp. $\nabla^L$) be a corresponding $\mathbb{Z}/k$ Euclidean (resp. Hermitian) connection on $V$ (resp. $L$). We make the assumption that $g^V$, $\nabla^V$, $g^L$, $\nabla^L$ are of product structure near $\partial X$ (cf. (2.2)).

By taking $\epsilon > 0$ sufficiently small, one can always find the metrics $g^{TX}$, $g^W$, $g^V$, $g^L$ and the connections $\nabla^W$, $\nabla^V$, $\nabla^L$ verifying the above assumptions.

The Clifford algebra bundle $C(TX)$ is the bundle of Clifford algebras over $X$ whose fibre at $x \in X$ is the Clifford algebra $C(T_xX)$ (cf. [14]). Let $C(V)$ be the Clifford algebra bundle of $(V, g^V)$.

Let $S(U, L)$ be the fundamental complex spinor bundle for $(U, L)$ (cf. [14, Appendix D]). We denote by $c(\cdot)$ the Clifford action of $C(TX)$, $C(V)$ on $S(U, L)$. Let $\{e_i\}_{i=1}^{2l}$ (resp. $\{f_j\}_{j=1}^{2p}$) be an oriented orthonormal basis of $(TX, g^{TX})$ (resp. $(V, g^V)$). There are two canonical ways to consider $S(U, L)$ as a $\mathbb{Z}_2$-graded vector bundle. Let

$$\begin{aligned} \tau_s &= (\sqrt{-1})^l c(e_1) \cdots c(e_{2l}), \\ \tau_e &= (\sqrt{-1})^{l+p} c(e_1) \cdots c(e_{2l}) c(f_1) \cdots c(f_{2p}) \end{aligned} \tag{2.3}$$

be two involutions of $S(U, L)$. Then $\tau_s^2 = \tau_e^2 = 1$. We decompose $S(U, L) = S_+(U, L) \oplus S_-(U, L)$ corresponding to $\tau_s$ (resp. $\tau_e$) such that $\tau_s|_{S_\pm(U,L)} = \pm 1$ (resp. $\tau_e|_{S_\pm(U,L)} = \pm 1$).

In the remaining part of this paper, we always fix an involution $\tau$ on $S(U, L)$, either $\tau_s$ or $\tau_e$, without further notice.

Let $\nabla^{S(U,L)}$ be the Hermitian connection on $S(U, L)$ induced by $\nabla^{TX} \oplus \nabla^V$ and $\nabla^L$ (cf. [14, Appendix D]). Then $\nabla^{S(U,L)}$ preserves the $\mathbb{Z}_2$-grading of $S(U, L)$. Let $\nabla^{S(U,L) \otimes W}$ be the Hermitian connection on $S(U, L) \otimes W$ obtained from the tensor product of $\nabla^{S(U,L)}$ and $\nabla^W$.

**Definition 2.2** *The twisted Spin$^c$ Dirac operator $D^X$ on $S(U,L)\otimes W$ over $X$ is defined by*

$$D^X = \sum_{i=1}^{2l} c(e_i)\nabla_{e_i}^{S(U,L)\otimes W} : \Gamma(X, S(U,L)\otimes W) \longrightarrow \Gamma(X, S(U,L)\otimes W). \quad (2.4)$$

*Denote by $D_{\pm}^X$ the restrictions of $D^X$ on $\Gamma(X, S_{\pm}(U,L)\otimes W)$.*

By [14], $D^X$ is a formally self-adjoint operator. To get an elliptic operator, we impose the boundary condition of Atiyah-Patodi-Singer type [1].

We first recall the canonical boundary operators (cf. [5, (1.4)]). For a first order differential operator $D : \Gamma(S(U,L)\otimes W) \longrightarrow \Gamma(S(U,L)\otimes W)$ on $X$, if there exists $\epsilon > 0$ sufficient small such that the following identity holds on $\mathscr{U}_\epsilon$,

$$D = c\left(\frac{\partial}{\partial r}\right)\left(\frac{\partial}{\partial r} + B\right), \quad (2.5)$$

with $B$ independent of $r$ and its restriction on $\Gamma(X, S(U,L)\otimes W)|_{\partial X}$ formally self-adjoint, then we will call $B$ the canonical boundary operator associated to $D$. When there is no confusion, we will also use $B$ to denote its restriction on $\Gamma(X, S(U,L)\otimes W)|_{\partial X}$.

We then recall the Atiyah-Patodi-Singer projection associated to a boundary operator (cf. [1]). For a first order formally self-adjoint elliptic differential operator $B : \Gamma(X, S(U,L)\otimes W)|_{\partial X} \longrightarrow \Gamma(X, S(U,L)\otimes W)|_{\partial X}$ on $\partial X$, let $\mathrm{Spec}\,(B)$ be the spectrum of $B$. For any $\lambda \in \mathrm{Spec}\,(B)$, let $E_\lambda$ be the eigenspace corresponding to $\lambda$. For $a \in \mathbb{R}$, let $P_{\geq a}$ be the orthogonal projection from the $L^2$-completion of $\Gamma(X, S(U,L)\otimes W)|_{\partial X}$ onto $\oplus_{\lambda \geq a} E_\lambda$. If we assume in addition that $B$ preserves the $\mathbb{Z}_2$-grading of $\Gamma(X, S(U,L)\otimes W)|_{\partial X}$, and let $B_\pm$ be the restrictions of $B$ on $\Gamma(X, S_\pm(U,L)\otimes W)|_{\partial X}$, then we will restrict $P_{\geq a}$ on the $L^2$-completions of $\Gamma(X, S_\pm(U,L)\otimes W)|_{\partial X}$ and denote them by $P_{\geq a,\pm}$. We call the particular projection $P_{\geq 0}$ (resp. $P_{\geq 0,\pm}$) the Atiyah-Patodi-Singer projection associated to $B$ (resp. $B_\pm$) to emphasize its role in [1].

Let $e_1 = \frac{\partial}{\partial r}$ be the inward unit normal vector field perpendicular to $\partial X$. Let $e_2, \cdots, e_{2l}$ be an oriented orthonormal basis of $T\partial X$ so that $e_1, e_2, \cdots, e_{2l}$ is an oriented orthonormal basis of $TX|_{\partial X}$. Then using parallel transport with respect to $\nabla^{TX}$ along the unit speed geodesics perpendicular to $\partial X$, $e_1, e_2, \cdots, e_{2l}$ forms an oriented orthonormal basis of $TX$ over $\mathscr{U}_\epsilon$.

**Definition 2.3** *Let $B^X : \Gamma(X, S(U,L)\otimes W)|_{\partial X} \longrightarrow \Gamma(X, S(U,L)\otimes W)|_{\partial X}$ be the differential operator on $\partial X$ defined by*

$$B^X = -\sum_{i=2}^{2l} c\left(\frac{\partial}{\partial r}\right) c(e_i)\nabla_{e_i}^{S(U,L)\otimes W}. \quad (2.6)$$

*Let $B_\pm^X$ be the restrictions of $B^X$ to $\Gamma(X, S_\pm(U,L)\otimes W)|_{\partial X}$.*

By [1], $B^X$ is a formally self-adjoint first order elliptic differential operator intrinsically defined on $\partial X$, which is the canonical boundary operator associated to $D^X$ and preserves the natural $\mathbb{Z}_2$-grading of $(S(U,L)\otimes W)|_{\partial X}$.

We now recall the Dirac type operator [5, Definition 1.1] as well as the boundary condition of Atiyah-Patodi-Singer type [1].

**Definition 2.4** *By a Dirac type operator on $\Gamma(X, S(U,L)\otimes W)$, we mean a first order differential operator $D : \Gamma(X, S(U,L)\otimes W) \longrightarrow \Gamma(X, S(U,L)\otimes W)$ with the canonical boundary operator $B$, such that $D - D^X$ is an odd self-adjoint element of zeroth order. We will also call the restrictions $D_\pm$ of $D$ to $\Gamma(X, S_\pm(U,L)\otimes W)$ a Dirac type operator.*

Let now $D$ be a $\mathbb{Z}/k$ Dirac type operator with canonical boundary operator $B$. Obviously, $B$ preserves the $\mathbb{Z}_2$-grading of $\Gamma(X, S(U,L)\otimes W)|_{\partial X}$,

Following [1], the boundary problem

$$\begin{aligned}(D_+, P_{\geq 0,+}):\ &\left\{ s \,\middle|\, s\in\Gamma(X, S_+(U,L)\otimes W), P_{\geq 0,+}(s\big|_{\partial X}) = 0 \right\} \\ &\longrightarrow \Gamma(X, S_-(U,L)\otimes W),\end{aligned} \tag{2.7}$$

defines an elliptic boundary problem whose adjoint is $(D_-, P_{>0,-})$. Moreover, it induces a Fredholm operator [1]. We will call the boundary problem $(D_+, P_{\geq 0,+})$ the Atiyah-Patodi-Singer boundary problem associated to $D_+$. Set

$$\text{APS-ind}(D) = \dim\ker(D_+, P_{\geq 0,+}) - \dim\ker(D_-, P_{>0,-}). \tag{2.8}$$

## 2.2 $\mathbb{Z}/k$ circle actions and a variation formula

**Definition 2.5** *We will call a circle action on $X$ a $\mathbb{Z}/k$ circle action if it preserves $\partial X$ and there exists a corresponding circle action on $Y$ such that these two actions are compatible with $\pi$. The circle action is said to be nontrivial if it is not equal to identity.*

In what follows we assume that $X$ admits a nontrivial $\mathbb{Z}/k$ circle action preserving the orientation and that the $\mathbb{Z}/k$ circle action on $X$ lifts to $\mathbb{Z}/k$ circle actions on $V$, $L$ and $W$, respectively. Without loss of generality, we may and we will assume that these $\mathbb{Z}/k$ circle actions preserve $g^{TX}$, $g^V$, $g^L$, $g^W$, $\nabla^V$, $\nabla^L$ and $\nabla^W$, respectively. We also assume that the $\mathbb{Z}/k$ circle actions on $TX$, $V$ and $L$ lift to a $\mathbb{Z}/k$ circle action on $S(U,L)$ and preserve its $\mathbb{Z}_2$-grading.

Let $\mathscr{E}$ be a $\mathbb{Z}/k$ $S^1$-equivariant vector bundle over $X$. Let $\mathscr{E}_Y$ be the $S^1$-equivariant vector bundle over $Y$ induced from $\mathscr{E}$ through the map $\pi : \partial X \to Y$. Recall that the circle action on $\Gamma(X, \mathscr{E})$ is defined by $(g\cdot s)(x) = g(s(g^{-1}x))$ for $g\in S^1$, $s\in\Gamma(X,\mathscr{E})$, $x\in X$. Similarly, the group $S^1$ acts on $\Gamma(X,\mathscr{E})|_{\partial X}$ and $\Gamma(Y, \mathscr{E}_Y)$. For $\xi\in\mathbb{Z}$, by the weight-$\xi$ subspace of $\Gamma(X,\mathscr{E})$ (resp. $\Gamma(X,\mathscr{E})|_{\partial X}$, $\Gamma(Y,\mathscr{E}_Y)$), we mean the subspace of $\Gamma(X,\mathscr{E})$ (resp. $\Gamma(X,\mathscr{E})|_{\partial X}$, $\Gamma(Y,\mathscr{E}_Y)$) on which $S^1$ acts as multiplication by $g^\xi$ for $g\in S^1$.

For any $\xi\in\mathbb{Z}$, let $\mathrm{E}_\xi^\pm$ (resp. $\mathrm{E}_{\xi,\partial}^\pm$, $\mathrm{E}_{Y,\xi}^\pm$) be the weight-$\xi$ subspaces of $\Gamma(X, S_\pm(U,L)\otimes W)$ (resp. $\Gamma(X, S_\pm(U,L)\otimes W)|_{\partial X}$, $\Gamma(Y, (S(U,L)\otimes W)_Y)$).

Let $D$ be a $\mathbb{Z}/k$ $S^1$-equivariant Dirac type operator on $\Gamma(S(U,L)\otimes W)$ with canonical boundary operator $B$ acting on $\Gamma(X, S(U,L)\otimes W)|_{\partial X}$. Let $P_{\geq 0,+}$ be the Atiyah-Patodi-Singer projection associated to $B_+$. For $\xi\in\mathbb{Z}$, let $D_{\pm,\xi}$ and $P_{\geq 0,+,\xi}$ (resp. $P_{>0,-,\xi}$) be the restrictions of $D_\pm$ and $P_{\geq 0,+}$ (resp. $P_{>0,-}$) on the corresponding weight-$\xi$ subspaces $\mathrm{E}_\xi^\pm$ and $\mathrm{E}_{\xi,\partial}^+$ (resp. $\mathrm{E}_{\xi,\partial}^-$) respectively. Then $(D_{+,\xi}, P_{\geq 0,+,\xi})$ forms an elliptic boundary problem. Set

$$\text{APS-ind}(D,\xi) = \dim\ker(D_{+,\xi}, P_{\geq 0,+,\xi}) - \dim\ker(D_{-,\xi}, P_{>0,-,\xi}) \ . \tag{2.9}$$

Let $\left\{D_t : \Gamma(X, S(U,L)\otimes W) \longrightarrow \Gamma(X, S(U,L)\otimes W) \,\middle|\, 0\leq t\leq 1\right\}$ be a one parameter family of $\mathbb{Z}/k$ $S^1$-equivariant Dirac type operators, with the canonical boundary operators $\left\{B_t \,\middle|\, 0\leq t\leq 1\right\}$ (cf. Definition 2.4). For any $t\in[0,1]$, let $D^Y_{t,+}$ be the induced operator from $B_{t,+}$ through the map $\pi : \partial X \to Y$, and let $B_{t,+,\xi}$ (resp. $D^Y_{t,+,\xi}$) be the restriction of $B_{t,+}$ (resp. $D^Y_{t,+}$) on the weight-$\xi$ subspace $\mathrm{E}^+_{\xi,\partial}$ (resp. $\mathrm{E}^+_{Y,\xi}$). We have the following variation formula.

**Theorem 2.6** (Compare with [5, Theorem 1.2]) *The following identity holds,*

$$\begin{aligned}\text{APS-ind}(D_1,\xi) - \text{APS-ind}(D_0,\xi) &= -\,\mathrm{sf}\left\{B_{t,+,\xi} \,\middle|\, 0\leq t\leq 1\right\}\\ &= -k\,\mathrm{sf}\left\{D^Y_{t,+,\xi} \,\middle|\, 0\leq t\leq 1\right\},\end{aligned} \tag{2.10}$$

*where* sf *is the notation for the spectral flow of* [2]. *In particular,*

$$\text{APS-ind}(D_1,\xi) \equiv \text{APS-ind}(D_0,\xi) \mod k\mathbb{Z}\,.$$

**Proof** The proof is the same as that of [5, Theorem 1.2].

## 2.3 A mod $k$ localization formula for $\mathbb{Z}/k$ circle actions

Let $\mathcal{H}$ be the canonical basis of $\mathrm{Lie}(S^1) = \mathbb{R}$, i.e., for $t\in\mathbb{R}$, $\exp(t\mathcal{H}) = e^{2\pi\sqrt{-1}t} \in S^1$. Let $H$ be the Killing vector field on $X$ corresponding to $\mathcal{H}$. Since the circle action on $X$ is of $\mathbb{Z}/k$, $H|_{\partial X}\subset T\partial X$ induces a Killing vector field $H_Y$ on $Y$. Let $X_H$ (resp. $Y_H$) be the zero set of $H$ (resp. $H_Y$) on $X$ (resp. $Y$). Then $X_H$ is a $\mathbb{Z}/k$ manifold and there is a canonical map $\pi_{X_H} : \partial X_H \to Y_H$ induced by $\pi$. In general, $X_H$ is not connected. We fix a connected component $X_{H,\alpha}$ of $X_H$, and we omit the subscript $\alpha$ if there is no confusion.

Clearly, $X_H$ intersects with $\partial X$ transversally. Let $g^{TX_H}$ be the metric on $X_H$ induced by $g^{TX}$. Then $g^{TX_H}$ is naturally of product structure near $\partial X_H$. In fact, by choosing $\epsilon>0$ small enough, we know $\mathscr{U}'_\epsilon = \mathscr{U}_\epsilon \cap X_H$ carries the metric naturally induced from $g^{TX}|_{\mathscr{U}_\epsilon}$.

Let $\widetilde{\pi} : N \to X_H$ be the normal bundle to $X_H$ in $X$, which is identified to be the orthogonal complement of $TX_H$ in $TX|_{X_H}$. Then $TX|_{X_H}$ admits a $\mathbb{Z}/k$ $S^1$-equivariant decomposition (cf. [20, (1.8)])

$$TX|_{X_H} = \bigoplus_{v\neq 0} N_v \oplus TX_H\,, \tag{2.11}$$

where $N_v$ is a $\mathbb{Z}/k$ complex vector bundle such that $g\in S^1$ acts on it by $g^v$ with $v\in\mathbb{Z}\backslash\{0\}$. We will regard $N$ as a $\mathbb{Z}/k$ complex vector bundle and write $N_{\mathbb{R}}$ for the underlying real vector bundle of $N$. Clearly, $N = \oplus_{v\neq 0} N_v$. For $v\neq 0$, let $N_{v,\mathbb{R}}$ denote the underlying real vector bundle of $N_v$.

Similarly, let

$$W|_{X_H} = \bigoplus_v W_v, \quad V|_{X_H} = \bigoplus_{v \neq 0} V_v \oplus V_0^{\mathbb{R}} \tag{2.12}$$

be the $\mathbb{Z}/k$ $S^1$-equivariant decompositions of the restrictions of $W$ and $V$ over $X_H$ respectively, where $W_v$ and $V_v$ ($v \in \mathbb{Z}$) are $\mathbb{Z}/k$ complex vector bundles over $X_H$ on which $g \in S^1$ acts by $g^v$, and $V_0^{\mathbb{R}}$ is the real subbundle of $V$ such that $S^1$ acts as identity. For $v \neq 0$, let $V_{v,\mathbb{R}}$ denote the underlying real vector bundle of $V_v$. Denote by $2p' = \dim V_0^{\mathbb{R}}$ and $2l' = \dim X_H$.

Let us write

$$L_F = L \otimes \left( \bigotimes_{v \neq 0} \det N_v \otimes \bigotimes_{v \neq 0} \det V_v \right)^{-1}. \tag{2.13}$$

Then $TX_H \oplus V_0^{\mathbb{R}}$ has a $\mathbb{Z}/k$ Spin$^c$-structure since $\omega_2(TX_H \oplus V_0^{\mathbb{R}}) = c_1(L_F)$ mod (2). Let $S(TX_H \oplus V_0^{\mathbb{R}}, L_F)$ be the fundamental spinor bundle for $(TX_H \oplus V_0^{\mathbb{R}}, L_F)$ as in Section 2.1.

Recall that $N_{v,\mathbb{R}}$ and $V_{v,\mathbb{R}}$ ($v \neq 0$) are canonically oriented by their complex structures. The decompositions (2.11), (2.12) induce the orientations of $TX_H$ and $V_0^{\mathbb{R}}$ respectively. Let $\{e_i\}_{i=1}^{2l'}$, $\{f_j\}_{j=1}^{2p'}$ be the corresponding oriented orthonormal basis of $(TX_H, g^{TX_H})$ and $(V_0^{\mathbb{R}}, g^{V_0^{\mathbb{R}}})$. There are two canonical ways to consider $S(TX_H \oplus V_0^{\mathbb{R}}, L_F)$ as a $\mathbb{Z}_2$-graded vector bundle . Let

$$\begin{aligned} \tau_s &= (\sqrt{-1})^{l'} c(e_1) \cdots c(e_{2l'}), \\ \tau_e &= (\sqrt{-1})^{l'+p'} c(e_1) \cdots c(e_{2l'}) c(f_1) \cdots c(f_{2p'}) \end{aligned} \tag{2.14}$$

be two involutions of $S(TX_H \oplus V_0^{\mathbb{R}}, L_F)$. Then $\tau_s^2 = \tau_e^2 = 1$. We decompose $S(TX_H \oplus V_0^{\mathbb{R}}, L_F) = S_+(TX_H \oplus V_0^{\mathbb{R}}, L_F) \oplus S_-(TX_H \oplus V_0^{\mathbb{R}}, L_F)$ corresponding to $\tau_s$ (resp. $\tau_e$) such that $\tau_s|_{S_\pm(TX_H \oplus V_0^{\mathbb{R}}, L_F)} = \pm 1$ (resp. $\tau_e|_{S_\pm(TX_H \oplus V_0^{\mathbb{R}}, L_F)} = \pm 1$).

Let $C(N_{\mathbb{R}})$ be the Clifford algebra bundle of $(N_{\mathbb{R}}, g^N)$. Then $\Lambda(\overline{N}^*)$ is a $C(N_{\mathbb{R}})$-Clifford module. Namely, for $e \in N$, let $e' \in \overline{N}^*$ correspond to $e$ by the metric $g^N$, and let

$$c(e) = \sqrt{2}\, e' \wedge\, , \quad c(\overline{e}) = -\sqrt{2}\, i_{\overline{e}}\, , \tag{2.15}$$

where $\wedge$ and $i$ denote the exterior and interior multiplications, respectively. Let $\tau^N$ be the involution on $\Lambda(\overline{N}^*)$ given by $\tau^N|_{\Lambda^{\mathrm{even/odd}}(\overline{N}^*)} = \pm 1$.

Similarly, we can define the Clifford action of $C(V_{v,\mathbb{R}})$ on the $C(V_{v,\mathbb{R}})$-Clifford module $\Lambda(\overline{V}_v^*)$ with the involution $\tau_v^V|_{\Lambda^{\mathrm{even/odd}}(\overline{V}_v^*)} = \pm 1$.

Upon restriction to $X_H$, one has the following $\mathbb{Z}/k$ isomorphisms of $\mathbb{Z}_2$-graded Clifford modules over $X_H$ (compare with [20, (1.49)]),

$$\big(S(U, L), \tau_s\big)|_{X_H} \simeq \big(S(TX_H \oplus V_0^{\mathbb{R}}, L_F), \tau_s\big) \,\widehat{\otimes}\, \big(\Lambda \overline{N}^*, \tau^N\big) \widehat{\otimes} \widehat{\bigotimes_{v \neq 0}} \big(\Lambda \overline{V}_v^*, \mathrm{id}\big), \tag{2.16}$$

where id denotes the trivial involution, and

$$\big(S(U,L),\tau_e\big)|_{X_H} \simeq \big(S(TX_H\oplus V_0^{\mathbb{R}},L_F),\tau_e\big)\,\widehat{\otimes}\,\big(\Lambda\overline{N}^*,\tau^N\big)\,\widehat{\otimes}\,\widehat{\bigotimes_{v\neq 0}}\big(\Lambda\overline{V}_v^*,\tau_v^V\big). \tag{2.17}$$

Here we denote by $\widehat{\otimes}$ the $\mathbb{Z}_2$-graded tensor product (cf. [14, pp. 11]). Furthermore, isomorphisms (2.16), (2.17) give the identifications of the canonical connections on the bundles (compare with [20, (1.13)]). We still denote the involution on $\big(S(TX_H\oplus V_0^{\mathbb{R}},L_F)$ by $\tau$.

Let $R$ be a $\mathbb{Z}/k$ Hermitian vector bundle over $X_H$ endowed with a $\mathbb{Z}/k$ Hermitian connection. We make the assumption that the Hermitian metric and the Hermitian connection are both of product structure near $\partial X_H$. We will denote by $D^{X_H}\otimes R$ the twisted Spin$^c$ Dirac operator on $S(TX_H\oplus V_0^{\mathbb{R}},L_F)\otimes R$ and by $D^{X_{H,\alpha}}\otimes R$ its restriction to $X_{H,\alpha}$ (cf. Definition 2.2).

We denote by $K(X_H)$ the $K$-group of $\mathbb{Z}/k$ complex vector bundles over $X_H$ (cf. [7, pp. 285]). We use the same notations as in [20, pp. 128],

$$\begin{aligned}\mathrm{Sym}_q(R) &= \sum_{n=0}^{+\infty} q^n\mathrm{Sym}^n(R)\in K(X_H)[[q]],\\ \Lambda_q(R) &= \sum_{n=0}^{+\infty} q^n\Lambda^n(R)\in K(X_H)[[q]],\end{aligned} \tag{2.18}$$

for the symmetric and exterior power operations in $K(X_H)[[q]]$, respectively.

Let $S^1$ act on $L|_{X_H}$ by sending $g\in S^1$ to $g^{l_c}$ ($l_c\in\mathbb{Z}$) on $X_H$. Then $l_c$ is locally constant on $X_H$. Following [20, (1.50)], we define the following elements in $K(X_H)[[q^{\frac{1}{2}}]]$,

$$\begin{aligned}R_\pm(q) &= q^{\frac{1}{2}\sum_v |v|\dim N_v-\frac{1}{2}\sum_v v\dim V_v+\frac{1}{2}l_c}\bigotimes_{v>0}\big(\mathrm{Sym}_{q^v}(N_v)\otimes\det N_v\big)\\ &\qquad\otimes\bigotimes_{v<0}\mathrm{Sym}_{q^{-v}}(\overline{N}_v)\otimes\bigotimes_{v\neq 0}\Lambda_{\pm q^v}(V_v)\otimes\Big(\sum_v q^vW_v\Big)\\ &= \sum_n R_{\pm,n}q^n\ ,\end{aligned} \tag{2.19}$$

$$\begin{aligned}R'_\pm(q) &= q^{-\frac{1}{2}\sum_v |v|\dim N_v-\frac{1}{2}\sum_v v\dim V_v+\frac{1}{2}l_c}\bigotimes_{v>0}\mathrm{Sym}_{q^{-v}}(\overline{N}_v)\\ &\qquad\otimes\bigotimes_{v<0}\big(\mathrm{Sym}_{q^v}(N_v)\otimes\det N_v\big)\otimes\bigotimes_{v\neq 0}\Lambda_{\pm q^v}(V_v)\otimes\Big(\sum_v q^vW_v\Big)\\ &= \sum_n R'_{\pm,n}q^n.\end{aligned} \tag{2.20}$$

As explained in [20, pp. 139], since $TX\oplus V\oplus L$ is spin, one gets

$$\sum_v v\dim N_v+\sum_v v\dim V_v+l_c\equiv 0 \mod (2). \tag{2.21}$$

Therefore, $R_{\pm,\xi}(q)$, $R'_{\pm,\xi}(q)\in K(X_H)[[q]]$.

Clearly each $R_{\pm,\xi}$, $R'_{\pm,\xi}$ ($\xi \in \mathbb{Z}$) is a $\mathbb{Z}/k$ vector bundle over $X_H$ carrying a canonically induced $\mathbb{Z}/k$ Hermitian metric and a canonically induced $\mathbb{Z}/k$ Hermitian connection, which are both of product structure near $\partial X_H$.

We now state a mod $k$ localization formula which generalizes [20, Theorem 1.2] to the case of $\mathbb{Z}/k$-manifolds. It also generalizes the $\mathbb{Z}/k$ equivariant index theorem in [27, Theorem 2.1] to the case of Spin$^c$-manifolds.

**Theorem 2.7** *For any $\xi \in \mathbb{Z}$, the following identities hold,*

$$\begin{aligned}\text{APS-ind}_{\tau_s}\big(D^X,\xi\big) &\equiv \sum_\alpha (-1)^{\sum_{0<v}\dim N_v}\,\text{APS-ind}_{\tau_s}\big(D^{X_{H,\alpha}}\otimes R_{+,\xi}\big)\\ &\equiv \sum_\alpha (-1)^{\sum_{v<0}\dim N_v}\,\text{APS-ind}_{\tau_s}\big(D^{X_{H,\alpha}}\otimes R'_{+,\xi}\big) \mod k\mathbb{Z}\ , \end{aligned} \tag{2.22}$$

$$\begin{aligned}\text{APS-ind}_{\tau_e}\big(D^X,\xi\big) &\equiv \sum_\alpha (-1)^{\sum_{0<v}\dim N_v}\,\text{APS-ind}_{\tau_e}\big(D^{X_{H,\alpha}}\otimes R_{-,\xi}\big)\\ &\equiv \sum_\alpha (-1)^{\sum_{v<0}\dim N_v}\,\text{APS-ind}_{\tau_e}\big(D^{X_{H,\alpha}}\otimes R'_{-,\xi}\big) \mod k\mathbb{Z}\ . \end{aligned} \tag{2.23}$$

**Proof** The proof will be given in Section 3.

## 2.4 A $\mathbb{Z}/k$ extension of Hattori's vanishing theorem

In this subsection, we assume that $TX$ has a $\mathbb{Z}/k$ $S^1$-equivariant almost complex structure $J$. Then one has the canonical splitting

$$TX\otimes_{\mathbb{R}}\mathbb{C} = T^{(1,0)}X\oplus T^{(0,1)}X, \tag{2.24}$$

where $T^{(1,0)}X$ and $T^{(0,1)}X$ are the eigenbundles of $J$ corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively.

Let $K_X = \det(T^{(1,0)}X)$ be the determinant line bundle of $T^{(1,0)}X$ over $X$. Then the complex spinor bundle $S(TX, K_X)$ for $(TX, K_X)$ is $\Lambda(T^{*(0,1)}X)$ (cf. [14, Appendix D]).

We suppose that $c_1(T^{(1,0)}X) = 0 \mod (N)$ ($N \in \mathbb{Z}, N \geq 2$). Then the complex line bundle $K_X^{1/N}$ is well defined over $X$. After replacing the $S^1$ action by its $N$-fold action, we can always assume that $S^1$ acts on $K_X^{1/N}$. For $s \in \mathbb{Z}$, let $D^X \otimes K_X^{s/N}$ be the twisted Spin$^c$-Dirac operator on $\Lambda(T^{*(0,1)}X)\otimes K_X^{s/N}$ defined as in (2.4).

Using Theorem 2.7, we can generalize the main result of Hattori [8] to the case of $\mathbb{Z}/k$ almost complex manifolds.

**Theorem 2.8** *Assume that $X$ is a connected $\mathbb{Z}/k$ almost complex manifold with a nontrivial $\mathbb{Z}/k$ circle action. If $c_1(T^{(1,0)}X) = 0 \mod (N)$ ($N \in \mathbb{Z}, N \geq 2$), then for $s \in \mathbb{Z}$, $-N < s < 0$, $D^X \otimes K_X^{s/N}$ has vanishing property in $\mathbb{Z}/k$ category. In particular, the following identity holds,*

$$\text{t-ind}\big(D^X\otimes K_X^{s/N}\big) = 0 \quad in \quad \mathbb{Z}/k\mathbb{Z}\ . \tag{2.25}$$

**Proof** Using the almost complex structure on $TX_H$ induced by the almost complex structure $J$ on $TX$ and by (2.11), we know

$$T^{(1,0)}X\big|_{X_H} = \bigoplus_{v\neq 0} N_v \oplus T^{(1,0)}X_H\,, \tag{2.26}$$

where $N_v$ are complex subbundles of $T^{(1,0)}X\big|_{X_H}$ on which $g\in S^1$ acts by multiplication by $g^v$.

We claim that for each $\xi\in\mathbb{Z}$, the following identity holds,

$$\text{APS-ind}\big(D^X\otimes K_X^{s/N},\xi\big)\equiv 0 \mod k\mathbb{Z}\,. \tag{2.27}$$

In fact, if $X_H=\emptyset$, the empty set, by Theorem 2.7, (2.27) is obvious.

When $X_H\neq\emptyset$, we see that $\sum_v |v|\dim N_v>0$ (i.e., at least one of the $N_v$'s is nonzero) on each connected component of $X_H$. Consider $R_+(q)$, $R'_+(q)$ of (2.19) and (2.20) for the case that $V=0$, $W=K_X^{s/N}$. We deduce that

$$\begin{aligned}
R_{+,\xi}=0 \quad &\text{if}\quad \xi<a_1=\inf_\alpha\Big(\frac{1}{2}\sum_v|v|\dim N_v+\Big(\frac{1}{2}+\frac{s}{N}\Big)\sum_v v\dim N_v\Big),\\
R'_{+,\xi}=0 \quad &\text{if}\quad \xi>a_2=\sup_\alpha\Big(-\frac{1}{2}\sum_v|v|\dim N_v+\Big(\frac{1}{2}+\frac{s}{N}\Big)\sum_v v\dim N_v\Big).
\end{aligned}$$

Since $-N<s<0$, we know $a_1>0$ and $a_2<0$. By using Theorem 2.7, we see that (2.27) holds for any $\xi\in\mathbb{Z}$.

Now Theorem 2.8 follows easily from (1.1), (1.3) and (2.27). The proof of Theorem 2.8 is completed.

**Remark 2.9** *From the proof of Theorem 2.8, one also deduces that if $X$ is a connected $\mathbb{Z}/k$ almost complex manifold with a nontrivial $\mathbb{Z}/k$ circle action, then $D^X$, $D^X\otimes K_X^{-1}$ are rigid in $\mathbb{Z}/k$ category.*

# 3 A proof of Theorem 2.7

In this section, following Zhang [27] and by making use of the analysis of Wu-Zhang [26] and Dai-Zhang [5] as well as Liu-Ma-Zhang [20], which in turn depend on the analytic localization techniques of Bismut-Lebeau [3], we present a proof of Theorem 2.7.

This section is organized as follows. In Section 3.1, we recall a result from [26] concerning the Witten deformation on flat spaces. In Section 3.2, we establish the Taylor expansions of $D^X$ and $c(H)$ (resp. $B^X$) near the fixed point set $X_H$ (resp. $\partial X_H$). In Section 3.3, following [5, Section 3(b)], we decompose the Dirac type operators under consideration to a sum of four operators and introduce a deformation of the Dirac type operators as well as their associated boundary operators. In Section 3.4, by using the techniques of [5, Section 3(c)], [20, Section 1.2] and [3, Section 9], we carry out various estimates for certain operators and prove the Fredholm property of the Atiyah-Patodi-Singer type boundary problem for the deformed operators introduced in Section 3.3. In Section 3.5, we complete the proof of Theorem 2.7.

### 3.1 Witten deformation on flat spaces

Recall that $\mathcal{H}$ is the canonical basis of $\mathrm{Lie}(S^1) = \mathbb{R}$. In this subsection, let $W$ be a complex vector space of dimension $n$ with an Hermitian form. Let $\rho$ be a unitary representation of the circle group $S^1$ on $W$ such that all the weights are nonzero. Suppose $W^{\pm}$ are the subspaces of $W$ corresponding to the positive and negative weights respectively, with $\dim_{\mathbb{C}} W^- = \nu$, $\dim_{\mathbb{C}} W^+ = n - \nu$. Let $z = \{z^i\}$ be the complex linear coordinates on $W$ such that the Hermitian structure on $W$ takes the standard form and $\rho$ is diagonal with weights $\lambda_i \in \mathbb{Z}\backslash\{0\}$ $(1 \le i \le n)$, and $\lambda_i < 0$ for $i \le \nu$. The Lie algebra action on $W$ is given by the vector field

$$H = 2\pi\sqrt{-1}\sum_{i=1}^{n} \lambda_i \left( z^i \frac{\partial}{\partial z^i} - \bar{z}^i \frac{\partial}{\partial \bar{z}^i} \right). \tag{3.1}$$

Set

$$K^{\pm}(W) = \mathrm{Sym}((W^{\pm})^*) \otimes \mathrm{Sym}(W^{\mp}) \otimes \det(W^{\mp}). \tag{3.2}$$

Let $E$ be a finite dimensional complex vector space with an Hermitian form and suppose $E$ carries a unitary representation of $S^1$.

Let $\overline{\partial}$ be the twisted Dolbeault operator acting on $\Omega^{0,*}(W,E)$, the set of smooth sections of $\Lambda(\overline{W}^*) \otimes E$ on $W$. Let $\overline{\partial}^*$ be the formal adjoint of $\overline{\partial}$. Let $D = \sqrt{2}(\overline{\partial} + \overline{\partial}^*)$. Let $c(H)$ be the Clifford action of $H$ on $\Lambda(\overline{W}^*)$ defined as in (2.15). Let $\mathscr{L}_H$ be the Lie derivative along $H$ acting on $\Omega^{0,*}(W,E)$.

The following result was proved in [26, Proposition 3.2].

**Proposition 3.1** 1. *A basis of the space of $L^2$-solutions of $D + \sqrt{-1}c(H)$ (resp. $D - \sqrt{-1}c(H)$) on the space of $C^\infty$ sections of $\Lambda(\overline{W}^*)$ is given by*

$$\Big(\prod_{i=1}^{\nu} z_i^{k_i}\Big)\Big(\prod_{i=\nu+1}^{n} \bar{z}_i^{k_i}\Big) e^{-\sum_{i=1}^{n}\pi|\lambda_i||z_i|^2} d\bar{z}_{\nu+1}\cdots d\bar{z}_n \quad (k_i \in \mathbb{N}) \tag{3.3}$$

*with weight $\sum_{i=1}^{\nu} k_i|\lambda_i| + \sum_{i=\nu+1}^{n}(k_i+1)|\lambda_i|$ (resp.*

$$\Big(\prod_{i=1}^{\nu} \bar{z}_i^{k_i}\Big)\Big(\prod_{i=\nu+1}^{n} z_i^{k_i}\Big) e^{-\sum_{i=1}^{n}\pi|\lambda_i||z_i|^2} d\bar{z}_1\cdots d\bar{z}_\nu \quad (k_i \in \mathbb{N}) \tag{3.4}$$

*with weight $-\sum_{i=\nu+1}^{n} k_i|\lambda_i| - \sum_{i=1}^{\nu}(k_i+1)|\lambda_i|$).*

*So the space of $L^2$-solution of a given weight of $D + \sqrt{-1}c(H)$ (resp. $D - \sqrt{-1}c(H)$) on the space of $C^\infty$ sections of $\Lambda(\overline{W}^*)\otimes E$ is finite dimensional. The direct sum of these weight spaces is isomorphic to $K^-(W)\otimes E$ (resp. $K^+(W)\otimes E$) as representations of $S^1$.*

2. *When restricted to an eigenspace of $\mathscr{L}_H$, the operator $D + \sqrt{-1}c(H)$ (resp. $D - \sqrt{-1}c(H)$) has discrete eigenvalues.*

### 3.2 A Taylor expansion of certain operators near the fixed-point set

Following [3, Section 8(e)], we now describe a coordinate system on $X$ near $X_H$. For $\varepsilon > 0$, set $\mathscr{B}_\varepsilon = \big\{Z \in N \big| \, |Z| < \varepsilon\big\}$. Since $X$ and $X_H$ are compact, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$, the exponential map

$$(y, Z) \in N \longmapsto \exp_y^X(Z) \in X$$

is a diffeomorphism from $\mathscr{B}_\varepsilon$ onto a tubular neighborhood $\mathscr{V}_\varepsilon$ of $X_H$ in $X$. From now on, we identify $\mathscr{B}_\varepsilon$ with $\mathscr{V}_\varepsilon$ and use the notation $x = (y, Z)$ instead of $x = \exp_y^X(Z)$. Finally, we identify $y \in X_H$ with $(y, 0) \in N$.

Let $\widetilde{\pi}^*((S(U,L) \otimes W)|_{X_H})$ be the vector bundle on $N$ obtained by pulling back $(S(U,L) \otimes W)|_{X_H}$ for $\widetilde{\pi} : N \to X_H$.

Let $g^{TX_H}$, $g^N$ be the corresponding metrics on $TX_H$ and $N$ induced by $g^{TX}$. Let $\mathrm{d}v_X$, $\mathrm{d}v_{X_H}$ and $\mathrm{d}v_N$ be the corresponding volume elements on $(TX, g^{TX})$, $(TX_H, g^{TX_H})$ and $(N, g^N)$. Let $k(y, Z)$ ($(y, Z) \in \mathscr{B}_\varepsilon$) be the smooth positive function defined by

$$\mathrm{d}v_X(y, Z) = k(y, Z)\mathrm{d}v_{X_H}(y)\mathrm{d}v_{N_y}(Z). \tag{3.5}$$

Then $k(y) = 1$ and $\frac{\partial k}{\partial Z}(y) = 0$ for $y \in X_H$. The latter follows from the well-known fact that $X_H$ is totally geodesic in $X$.

For $x = (y, Z) \in \mathscr{V}_{\varepsilon_0}$, we will identify $S(U,L)_x$ with $S(U,L)_y$ and $W_x$ with $W_y$ by the parallel transport with respect to the $S^1$-invariant connections $\nabla^{S(U,L)}$ and $\nabla^W$ respectively, along the geodesic $t \longmapsto (y, tZ)$. The induced identification of $(S(U,L) \otimes W)_x$ with $(S(U,L) \otimes W)_y$ preserves the metric and the $\mathbb{Z}_2$-grading, and moreover, is $S^1$-equivariant. Consequently, $D^X$ can be considered as an operator acting on the sections of the bundle $\widetilde{\pi}^*((S(U,L) \otimes W)|_{X_H})$ over $\mathscr{B}_{\varepsilon_0}$ commuting with the circle action.

For $\varepsilon > 0$, let $\mathbf{E}(\varepsilon)$ (resp. $\mathbf{E}$) be the set of smooth sections of $\widetilde{\pi}^*((S(U,L) \otimes W)|_{X_H})$ on $\mathscr{B}_\varepsilon$ (resp. on the total space of $N$). If $f, g \in \mathbf{E}$ have compact supports, we will write

$$\langle f, g \rangle = \left(\frac{1}{2\pi}\right)^{\dim X} \int_{X_H} \left( \int_N \langle f, g \rangle (y, Z) \mathrm{d}v_{N_y}(Z) \right) \mathrm{d}v_{X_H}(y). \tag{3.6}$$

Then $k^{1/2} D^X k^{-1/2}$ is a (formally) self-adjoint operator on $\mathbf{E}$.

The connection $\nabla^N$ on $N$ induces a splitting $TN = N \oplus T^H N$, where $T^H N$ is the horizontal part of $TN$ with respect to $\nabla^N$. Moreover, since $X_H$ is totally geodesic, this splitting, when restricted to $X_H$, is preserved by the connection $\nabla^{TX}$ on $TX|_{X_H}$. Let $\widetilde{\nabla}$ be the connection on $(S(U,L) \otimes W)|_{X_H}$ induced by the restriction of $\nabla^{S(U,L) \otimes W}$ to $X_H$. We denote by $\widetilde{\pi}^* \widetilde{\nabla}$ the pulling back of the connection $\widetilde{\nabla}$ on $(S(U,L) \otimes W)|_{X_H}$ to the bundle $\widetilde{\pi}^*((S(U,L) \otimes W)|_{X_H})$.

We choose a local orthonormal basis of $TX$ such that $e_1, \cdots, e_{2l'}$ form a basis of $TX_H$, and $e_{2l'+1}, \cdots, e_{2l}$, that of $N_\mathbb{R}$. Denote the horizontal lift of $e_i$ $(1 \leq i \leq 2l')$ to $T^H N$ by $e_i^H$. We define

$$D^H = \sum_{i=1}^{2l'} c(e_i)(\widetilde{\pi}^* \widetilde{\nabla})_{e_i^H}, \quad D^N = \sum_{i=2l'+1}^{2l} c(e_i)(\widetilde{\pi}^* \widetilde{\nabla})_{e_i}. \tag{3.7}$$

Clearly, $D^N$ acts along the fibers of $N$. Let $\overline{\partial}^N$ be the $\overline{\partial}$-operator along the fibers of $N$, and let $\overline{\partial}^{N*}$ be its formal adjoint with respect to (3.6). It is easy to see that $D^N = \sqrt{2}\big(\overline{\partial}^N + \overline{\partial}^{N*}\big)$. Both $D^N$ and $D^H$ are formally self-adjoint with respect to (3.6).

For $T>0$, we define a scaling $f \in \mathbf{E}(\varepsilon_0) \to S_T f \in \mathbf{E}(\varepsilon_0\sqrt{T})$ by

$$S_T f(y,Z) = f\left(y, \frac{Z}{\sqrt{T}}\right), \quad (y,Z) \in \mathscr{B}_{\varepsilon_0\sqrt{T}} \quad . \tag{3.8}$$

For a first order differential operator

$$Q_T = \sum_{i=1}^{2l'} a_T^i(y,Z)(\widetilde{\pi}^*\widetilde{\nabla})_{e_i^H} + \sum_{i=2l'+1}^{2l} b_T^i(y,Z)(\widetilde{\pi}^*\widetilde{\nabla})_{e_i} + c_T(y,Z) \tag{3.9}$$

acting on $\mathbf{E}(\varepsilon_0\sqrt{T})$, where $a_T^i, b_T^i$, and $c_T$ are endomorphisms of $\widetilde{\pi}^*((S(U,L)\otimes W)|_{X_H})$ which depend smoothly on $(y,Z)$, we write

$$Q_T = O\big(|Z|^2\partial^N + |Z|\partial^H + |Z| + |Z|^p\big), \tag{3.10}$$

if there is a constant $C>0$, $p\in\mathbb{N}$ such that for any $T\geq 1$, $(y,Z)\in\mathscr{B}_{\varepsilon_0\sqrt{T}}$, we have

$$\begin{aligned} &|a_T^i(y,Z)| \leq C|Z| \quad (1\leq i\leq 2l'),\\ &|b_T^i(y,Z)| \leq C|Z|^2 \quad (2l'+1\leq i\leq 2l),\\ &|c_T(y,Z)| \leq C(|Z|+|Z|^p)\,. \end{aligned} \tag{3.11}$$

Let $\mathbf{E}_\partial$ be the set of smooth sections of $\widetilde{\pi}^*((S(U,L)\otimes W)|_{X_H})$ over $N|_{\partial X_H}$. On the boundary of $X_H$, we choose the local orthonormal basis as in Definition 2.3. Similarly as in (2.6), we define

$$B^H = -\sum_{i=2}^{2l'} c\left(\frac{\partial}{\partial r}\right) c(e_i)(\widetilde{\pi}^*\widetilde{\nabla})_{e_i^H}, \quad B^N = -c\left(\frac{\partial}{\partial r}\right) D^N|_{\partial X_H} \tag{3.12}$$

on $\mathbf{E}_\partial$ (compare with (3.7)).

Let $J_H$ be the representation of $\mathrm{Lie}(S^1)$ on $N$. Then $Z\to J_H Z$ is a Killing vector field on $N$. We have the following analogue of [3, Theorem 8.18], [20, Proposition 1.2] and [26, Proposition 3.3].

**Proposition 3.2** *As $T\to+\infty$,*

$$\begin{aligned} S_T k^{1/2} D^X k^{-1/2} S_T^{-1} &= \sqrt{T} D^N + D^H + \frac{1}{\sqrt{T}} O(|Z|^2\partial^N + |Z|\partial^H + |Z|),\\ S_T k^{1/2} c(H) k^{-1/2} S_T^{-1} &= \frac{1}{\sqrt{T}} c(J_H Z) + \frac{1}{\sqrt{T^3}} O(|Z|^3),\\ S_T k^{1/2} B^X k^{-1/2} S_T^{-1} &= \sqrt{T} B^N + B^H + \frac{1}{\sqrt{T}} O(|Z|^2\partial^N + |Z|\partial^H + |Z|). \end{aligned}$$

### 3.3 A decomposition of Dirac type operators under consideration and the associated deformation

For $p \geq 0$, let $\mathrm{E}^p$ (resp. $\mathrm{E}^p_\partial$, $\mathbf{E}^p$, $\mathrm{F}^p$, $\mathrm{F}^p_\partial$) be the set of sections of the bundles $S(U,L)\otimes W$ over $X$ (resp. $(S(U,L)\otimes W)|_{\partial X}$ over $\partial X$, $\widetilde{\pi}^*((S(U,L)\otimes W)|_{X_H})$ over $N$, $S(TX_H\oplus V_0^{\mathbb{R}}, L_F)\otimes K^-(N)\otimes \big(\widehat{\otimes}_{v\neq 0}\Lambda V_v\otimes W\big)|_{X_H}$ over $X_H$, $\big(S(TX_H\oplus V_0^{\mathbb{R}}, L_F)\otimes K^-(N)\otimes \widehat{\otimes}_{v\neq 0}\Lambda V_v\otimes W\big)|_{\partial X_H}$ over $\partial X_H$) which lie in the $p$-th Sobolev spaces. The group $S^1$ acts on all these spaces (cf. Section 2.2). For any $\xi\in\mathbb{Z}$, let $\mathrm{E}^p_\xi$, $\mathrm{E}^p_{\xi,\partial}$ $\mathbf{E}^p_\xi$, $\mathrm{F}^p_\xi$ and $\mathrm{F}^p_{\xi,\partial}$ be the corresponding weight-$\xi$ subspaces, respectively.

Recall that the constant $\varepsilon_0>0$ is defined in last subsection. We now take $\varepsilon\in(0,\frac{\varepsilon_0}{2}]$. Let $\rho:\mathbb{R}\to[0,1]$ be a smooth function such that

$$\rho(a)=\begin{cases}1 & \text{if } a\leq \frac{1}{2},\\ 0 & \text{if } a\geq 1.\end{cases} \tag{3.13}$$

For $Z\in N$, set $\rho_\varepsilon(Z)=\rho(\frac{|Z|}{\varepsilon})$.

By Proposition 3.1, the solution space of the operator $D^N+\sqrt{-1}Tc(J_HZ)$ along the fiber $N_y$ ($y\in X_H$) is the $L^2$ completion of $K^-(N_y)\otimes(\widehat{\otimes}_{v\neq 0}\Lambda V_v\otimes W)_y$. They form an infinite dimensional Hermitian complex vector bundle $K^-(N)\otimes(\widehat{\otimes}_{v\neq 0}\Lambda V_v\otimes W)|_{X_H}$ over $X_H$, with the Hermitian connection induced from those on $N$, $V|_{X_H}\to X_H$ and $W|_{X_H}\to X_H$. Let $\theta$ be the isomorphism from $L^2(X_H, K^-(N)\otimes(\widehat{\otimes}_{v\neq 0}\Lambda V_v\otimes W)|_{X_H})$ to $L^2(N,\widetilde{\pi}^*((\Lambda\overline{N}^*\otimes\widehat{\otimes}_{v\neq 0}\Lambda V_v\otimes W)|_{X_H}))$ given by Proposition 3.1.

Let $\alpha\in\Gamma\big(X_H, S(TX_H\oplus V_0^{\mathbb{R}}, L_F)\big)$, $\phi\in L^2(X_H, K^-(N)\otimes(\widehat{\otimes}_{v\neq 0}\Lambda V_v\otimes W)|_{X_H})$, $\sigma=\alpha\otimes\phi$. We define a linear map

$$I_{T,\xi}:\mathrm{F}^p_\xi\longrightarrow \mathbf{E}^p_\xi,\qquad \sigma\longmapsto T^{\frac{\dim N_{\mathbb{R}}}{2}}\rho_\varepsilon(Z)\,\widetilde{\pi}^*\alpha\wedge S_T^{-1}(\theta\phi). \tag{3.14}$$

In general, there exist $c(\varepsilon)>0$ and $C>0$, such that $c(\varepsilon)<\|I_{T,\xi}\|<C$.

Let the image of $I_{T,\xi}$ from $\mathrm{F}^p_\xi$ be $\mathbf{E}^p_{T,\xi}=I_{T,\xi}\mathrm{F}^p_\xi\subseteq\mathbf{E}^p_\xi$. Denote the orthogonal complement of $\mathbf{E}^0_{T,\xi}$ in $\mathbf{E}^0_\xi$ by $\mathbf{E}^{0,\perp}_{T,\xi}$, and let $\mathbf{E}^{p,\perp}_{T,\xi}=\mathbf{E}^p_\xi\cap\mathbf{E}^{0,\perp}_{T,\xi}$. Let $p_{T,\xi}$ and $p^\perp_{T,\xi}$ be the orthogonal projections from $\mathbf{E}^0_\xi$ to $\mathbf{E}^0_{T,\xi}$ and $\mathbf{E}^{0,\perp}_{T,\xi}$ respectively.

We denote by $\big(\big(\widehat{\bigotimes}_{v\neq 0}\Lambda V_v\big)\otimes\big(\bigoplus_v W_v\big)\big)_{\xi-\frac{1}{2}\sum_v|v|\dim N_v}$ the subbundle of $\big(\widehat{\bigotimes}_{v\neq 0}\Lambda V_v\big)\otimes\big(\bigoplus_v W_v\big)$ whose weight equals to $\xi-\frac{1}{2}\sum_v|v|\dim N_v$ with respect to the given circle action. Let $q_\xi$ be the orthogonal bundle projection from the vector bundle

$$\Big(\widehat{\bigotimes}_{v\neq 0}\Lambda\overline{N}_v^*\Big)\,\widehat{\otimes}\,\Big(\widehat{\bigotimes}_{v\neq 0}\Lambda V_v\Big)\otimes\Big(\bigoplus_v W_v\Big)\longrightarrow X_H$$

to its subbundle

$$\bigotimes_{v>0}\det N_v\,\widehat{\otimes}\,\Big(\Big(\widehat{\bigotimes}_{v\neq 0}\Lambda V_v\Big)\otimes\Big(\bigoplus_v W_v\Big)\Big)_{\xi-\frac{1}{2}\sum_v|v|\dim N_v}\longrightarrow X_H\,.$$

We now proceed to deduce a formula which computes $p_{T,\xi}s$ for $s\in\mathrm{E}^0_\xi$ explicitly under a local unitary trivialization of $N$.

For $y_0 \in X_H$, on a small neighborhood $\mathscr{V}_{y_0} \subset X_H$ of $y_0$, choose a unitary trivialization $N|_{\mathscr{V}_{y_0}} \cong \mathscr{V}_{y_0} \times \mathbb{C}^n = \{(y, Z) \,|\, y \in \mathscr{V}_{y_0}, Z = (z_1, \cdots, z_n) \in \mathbb{C}^n\}$ such that for $t \in \mathbb{R}$,

$$\exp(t\mathcal{H}) \cdot \frac{\partial}{\partial z_i} = e^{2\pi\sqrt{-1}\lambda_i t} \frac{\partial}{\partial z_i}.$$

Without loss of generality, we assume that $\lambda_i < 0$ for $i \le \nu$ and $\lambda_i > 0$ for $\nu < i \le n$. For any $T > 0$, $\overrightarrow{k} = (k_1, \cdots, k_n) \in \mathbb{N}^n$, and $(y, Z) \in \mathscr{V}_{y_0} \times \mathbb{C}^n$, set

$$f_{T,\overrightarrow{k}}(Z) = (\prod_{i=1}^{\nu} z_i^{k_i})(\prod_{i=\nu+1}^{n} \bar{z}_i^{k_i}) e^{-T\sum_{i=1}^{n} \pi|\lambda_i||z_i|^2},$$

$$\alpha_{T,\overrightarrow{k}}(y) = \int_{N_{\mathbb{R},y}} \rho_\varepsilon^2(Z) \prod_{i=1}^{n} \left( |z_i|^{2k_i} e^{-2T\pi|\lambda_i||z_i|^2} \right) \frac{dv_N}{(2\pi)^{\dim N_{\mathbb{R}}}}.$$

Computing directly, we have for $s \in \mathrm{E}_\xi^0$ that (compare with [3, Proposition 9.2])

$$\begin{aligned} p_{T,\xi} s(y, Z) = & \sum_{\overrightarrow{k}, \xi_2, \ s.t. \ \sum_{i=1}^{n} k_i |\lambda_i| + \xi_2 = \xi} \alpha_{T,\overrightarrow{k}}^{-1} \rho_\varepsilon(Z) f_{T,\overrightarrow{k}}(Z) \\ & \cdot q_{\xi_2} \int_{N_{\mathbb{R},y}} \rho_\varepsilon(Z') \overline{f_{T,\overrightarrow{k}}(Z')} s(y, Z') \frac{dv_N(Z')}{(2\pi)^{\dim N_{\mathbb{R}}}}. \end{aligned} \tag{3.15}$$

Using (3.15), we get the following analogue of [3, Proposition 9.3].

**Proposition 3.3** *There exists $C > 0$ such that if $T \ge 1$, $\sigma \in \mathrm{F}_\xi^1$, then*

$$\|I_{T,\xi}\sigma\|_{\mathbf{E}_\xi^1} \le C(\|\sigma\|_{\mathrm{F}_\xi^1} + \sqrt{T}\|\sigma\|_{\mathrm{F}_\xi^0}). \tag{3.16}$$

*There exists $C > 0$ such that for any $T \ge 1$, any $s \in \mathbf{E}_\xi^1$, then*

$$\|p_{T,\xi} s\|_{\mathbf{E}_\xi^1} \le C(\|s\|_{\mathbf{E}_\xi^1} + \sqrt{T}\|s\|_{\mathbf{E}_\xi^0}). \tag{3.17}$$

*Given $\gamma > 0$, there exists $C' > 0$ such that for $T \ge 1$, for $s \in \mathbf{E}_\xi^0$, then*

$$\|p_{T,\xi}|Z|^\gamma s\|_{\mathbf{E}_\xi^0} \le \frac{C'}{T^{\frac{\gamma}{2}}} \|s\|_{\mathbf{E}_\xi^0}. \tag{3.18}$$

Since we have the identification of the bundles

$$\begin{aligned} & (S(U, L) \otimes W)|_{\mathscr{V}_{\varepsilon_0}} \simeq \\ & \qquad \widetilde{\pi}^* \left( S(TX_H \oplus V_0^{\mathbb{R}}, L_F) \otimes \Lambda(\overline{N}^*) \otimes \left( \widehat{\otimes}_{v \neq 0} \Lambda V_v \otimes W \right)|_{X_H} \right) \Big|_{\mathscr{B}_{\varepsilon_0}}, \end{aligned}$$

we can consider $k^{-1/2} I_{T,\xi}\sigma$ as an element of $\mathrm{E}_\xi^p$ for $\sigma \in \mathrm{F}_\xi^p$. Set

$$J_{T,\xi} = k^{-1/2} I_{T,\xi}. \tag{3.19}$$

We denote by $J_{T,\xi,\partial} : \mathrm{F}_{\xi,\partial}^p \to \mathrm{E}_{\xi,\partial}^p$ the restriction of $J_{T,\xi}$ on the boundary. Let $\mathrm{E}_{T,\xi}^p = J_{T,\xi}\mathrm{F}_\xi^p$ (resp. $\mathrm{E}_{T,\xi,\partial}^p = J_{T,\xi,\partial}\mathrm{F}_{\xi,\partial}^p$) be the image of $J_{T,\xi}$ (resp. $J_{T,\xi,\partial}$).

Denote the orthogonal complement of $\mathrm{E}^0_{T,\xi}$ (resp. $\mathrm{E}^0_{T,\xi,\partial}$) in $\mathrm{E}^0_\xi$ (resp. $\mathrm{E}^0_{\xi,\partial}$) by $\mathrm{E}^{0,\perp}_{T,\xi}$ (resp. $\mathrm{E}^{0,\perp}_{T,\xi,\partial}$) and let $\mathrm{E}^{p,\perp}_{T,\xi} = \mathrm{E}^p_\xi \cap \mathrm{E}^{0,\perp}_{T,\xi}$ (resp. $\mathrm{E}^{p,\perp}_{T,\xi,\partial} = \mathrm{E}^p_{\xi,\partial} \cap \mathrm{E}^{0,\perp}_{T,\xi,\partial}$). Let $\bar{p}_{T,\xi}$ (resp. $\bar{p}_{T,\xi,\partial}$) and $\bar{p}^\perp_{T,\xi}$ (resp. $\bar{p}^\perp_{T,\xi,\partial}$) be the orthogonal projections from $\mathrm{E}^0_\xi$ (resp. $\mathrm{E}^0_{\xi,\partial}$) to $\mathrm{E}^0_{T,\xi}$ (resp. $\mathrm{E}^0_{T,\xi,\partial}$) and $\mathrm{E}^{0,\perp}_{T,\xi}$ (resp. $\mathrm{E}^{0,\perp}_{T,\xi,\partial}$) respectively. It is clear that $\bar{p}_{T,\xi} = k^{-1/2} p_{T,\xi} k^{1/2}$ (resp. $\bar{p}_{T,\xi,\partial} = k^{-1/2} p_{T,\xi,\partial} k^{1/2}$).

For any (possibly unbounded) operator $A$ (resp. $B$) on $\mathrm{E}^0_\xi$ (resp. $\mathrm{E}^0_{\xi,\partial}$), we write

$$A = \begin{pmatrix} A^{(1)} & A^{(2)} \\ A^{(3)} & A^{(4)} \end{pmatrix} \qquad ( \text{ resp. } B = \begin{pmatrix} B^{(1)} & B^{(2)} \\ B^{(3)} & B^{(4)} \end{pmatrix} \ ) \tag{3.20}$$

according to the decomposition $\mathrm{E}^0_\xi = \mathrm{E}^0_{T,\xi} \bigoplus \mathrm{E}^{0,\perp}_{T,\xi}$ (resp. $\mathrm{E}^0_{\xi,\partial} = \mathrm{E}^0_{T,\xi,\partial} \bigoplus \mathrm{E}^{0,\perp}_{T,\xi,\partial}$), i.e.,

$$\begin{aligned} A^{(1)} &= \bar{p}_{T,\xi} A\, \bar{p}_{T,\xi} \ , \quad A^{(2)} = \bar{p}_{T,\xi} A\, \bar{p}^\perp_{T,\xi} \ , \\ A^{(3)} &= \bar{p}^\perp_{T,\xi} A\, \bar{p}_{T,\xi} \ , \quad A^{(4)} = \bar{p}^\perp_{T,\xi} A\, \bar{p}^\perp_{T,\xi} \ . \end{aligned} \tag{3.21}$$

$$( \text{ resp.} \qquad \begin{aligned} B^{(1)} &= \bar{p}_{T,\xi,\partial} B\, \bar{p}_{T,\xi,\partial} \ , \quad B^{(2)} = \bar{p}_{T,\xi,\partial} B\, \bar{p}^\perp_{T,\xi,\partial} \ , \\ B^{(3)} &= \bar{p}^\perp_{T,\xi,\partial} B\, \bar{p}_{T,\xi,\partial} \ , \quad B^{(4)} = \bar{p}^\perp_{T,\xi,\partial} B\, \bar{p}^\perp_{T,\xi,\partial} \ . \quad ) \end{aligned} \tag{3.22}$$

For $T > 0$, set

$$D_T = D^X + \sqrt{-1} T c(H), \quad B_T = B^X - \sqrt{-1} T c\left(\frac{\partial}{\partial r}\right) c(H). \tag{3.23}$$

Then $B_T$ is the canonical boundary operator associated to the Dirac type operator $D_T$ in the sense of (2.5). Let $D_{T,\xi}$ and $B_{T,\xi}$ be the restrictions of $D_T$ and $B_T$ on $\mathrm{E}^0_\xi$ and $\mathrm{E}^0_{\xi,\partial}$, respectively.

We now introduce a deformation of $D_{T,\xi}$ (resp. $B_{T,\xi}$) according to the decomposition (3.21) (resp. (3.22)).

**Definition 3.4** (cf. [5, Definition 3.2], [20, (1.39)]) *For any $T > 0$, $u \in [0,1]$, set*

$$\begin{aligned} D_{T,\xi}(u) &= D^{(1)}_{T,\xi} + D^{(4)}_{T,\xi} + u\big(D^{(2)}_{T,\xi} + D^{(3)}_{T,\xi}\big) \, , \\ B_{T,\xi}(u) &= B^{(1)}_{T,\xi} + B^{(4)}_{T,\xi} + u\big(B^{(2)}_{T,\xi} + B^{(3)}_{T,\xi}\big) \, . \end{aligned} \tag{3.24}$$

One verifies that $B_{T,\xi}(u)$ is the canonical boundary operator associated to $D_{T,\xi}(u)$ in the sense of (2.5).

## 3.4 Various estimates of the operators as $T \to +\infty$

We continue the discussion in the previous subsection. Corresponding to the involution $\tau$ on $S(U,L)$, for $\tau = \tau_s$ (resp. $\tau = \tau_e$), let $D^{X_H}_\xi$ be the restriction of the twisted Spin$^c$ Dirac operator $D^{X_H} \otimes R_+(1)$ (resp. $D^{X_H} \otimes R_-(1)$) on $\mathrm{F}^0_\xi$, and let $B^{X_H}_\xi$ be the restriction of the canonical boundary operator associated to $D^{X_H} \otimes R_+(1)$ (resp. $D^{X_H} \otimes R_-(1)$) on $\mathrm{F}^0_{\xi,\partial}$.

With (3.15), (3.23) and Propositions 3.1, 3.2, 3.3 at our hands, by proceeding exactly as in [3, Sections 8 and 9], we can show that the following estimates for $B^{(i)}_{T,\xi}$ $(1 \le i \le 4)$ hold.

**Proposition 3.5** (Compare with [5, Proposition 3.3]) *There exists $\varepsilon > 0$ such that*

(i) *As $T \longrightarrow +\infty$,*

$$J^{-1}_{T,\xi,\partial} B^{(1)}_{T,\xi} J_{T,\xi,\partial} = B^{X_H}_{\xi} + O\left(\frac{1}{\sqrt{T}}\right), \tag{3.25}$$

*where $O(\frac{1}{\sqrt{T}})$ denotes a first order differential operator whose coefficients are dominated by $\frac{C}{\sqrt{T}}$ $(C > 0)$.*

(ii) *There exist $C_1 > 0$, $C_2 > 0$ and $T_0 > 0$ such that for any $T \ge T_0$, any $s \in \mathrm{E}^{1,\perp}_{T,\xi,\partial}$ , $s' \in \mathrm{E}^{1}_{T,\xi,\partial}$ , then*

$$\begin{aligned} \left\|B^{(2)}_{T,\xi} s\right\|_{\mathrm{E}^0_{\xi,\partial}} &\le C_1\Big(\frac{1}{\sqrt{T}}\|s\|_{\mathrm{E}^1_{\xi,\partial}} + \|s\|_{\mathrm{E}^0_{\xi,\partial}}\Big), \\ \left\|B^{(3)}_{T,\xi} s'\right\|_{\mathrm{E}^0_{\xi,\partial}} &\le C_1\Big(\frac{1}{\sqrt{T}}\|s'\|_{\mathrm{E}^1_{\xi,\partial}} + \|s'\|_{\mathrm{E}^0_{\xi,\partial}}\Big), \end{aligned} \tag{3.26}$$

*and*

$$\left\|B^{(4)}_{T,\xi} s\right\|_{\mathrm{E}^0_{\xi,\partial}} \ge C_2\Big(\|s\|_{\mathrm{E}^1_{\xi,\partial}} + \sqrt{T}\,\|s\|_{\mathrm{E}^0_{\xi,\partial}}\Big). \tag{3.27}$$

From here, proceeding as in [5, Section 3(c)], we can deduce that there exists $T_1 > 0$ such that for $T \ge T_1$, each $B_{T,\xi}(u)$, for $u \in [0,1]$, is self-adjoint, elliptic and has discrete eigenvalues with finite multiplicity. Let $P_{T,\xi}(u)$ denote the Atiyah-Patodi-Singer projection associated to $B_{T,\xi}(u)$. In the following proposition, we will show that the boundary problems $(D_{T,\xi}(u), P_{T,\xi}(u))$, $u \in [0,1]$, are Fredholm when $T$ is large enough.

For any $T \ge T_1$ and $u \in [0,1]$, let

$$D_{\mathrm{APS},T,\xi}(u) : \left\{ s \in \mathrm{E}^1_{\xi} \,\middle|\, P_{T,\xi}(u)(s|_{\partial X}) = 0 \right\} \longrightarrow \mathrm{E}^0_{\xi}$$

be the uniquely determined extension of $D_{T,\xi}(u)$.

**Proposition 3.6** (Compare with [5, Proposition 3.5]) *There exists $T_2 > 0$ such that for any $u \in [0,1]$ and $T \ge T_2$, $D_{\mathrm{APS},T,\xi}(u)$ is a Fredholm operator.*

To prove Proposition 3.6, we modify the process in [5, Section 3(d)]. For the case where $s$ is supported in $X \backslash \mathscr{U}_{\epsilon'}$ $(0 < \epsilon' < \epsilon)$, we need an analogue of [5, Lemma 3.7]. As a matter of fact, using (3.15), (3.23) as well as Propositions 3.1, 3.2, 3.3 and proceeding exactly as in [3, Sections 8 and 9], we deduce the following interior estimates.

**Proposition 3.7** *There exists $\varepsilon > 0$ such that*

(i) *As* $T \longrightarrow +\infty$,

$$J_{T,\xi}^{-1}\, D_{T,\xi}^{(1)} J_{T,\xi} = D_{\xi}^{X_H} + O\left(\frac{1}{\sqrt{T}}\right), \tag{3.28}$$

*where* $O(\frac{1}{\sqrt{T}})$ *denotes a first order differential operator whose coefficients are dominated by* $\frac{C}{\sqrt{T}}$ $(C>0)$.

(ii) *There exist* $C_1' > 0$, $C_2' > 0$ *and* $T_0' > 0$ *such that for any* $T \geq T_0'$, *any* $s \in \mathrm{E}_{T,\xi}^{1,\perp}$, $s' \in \mathrm{E}_{T,\xi}^{1}$, *then*

$$\begin{aligned} \big\| D_{T,\xi}^{(2)} s \big\|_{\mathrm{E}_\xi^0} &\leq C_1' \left( \frac{\|s\|_{\mathrm{E}_\xi^1}}{\sqrt{T}} + \|s\|_{\mathrm{E}_\xi^0} \right), \\ \big\| D_{T,\xi}^{(3)} s' \big\|_{\mathrm{E}_\xi^0} &\leq C_1' \left( \frac{\|s'\|_{\mathrm{E}_\xi^1}}{\sqrt{T}} + \|s'\|_{\mathrm{E}_\xi^0} \right), \end{aligned} \tag{3.29}$$

*and*

$$\big\| D_{T,\xi}^{(4)} s \big\|_{\mathrm{E}_\xi^0} \geq C_2' \Big( \|s\|_{\mathrm{E}_\xi^1} + \sqrt{T}\, \|s\|_{\mathrm{E}_\xi^0} \Big). \tag{3.30}$$

With Proposition 3.7 at our hands, we can complete the proof of Proposition 3.6 in the same way as in the proof of [5, Proposition 3.5] by applying the gluing argument in [3, pp. 115-117].

### 3.5 A proof of Theorem 2.7

Let $D_\xi^{Y_H}$ (resp. $D_{\xi,\pm}^{Y_H}$) be the induced operator from $B_\xi^{X_H}$ (resp. $B_{\xi,\pm}^{X_H}$) through $\pi_{X_H}$. For any $\xi \in \mathbb{Z}$, choose $a_\xi > 0$ be such that

$$\mathrm{Spec}(D_{\xi,+}^{Y_H}) \cap [-2a_\xi, 2a_\xi] \subseteq \{0\}. \tag{3.31}$$

By restricting $B_{T,\xi}$ on $\Gamma(X, S_+(U,L)\otimes W)|_{\partial X}$, we know the estimates in Proposition 3.5 still hold for $B_{T,\xi,+}$. Using the techniques in [3, Section 9], one gets the following analogue of [3, (9.156)].

**Lemma 3.8** *There exist* $T_3 > 0$ *such that for any* $T \geq T_3$,

$$\#\big\{\lambda \in \mathrm{Spec}(B_{T,\xi,+}) \,\big|\, \lambda \in \big[-a_\xi, a_\xi\big]\big\} = \dim\big(\ker B_{\xi,+}^{X_H}\big) = k \dim\big(\ker D_{\xi,+}^{Y_H}\big).$$

We first assume that $D_{\xi,+}^{Y_H}$ is invertible, then $B_{\xi,+}^{X_H}$ is invertible. Moreover, we have the following analogue of [5, Proposition 3.8].

**Proposition 3.9** *If* $D_{\xi,+}^{Y_H}$ *is invertible, then there exists* $T_4 > 0$ *such that for any* $T \geq T_4$, $u \in [0,1]$, *the boundary operator* $B_{T,\xi,+}(u)$ *is invertible.*

By Propositions 3.6 and 3.9, we have a continuous family of Fredholm operators $\{D_{\mathrm{APS},T,\xi}(u)\}_{0\leq u\leq 1}$ when $T$ is large enough. Furthermore, by Proposition 3.9 and Green's formula, we know that the operators $D_{\mathrm{APS},T,\xi}(u)$, $0 \leq u \leq 1$, are self-adjoint. Under the assumption of Proposition 3.9, we have

$$\text{APS-ind}\, D_{T,\xi}(u) = \mathrm{Tr}\left[\tau|_{\ker(D_{\mathrm{APS},T,\xi}(u))}\right].$$

**Theorem 3.10** (Compare with [20, (1.43)]) *If $D^{Y_H}_{\xi,+}$ is invertible, then there exists $T_5>0$ such that for any $T\geq T_5$, the following identity holds,*

$$\text{APS-ind}(D_{T,\xi})=\sum_\alpha(-1)^{\sum_{0<v}\dim N_v}\,\text{APS-ind}\big(D^{X_{H,\alpha}}_\xi\big)\ . \tag{3.32}$$

**Proof** From above discussions and the homotopy invariance of the index of Fredholm operators, we get that

$$\text{APS-ind}(D_{T,\xi})=\text{APS-ind}\big(D_{T,\xi}(0)\big). \tag{3.33}$$

Let $P_{T,\xi,1}$ (resp. $P_{T,\xi,4}$) be the Atiyah-Patodi-Singer projection associated to $B^{(1)}_{T,\xi}$ (resp. $B^{(4)}_{T,\xi}$) acting on $\mathrm{E}^0_{T,\xi,\partial}$ (resp. $\mathrm{E}^{0,\perp}_{T,\xi,\partial}$). Using Proposition 3.5 and proceeding as in the proof of [5, Proposition 3.5], one sees easily that the boundary problems $(D^{(1)}_{T,\xi},P_{T,\xi,1})$ and $(D^{(4)}_{T,\xi},P_{T,\xi,4})$ are both Fredholm. Furthermore, we can deduce that for $T$ large enough,

$$\text{APS-ind}\big(D^{(4)}_{T,\xi}\big)=0. \tag{3.34}$$

On the other hand, for $T$ large enough and $u\in[0,1]$, set

$$\begin{aligned}D^{X_H}_{T,\xi}(u)&=u\,D^{X_H}_\xi+(1-u)\,J^{-1}_{T,\xi}\,D^{(1)}_{T,\xi}\,J_{T,\xi}\ ,\\ B^{X_H}_{T,\xi}(u)&=u\,B^{X_H}_\xi+(1-u)\,J^{-1}_{T,\xi,\partial}\,B^{(1)}_{T,\xi}\,J_{T,\xi,\partial}\ .\end{aligned} \tag{3.35}$$

Following the proof of [5, Proposition 3.8], one sees that when $T$ is large enough, $B^{X_H}_{T,\xi}(u)$ is invertible for every $u\in[0,1]$.

We denote by $P^{X_H}_{T,\xi}(u)$ the Atiyah-Patodi-Singer projection associated to $B^{X_H}_{T,\xi}(u)$. Using (3.25), (3.28) and the same argument as in the proof of [5, Proposition 3.5], one sees that when $T$ is large enough, $\big(D^{X_H}_{T,\xi}(u),P^{X_H}_{T,\xi}(u)\big)$, $u\in[0,1]$, form a continuous family of formally self-adjoint Fredholm boundary problems. Thus by the homotopy invariance of the index of Fredholm operators, one gets

$$\text{APS-ind}(D^{X_H}_\xi)=\text{APS-ind}\big(J^{-1}_{T,\xi}\,D^{(1)}_{T,\xi}\,J_{T,\xi}\big). \tag{3.36}$$

From (2.16), (2.17), (3.14) and (3.19), one gets

$$J^{-1}_{T,\xi}\circ\tau\circ J_{T,\xi}=(-1)^{\sum_{0<v}\dim N_v}\tau,\quad \text{where }\tau=\tau_s\text{ or }\tau_e\ . \tag{3.37}$$

From (3.24) and (3.33)-(3.37), one finds

$$\text{APS-ind}(D_{T,\xi})=\sum_\alpha(-1)^{\sum_{0<v}\dim N_v}\,\text{APS-ind}\big(D^{X_{H,\alpha}}_\xi\big)\ . \tag{3.38}$$

The proof of Theorem 3.10 is completed.

In general, $\dim\ker\big(D^{Y_H}_{\xi,+}\big)$ need not be zero. To control the eigenvalues of $B_{T,\xi,+}$ lying in $[-a_\xi,a_\xi]$, we use the method in [5, Section 4(a)] to perturb the Dirac operators under consideration.

Let $\epsilon > 0$ be sufficiently small so that there exists an $S^1$-invariant smooth function $f : X \longrightarrow \mathbb{R}$ such that $f \equiv 1$ on $\mathscr{U}_{\epsilon/3}$ and $f \equiv 0$ outside of $\mathscr{U}_{2\epsilon/3}$.

Let $D^{X_H}_{\xi,-a_\xi}$ be the Dirac type operator defined by

$$D^{X_H}_{\xi,-a_\xi} = D^{X_H}_{\xi} - a_\xi f c\left(\frac{\partial}{\partial r}\right), \tag{3.39}$$

where for $\tau = \tau_s$ (resp. $\tau_e$), $D^{X_H}_{\xi,-a_\xi}$ is considered as a differential operator acting on $\Gamma\big(X_H, S(TX_H \oplus V_0^{\mathbb{R}}, L_F) \otimes R_{+,\xi}\big)$ (resp. $\Gamma\big(X_H, S(TX_H \oplus V_0^{\mathbb{R}}, L_F) \otimes R_{-,\xi}\big)$).

By Theorem 2.6, we get

$$\text{APS-ind}\big(D^{X_{H,\alpha}}_{\xi,-a_\xi}\big) - \text{APS-ind}\big(D^{X_{H,\alpha}}_{\xi}\big) = -\,\mathrm{sf}\big\{B^{X_{H,\alpha}}_{\xi,+} - a_\xi t \,\big|\, 0 \le t \le 1\big\}\ . \tag{3.40}$$

By (3.31), the right hand side of (3.40) is equal to zero.

For any $T \in \mathbb{R}$, let $D_{T,-a_\xi} : \Gamma(X, S(U,L) \otimes W) \longrightarrow \Gamma(X, S(U,L) \otimes W)$ be the Dirac type operator defined by

$$D_{T,-a_\xi} = D_T - a_\xi f c\left(\frac{\partial}{\partial r}\right)\ . \tag{3.41}$$

Let $D_{T,\xi,-a_\xi}$ be its restriction to the weight-$\xi$ subspace.

Let $B^{X_H}_{\xi,-a_\xi}$ be the canonical boundary operator of $D^{X_H}_{\xi,-a_\xi}$ in the sense of (2.5). Since $D^{Y_H}_{\xi} - a_\xi$, which is the induced operator from $B^{X_H}_{\xi,-a_\xi}$ through $\pi_{X_H}$, is invertible, by the proof of Theorem 3.10, we get when $T$ is large enough,

$$\text{APS-ind}(D_{T,\xi,-a_\xi}) = \sum_{\alpha} (-1)^{\sum_{0<v} \dim N_v}\, \text{APS-ind}\big(D^{X_{H,\alpha}}_{\xi,-a_\xi}\big)\ . \tag{3.42}$$

By Theorem 2.6, we deduce that, for $\xi \in \mathbb{Z}$,

$$\text{APS-ind}(D_{T,\xi,-a_\xi}) \equiv \text{APS-ind}(D_{T,\xi}) \mod k\mathbb{Z}. \tag{3.43}$$

From (3.40), (3.42) and (3.43), we get

$$\text{APS-ind}(D_{T,\xi}) \equiv \sum_{\alpha} (-1)^{\sum_{0<v} \dim N_v}\, \text{APS-ind}\big(D^{X_{H,\alpha}}_{\xi}\big) \mod k\mathbb{Z}\ . \tag{3.44}$$

On the other hand, by Theorem 2.6, one knows the mod $k$ invariance of APS-ind$(D_{T,\xi})$ with respect to $T \in \mathbb{R}$, from which and (3.32), (3.44), one gets

$$\text{APS-ind}(D, \xi) \equiv \sum_{\alpha} (-1)^{\sum_{0<v} \dim N_v}\, \text{APS-ind}\big(D^{X_{H,\alpha}}_{\xi}\big) \mod k\mathbb{Z}\ . \tag{3.45}$$

By taking $\tau = \tau_s$ (resp. $\tau_e$), we get the first equation of (2.22) (resp. (2.23)). To get the second equation of (2.22) (resp. (2.23)), we only need to apply the first equation of (2.22) (resp. (2.23)) to the case where the circle action on $X$ defined by the inverse of the original circle action on $X$.

The proof of Theorem 2.7 is completed.

# 4 Rigidity and vanishing theorems on $\mathbb{Z}/k$ Spin$^c$ manifolds

In this section, combining the $S^1$-equivariant index theorem we have established in Section 2 with the methods of [19], we prove the rigidity and vanishing theorems for $\mathbb{Z}/k$ Spin$^c$ manifolds, which generalize [16, Theorems A and B]. As will be pointed out in Remark 4.3, when applied to $\mathbb{Z}/k$ spin manifolds, our results provide a resolution to a conjecture of Devote [6]. Both the statement of the main results and their proof are inspired by the corresponding results as well as their proof for closed manifolds in [19, 20]. As explained in Section 2.1, when we regard the considered $\mathbb{Z}/k$ manifold as a quotient space which has the homotopy type of a CW complex, by using splitting principle [11, Chapter 17], we can apply the topological arguments in [19, 20] in our $\mathbb{Z}/k$ context with little modification. Thus we will only indicate the main steps of the proof of our results.

This section is organized as follows. In Section 4.1, we state our main results, the rigidity and vanishing theorems for $\mathbb{Z}/k$ Spin$^c$ manifolds. In Section 4.2, we present two recursive formulas which will be used to prove our main results stated in Section 4.1. In Section 4.3, we prove the rigidity and vanishing theorems for $\mathbb{Z}/k$ Spin$^c$ manifolds.

## 4.1 Rigidity and vanishing theorems

Let $X$ be a $2l$-dimensional $\mathbb{Z}/k$-manifold, which admits a nontrivial $\mathbb{Z}/k$ circle action. We assume that $TX$ has a $\mathbb{Z}/k$ $S^1$-equivariant Spin$^c$ structure. Let $V$ be an even dimensional $\mathbb{Z}/k$ real vector bundle over $X$. We assume that $V$ has a $\mathbb{Z}/k$ $S^1$-equivariant spin structure. Let $W$ be a $\mathbb{Z}/k$ $S^1$-equivariant complex vector bundle of rank $r$ over $X$. Let $K_W = \det(W)$ be the determinant line bundle of $W$, which is obviously a $\mathbb{Z}/k$ complex line bundle.

Let $K_X$ be the $\mathbb{Z}/k$ complex line bundle over $X$ induced by the Spin$^c$ structure of $TX$. Let $S(TX, K_X)$ be the complex spinor bundle of $(TX, K_X)$ as in Section 2.1. Let $S(V) = S^+(V) \oplus S^-(V)$ be the spinor bundle of $V$.

Let $K(X)$ be the $K$-group of $\mathbb{Z}/k$ complex vector bundles over $X$ (cf. [7, pp. 285]). We define the following elements in $K(X)[[q^{1/2}]]$ (cf. [19, (2.1)])

$$\begin{aligned}
R_1(V) &= \Big(S^+(V) + S^-(V)\Big) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}(V) \ , \\
R_2(V) &= \Big(S^+(V) - S^-(V)\Big) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{-q^n}(V) \ , \\
R_3(V) &= \bigotimes_{n=1}^{\infty} \Lambda_{-q^{n-1/2}}(V) \ , \\
R_4(V) &= \bigotimes_{n=1}^{\infty} \Lambda_{q^{n-1/2}}(V) \ .
\end{aligned} \tag{4.1}$$

For $N \in \mathbb{N}$, let $y = e^{2\pi i/N} \in \mathbb{C}$ be an $N$th root of unity. Set

$$Q_y(W) = \bigotimes_{n=0}^{\infty} \Lambda_{-y^{-1}\cdot q^n}(\overline{W}) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{-y\cdot q^n}(W) \in K(X)[[q]] \ . \tag{4.2}$$

Then there exist $Q_\ell(W) \in K(X)[[q]]$, $0 \leq \ell < N$ such that

$$Q_y(W) = \sum_{\ell=0}^{N-1} y^\ell Q_\ell(W). \tag{4.3}$$

Let $H^*_{S^1}(X, \mathbb{Z}) = H^*(X \times_{S^1} ES^1, \mathbb{Z})$ denote the $S^1$-equivariant cohomology group of $X$, where $ES^1$ is the universal $S^1$-principal bundle over the classifying space $BS^1$ of $S^1$. So $H^*_{S^1}(X, \mathbb{Z})$ is a module over $H^*(BS^1, \mathbb{Z})$ induced by the projection $\overline{\pi} : X \times_{S^1} ES^1 \to BS^1$. Let $p_1(\cdot)_{S^1}$ and $\omega_2(\cdot)_{S^1}$ denote the first $S^1$-equivariant pontrjagin class and the second $S^1$-equivariant Stiefel-Whitney class, respectively. As $V \times_{S^1} ES^1$ is spin over $X \times_{S^1} ES^1$, one knows that $\frac{1}{2}p_1(V)_{S^1}$ is well defined in $H^*_{S^1}(X, \mathbb{Z})$ (cf. [24, pp. 456-457]). Recall that

$$H^*(BS^1, \mathbb{Z}) = \mathbb{Z}[[u]] \tag{4.4}$$

with $u$ a generator of degree 2.

In the following, we denote by $D^X \otimes R$ the twisted Spin$^c$ Dirac operator acting on $S(TX, K_X) \otimes R$ (cf. Definition 2.2). Furthermore, for $m \in \frac{1}{2}\mathbb{Z}$, $h \in \mathbb{Z}$ and $R(q) = \sum_{m \in \frac{1}{2}\mathbb{Z}} q^m R_m \in K_{S^1}(X)[[q^{1/2}]]$, we will also denote APS-ind$(D^X \otimes R_m, h)$ (cf. (2.9)) by APS-ind$(D^X \otimes R(q), m, h)$.

Now we can state the main results of this paper as follows, which generalize [16, Theorems A and B] to the case of $\mathbb{Z}/k$ Spin$^c$ manifolds.

**Theorem 4.1** *Assume that $\omega_2(W)_{S^1} = \omega_2(TX)_{S^1}$, $\frac{1}{2}p_1(V + W - TX)_{S^1} = e \cdot \overline{\pi}^* u^2$ ($e \in \mathbb{Z}$) in $H^*_{S^1}(X, \mathbb{Z})$, and $c_1(W) = 0 \mod (N)$. For $0 \leq \ell < N$, $i = 1$, 2, 3, 4, consider the $S^1$-equivariant twisted Spin$^c$ Dirac operators*

$$D^X \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes \bigotimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TX) \otimes R_i(V) \otimes Q_\ell(W) \ . \tag{4.5}$$

(i) *If $e = 0$, then these operators are rigid in $\mathbb{Z}/k$ category.*

(ii) *If $e < 0$, then they have vanishing properties in $\mathbb{Z}/k$ category.*

**Remark 4.2** (Compare with [19, Remark 2.1]) *As $\omega_2(W)_{S^1} = \omega_2(TX)_{S^1}$, $c_1(K_W \otimes K_X^{-1})_{S^1} = 0 \mod (2)$. We note that in our case, $X \times_{S^1} ES^1$ has the homotopy type of a CW complex* [21]*. By* [9, Corollary 1.2]*, the circle action on $X$ can be lifted to $(K_W \otimes K_X^{-1})^{1/2}$ and is compatible with the circle action on $K_W \otimes K_X^{-1}$.*

**Remark 4.3** *If $X$ is a $\mathbb{Z}/k$ spin manifold, by taking $V = TX$, $W = 0$ and $i = 3$ in Theorem 4.1, we resolve a conjecture of* [6].

Actually, as in [19], our proof of Theorem 4.1 works under the following slightly weaker hypothesis. Let us first explain some notations.

For each $n > 1$, consider $\mathbb{Z}_n \subset S^1$, the cyclic subgroup of order $n$. We have the $\mathbb{Z}_n$-equivariant cohomology of $X$ defined by $H^*_{\mathbb{Z}_n}(X,\mathbb{Z}) = H^*(X \times_{\mathbb{Z}_n} ES^1, \mathbb{Z})$, and there is a natural "forgetful" map $\alpha(S^1, \mathbb{Z}_n) : X \times_{\mathbb{Z}_n} ES^1 \to X \times_{S^1} ES^1$ which induces a pullback $\alpha(S^1,\mathbb{Z}_n)^* : H^*_{S^1}(X,\mathbb{Z}) \to H^*_{\mathbb{Z}_n}(X,\mathbb{Z})$. We denote by $\alpha(S^1, 1)$ the arrow which forgets the $S^1$-action. Thus $\alpha(S^1,1)^* : H^*_{S^1}(X,\mathbb{Z}) \to H^*(X,\mathbb{Z})$ is induced by the inclusion of $X$ into $X \times_{S^1} ES^1$ as a fiber over $BS^1$.

Finally, note that if $\mathbb{Z}_n$ acts trivially on a space $M$, then there is a new arrow $t^* : H^*(M,\mathbb{Z}) \to H^*_{\mathbb{Z}_n}(M,\mathbb{Z})$ induced by the projection $t : M \times_{\mathbb{Z}_n} ES^1 = M \times B\mathbb{Z}_n \to M$.

Let $\mathbb{Z}_\infty = S^1$. For each $1 < n \leq +\infty$, let $i : X(n) \to X$ be the inclusion of the fixed point set of $\mathbb{Z}_n \subset S^1$ in $X$, and so $i$ induces $i_{S^1} : X(n) \times_{S^1} ES^1 \to X \times_{S^1} ES^1$.

In the rest of this paper, we use the same assumption as in [19, (2.4)]. Suppose that there exists some integer $e \in \mathbb{Z}$ such that for $1 < n \leq +\infty$,

$$\begin{aligned} &\alpha(S^1,\mathbb{Z}_n)^* \circ i^*_{S^1}\Big(\frac{1}{2}p_1(V+W-TX)_{S^1} - e\cdot \overline{\pi}^* u^2\Big) \\ &\qquad = t^* \circ \alpha(S^1,1)^* \circ i^*_{S^1}\Big(\frac{1}{2}p_1(V+W-TX)_{S^1}\Big)\ . \end{aligned} \tag{4.6}$$

Remark that the relation (4.6) clearly follows from the hypothesis of Theorem 4.1 by pulling back and forgetting. Thus it is a weaker hypothesis.

Let $G_y$ be the multiplicative group generated by $y$. Following Witten [25], we consider the action of $y_0 \in G_y$ on $W$ (resp. $\overline{W}$) by multiplication by $y_0$ (resp. $y_0^{-1}$) on $W$ (resp. $\overline{W}$). Set

$$Q(W) = \bigotimes_{n=0}^{\infty} \Lambda_{-q^n}(\overline{W}) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{-q^n}(W) \in K(X)[[q]]\ . \tag{4.7}$$

Then the actions of $G_y$ on $W$ and $\overline{W}$ naturally induce the action of $G_y$ on $Q(W)$. Clearly, $y \cdot Q(W) = Q_y(W)$. By (4.3), we know that for $0 \leq \ell < N$,

$$y_0 \cdot Q_\ell(W) = y_0^\ell Q_\ell(W), \quad \text{where } \ y_0 \in G_y. \tag{4.8}$$

In what follows, for $m \in \frac{1}{2}\mathbb{Z}$, $0 \leq \ell < N$, $h \in \mathbb{Z}$ and $R(q) \in K_{S^1}(X)[[q^{1/2}]]$, we will denote APS-ind$(D^X \otimes R(q) \otimes Q_\ell(W), m, h)$ by APS-ind$(D^X \otimes R(q) \otimes Q(W), m, \ell, h)$.

We can now state a slightly more general version of Theorem 4.1.

**Theorem 4.4** *Under the hypothesis* (4.6)*, consider the* $S^1 \times G_y$*-equivariant twisted Spin*$^c$ *Dirac operators*

$$D^X \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes \bigotimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TX) \otimes R_i(V) \otimes Q(W)\ . \tag{4.9}$$

(i) *If $e=0$, for $m\in\frac{1}{2}\mathbb{Z}$, $h\in\mathbb{Z}$, $h\neq 0$, $0\le \ell<N$, one has*

$$\begin{aligned}\text{APS-ind}\Big(D^X\otimes(K_W\otimes K_X^{-1})^{1/2}\otimes\bigotimes_{n=1}^{\infty}\operatorname{Sym}_{q^n}(TX)\\ \otimes R_i(V)\otimes Q(W),m,\ell,h\Big)\equiv 0\mod k\mathbb{Z}\ .\end{aligned}\tag{4.10}$$

(ii) *If $e<0$, for $m\in\frac{1}{2}\mathbb{Z}$, $h\in\mathbb{Z}$, $0\le \ell<N$, one has*

$$\begin{aligned}\text{APS-ind}\Big(D^X\otimes(K_W\otimes K_X^{-1})^{1/2}\otimes\bigotimes_{n=1}^{\infty}\operatorname{Sym}_{q^n}(TX)\\ \otimes R_i(V)\otimes Q(W),m,\ell,h\Big)\equiv 0\mod k\mathbb{Z}\ .\end{aligned}\tag{4.11}$$

*In particular, one has*

$$\begin{aligned}\text{APS-ind}\Big(D^X\otimes(K_W\otimes K_X^{-1})^{1/2}\otimes\bigotimes_{n=1}^{\infty}\operatorname{Sym}_{q^n}(TX)\\ \otimes R_i(V)\otimes Q(W),m,\ell\Big)\equiv 0\mod k\mathbb{Z}\ .\end{aligned}\tag{4.12}$$

The rest of this paper is devoted to a proof of Theorem 4.4.

### 4.2 Several intermediate results

Recall that $X_H=\{X_{H,\alpha}\}$ be the fixed point set of the circle action. As in [19, pp. 940], we may and we will assume that

$$\begin{aligned}TX|_{X_H}&=TX_H\oplus\bigoplus_{v>0}N_v,\\ TX|_{X_H}\otimes_{\mathbb{R}}\mathbb{C}&=TX_H\otimes_{\mathbb{R}}\mathbb{C}\oplus\bigoplus_{v>0}\big(N_v\oplus\overline{N}_v\big),\end{aligned}\tag{4.13}$$

where $N_v$ is the complex vector bundles on which $S^1$ acts by sending $g$ to $g^v$. We also assume that

$$V|_{X_H}=V_0^{\mathbb{R}}\oplus\bigoplus_{v>0}V_v\ ,\qquad W|_{X_H}=\bigoplus_{v}W_v\ ,\tag{4.14}$$

where $V_v$, $W_v$ are complex vector bundles on which $S^1$ acts by sending $g$ to $g^v$, and $V_0^{\mathbb{R}}$ is a real vector bundle on which $S^1$ acts as identity.

By (4.13), as in (2.16) or (2.17), there is a natural $\mathbb{Z}/k$ isomorphism between the $\mathbb{Z}_2$-graded $C(TX)$-Clifford modules over $X_H$,

$$S(TX,K_X)|_{X_H}\simeq S\Big(TX_H,K_X\otimes_{v>0}(\det N_v)^{-1}\Big)\,\widehat{\otimes}\,\widehat{\bigotimes}_{v>0}\Lambda N_v\ .\tag{4.15}$$

For a $\mathbb{Z}/k$ complex vector bundle $R$ over $X_H$, let $D^{X_H}\otimes R$, $D^{X_{H,\alpha}}\otimes R$ be the twisted Spin$^c$ Dirac operators on $S(TX_H,K_X\otimes_{v>0}(\det N_v)^{-1})\otimes R$ over $X_H$, $X_{H,\alpha}$, respectively (cf. Definition 2.2).

For $i = 1, 2, 3, 4$, we set

$$R^i = (K_W \otimes K_X^{-1})^{1/2} \otimes R_i(V) \otimes Q(W). \tag{4.16}$$

Then by Theorem 2.7, we can express the global Atiyah-Patodi-Singer index via the Atiyah-Patodi-Singer indices on the fixed point set up to $k\mathbb{Z}$.

**Proposition 4.5** (Compare with [19, Proposition 2.1]) *For $m \in \frac{1}{2}\mathbb{Z}$, $h \in \mathbb{Z}$, $1 \le i \le 4$, $0 \le \ell < N$, we have*

$$\begin{aligned}
&\text{APS-ind}\Big(D^X \otimes \otimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TX) \otimes R^i, m, \ell, h\Big) \\
&\equiv \sum_{\alpha} (-1)^{\sum_{v>0} \dim N_v} \,\text{APS-ind}\Big(D^{X_{H,\alpha}} \otimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TX) \otimes R^i \\
&\qquad\qquad \otimes \mathrm{Sym}\left(\oplus_{v>0} N_v\right) \otimes_{v>0} \det N_v, m, \ell, h\Big) \mod k\mathbb{Z}\ .
\end{aligned} \tag{4.17}$$

To simplify the notations, we use the same convention as in [19, pp. 945]. For $n_0 \in \mathbb{N}^*$, we define a number operator $P$ on $K_{S^1}(X)[[q^{\frac{1}{n_0}}]]$ in the following way: if $R(q) = \oplus_{n \in \frac{1}{n_0}\mathbb{Z}} R_n q^n \in K_{S^1}(X)[[q^{\frac{1}{n_0}}]]$, then $P$ acts on $R(q)$ by multiplication by $n$ on $R_n$. From now on, we simply denote $\mathrm{Sym}_{q^n}(TX)$, $\Lambda_{q^n}(V)$ and $\Lambda_{q^n}(W)$ by $\mathrm{Sym}(TX_n)$, $\Lambda(V_n)$ and $\Lambda(W_n)$, respectively. In this way, $P$ acts on $TX_n$, $V_n$ and $W_n$ by multiplication by $n$, and the actions of $P$ on $\mathrm{Sym}(TX_n)$, $\Lambda(V_n)$ and $\Lambda(W_n)$ are naturally induced by the corresponding actions of $P$ on $TX_n$, $V_n$ and $W_n$. So the eigenspace of $P = n$ is just given by the coefficient of $q^n$ of the corresponding element $R(q)$. For $R(q) = \oplus_{n \in \frac{1}{n_0}\mathbb{Z}} R_n q^n \in K_{S^1}(X)[[q^{\frac{1}{n_0}}]]$, we will also denote $\text{APS-ind}\big(D^X \otimes R_m, h\big)$ by $\text{APS-ind}\big(D^X \otimes R(q), m, h\big)$.

Recall that $H$ is the Killing vector field on $X$ corresponding to $\mathcal{H}$, the canonical basis of $\mathrm{Lie}(S^1)$. If $E$ is a $\mathbb{Z}/k$ $S^1$-equivariant vector bundle over $X$, let $\mathscr{L}_H$ denote the corresponding Lie derivative along $H$ acting on $\Gamma(X_H, E|_{X_H})$. The weight of the circle action on $\Gamma(X_H, E|_{X_H})$ is given by the action

$$\boldsymbol{J}_H = \frac{1}{2\pi\sqrt{-1}} \mathscr{L}_H.$$

Recall that the $\mathbb{Z}_2$-grading on $S(TX, K_X) \otimes_{n=1}^{\infty} \mathrm{Sym}(TX_n)$ is induced by the $\mathbb{Z}_2$-grading on $S(TX, K_X)$. Write

$$\begin{aligned}
&\mathcal{F}^0(X) = \bigotimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TX) \otimes \mathrm{Sym}(\oplus_{v>0} N_v) \otimes_{v>0} \det N_v\ , \\
&F_V^1 = S(V) \otimes \bigotimes_{n=1}^{\infty} \Lambda(V_n)\ , \qquad F_V^2 = \bigotimes_{n \in \mathbb{N} + \frac{1}{2}} \Lambda(V_n)\ , \\
&Q^1(W) = \bigotimes_{n=0}^{\infty} \Lambda(\overline{W}_n) \otimes \bigotimes_{n=1}^{\infty} \Lambda(W_n)\ .
\end{aligned} \tag{4.18}$$

There are two natural $\mathbb{Z}_2$-gradings on $F_V^1$, $F_V^2$ (resp. $Q^1(W)$). The first grading is induced by the $\mathbb{Z}_2$-grading of $S(V)$ and the forms of homogeneous degrees in

$\otimes_{n=1}^{\infty}\Lambda(V_n)$, $\otimes_{n\in\mathbb{N}+\frac{1}{2}}\Lambda(V_n)$ (resp. $Q^1(W)$). We define $\tau_e|_{F_V^{i\pm}} = \pm 1$ ($i = 1, 2$) (resp. $\tau_1|_{Q^1(W)^\pm} = \pm 1$) to be the involution defined by this $\mathbb{Z}_2$-grading. The second grading is the one for which $F_V^i$ ($i = 1, 2$) are purely even, i.e., $F_V^{i+} = F_V^i$. We denote by $\tau_s = \mathrm{id}$ the involution defined by this $\mathbb{Z}_2$-grading. Then the coefficient of $q^n$ ($n \in \frac{1}{2}\mathbb{Z}$) in (4.1) of $R_1(V)$ or $R_2(V)$ (resp. $R_3(V)$, $R_4(V)$ or $Q(W)$) is exactly the $\mathbb{Z}_2$-graded $\mathbb{Z}/k$ vector subbundle of $(F_V^1, \tau_s)$ or $(F_V^1, \tau_e)$ (resp. $(F_V^2, \tau_e)$, $(F_V^2, \tau_s)$ or $(Q^1(W), \tau_1)$), on which $P$ acts by multiplication by $n$.

Furthermore, we denote by $\tau_e$ (resp. $\tau_s$) the $\mathbb{Z}_2$-grading on $S(TX, K_X) \otimes \otimes_{n=1}^{\infty}\mathrm{Sym}(TX_n) \otimes F_V^i$ ($i = 1, 2$) induced by the above $\mathbb{Z}_2$-gradings. We will denote by $\tau_{e1}$ (resp. $\tau_{s1}$) the $\mathbb{Z}_2$-grading on $S(TX, K_X) \otimes \otimes_{n=1}^{\infty}\mathrm{Sym}(TX_n) \otimes F_V^i \otimes Q^1(W)$ ($i = 1, 2$) defined by

$$\tau_{e1} = \tau_e \widehat{\otimes} \tau_1 \ , \quad \tau_{s1} = \tau_s \widehat{\otimes} \tau_1 \ . \tag{4.19}$$

Let $h^{V_v}$ be the Hermitian metric on $V_v$ induced by the metric $h^V$ on $V$. In the following, we identity $\Lambda V_v$ with $\Lambda \overline{V}_v^*$ by using the Hermitian metric $h^{V_v}$ on $V_v$. By (4.14), as in (4.15), there is a natural $\mathbb{Z}/k$ isomorphism between the $\mathbb{Z}_2$-graded $C(V)$-Clifford modules over $X_H$,

$$S(V)|_{X_H} \simeq S\Big(V_0^{\mathbb{R}}, \otimes_{v>0}(\det V_v)^{-1}\Big) \otimes \widehat{\bigotimes}_{v>0} \Lambda V_v \ . \tag{4.20}$$

Let $V_0 = V_0^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. By using the above notations, we rewrite (4.18) on the fixed point set $X_H$,

$$\begin{aligned}
\mathcal{F}^0(X) &= \bigotimes_{n=1}^{\infty}\mathrm{Sym}(TX_H) \otimes \bigotimes_{n=1}^{\infty}\mathrm{Sym}\Big(\oplus_{v>0}\big(N_{v,n} \oplus \overline{N}_{v,n}\big)\Big) \\
&\qquad\qquad \otimes \mathrm{Sym}(\oplus_{v>0}N_v) \otimes_{v>0} \det N_v \ , \\
F_V^1 &= \bigotimes_{n=1}^{\infty}\Lambda\Big(V_{0,n} \oplus \oplus_{v>0}(V_{v,n} \oplus \overline{V}_{v,n})\Big) \\
&\qquad\qquad \otimes S\Big(V_0^{\mathbb{R}}, \otimes_{v>0}(\det V_v)^{-1}\Big) \otimes_{v>0} \Lambda V_{v,0} \ , \\
F_V^2 &= \bigotimes_{n\in\mathbb{N}+\frac{1}{2}} \Lambda\Big(V_{0,n} \oplus \oplus_{v>0}(V_{v,n} \oplus \overline{V}_{v,n})\Big) \ , \\
Q^1(W) &= \bigotimes_{n=0}^{\infty}\Lambda(\oplus_v \overline{W}_{v,n}) \otimes \bigotimes_{n=1}^{\infty}\Lambda(\oplus_v W_{v,n}) \ .
\end{aligned} \tag{4.21}$$

We introduce the same shift operators as in [19, Section 3.2], which follow [24] in spirit. For $p \in \mathbb{N}$, we set

$$\begin{aligned}
&r_* : N_{v,n} \to N_{v,n+pv} \ , \quad r_* : \overline{N}_{v,n} \to \overline{N}_{v,n-pv} \ , \\
&r_* : V_{v,n} \to V_{v,n+pv} \ , \quad r_* : \overline{V}_{v,n} \to \overline{V}_{v,n-pv} \ , \\
&r_* : W_{v,n} \to W_{v,n+pv} \ , \quad r_* : \overline{W}_{v,n} \to \overline{W}_{v,n-pv} \ .
\end{aligned} \tag{4.22}$$

Furthermore, for $p \in \mathbb{N}$, we introduce the following elements in $K_{S^1}(X_H)[[q]]$ (cf. [19, (3.6)]),

$$\begin{aligned}
\mathcal{F}_p(X) &= \bigotimes_{n=1}^{\infty} \mathrm{Sym}(TX_H) \otimes \bigotimes_{v>0} \Big( \bigotimes_{n=1}^{\infty} \mathrm{Sym}\,(N_{v,n}) \bigotimes_{n>pv} \mathrm{Sym}\,(\overline{N}_{v,n}) \Big) \ , \\
\mathcal{F}'_p(X) &= \bigotimes_{v>0} \bigotimes_{0\le n\le pv} \Big( \mathrm{Sym}\,(N_{v,-n}) \otimes \det(N_v) \Big) \ , \\
\mathcal{F}^{-p}(X) &= \mathcal{F}_p(X) \otimes \mathcal{F}'_p(X) \ .
\end{aligned} \tag{4.23}$$

Note that when $p = 0$, $\mathcal{F}^{-p}(X)$ is exactly the $\mathcal{F}^0(X)$ in (4.21). The $\mathbb{Z}_2$-grading on $S(TX_H, K_X \otimes_{v>0} (\det N_v)^{-1}) \otimes \mathcal{F}^{-p}(X)$ is induced by the $\mathbb{Z}_2$-grading on $S(TX_H, K_X \otimes_{v>0} (\det N_v)^{-1})$.

As in [19, (2.9)], we write

$$\begin{aligned}
L(N) &= \otimes_{v>0} (\det N_v)^v \ , \quad L(V) = \otimes_{v>0} (\det V_v)^v \ , \\
L(W) &= \otimes_{v\neq 0} (\det W_v)^v \ , \quad L = L(N)^{-1} \otimes L(V) \otimes L(W) \ .
\end{aligned} \tag{4.24}$$

Using the similar $\mathbb{Z}/k$ $S^1$-equivariant isomorphism of complex vector bundles as in [20, (3.14)] and the similar $\mathbb{Z}/k$ $G_y \times S^1$-equivariant isomorphism of complex vector bundles as in [19, (3.15) and (3.16)], by direct calculation, we deduce the following proposition.

**Proposition 4.6** (cf. [19, Proposition 3.1]) *For $p \in \mathbb{Z}$, $p > 0$, $i = 1, 2$, there are natural $\mathbb{Z}/k$ isomorphisms of vector bundles over $X_H$,*

$$r_*(\mathcal{F}^{-p}(X)) \simeq \mathcal{F}^0(X) \otimes L(N)^p \ , \quad r_*(F_V^i) \simeq F_V^i \otimes L(V)^p \ . \tag{4.25}$$

*For any $p \in \mathbb{Z}$, $p > 0$, there is a natural $\mathbb{Z}/k$ $G_y \times S^1$-equivariant isomorphism of vector bundles over $X_H$,*

$$r_*(Q^1(W)) \simeq Q^1(W) \otimes L(W)^{-p}. \tag{4.26}$$

On $X_H$, as in [19, (2.8)], we write

$$\begin{aligned}
e(N) &= \sum_{v>0} v^2 \dim N_v \ , \quad d'(N) = \sum_{v>0} v \dim N_v \ , \\
e(V) &= \sum_{v>0} v^2 \dim V_v \ , \quad d'(V) = \sum_{v>0} v \dim V_v \ , \\
e(W) &= \sum_{v} v^2 \dim W_v \ , \quad d'(W) = \sum_{v} v \dim W_v \ .
\end{aligned} \tag{4.27}$$

Then $e(N)$, $e(V)$, $e(W)$, $d'(N)$, $d'(V)$ and $d'(W)$ are locally constant functions on $X_H$.

Take $\mathbb{Z}_\infty = S^1$ in the hypothesis (4.6). By using splitting principle [11, Chapter 17], we get the same identities as in [19, (2.11)],

$$c_1(L) = 0 \ , \quad e(V) + e(W) - e(N) = 2e \ . \tag{4.28}$$

As indicated in Section 2.1, (4.28) means $L$ is a trivial complex line bundle over each component $X_{H,\alpha}$ of $X_H$, and $S^1$ acts on $L$ by sending $g$ to $g^{2e}$, and $G_y$ acts on $L$ by sending $y$ to $y^{d'(W)}$.

The following proposition is deduced from Proposition 4.6.

**Proposition 4.7** (cf. [19, Proposition 3.2]) *For $p \in \mathbb{Z}$, $p > 0$, $i = 1, 2$, the $\mathbb{Z}/k$ $G_y$-equivariant isomorphism of vector bundles over $X_H$ induced by* (4.25), (4.26),

$$
\begin{aligned}
r_* \ : S(TX_H, K_X \otimes_{v>0} (\det N_v)^{-1}) &\otimes (K_W \otimes K_X^{-1})^{1/2} \\
&\otimes \mathcal{F}^{-p}(X) \otimes F_V^i \otimes Q^1(W) \\
\longrightarrow \ S(TX_H, K_X \otimes_{v>0} (\det N_v)^{-1}) &\otimes (K_W \otimes K_X^{-1})^{1/2} \\
&\otimes \mathcal{F}^0(X) \otimes F_V^i \otimes Q^1(W) \otimes L^{-p} \ ,
\end{aligned}
\tag{4.29}
$$

*verifies the following identities*

$$
\begin{aligned}
r_*^{-1} \cdot \boldsymbol{J}_H \cdot r_* &= \boldsymbol{J}_H \ , \\
r_*^{-1} \cdot P \cdot r_* &= P + p\boldsymbol{J}_H + p^2 e - \frac{1}{2}p^2 e(N) - \frac{p}{2}d'(N) \ .
\end{aligned}
\tag{4.30}
$$

*For the $\mathbb{Z}_2$-gradings, we have*

$$
r_*^{-1}\tau_e r_* = \tau_e \ , \quad r_*^{-1}\tau_s r_* = \tau_s \ , \quad r_*^{-1}\tau_1 r_* = (-1)^{pd'(W)}\tau_1 \ . \tag{4.31}
$$

Then we get the following recursive formula.

**Theorem 4.8** (Compare with [19, Theorem 2.4]) *For each $\alpha$, $m \in \frac{1}{2}\mathbb{Z}$, $1 \leq \ell < N$, $h$, $p \in \mathbb{Z}$, $p > 0$, the following identity holds,*

$$
\begin{aligned}
&\text{APS-ind}\Big(D^{X_{H,\alpha}} \otimes \mathcal{F}^{-p}(X) \otimes R^i, m + \frac{1}{2}p^2 e(N) + \frac{p}{2}d'(N), \ell, h\Big) \\
&= (-1)^{pd'(W)} \,\text{APS-ind}\Big(D^{X_{H,\alpha}} \otimes \mathcal{F}^0(X) \otimes R^i \otimes L^{-p}, m + ph + p^2 e, \ell, h\Big) \ .
\end{aligned}
\tag{4.32}
$$

Now we state another recursive formula whose proof will be presented in Section 5.

**Theorem 4.9** (Compare with [19, Theorem 2.3]) *For $1 \leq i \leq 4$, $m \in \frac{1}{2}\mathbb{Z}$, $1 \leq \ell < N$, $h$, $p \in \mathbb{Z}$, $p > 0$, we have the following identity,*

$$
\begin{aligned}
&\sum_\alpha (-1)^{\sum_{v>0} \dim N_v} \,\text{APS-ind}\Big(D^{X_{H,\alpha}} \otimes \mathcal{F}^0(X) \otimes R^i, m, \ell, h\Big) \\
&\equiv \sum_\alpha (-1)^{pd'(N) + \sum_{v>0} \dim N_v} \,\text{APS-ind}\Big(D^{X_{H,\alpha}} \otimes \mathcal{F}^{-p}(X) \otimes R^i, \\
&\qquad\qquad m + \tfrac{1}{2}p^2 e(N) + \tfrac{p}{2}d'(N), \ell, h\Big) \mod k\mathbb{Z} \ .
\end{aligned}
\tag{4.33}
$$

### 4.3 A proof of Theorem 4.4

Recall we assume in Theorem 4.1 that $c_1(W) \equiv 0 \bmod (N)$. Then by [10, Section 8] and [19, Lemma 2.1], $d'(W) \bmod (N)$ is constant on each connected component $X_{H,\alpha}$ of $X_H$. So we can extend $L$ to a trivial complex line bundle over $X$, and we extend the $S^1$-action on it by sending $g$ on the canonical section 1 of $L$ to $g^{2e} \cdot 1$, and $G_y$ acts on $L$ by sending $y$ to $y^{d'(W)}$.

As $\frac{1}{2}p_1(TX-W)_{S^1} \in H^*_{S^1}(X,\mathbb{Z})$ is well defined, one has the same identity as in [19, (2.27)],

$$d'(N) + d'(W) \equiv 0 \mod (2). \tag{4.34}$$

From Proposition 4.5, Theorems 4.8, 4.9 and (4.34), for $1 \le i \le 4$, $m \in \frac{1}{2}\mathbb{Z}$, $1 \le \ell < N$, $h$, $p \in \mathbb{Z}$, $p > 0$, we get the following identity (compare with [19, (2.28)]),

$$\begin{aligned}
&\text{APS-ind}\big(D^X \otimes \bigotimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TX) \otimes R^i, m, \ell, h\big) && (4.35)\\
&\equiv \text{APS-ind}\big(D^X \otimes \bigotimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TX) \otimes R^i \otimes L^{-p}, m', \ell, h\big) \mod k\mathbb{Z}\ ,
\end{aligned}$$

with

$$m' = m + ph + p^2 e. \tag{4.36}$$

By (4.1), (4.2), if $m < 0$ or $m' < 0$, then either side of (4.35) is identically zero, which completes the proof of Theorem 4.4. In fact,

(i) Assume that $e = 0$. Let $h \in \mathbb{Z}$, $m_0 \in \frac{1}{2}\mathbb{Z}$, $h \neq 0$ be fixed. If $h > 0$, we take $m' = m_0$, then for $p$ large enough, we get $m < 0$ in (4.35). If $h < 0$, we take $m = m_0$, then for $p$ large enough, we get $m' < 0$ in (4.35).

(ii) Assume that $e < 0$. For $h \in \mathbb{Z}$, $m_0 \in \frac{1}{2}\mathbb{Z}$, we take $m = m_0$, then for $p$ large enough, we get $m' < 0$ in (4.35).

The proof of Theorem 4.4 is completed.

# 5 A proof of Theorem 4.9

In this section, following [19, Section 4], we present a proof of Theorem 4.9.

This section is organized as follows. In Section 5.1, we first introduce the same refined shift operators as in [19, Section 4.2]. In Section 5.2, we construct the twisted Spin$^c$ Dirac operator on $X(n_j)$, the fixed point set of the naturally induced $\mathbb{Z}_{n_j}$-action on $X$. In Section 5.3, by applying the $S^1$-equivariant index theorem we have established in Section 2, we prove Theorem 4.9.

## 5.1 The refined shift operators

We first introduce a partition of $[0,1]$ as in [19, pp. 942–943]. Set $J = \big\{v \in \mathbb{N} \,\big|\, \text{there exists } \alpha \text{ such that } N_v \neq 0 \text{ on } X_{H,\alpha}\big\}$ and

$$\Phi = \big\{\beta \in (0,1] \,\big|\, \text{there exists } v \in J \text{ such that } \beta v \in \mathbb{Z}\big\}\ . \tag{5.1}$$

We order the elements in $\Phi$ so that $\Phi = \big\{\beta_i \,\big|\, 1 \le i \le J_0, J_0 \in \mathbb{N} \text{ and } \beta_i < \beta_{i+1}\big\}$. Then for any integer $1 \le i \le J_0$, there exist $p_i$, $n_i \in \mathbb{N}$, $0 < p_i \le n_i$, with $(p_i, n_i) = 1$ such that

$$\beta_i = p_i/n_i\ . \tag{5.2}$$

Clearly, $\beta_{J_0} = 1$. We also set $p_0 = 0$ and $\beta_0 = 0$.

For $0 \leq j \leq J_0$, $p \in \mathbb{N}^*$, we write

$$\begin{aligned} I_j^p &= \left\{(v,n) \in \mathbb{N}\times\mathbb{N} \,\middle|\, v \in J,\ (p-1)v < n \leq pv,\ \frac{n}{v} = p+1+\frac{p_j}{n_j}\right\} , \\ \overline{I}_j^p &= \left\{(v,n) \in \mathbb{N}\times\mathbb{N} \,\middle|\, v \in J,\ (p-1)v < n \leq pv,\ \frac{n}{v} > p+1+\frac{p_j}{n_j}\right\} . \end{aligned} \tag{5.3}$$

Clearly, $I_0^p = \emptyset$, the empty set. We define $\mathcal{F}_{p,j}(X)$ as in [19, (2.21)], which are analogous with (4.23). More specifically, we set

$$\begin{aligned} \mathcal{F}_{p,j}(X) = &\bigotimes_{n=1}^{\infty} \mathrm{Sym}\,(TX_H) \otimes \bigotimes_{v>0} \left( \bigotimes_{n=1}^{\infty} \mathrm{Sym}\,(N_{v,n}) \otimes \bigotimes_{n>(p-1)v+\frac{p_j}{n_j}v} \mathrm{Sym}\,(\overline{N}_{v,n}) \right) \\ &\bigotimes_{v>0} \bigotimes_{0\leq n\leq (p-1)v+\left[\frac{p_j}{n_j}v\right]} \Big(\mathrm{Sym}\,(N_{v,-n}) \otimes \det N_v\Big) \\ = \mathcal{F}_p(X) \otimes \mathcal{F}'_{p-1}(X) \otimes &\bigotimes_{(v,n)\in\cup_{i=0}^{j} I_i^p} \Big(\mathrm{Sym}\,(N_{v,-n}) \otimes \det N_v\Big) \bigotimes_{(v,n)\in\overline{I}_j^p} \mathrm{Sym}\,(\overline{N}_{v,n}) , \end{aligned} \tag{5.4}$$

where we use the notation that for $s \in \mathbb{R}$, $[s]$ denotes the greatest integer which is less than or equal to $s$. Then

$$\mathcal{F}_{p,0}(X) = \mathcal{F}^{-p+1}(X) , \quad \mathcal{F}_{p,J_0}(X) = \mathcal{F}^{-p}(X) . \tag{5.5}$$

From the construction of $\beta_i$, we know that for $v \in J$, there is no integer in $\left(\frac{p_{j-1}}{n_{j-1}}v, \frac{p_j}{n_j}v\right)$. Furthermore,

$$\begin{aligned} \left[\frac{p_{j-1}}{n_{j-1}}v\right] &= \left[\frac{p_j}{n_j}v\right] - 1 \ \text{ if } \ v \equiv 0 \mod (n_j) , \\ \left[\frac{p_{j-1}}{n_{j-1}}v\right] &= \left[\frac{p_j}{n_j}v\right] \ \text{ if } \ v \not\equiv 0 \mod (n_j) . \end{aligned} \tag{5.6}$$

We use the same shift operators $r_{j*}$, $1 \leq j \leq J_0$ as in [19, (4.21)], which refine the shift operator $r_*$ defined in (4.22). For $p \in \mathbb{N}^*$, set

$$\begin{aligned} r_{j*} : N_{v,n} \to N_{v,n+(p-1)v+p_jv/n_j} , \quad & r_{j*} : \overline{N}_{v,n} \to \overline{N}_{v,n-(p-1)v-p_jv/n_j} , \\ r_{j*} : V_{v,n} \to V_{v,n+(p-1)v+p_jv/n_j} , \quad & r_{j*} : \overline{V}_{v,n} \to \overline{V}_{v,n-(p-1)v-p_jv/n_j} , \\ r_{j*} : W_{v,n} \to W_{v,n+(p-1)v+p_jv/n_j} , \quad & r_{j*} : \overline{W}_{v,n} \to \overline{W}_{v,n-(p-1)v-p_jv/n_j} . \end{aligned} \tag{5.7}$$

For $1 \leq j \leq J_0$, we define $\mathcal{F}(\beta_j)$, $F_V^1(\beta_j)$, $F_V^2(\beta_j)$ and $Q_W(\beta_j)$ as in [19, (4.13)].

$$\begin{aligned} \mathcal{F}(\beta_j) = &\bigotimes_{0<n\in\mathbb{Z}} \mathrm{Sym}\,(TX_{H,n}) \otimes \bigotimes_{v>0, v\equiv 0, \frac{n_j}{2} \bmod (n_j)} \ \bigotimes_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v} \mathrm{Sym}\,(N_{v,n} \oplus \overline{N}_{v,n}) \\ &\otimes \bigotimes_{0<v'<n_j/2} \mathrm{Sym} \left( \bigoplus_{v\equiv v', -v' \bmod (n_j)} \left( \bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v} N_{v,n} \bigoplus_{0<n\in\mathbb{Z}-\frac{p_j}{n_j}v} \overline{N}_{v,n} \right) \right) , \end{aligned}$$

$$F_V^1(\beta_j) = \Lambda\left( \bigoplus_{0<n\in\mathbb{Z}} V_{0,n} \bigoplus_{v>0, v\equiv 0, \frac{n_j}{2} \bmod (n_j)} \Big( \bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v} V_{v,n} \bigoplus_{0<n\in\mathbb{Z}-\frac{p_j}{n_j}v} \overline{V}_{v,n} \Big) \right.$$

$$\left. \bigoplus_{0<v'<n_j/2} \left( \bigoplus_{v\equiv v', -v' \bmod (n_j)} \Big( \bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v} V_{v,n} \bigoplus_{0<n\in\mathbb{Z}-\frac{p_j}{n_j}v} \overline{V}_{v,n} \Big) \right) \right) ,$$

$$F_V^2(\beta_j) = \Lambda\left( \bigoplus_{0<n\in\mathbb{Z}+\frac{1}{2}} V_{0,n} \bigoplus_{v>0,\, v\equiv 0, \frac{n_j}{2} \bmod (n_j)} \Big( \bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v+\frac{1}{2}} V_{v,n} \bigoplus_{0<n\in\mathbb{Z}-\frac{p_j}{n_j}v+\frac{1}{2}} \overline{V}_{v,n} \Big) \right.$$

$$\left. \bigoplus_{0<v'<n_j/2} \left( \bigoplus_{v\equiv v', -v' \bmod (n_j)} \Big( \bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v+\frac{1}{2}} V_{v,n} \bigoplus_{0<n\in\mathbb{Z}-\frac{p_j}{n_j}v+\frac{1}{2}} \overline{V}_{v,n} \Big) \right) \right) ,$$

$$Q_W(\beta_j) = \Lambda\left( \bigoplus_{v} \Big( \bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v} W_{v,n} \bigoplus_{0\leq n\in\mathbb{Z}-\frac{p_j}{n_j}v} \overline{W}_{v,n} \Big) \right) . \tag{5.8}$$

Using the definition of $r_{j*}$ and computing directly, we get an analogue of Proposition 4.6 as follows.

**Proposition 5.1** (cf. [19, Proposition 4.1]) *There are natural $\mathbb{Z}/k$ isomorphisms of vector bundles over $X_H$,*

$$r_{j*}(\mathcal{F}_{p,j-1}(X)) \simeq \mathcal{F}(\beta_j) \otimes \bigotimes_{v>0,\ v\equiv 0 \bmod (n_j)} \mathrm{Sym}\,(\overline{N}_{v,0})$$
$$\otimes \bigotimes_{v>0} (\det N_v)^{\left[\frac{p_j}{n_j}v\right]+(p-1)v+1} \otimes \bigotimes_{v>0,\ v\equiv 0 \bmod (n_j)} (\det N_v)^{-1} ,$$

$$r_{j*}(\mathcal{F}_{p,j}(X)) \simeq \mathcal{F}(\beta_j) \otimes \bigotimes_{v>0,\ v\equiv 0 \bmod (n_j)} \mathrm{Sym}(N_{v,0})$$
$$\otimes \bigotimes_{v>0} (\det N_v)^{\left[\frac{p_j}{n_j}v\right]+(p-1)v+1} ,$$

$$r_{j*}(F_V^1) \simeq S\Big(V_0^{\mathbb{R}}, \otimes_{v>0}(\det V_v)^{-1}\Big) \otimes F_V^1(\beta_j)$$
$$\otimes \bigotimes_{v>0,\ v\equiv 0 \bmod (n_j)} \Lambda(V_{v,0}) \otimes \bigotimes_{v>0} (\det \overline{V}_v)^{\left[\frac{p_j}{n_j}v\right]+(p-1)v} ,$$

$$r_{j*}(F_V^2) \simeq F_V^2(\beta_j) \otimes \bigotimes_{v>0,\ v\equiv \frac{n_j}{2} \bmod (n_j)} \Lambda(V_{v,0}) \otimes \bigotimes_{v>0} (\det \overline{V}_v)^{\left[\frac{p_j}{n_j}v+\frac{1}{2}\right]+(p-1)v} .$$

*There is a natural $\mathbb{Z}/k$ $G_y \times S^1$-equivariant isomorphism of vector bundles over $X_H$,*

$$r_{j*}(Q^1(W)) \simeq Q_W(\beta_j) \otimes \bigotimes_{v>0} (\det \overline{W}_v)^{\left[\frac{p_j}{n_j}v\right]+(p-1)v+1}$$
$$\otimes \bigotimes_{v>0,\ v\equiv 0 \mod (n_j)} (\det \overline{W}_v)^{-1} \otimes \bigotimes_{v<0} (\det W_v)^{\left[-\frac{p_j}{n_j}v\right]-(p-1)v} .$$

## 5.2 The Spin$^c$ Dirac operators on $X(n_j)$

Recall that there is a nontrivial $\mathbb{Z}/k$ circle action on $X$ which can be lifted to the $\mathbb{Z}/k$ circle actions on $V$ and $W$.

For $n \in \mathbb{N}^*$, let $\mathbb{Z}_n \subset S^1$ denote the cyclic subgroup of order $n$. Let $X(n_j)$ be the fixed point set of the induced $\mathbb{Z}_{n_j}$ action on $X$. Let $N(n_j) \to X(n_j)$ be the normal bundle to $X(n_j)$ in $X$. As in [4, pp. 151] (see also [19, Section 4.1], [20, Section 4.1] or [24]), we see that $N(n_j)$ and $V$ can be decomposed, as $\mathbb{Z}/k$ real vector bundles over $X(n_j)$, into

$$\begin{aligned} N(n_j) &= \bigoplus_{0<v<n_j/2} N(n_j)_v \oplus N(n_j)^{\mathbb{R}}_{n_j/2} , \\ V|_{X(n_j)} &= V(n_j)^{\mathbb{R}}_0 \oplus \bigoplus_{0<v<n_j/2} V(n_j)_v \oplus V(n_j)^{\mathbb{R}}_{n_j/2} , \end{aligned} \tag{5.9}$$

where $V(n_j)^{\mathbb{R}}_0$ is the $\mathbb{Z}/k$ real vector bundle on which $\mathbb{Z}_{n_j}$ acts by identity, and $N(n_j)^{\mathbb{R}}_{n_j/2}$ (resp. $V(n_j)^{\mathbb{R}}_{n_j/2}$) is defined to be zero if $n_j$ is odd. Moreover, for $0 < v < n_j/2$, $N(n_j)_v$ (resp. $V(n_j)_v$) admits unique $\mathbb{Z}/k$ complex structure such that $N(n_j)_v$ (resp. $V(n_j)_v$) becomes a $\mathbb{Z}/k$ complex vector bundle on which $g \in \mathbb{Z}_{n_j}$ acts by $g^v$. We also denote by $V(n_j)_0$, $V(n_j)_{n_j/2}$ and $N(n_j)_{n_j/2}$ the corresponding complexification of $V(n_j)^{\mathbb{R}}_0$, $V(n_j)^{\mathbb{R}}_{n_j/2}$ and $N(n_j)^{\mathbb{R}}_{n_j/2}$.

Similarly, we also have the following $\mathbb{Z}_{n_j}$-equivariant decomposition of $W$, as $\mathbb{Z}/k$ complex vector bundles over $X(n_j)$,

$$W = \bigoplus_{0\le v<n_j} W(n_j)_v , \tag{5.10}$$

where for $0 \le v < n_j$, $g \in \mathbb{Z}_{n_j}$ acts on $W(n_j)_v$ by sending $g$ to $g^v$.

By [19, Lemma 4.1] (see also [4, Lemmas 9.4 and 10.1] or [24, Lemma 5.1]), we know that the $\mathbb{Z}/k$ vector bundles $TX(n_j)$ and $V(n_j)^{\mathbb{R}}_0$ are orientable and even dimensional. Thus $N(n_j)$ is orientable over $X(n_j)$. By (5.9), $V(n_j)^{\mathbb{R}}_{n_j/2}$ and $N(n_j)^{\mathbb{R}}_{n_j/2}$ are also orientable and even dimensional. In what follows, we fix the orientations of $N(n_j)^{\mathbb{R}}_{n_j/2}$ and $V(n_j)^{\mathbb{R}}_{n_j/2}$. Then $TX(n_j)$ and $V(n_j)^{\mathbb{R}}_0$ are naturally oriented by (5.9) and the orientations of $TX$, $V$, $N(n_j)^{\mathbb{R}}_{n_j/2}$ and $V(n_j)^{\mathbb{R}}_{n_j/2}$.

By (4.13), (4.14), (5.9) and (5.10), upon restriction to $X_H$, we get the following identifications of $\mathbb{Z}/k$ complex vector bundles (cf. [19, (4.9) and (4.12)]), for $0 < v \leq n_j/2$,

$$\begin{aligned} N(n_j)_v &= \bigoplus_{v'>0,\, v'\equiv v \,\mathrm{mod}(n_j)} N_{v'} \oplus \bigoplus_{v'>0,\, v'\equiv -v \,\mathrm{mod}(n_j)} \overline{N}_{v'} \ , \\ V(n_j)_v &= \bigoplus_{v'>0,\, v'\equiv v \,\mathrm{mod}(n_j)} V_{v'} \oplus \bigoplus_{v'>0,\, v'\equiv -v \,\mathrm{mod}(n_j)} \overline{V}_{v'} \ , \end{aligned} \tag{5.11}$$

for $0 \leq v < n_j$,

$$W(n_j)_v = \bigoplus_{v'>0,\, v'\equiv v \,\mathrm{mod}(n_j)} W_{v'} \ . \tag{5.12}$$

Also we get the following identifications of $\mathbb{Z}/k$ real vector bundles over $X_H$ (cf. [19, (4.11)]),

$$\begin{aligned} TX(n_j) &= TX_H \oplus \bigoplus_{v>0,\, v\equiv 0 \,\mathrm{mod}\,(n_j)} N_v \ , \quad N(n_j)^{\mathbb{R}}_{n_j/2} = \bigoplus_{v>0,\, v\equiv \frac{n_j}{2} \,\mathrm{mod}\,(n_j)} N_v \ , \\ V(n_j)^{\mathbb{R}}_0 &= V^{\mathbb{R}}_0 \oplus \bigoplus_{v>0,\, v\equiv 0 \,\mathrm{mod}(n_j)} V_v \ , \quad V(n_j)^{\mathbb{R}}_{n_j/2} = \bigoplus_{v>0,\, v\equiv \frac{n_j}{2} \,\mathrm{mod}(n_j)} V_v \ . \end{aligned}$$

Moreover, we have the identifications of $\mathbb{Z}/k$ complex vector bundles over $X_H$ as follows,

$$\begin{aligned} TX(n_j)\otimes_{\mathbb{R}}\mathbb{C} &= TX_H\otimes_{\mathbb{R}}\mathbb{C} \oplus \bigoplus_{v>0,\, v\equiv 0 \,\mathrm{mod}(n_j)} (N_v\oplus \overline{N}_v) \ , \\ V(n_j)_0 &= V^{\mathbb{R}}_0\otimes_{\mathbb{R}}\mathbb{C} \oplus \bigoplus_{v>0,\, v\equiv 0 \,\mathrm{mod}(n_j)} (V_v\oplus \overline{V}_v) \ . \end{aligned} \tag{5.13}$$

As $(p_j, n_j) = 1$, we know that, for $v \in \mathbb{Z}$, $p_j v/n_j \in \mathbb{Z}$ if and only if $v/n_j \in \mathbb{Z}$. Also, $p_j v/n_j \in \mathbb{Z} + \frac{1}{2}$ if and only if $v/n_j \in \mathbb{Z} + \frac{1}{2}$. Remark if $v \equiv -v' \ \mathrm{mod}(n_j)$, then $\{n \,|\, 0 < n \in \mathbb{Z} + \frac{p_j}{n_j} v\} = \{n \,|\, 0 < n \in \mathbb{Z} - \frac{p_j}{n_j} v'\}$. Using the identifications (5.11), (5.12) and (5.13), we can rewrite $\mathcal{F}(\beta_j)$, $F^1_V(\beta_j)$, $F^2_V(\beta_j)$ and $Q_W(\beta_j)$ defined in (5.8) as follows (cf. [19, (4.7)]),

$$\begin{aligned} \mathcal{F}(\beta_j) = \bigotimes_{0<n\in\mathbb{Z}} \mathrm{Sym}\left(TX(n_j)_n\right) \otimes \bigotimes_{0<v<n_j/2} \mathrm{Sym}\Big( \bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v} N(n_j)_{v,n} \\ \oplus \bigoplus_{0<n\in\mathbb{Z}-\frac{p_j}{n_j}v} \overline{N(n_j)}_{v,n}\Big) \oplus \bigoplus_{0<n\in\mathbb{Z}+\frac{1}{2}} \mathrm{Sym}\left(N(n_j)_{n_j/2,n}\right) \ , \end{aligned} \tag{5.14}$$

$$\begin{aligned} F^1_V(\beta_j) = \Lambda\Big( \bigoplus_{0<n\in\mathbb{Z}} V(n_j)_{0,n} \oplus \bigoplus_{0<v<n_j/2} \Big( \bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v} V(n_j)_{v,n} \\ \oplus \bigoplus_{0<n\in\mathbb{Z}-\frac{p_j}{n_j}v} \overline{V(n_j)}_{v,n}\Big) \bigoplus_{0<n\in\mathbb{Z}+\frac{1}{2}} V(n_j)_{n_j/2,n}\Big) \ , \end{aligned} \tag{5.15}$$

$$F_V^2(\beta_j) = \Lambda\Big( \bigoplus_{0<n\in\mathbb{Z}} V(n_j)_{n_j/2,n} \oplus \bigoplus_{0<v<n_j/2} \Big( \bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v+\frac{1}{2}} V(n_j)_{v,n} \oplus \bigoplus_{0<n\in\mathbb{Z}-\frac{p_j}{n_j}v+\frac{1}{2}} \overline{V(n_j)}_{v,n}\Big) \bigoplus_{0<n\in\mathbb{Z}+\frac{1}{2}} V(n_j)_{0,n}\Big) , \tag{5.16}$$

$$Q_W(\beta_j) = \Lambda\Big( \bigoplus_{0\le v<n_j} \Big( \bigoplus_{0<n\in\mathbb{Z}+\frac{p_j}{n_j}v} W(n_j)_{v,n} \oplus \bigoplus_{0\le n\in\mathbb{Z}-\frac{p_j}{n_j}v} \overline{W(n_j)}_{v,n}\Big)\Big) . \tag{5.17}$$

Thus $\mathcal{F}(\beta_j)$, $F_V^1(\beta_j)$, $F_V^2(\beta_j)$ and $Q_W(\beta_j)$ can be extended to $\mathbb{Z}/k$ vector bundles over $X(n_j)$.

We now define the Spin$^c$ Dirac operators on $X(n_j)$ following [19, Section 4.1].

Consider the hypothesis in (4.6). By splitting principle [11, Chapter 17] and computing as in [4, Lemmas 11.3 and 11.4], we get

$$\Big( \sum_{0<v<\frac{n_j}{2}} v\cdot c_1\Big(V(n_j)_v + W(n_j)_v - W(n_j)_{n_j-v} - N(n_j)_v\Big) + r(n_j)\cdot \frac{n_j}{2}\cdot \omega_2\Big(W(N_j)_{n_j/2} + V(n_j)_{n_j/2} - N(n_j)_{n_j/2}\Big)\Big)\cdot u_{n_j} = 0 , \tag{5.18}$$

where $r(n_j) = \frac{1}{2}(1+(-1)^{n_j})$, and $u_{n_j}\in H^2(B\mathbb{Z}_{n_j},\mathbb{Z})\simeq \mathbb{Z}_{n_j}$ is the generator of $H^*(B\mathbb{Z}_{n_j},\mathbb{Z})\simeq \mathbb{Z}[u_{n_j}]/(n_j\cdot u_{n_j})$ . Then by (5.18), we know that

$$\sum_{0<v<\frac{n_j}{2}} v\cdot c_1\Big(V(n_j)_v + W(n_j)_v - W(n_j)_{n_j-v} - N(n_j)_v\Big) + r(n_j)\cdot \frac{n_j}{2}\cdot \omega_2\Big(W(n_j)_{n_j/2} + V(n_j)_{n_j/2} - N(n_j)_{n_j/2}\Big)$$

is divided by $n_j$. Therefore, we have

**Lemma 5.2** (cf. [19, Lemma 4.2]) *Assume that* (4.6) *holds. Let*

$$L(n_j) = \bigotimes_{0<v<n_j/2} \Big(\det(N(n_j)_v)\otimes \det(\overline{V(n_j)_v}) \otimes \det(\overline{W(n_j)_v})\otimes \det(W(n_j)_{n_j-v})\Big)^{(r(n_j)+1)v} \tag{5.19}$$

*be the complex line bundle over $X(n_j)$. Then we have*

(i) *$L(n_j)$ has an $n_j^{\text{th}}$ root over $X(n_j)$.*

(ii) *Let $U_1 = TX(n_j)\oplus V(n_j)_0^{\mathbb{R}}$, $U_2 = TX(n_j)\oplus V(n_j)_{n_j/2}^{\mathbb{R}}$. Let*

$$L_1 = K_X\otimes \bigotimes_{0<v<n_j/2} \Big(\det(N(n_j)_v)\otimes \det(\overline{V(n_j)_v})\Big) \otimes \det\Big(W(n_j)_{n_j/2}\Big)\otimes L(n_j)^{r(n_j)/n_j} ,$$

$$L_2 = K_X\otimes \bigotimes_{0<v<n_j/2} \Big(\det(N(n_j)_v)\Big)\otimes \det\Big(W(n_j)_{n_j/2}\Big)\otimes L(n_j)^{r(n_j)/n_j}.$$

*Then $U_1$ (resp. $U_2$) has a $\mathbb{Z}/k$ Spin$^c$ structure defined by $L_1$ (resp. $L_2$).*

Remark that in order to define an $S^1$ (resp. $G_y$) action on $L(n_j)^{r(n_j)/n_j}$, we must replace the $S^1$ (resp. $G_y$) action by its $n_j$-fold action. Here by abusing notation, we still say an $S^1$ (resp. $G_y$) action without causing any confusion.

In what follows, by $D^{X(n_j)}$ we mean the $S^1$-equivariant Spin$^c$ Dirac operator on $S(U_1, L_1)$ or $S(U_2, L_2)$ over $X(n_j)$ (cf. Definition 2.2).

Corresponding to (2.13), by (5.11), we denote by

$$S(U_1,L_1)' = S\Big(TX_H \oplus V_0^{\mathbb{R}}, L_1 \otimes \bigotimes_{v>0,\, v\equiv 0 \bmod (n_j)} (\det N_v \otimes \det V_v)^{-1}\Big) \otimes \bigotimes_{v>0,\, v\equiv 0 \bmod (n_j)} \Lambda V_v \ , \tag{5.20}$$

$$S(U_2,L_2)' = S\Big(TX_H, L_2 \otimes \bigotimes_{v>0, v\equiv 0 \bmod (n_j)} (\det N_v)^{-1} \otimes \bigotimes_{v>0,\, v\equiv \frac{n_j}{2} \bmod (n_j)} (\det V_v)^{-1}\Big) \otimes \bigotimes_{v>0,\, v\equiv \frac{n_j}{2} \bmod (n_j)} \Lambda V_v \ . \tag{5.21}$$

Then by (2.16) and (2.17), for $i = 1, 2$, we have the following isomorphisms of Clifford modules over $X_H$,

$$S(U_i, L_i) \simeq S(U_i, L_i)' \otimes \bigotimes_{v>0,\, v\equiv 0 \bmod (n_j)} \Lambda N_v. \tag{5.22}$$

We define the $\mathbb{Z}_2$-gradings on $S(U_i, L_i)'$ ($i = 1, 2$) induced by the $\mathbb{Z}_2$-gradings on $S(U_i, L_i)$ ($i = 1, 2$) and on $\otimes_{v>0,\, v\equiv 0 \bmod (n_j)} \Lambda N_v$ such that the isomorphisms (5.22) preserve the $\mathbb{Z}_2$-gradings.

As in [19, pp. 952], we introduce formally the following $\mathbb{Z}/k$ complex line bundles over $X_H$,

$$L_1' = \Big(L_1^{-1} \otimes \bigotimes_{v>0,\, v\equiv 0 \bmod (n_j)} (\det N_v \otimes \det V_v) \bigotimes_{v>0} (\det N_v \otimes \det V_v)^{-1} \otimes K_X\Big)^{\frac{1}{2}},$$

$$L_2' = \Big(L_2^{-1} \otimes \bigotimes_{v>0,\, v\equiv 0 \bmod (n_j)} \det N_v \bigotimes_{v>0,\, v\equiv \frac{n_j}{2} \bmod (n_j)} \det V_v \bigotimes_{v>0} (\det N_v)^{-1} \otimes K_X\Big)^{\frac{1}{2}}.$$

In fact, from (2.16), (2.17), Lemma 5.2 and the assumption that $V$ is spin, one verifies easily that $c_1(L_i'^2) = 0 \mod (2)$ for $i = 1, 2$, which implies that $L_1'$ and $L_2'$ are well defined $\mathbb{Z}/k$ complex line bundles over $X_H$ (cf. Section 2.1).

Then by (5.20), (5.21) and the definitions of $L_1$, $L_2$, $L_1'$ and $L_2'$, we get the following identifications of $\mathbb{Z}/k$ Clifford modules over $X_H$ (cf. [19, (4.19)]),

$$S(U_1, L_1)' \otimes L_1' = S\big(TX_H, K_X \otimes_{v>0} (\det N_v)^{-1}\big) \otimes S\big(V_0^{\mathbb{R}}, \otimes_{v>0} (\det V_v)^{-1}\big) \otimes \bigotimes_{v>0,\, v\equiv 0 \bmod (n_j)} \Lambda(V_v) \ , \tag{5.23}$$

$$S(U_2, L_2)' \otimes L_2' = S\big(TX_H, K_X \otimes_{v>0} (\det N_v)^{-1}\big) \otimes \bigotimes_{v>0,\, v\equiv \frac{n_j}{2} \bmod (n_j)} \Lambda(V_v) \ . \tag{5.24}$$

**Lemma 5.3** (cf. [19, Lemma 4.3]) *Let us write*

$$L(\beta_j)_1 = L_1' \otimes \bigotimes_{v>0} (\det N_v)^{\left[\frac{p_j}{n_j}v\right]+(p-1)v+1} \otimes \bigotimes_{v>0} (\det \overline{V}_v)^{\left[\frac{p_j}{n_j}v\right]+(p-1)v}$$
$$\otimes \bigotimes_{v>0,\, v\equiv 0 \,\mathrm{mod}(n_j)} (\det N_v)^{-1} \otimes \bigotimes_{v<0} (\det W_v)^{\left[-\frac{p_j}{n_j}v\right]-(p-1)v}$$
$$\otimes \bigotimes_{v>0} (\det \overline{W}_v)^{\left[\frac{p_j}{n_j}v\right]+(p-1)v+1} \otimes \bigotimes_{v>0,\, v\equiv 0 \,\mathrm{mod}(n_j)} (\det \overline{W}_v)^{-1} ,$$
$$L(\beta_j)_2 = L_2' \otimes \bigotimes_{v>0} (\det N_v)^{\left[\frac{p_j}{n_j}v\right]+(p-1)v+1} \otimes \bigotimes_{v>0} (\det \overline{V}_v)^{\left[\frac{p_j}{n_j}v+\frac{1}{2}\right]+(p-1)v}$$
$$\otimes \bigotimes_{v>0,\, v\equiv 0 \,\mathrm{mod}(n_j)} (\det N_v)^{-1} \otimes \bigotimes_{v<0} (\det W_v)^{\left[-\frac{p_j}{n_j}v\right]-(p-1)v}$$
$$\otimes \bigotimes_{v>0} (\det \overline{W}_v)^{\left[\frac{p_j}{n_j}v\right]+(p-1)v+1} \otimes \bigotimes_{v>0,\, v\equiv 0 \,\mathrm{mod}(n_j)} (\det \overline{W}_v)^{-1} .$$

*Then $L(\beta_j)_1$ and $L(\beta_j)_2$ can be extended naturally to $\mathbb{Z}/k$ $G_y \times S^1$-equivariant complex line bundles over $X(n_j)$ which we will still denote by $L(\beta_j)_1$ and $L(\beta_j)_2$ respectively.*

Now we compare the $\mathbb{Z}_2$-gradings in (5.23). Set

$$\begin{aligned} \Delta(n_j, N) &= \sum_{\frac{n_j}{2}<v'<n_j} \sum_{0<v,\, v\equiv v' \,\mathrm{mod}\,(n_j)} \dim N_v + o\Big(N(n_j)^{\mathbb{R}}_{\frac{n_j}{2}}\Big) , \\ \Delta(n_j, V) &= \sum_{\frac{n_j}{2}<v'<n_j} \sum_{0<v,\, v\equiv v' \,\mathrm{mod}\,(n_j)} \dim V_v + o\Big(V(n_j)^{\mathbb{R}}_{\frac{n_j}{2}}\Big) , \end{aligned} \tag{5.25}$$

with $o(N(n_j)^{\mathbb{R}}_{\frac{n_j}{2}})$ (resp. $o(V(n_j)^{\mathbb{R}}_{\frac{n_j}{2}})$ equals 0 or 1, depending on whether the given orientation on $N(n_j)^{\mathbb{R}}_{\frac{n_j}{2}}$ (resp. $V(n_j)^{\mathbb{R}}_{\frac{n_j}{2}}$) agrees or disagrees with the complex orientation of $\oplus_{v>0,\, v\equiv \frac{n_j}{2} \,\mathrm{mod}\,(n_j)} N_v$ (resp. $\oplus_{v>0,\, v\equiv \frac{n_j}{2} \,\mathrm{mod}\,(n_j)} V_v$).

By [19, pp. 953], we know that for the $\mathbb{Z}_2$-gradings induced by $\tau_s$, the differences of the $\mathbb{Z}_2$-gradings of (5.23) and (5.24) are both $(-1)^{\Delta(n_j,N)}$; for the $\mathbb{Z}_2$-gradings induced by $\tau_e$, the difference of the $\mathbb{Z}_2$-gradings of (5.23) (resp. (5.24)) is $(-1)^{\Delta(n_j,N)+\Delta(n_j,V)}$ (resp. $(-1)^{\Delta(n_j,N)+o\left(V(n_j)^{\mathbb{R}}_{n_j/2}\right)}$ ).

To simplify the notations, we introduce the same functions as in [19, (4.30)], which are locally constant on $X_H$,

$$\begin{aligned}\varepsilon(W) = -\frac{1}{2}\sum_{v>0}(\dim W_v)\cdot \Big(\big(\big[\tfrac{p_j}{n_j}v\big] + (p-1)v\big)\big(\big[\tfrac{p_j}{n_j}v\big] + (p-1)v+1\big)\\
-\big(\tfrac{p_j}{n_j}v + (p-1)v\big)\big(2\big(\big[\tfrac{p_j}{n_j}v\big] + (p-1)v\big)+1\big)\Big)\\
-\frac{1}{2}\sum_{v<0}(\dim W_v)\cdot \Big(\big(\big[-\tfrac{p_j}{n_j}v\big] - (p-1)v\big)\big(\big[-\tfrac{p_j}{n_j}v\big] - (p-1)v+1\big)\\
+\big(\tfrac{p_j}{n_j}v + (p-1)v\big)\big(2\big(\big[-\tfrac{p_j}{n_j}v\big] - (p-1)v\big)+1\big)\Big)\ ,\end{aligned} \tag{5.26}$$

$$\begin{aligned}\varepsilon_1 = \frac{1}{2}\sum_{v>0}(\dim N_v - \dim V_v)\Big(\big(\big[\tfrac{p_j}{n_j}v\big] + (p-1)v\big)\big(\big[\tfrac{p_j}{n_j}v\big] + (p-1)v+1\big)\\
-\big(\tfrac{p_j}{n_j}v + (p-1)v\big)\big(2\big(\big[\tfrac{p_j}{n_j}v\big] + (p-1)v\big)+1\big)\Big)\ ,\end{aligned} \tag{5.27}$$

$$\begin{aligned}\varepsilon_2 = \frac{1}{2}\sum_{v>0}(\dim N_v)\cdot \Big(\big(\big[\tfrac{p_j}{n_j}v\big] + (p-1)v\big)\big(\big[\tfrac{p_j}{n_j}v\big] + (p-1)v+1\big)\\
-\big(\tfrac{p_j}{n_j}v + (p-1)v\big)\big(2\big(\big[\tfrac{p_j}{n_j}v\big] + (p-1)v\big)+1\big)\Big)\\
-\frac{1}{2}\sum_{v>0}(\dim V_v)\cdot \Big(\big(\big[\tfrac{p_j}{n_j}+\tfrac{1}{2}\big] + (p-1)v\big)^2\\
-2\big(\tfrac{p_j}{n_j}v + (p-1)v\big)\big(\big[\tfrac{p_j}{n_j}+\tfrac{1}{2}\big] + (p-1)v\big)\Big)\ .\end{aligned} \tag{5.28}$$

As in [19, (2.23)], for $0 \le j \le J_0$, we set

$$\begin{aligned}e(p,\beta_j,N) &= \frac{1}{2}\sum_{v>0}(\dim N_v)\cdot \big(\big[\tfrac{p_j}{n_j}v\big] + (p-1)v\big)\big(\big[\tfrac{p_j}{n_j}v\big]+(p-1)v+1\big)\ ,\\
d'(p,\beta_j,N) &= \sum_{v>0}(\dim N_v)\cdot \big(\big[\tfrac{p_j}{n_j}v\big] + (p-1)v\big)\ .\end{aligned} \tag{5.29}$$

Then $e(p,\beta_j,N)$ and $d'(p,\beta_j,N)$ are locally constant functions on $X_H$. In particular, we have

$$\begin{aligned}e(p,\beta_0,N) &= \frac{1}{2}(p-1)^2 e(N) + \frac{1}{2}(p-1)d'(N)\ ,\\
e(p,\beta_{J_0},N) &= \frac{1}{2}p^2 e(N) + \frac{1}{2}p\, d'(N)\ ,\\
d'(p,\beta_{J_0},N) &= d'(p+1,\beta_0,N) = p\, d'(N)\ .\end{aligned} \tag{5.30}$$

By Proposition 5.1, (5.23) and Lemma 5.3, we deduce an analogue of Proposition 4.7.

**Proposition 5.4** (cf. [19, Proposition 4.2]) *For $i = 1, 2$, the $\mathbb{Z}/k$ $G_y$-equivariant* *isomorphisms of complex vector bundles over $X_H$,*

$$
\begin{aligned}
r_{i1} :\ & S(TX_H, K_X \otimes_{v>0} (\det N_v)^{-1}) \otimes (K_W \otimes K_X^{-1})^{1/2} \\
& \quad \otimes \mathcal{F}_{p,j-1}(X) \otimes F_V^i \otimes Q^1(W) \\
& \longrightarrow S(U_i, L_i) \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes \mathcal{F}(\beta_j) \otimes F_V^i(\beta_j) \\
& \quad \otimes Q_W(\beta_j) \otimes L(\beta_j)_i \otimes \bigotimes_{v>0,\, v\equiv 0 \bmod (n_j)} \mathrm{Sym}\,(\overline{N}_{v,0}) \ , \\
r_{i2} :\ & S(TX_H, K_X \otimes_{v>0} (\det N_v)^{-1}) \otimes (K_W \otimes K_X^{-1})^{1/2} \\
& \quad \otimes \mathcal{F}_{p,j}(X) \otimes F_V^i \otimes Q^1(W) \\
& \longrightarrow S(U_i, L_i) \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes \mathcal{F}(\beta_j) \otimes F_V^i(\beta_j) \\
& \quad \otimes Q_W(\beta_j) \otimes L(\beta_j)_i \otimes \bigotimes_{v>0,\, v\equiv 0 \bmod (n_j)} \big(\mathrm{Sym}\,(N_{v,0}) \otimes \det N_v\big)
\end{aligned}
$$

*have the following properties:*

(i) *for $i = 1, 2$, $\gamma = 1, 2$,*

$$
\begin{aligned}
r_{i\gamma}^{-1} \cdot \boldsymbol{J}_H \cdot r_{i\gamma} &= \boldsymbol{J}_H \ , \\
r_{i\gamma}^{-1} \cdot P \cdot r_{i\gamma} &= P + \Big( \frac{p_j}{n_j} + (p-1) \Big) \boldsymbol{J}_H + \varepsilon_{i\gamma} \ ,
\end{aligned}
\tag{5.31}
$$

*where $\varepsilon_{i1} = \varepsilon_i + \varepsilon(W) - e(p, \beta_{j-1}, N)$, $\varepsilon_{i2} = \varepsilon_i + \varepsilon(W) - e(p, \beta_j, N)$.*

(ii) *Recall that $o\big(V(n_j)^{\mathbb{R}}_{\frac{n_j}{2}}\big)$ is defined in* (5.25)*. Let*

$$
\begin{aligned}
\mu_1 &= - \sum_{v>0} \big[\tfrac{p_j}{n_j} v\big] \dim V_v + \Delta(n_j, N) + \Delta(n_j, V) \quad \bmod (2), \\
\mu_2 &= - \sum_{v>0} \big[\tfrac{p_j}{n_j} v + \tfrac{1}{2}\big] \dim V_v + \Delta(n_j, N) + o\big(V(n_j)^{\mathbb{R}}_{\frac{n_j}{2}}\big) \quad \bmod (2), \\
\mu_3 &= \Delta(n_j, N) \quad \bmod (2), \\
\mu_4 &= \sum_{v} \big( \big[\tfrac{p_j}{n_j} v\big] + (p-1) v \big) \dim W_v + \dim W + \dim W(n_j)_0 \quad \bmod (2).
\end{aligned}
$$

*Then for $i = 1, 2$, $\gamma = 1, 2$, we have*

$$
\begin{aligned}
r_{i\gamma}^{-1} \tau_e r_{i\gamma} &= (-1)^{\mu_i} \tau_e \ , \quad r_{i\gamma}^{-1} \tau_s r_{i\gamma} = (-1)^{\mu_3} \tau_s \ , \\
r_{i\gamma}^{-1} \tau_1 r_{i\gamma} &= (-1)^{\mu_4} \tau_1 \ .
\end{aligned}
\tag{5.32}
$$

### 5.3 A proof of Theorem 4.9

Let $X'$ be a connected component of $X(n_j)$. By [19, Lemmas 4.4, 4.5, 4.6], we know that for $i = 1, 2$, $k = 1, 2, 3$, the following functions are independent on the connected components of $X_H$ in $X'$,

$$
\begin{aligned}
& \varepsilon_i + \varepsilon(W) \quad \bmod (2) \ , \quad d'(p, \beta_j, N) + \mu_k + \mu_4 \quad \bmod (2) \ , \\
& \sum_{v>0} \big[\tfrac{p_j}{n_j} v\big] \dim V_v + \Delta(n_j, V) \quad \bmod (2) \ , \\
& \sum_{v>0} \big[\tfrac{p_j}{n_j} v + \tfrac{1}{2}\big] \dim V_v + o\big(V(n_j)^{\mathbb{R}}_{\frac{n_j}{2}}\big) \quad \bmod (2) \ ,
\end{aligned}
$$

which implies that $d'(p,\beta_{j-1},N)+\sum_{0<v}\dim N_v+\mu_k+\mu_4 \mod (2)$ $(k=1,2,3)$ are constant functions on each connected components of $X(n_j)$.

By (5.14), (5.15), (5.16), (5.17) and Lemma 5.3, we know that the Dirac operator $D^{X(n_j)}\otimes\mathcal{F}(\beta_j)\otimes F_V^i(\beta_j)\otimes Q_W(\beta_j)\otimes L(\beta_j)_i$ $(i=1,2)$ is well defined on $X(n_j)$. Observe that the two equalities in Theorem 2.7 are both compatible with the $G_y$ action. Thus, by using Proposition 5.4 and applying both the first and the second equalities of Theorem 2.7 to each connected component of $X(n_j)$ separately, we deduce that for $i=1,2$, $1\le j\le J_0$, $m\in\frac{1}{2}\mathbb{Z}$, $1\le\ell<N$, $h\in\mathbb{Z}$, $\tau=\tau_{e1}$ or $\tau_{s1}$,

$$\begin{aligned}
&\sum_\alpha(-1)^{d'(p,\beta_{j-1},N)+\sum_{v>0}\dim N_v}\,\text{APS-ind}_\tau\Big(D^{X_{H,\alpha}}\otimes(K_W\otimes K_X^{-1})^{1/2}\\
&\qquad\otimes\mathcal{F}_{p,j-1}(X)\otimes F_V^i\otimes\ Q^1(W),m+e(p,\beta_{j-1},N),\ell,h\Big)\\
&\equiv\sum_\beta(-1)^{d'(p,\beta_{j-1},N)+\sum_{v>0}\dim N_v+\mu}\,\text{APS-ind}_\tau\Big(D^{X(n_j)}\otimes(K_W\otimes K_X^{-1})^{1/2}\\
&\qquad\otimes\mathcal{F}(\beta_j)\otimes F_V^i(\beta_j)\otimes Q_W(\beta_j)\otimes L(\beta_j)_i,m+\varepsilon_i\\
&\qquad\qquad+\varepsilon(W)+(\tfrac{p_j}{n_j}+(p-1))h,\ell,h\Big)\\
&\equiv\sum_\alpha(-1)^{d'(p,\beta_j,N)+\sum_{v>0}\dim N_v}\,\text{APS-ind}_\tau\Big(D^{X_{H,\alpha}}\otimes(K_W\otimes K_X^{-1})^{1/2}\\
&\qquad\otimes\mathcal{F}_{p,j}(X)\otimes F_V^i\otimes Q^1(W),m+e(p,\beta_j,N),\ell,h\Big)\mod k\mathbb{Z}\,,
\end{aligned}\tag{5.33}$$

where $\sum_\beta$ means the sum over all the connected components of $X(n_j)$. In (5.33), if $\tau=\tau_{s1}$, then $\mu=\mu_3+\mu_4$; if $\tau=\tau_{e1}$, then $\mu=\mu_i+\mu_4$. Combining (5.30) with (5.33), we get (4.33).

The proof of Theorem 4.9 is completed.

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