CONVEX ORDERINGS ON POSITIVE ROOTS, KASHIWARA CRYSTALS AND STRING CONES

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Abstract. These are notes for the talk given by the third author at the workshop "Algebraic Lie Theory and Representation Theory" in Sugadaira, Japan. We explain parts of Sections 1, 2 and 4 of [GKS]. Namely we derive a formula to compute the crystal structures on Lusztig's parametrizations of the canonical basis in type A. As an application we show that Lusztig's parametrizations and Kashiwara's parametrizations are dual to each other in a suitable sense.

Acknowledgment

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1. Rhombic tilings and Lusztig data

1.1. **Notations.** Let $\mathfrak{g} = \mathrm{sl_n}(\mathbb{C})$ and $\mathfrak{h} \subset \mathfrak{g}$ its Cartan subalgebra consisting of the diagonal matrices in $\mathfrak{g}$. We abbreviate $[n] := \{1, 2, \dots, n\}$ and define for $s \in [n]$ the function $\epsilon_s \in \mathfrak{h}^*$ such that $\epsilon_s(\mathrm{diag}(h_1, h_2, \dots, h_n)) = h_s$. The set of positive roots of $\mathfrak{g}$, denoted by Φ^+, is given by

$$\Phi^+ = \{\epsilon_s - \epsilon_t \mid 1 \le s < t \le n\}.$$

We set $N := \frac{n(n-1)}{2}$ to be the cardinality of Φ^+. For $a \in [n-1]$, we denote by $\alpha_a = \epsilon_a - \epsilon_{a+1}$ the simple roots of $\mathfrak{g}$ and write $\alpha_{s,t} = \epsilon_s - \epsilon_{t+1}$. The fundamental weights of $\mathfrak{g}$ are given as $\omega_a = \sum_{s \in [a]} \epsilon_s$.

We denote by U_q^- the negative part of the quantized enveloping algebra of $\mathfrak{g}$ with generators F_a, its crystal basis by $B(\infty)$ and the corresponding Kashiwara-operators by f_a, e_a.

Let W be the Weyl group of $\mathfrak{g}$ which is isomorphic to $\mathfrak{S}_n$ in this case. Here $\mathfrak{S}_n$ is the symmetric group in n letters. The group W is generated by the simple reflections σ_i ($i \in [n-1]$) interchanging i and $i+1$ with the following relations

$$\begin{aligned} \sigma_i^2 &= id, \\ \sigma_{i_1}\sigma_{i_2} &= \sigma_{i_2}\sigma_{i_1} \qquad \text{for } |i_1 - i_2| \ge 2 \quad \text{(commutation relation)}, \\ \sigma_{i_1}\sigma_{i_2}\sigma_{i_1} &= \sigma_{i_2}\sigma_{i_1}\sigma_{i_2} \qquad \text{for } |i_1 - i_2| = 1 \quad \text{(braid relation)}. \end{aligned}$$

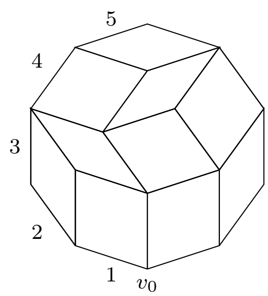
The Weyl group W has a unique longest element w_0 of length $N = |\Phi^+|$. For a reduced expression $w_0 = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_N}$ we write $\mathbf{i} := (i_1, i_2, \dots, i_N)$ and call $\mathbf{i}$ a *reduced word* (for w_0). We define a total ordering $\le$ on Φ^+ to be *convex* if whenever $\beta_1, \beta_2, \beta_1 + \beta_2 \in \Phi^+$ we have either $\beta_1 \le \beta_1 + \beta_2 \le \beta_2$ or $\beta_2 \le \beta_1 + \beta_2 \le \beta_1$. The set of total convex ordering is in bijection with the set of reduced words for w_0. For a reduced word $\mathbf{i} = (i_1, i_2, \ldots, i_N)$ the total ordering $\leq_{\mathbf{i}}$ on Φ^+ given by

$$\alpha_{i_1} \leq_{\mathbf{i}} s_{i_1}(\alpha_{i_2}) \leq_{\mathbf{i}} \ldots \leq_{\mathbf{i}} s_{i_1}s_{i_2}\cdots s_{i_{N-1}}(\alpha_{i_N}) \tag{1}$$

is convex and every convex ordering on Φ^+ arises as in (1).

1.2. **Rhombic tilings and reduced words.** Following [E97] we introduce geometric objects called rhombic tilings, which are associated to convex orderings on Φ^+. Let P_0 be a regular $2n$-gon with side length 1 and lowest vertex v_0. A *(rhombic) tiling* $\mathcal{T}$ is defined to be a tiling of P_0 into rhombi (*tiles*) with side length 1. We label the edges on the left boundary of P_0 consecutively by $1, 2, \ldots, n$ starting with the edge of the left boundary adjacent to the lowest vertex v_0 of P_0 and proceeding clockwise. We further assign a labeling to any edge in $\mathcal{T}$ by imposing that parallel edges have the same label.

Example 1.1. We give an example of a tiling $\mathcal{T}$ for $n = 5$. All labels of edges not lying on the left boundary of $\mathcal{T}$ are determined by the rule that parallel edges get the same label. We thus omit all labels for edges which do not lie on the left boundary.



□

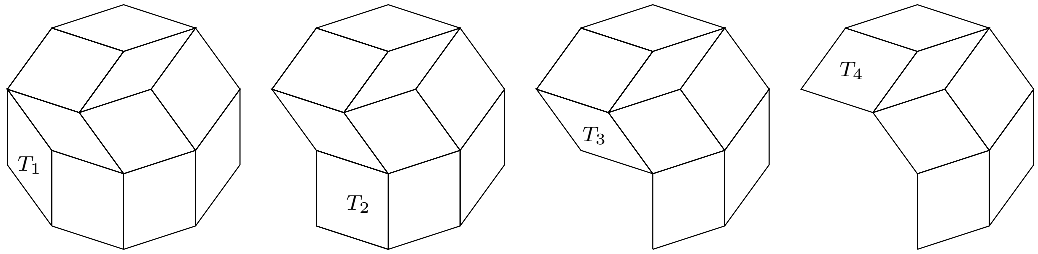
For every $1 \leq k < \ell \leq \frac{n(n-1)}{2}$ there exists a unique tile T of $\mathcal{T}$ such that the edges of T have labels k and ℓ. We denote this tile by $[k, \ell]$.

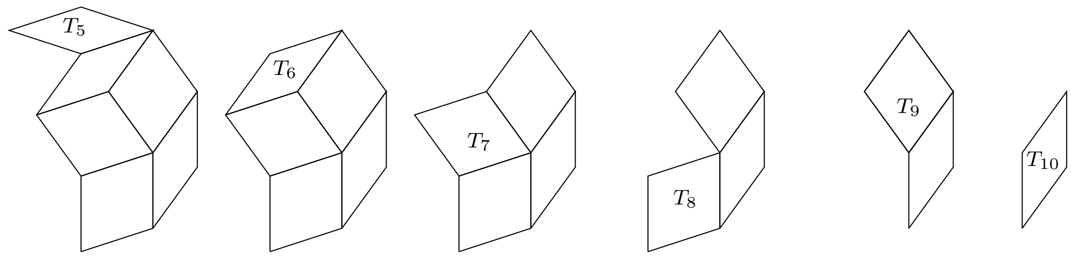
Definition 1.2. We define two total orderings on the set of tiles of a given tiling $\mathcal{T}$ via a choice of *left shelling*. In the first step of the left (resp. right) shelling of $\mathcal{T}$ we pick a tile $T =: T_1$ (resp. $T =: T_1'$) of the tiling $\mathcal{T}$ which intersects the left boundary of $\mathcal{T}$ with two edges (such a tile always exists by [E97, Lemma 2.1]). We cut away this tile from $\mathcal{T}$ to obtain a subtiling $\mathcal{T}_1$ with a new left (resp. right) boundary and pick a tile $T =: T_2$ which intersects the left (resp. right) boundary of $\mathcal{T}_1$ with two edges. We cut away T_2 from $\mathcal{T}_1$ thereby obtaining a subtiling $\mathcal{T}_2$. We proceed inductively with the thereby obtained left boundary until all tiles of $\mathcal{T}$ are cut away. We set

$$T_1 \leq_{\text{left}} T_2 \leq_{\text{left}} \ldots \leq_{\text{left}} T_N,$$

Note that the left (resp. right) shelling is not unique in general as there might be more than one tile with two edges on the left boundary in each step.

Example 1.3. A choice of left shelling for the tiling $\mathcal{T}$ given in Example 1.1 is obtained as follows.





We therefore have:

$$T_1 \leq_{\text{left}} T_2 \leq_{\text{left}} T_3 \leq_{\text{left}} T_4 \leq_{\text{left}} T_5 \leq_{\text{left}} T_6 \leq_{\text{left}} T_7 \leq_{\text{left}} T_8 \leq_{\text{left}} T_9 \leq_{\text{left}} T_{10}.$$

Swapping the steps in which T_8 and T_9 are picked yields another choice of left shelling for the given tiling inducing the total ordering:

$$T_1 \leq_{\text{left}'} T_2 \leq_{\text{left}'} T_3 \leq_{\text{left}'} T_4 \leq_{\text{left}'} T_5 \leq_{\text{left}'} T_6 \leq_{\text{left}'} T_7 \leq_{\text{left}'} T_9 \leq_{\text{left}'} T_8 \leq_{\text{left}'} T_{10}.$$

□

We identify the set of tiles of a given tiling $\mathcal{T}$ with the set of positive roots Φ^+ of $\mathfrak{g}$ via the map which is for $s < t$ defined by

$$[s,t] \mapsto \alpha_{s,t-1} := \epsilon_s - \epsilon_t. \tag{2}$$

By [E97, Theorem 2.2] there exists a unique tiling $\mathcal{T}_{\mathbf{i}}$ for each reduced word $\mathbf{i}$ of w_0 such that under the identification (2) the total ordering $\leq_{\mathbf{i}}$ is a choice of left ordering on $\mathcal{T}_{\mathbf{i}}$. Further $\mathcal{T}_{\mathbf{i}} = \mathcal{T}_{\mathbf{j}}$ for another reduced word $\mathbf{j}$ if and only if $\mathbf{j}$ is obtained from $\mathbf{i}$ only by applying commutation relations.

Example 1.4. Let $\mathbf{i} = (2,1,2,3,4,3,2,1,3,2)$ and $\mathbf{j} = (2,1,2,3,4,3,2,3,1,2)$. The tiling given in Example 1.3 is associated to the reduced word $\mathbf{i}$. We have $\mathcal{T} = \mathcal{T}_{\mathbf{i}} = \mathcal{T}_{\mathbf{j}}$. Note that $\mathbf{i}$ and $\mathbf{j}$ are commutation equivalent. Further under the identification of the set of tiles of $\mathcal{T}$ with the set of positive roots Φ^+, the total order $\leq_{\text{left}}$ of $\mathcal{T}$ equals the total order $\leq_{\mathbf{i}}$ of Φ^+ and the total order $\leq \text{left}'$ of $\mathcal{T}$ equals the total order $\leq_{\mathbf{j}}$ of Φ^+. □

1.3. **Lusztig's parametrization of the canonical basis.** Using braid group operators Lusztig defined in [L90] a PBW-type $B_{\mathbf{i}}$ basis of U_q^- for each reduced word $\mathbf{i} = (i_1, i_2, \dots, i_N)$. Namely let $\{\beta_1, \beta_2, \dots, \beta_N\} = \Phi^+$, $\beta_1 \leq_{\mathbf{i}} \beta_2 \leq_{\mathbf{i}} \dots \leq_{\mathbf{i}} \beta_N$, where $F_{\mathbf{i},\beta_j} = T_{i_1} T_{i_2} \cdots T_{i_{j-1}} F_j$ is given via the braid group action T_i defined in [Lu90, Section1.3]. Further $X^{(m)}$ is the q-divided power defined by $X^{(m)} := \frac{X^m}{[m][m-1]\cdots[2]}$ and $[m] := q^{m-1} + q^{m-2} + \dots + q^{-m+1}$. The PBW-type basis $B_{\mathbf{i}}$ is defined as

$$B_{\mathbf{i}} = \left\{ F_{\mathbf{i},\beta_1}^{(x_{\beta_1})} F_{\mathbf{i},\beta_2}^{(x_{\beta_2})} \cdots F_{i,\beta_N}^{(x_{\beta_N})} \mid (x_{\beta_1}, x_{\beta_2}, \dots, x_{\beta_N}) \in \mathbb{N}^N \right\},$$

and is in natural bijection with the canonical bases $\mathbf{B}$ of U_q^- (see [L90, Proposition 2.3, Theorem 3.2]).

Definition 1.5. We call $x = (x_{\beta_1}, x_{\beta_2}, \dots, x_{\beta_N}) \in \mathbb{N}^N$, the $\mathbf{i}$-*Lusztig datum* of the element $F_{\mathbf{i},\beta_1}^{(x_{\beta_1})} F_{\mathbf{i},\beta_2}^{(x_{\beta_2})} \cdots F_{i,\beta_N}^{(x_{\beta_N})} \in B_{\mathbf{i}}$. Via the identification of the positive roots with tiles in $\mathcal{T}_{\mathbf{i}}$ as in (2), we write $x = (x_T) \in \mathbb{N}^{\mathcal{T}_{\mathbf{i}}}$.

Let $\mathbf{i}$ and $\mathbf{j}$ be two reduced words for w_0. A piecewise-linear bijection $\Phi_{\mathbf{j}}^{\mathbf{i}} : \mathbb{N}^{\mathcal{T}_{\mathbf{i}}} \to \mathbb{N}^{\mathcal{T}_{\mathbf{j}}}$ from the set of $\mathbf{i}$-Lusztig data to the set of $\mathbf{j}$-Lusztig data is obtained in [L90, Section 2.1] by making use of the fact that any reduced word $\mathbf{j}$ can be obtained from any other reduced word $\mathbf{j}$ via applying a sequence of commutation- and braid relations.

The piecewise-linear bijections $\Phi_{\mathbf{j}}^{\mathbf{i}}$ are used to equip the set $\mathbb{N}^{\mathcal{T}_{\mathbf{i}}}$ with a crystal structure isomorphic to $B(\infty)$ ([L93, BZ01]) as follows.

Definition 1.6. Let $a \in [n-1]$ and let $\mathbf{i} = (i_1, i_2, \dots, i_N)$. We define $f_a x, e_a x \in \mathbb{N}^{\mathcal{T}_\mathbf{i}}$ and $\varepsilon_a(x) \in \mathbb{N}$ as follows. If $a = i_1$ then for $x \in \mathbb{N}^{\mathcal{T}_\mathbf{i}}$, $T \in \mathcal{T}_\mathbf{i}$

$$(f_a x)_T = \begin{cases} x_T + 1 & \text{if } T = [a, a+1] \\ x_T & \text{else,} \end{cases}$$

$$(e_a x)_T = \begin{cases} 0 & \text{if } x_{[a,a+1]} = 0, \\ x_T - 1 & \text{if } T = [a, a+1] \text{ and } x_{[a,a+1]} > 0, \\ x_T & \text{else,} \end{cases}$$

$$\varepsilon_a(x) = x_{[a,a+1]}.$$

If $a \neq i_1$, let $\mathbf{j} = (j_1, j_2, \dots, j_N)$ be any reduced word for w_0 such that $j_1 = a$. We define

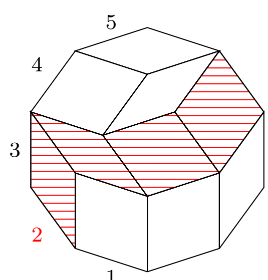
$$f_a x = \Phi_\mathbf{i}^\mathbf{j} \, f_a \, \Phi_\mathbf{j}^\mathbf{i} \, x, \qquad e_a x = \Phi_\mathbf{i}^\mathbf{j} \, e_a \, \Phi_\mathbf{j}^\mathbf{i} \, x \qquad \varepsilon_a(x) = \varepsilon_a(\Phi_\mathbf{j}^\mathbf{i} \, x).$$

2. The Crossing Formula

2.1. The poset of crossings.

Definition 2.1. For $s \in [n]$ we define the *s-strip* $\mathcal{L}_s$ to be the set of tiles with edges labeled by s. If we order the tiles of $\mathcal{L}_s$ in a sequence $(\gamma_i)_{1 \le i \le n-1}$ such that γ_1 intersects the left boundary with two edges and γ_i and γ_{i+1} share an edge for all $i \in [n-2]$ we say that γ_{i+1} *follows* γ_i *in inbound direction.* If γ_1 intersects the right boundary with two edges and γ_i and γ_{i+1} share an edge for all $i \in [n-2]$ we say that γ_{i+1} *follows* γ_i *in outbound direction.*

Example 2.2. We continue with Example 1.1 and depict the 2-strip $\mathcal{L}_2$ in $\mathcal{T}$.



The associated sequence in inbound direction is given by $([2,3],[2,1],[2,5],[2,4])$ and the associated sequence in inbound direction is given by $([2,4],[2,5],[2,1],[2,3])$. □

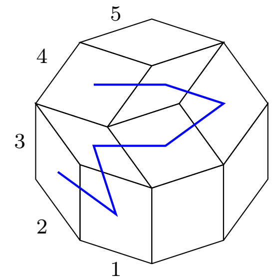
The main objects which encode the actions of the Kashiwara operators are *a-crossings* ($a \in [n-1]$) which are defined to be sequences $(\gamma_i)_{1 \le i \le m}$ of tiles in $\mathcal{T}$ such that

- γ_1 is the unique tile in $\mathcal{L}_a$ which intersects the left boundary with two edges and γ_m is the unique tile in $\mathcal{L}_{a+1}$ which intersects the left boundary with two edges,
- γ_i and γ_{i+1} share an edge for each $i \in \{1, 2, \dots, m-1\}$,
- if γ_i and γ_{i+1} lie both in the same s-strip $\mathcal{L}_s$, then γ_{i+1} follows γ_i in inbound direction (resp. outbound direction) if $s \le a$ (resp. $a+1 \le s$).

We denote the set of a-crossings by Γ_a.

For an a-crossing $(\gamma_i)_{1 \le i \le m} \in \Gamma_a$ we say that for $i \in \{2, 3, \dots, m-1\}$ the tile γ_i is a *station* if γ_{i-1} and γ_{i+1} do not lie on the same strip. We say γ_1 is a *station* if $\gamma_2 \notin \mathcal{L}_a$ and γ_m is a *station* if $\gamma_{m-1} \notin \mathcal{L}_{a+1}$.

Example 2.3. We continue with Example 1.1 and depict a 3-crossing γ in $\mathcal{T}$ given by the tiles surrounding the line drawn below, that is $\gamma = ([2,3],[3,1],[1,2],[2,5],[2,4],[4,5],[4,1])$. The set of stations of γ is given by $\{[1,3],[1,2],[2,4]\}$.



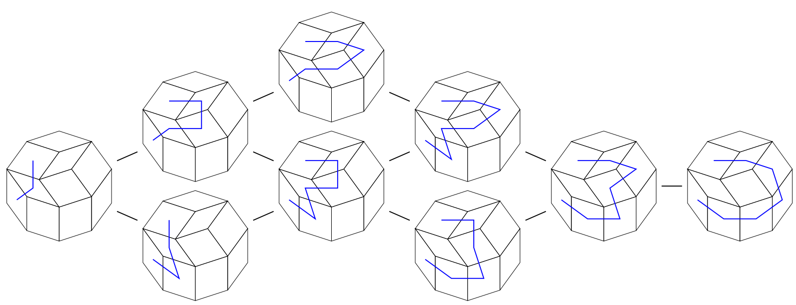
□

Following [GKS, Proposition 2.2] we equip the set Γ_a of a-crossings with a poset structure as follows.The set $\mathcal{T} \setminus \{T \mid T \in \gamma\}$ is partitioned into the set of tiles lying on the left of γ and the set of tiles lying on the right of γ. We denote the set consisting of those $T \in \mathcal{T}$ which lie left of γ or on γ by $\overline{\gamma}$. We define for $\gamma, \lambda \in \Gamma_a$

$$\mathring{\gamma} := \overline{\gamma} - \gamma,$$
$$\gamma \leq \lambda :\Leftrightarrow \overline{\gamma} \subseteq \overline{\lambda} \wedge \mathring{\gamma} \subseteq \mathring{\lambda}.$$

We illustrate the poset $(\Gamma_a, \leq)$ in an example.

Example 2.4. We depict the poset of 3-crossings in $\mathcal{T}$ as in Example 1.1. Each crossing is given by the tiles surrounding one of the blue lines.



□

2.2. **The formula.** Let $a \in [n-1]$. To state the formula for the Kashiwara operators f_a and e_a on $\mathbb{N}^{\mathcal{T}}$ we introduce the following notions. Let $\gamma = (\gamma_i)_{1\leq i\leq m} \in \Gamma_a$ be an a-crossing. If for $i \in \{2, 3, \dots, m-1)$ the tile γ_i is a station of γ we define

$$\operatorname{sgn}(\gamma_i) = \begin{cases} 1 & \text{if } \gamma_{i-1} = [s,t],\ \gamma_i = [t,u],\ \gamma_{i+1} = [u,v] \text{ and } t < u, \\ -1 & \text{if } \gamma_{i-1} = [s,t],\ \gamma_i = [t,u],\ \gamma_{i+1} = [u,v] \text{ and } u < t. \end{cases}$$

If γ_1 is a station of γ, we define

$$\operatorname{sgn}(\gamma_1) = \begin{cases} 1 & \text{if } \gamma_1 = [a,s],\ \gamma_2 = [s,t], \text{ and } a < s, \\ -1 & \text{if } \gamma_1 = [a,s],\ \gamma_2 = [s,t], \text{ and } s < a. \end{cases}$$

If γ_m is a station of γ, we define

$$\operatorname{sgn}(\gamma_m) = \begin{cases} 1 & \text{if } \gamma_{m-1} = [s,t],\ \gamma_m = [t,a+1], \text{ and } t < a+1, \\ -1 & \text{if } \gamma_{m-1} = [s,t],\ \gamma_m = [t,a+1], \text{ and } a+1 < t. \end{cases}$$

We define the vector $[\gamma] \in \mathbb{Z}^{\mathcal{T}}$ associated to γ by

$$[\gamma]_T := \begin{cases} \mathrm{sgn}(T) & \text{if } T \text{ is a station of } \gamma, \\ 0 & \text{else.} \end{cases}$$

We further define for a tile $T = [s,t] \in \mathcal{T}$

$$\epsilon_a([s,t]) := \begin{cases} 1 & \text{if } s \le a < a+1 \le t, \\ -1 & \text{else} \end{cases} \qquad ([s,t] \in \mathcal{T}),$$

$$F_a(x,\gamma) = \sum_{\substack{T\in\gamma \\ \epsilon_a(T)=1}} x_T - \sum_{\substack{T\in\gamma \\ \epsilon_a(T)=-1 \\ [\gamma]_T=0}} x_T, \qquad \left(x \in \mathbb{N}^{\mathcal{T}}\right).$$

Example 2.5. For the 3-crossing γ given in Example 2.3 we have the following. The stations of γ get assigned the following signs:

$$\mathrm{sgn}([1,3]) = -1, \quad \mathrm{sgn}[1,2] = 1, \quad \mathrm{sgn}[2,4] = 1$$

and therefore $[\gamma] \in \mathbb{Z}^{\mathcal{T}}$ is given by

$$[\gamma]_T = \begin{cases} 1 & \text{if } T \in \{[1,2],[2,4]\}, \\ -1 & \text{if } T = [1,3], \\ 0 & \text{if } T \notin \{[1,2],[1,3],[2,4]\}. \end{cases}.$$

Recall from Example 1.4 that $\mathcal{T} = \mathcal{T}_{\mathbf{i}}$ with $\mathbf{i} = (2,1,2,3,4,3,2,1,3,2)$. For an $\mathbf{i}$-Lusztig datum $x \in \mathbb{N}^{\mathcal{T}}$, we have

$$F_3(x,\gamma) = -x_{[2,3]} + x_{[2,5]} + x_{[2,4]} - x_{[4,5]} + x_{[1,4]}.$$

□

The following statement is proved in [GKS, Theorem 2.11]

Theorem 2.6 (Crossing Formula)**.** *Let* $\mathbf{i}$ *be a reduced word and* $\mathcal{T}$ *the tiling associated to* $\mathbf{i}$*. For* $a \in [n-1]$ *and an* $\mathbf{i}$*-Lusztig datum* $x \in \mathbb{N}^{\mathcal{T}}$ *we have*

$$f_a(x) - x = [\gamma^x],$$

where γ^x *is the* $\preceq$*-maximal element in* Γ_a *such that*

$$F_a(x,\gamma^x) = \max_{\gamma\in\Gamma_a} F_a(x,\gamma). \tag{3}$$

Furthermore we have

$$\varepsilon_a(x) = \max_{\gamma\in\Gamma_a} F_a(x,\gamma).$$

Remark 2.7. Theorem 2.6 recovers the explicit description from [R97] of the crystal structure on $\mathbf{i}$-Lusztig data in the special case of reduced words adapted to a quiver of type A.

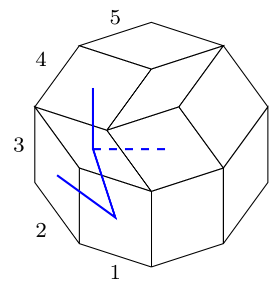
2.3. **The set of Reineke vectors.**

Definition 2.8. We say $\gamma = (\gamma_i)_{1\le i\le m} \in \Gamma_a$ is a *Reineke a-crossing* if it satisfies the following: For any $\gamma_i = [s,t]$ such that γ_{i-1}, γ_i and γ_{i+1} lie in the same strip $\mathcal{L}_s$ we have

$$\begin{aligned} s > t \quad & \text{if } t \le a \\ s < t \quad & \text{if } a+1 \le t. \end{aligned}$$

The set of all Reineke a-crossings is denoted by $\mathcal{R}_a$.

Example 2.9. In the poset Γ_3 given in Example 2.4 all 3-crossings satisfy the condition of Definition 2.8 except $\gamma = (3,1,4)$ since we have $1 < 2 \leq 3$.



□

We have the following theorem by [GKS, Theorem 2.13]..

Theorem 2.10. *For any $a \in [n-1]$ we have*

$$\{f_a x - x \mid x \in \mathbb{N}^{\mathcal{T}}\} = \{[\gamma] \mid \gamma \in \mathcal{R}_a\}.$$

We define

$$\mathbf{R}_a = \mathbf{R}_a(\mathcal{T}) := \{f_a x - x \mid x \in \mathbb{N}^{\mathcal{T}}\}.$$

Definition 2.11. The subset $\mathbf{R}_a \subset \mathbb{Z}^{\mathcal{T}}$ is called the set of *a-Reineke vectors*.

3. Application 2: A duality between Lusztig's and Kashiwara's parametrization

3.1. **Kashiwara's parametrizations.** We recall *Kashiwara's parametrization* of the dual canonical basis $\mathbb{B}^{\text{dual}}$ of U_q^- corresponding to the reduced word $\mathbf{i} = (i_1, \ldots, i_N)$ ([K93]). The parametrizing set is the set of $\mathbf{i}$-string data of elements of the crystal $B(\infty)$. An $\mathbf{i}$-*string datum* $\mathfrak{s}_{\mathbf{i}}(\mathrm{b})$ of $\mathrm{b} \in B(\infty)$ is a tuple $\mathfrak{s}_{\mathbf{i}}(\mathrm{b}) = (x_1, x_2, \ldots, x_N) \in \mathbb{N}^N$ defined inductively as follows:

$$\begin{aligned} x_1 &= \max\left\{k \in \mathbb{N} \mid e_{i_1}^k \,\mathrm{b} \neq 0\right\}, \\ x_2 &= \max\left\{k \in \mathbb{N} \mid e_{i_2}^k e_{i_1}^{x_1} \,\mathrm{b} \neq 0\right\}, \\ &\vdots \\ x_N &= \max\left\{k \in \mathbb{N} \mid e_{i_N}^k e_{i_{N-1}}^{x_{N-1}} \cdots e_{i_1}^{x_1} \,\mathrm{b} \neq 0\right\}. \end{aligned}$$

By [BZ93], [L98] the set

$$\mathbb{S}_{\mathbf{i}} := \{\mathfrak{s}_{\mathbf{i}}(\mathrm{b}) \mid \mathrm{b} \in B(\infty)\} \in \mathbb{N}^N$$

is a polyhedral cone called the *string cone associated to* $\mathbf{i}$.

3.2. **A Duality of cones.** In this section we derive a duality between Lusztig's and Kashiwara's parametrizations of $B(\infty)$.

For a reduced word $\mathbf{i} = (i_1, i_2, \ldots, i_N)$ for w_0 we denote by $\mathbf{i}^* = (i'_1, i'_2, \ldots, i'_N)$ the reduced word for w_0 given by $i'_k = n - i_k$ for all $1 \leq k \leq n-1$. We further denote by $\mathbf{R}^\vee(\mathcal{T}_{\mathbf{i}})$ the set of Reineke vectors of $\mathcal{T}_{\mathbf{i}}$ with coordinates relabeled with respect to the total ordering $\leq_{\mathbf{i}^*}$ of $\mathcal{T}_{\mathbf{i}}$ under the identification of tiles in $\mathcal{T}_{\mathbf{i}}$ with positive roots Φ^+ as in (2).

For a non-empty subset $C \subset \mathbb{R}^N$ we define the *polar cone* C^{pol} by

$$C^{\text{pol}} := \{x \in \mathbb{R}^N \mid \forall y \in C : \langle x, y\rangle \geq 0\}.$$

We have the following duality showing that the string cone inequalities are controlled by the crystal operations on Lusztig's parametrization.

Theorem 3.1. [GKS, Theorem 4.3] *Let* $\mathbf{i}$ *be a reduced word for* w_0 *and let* $\mathcal{T}_{\mathbf{i}}$ *be the tiling associated to* $\mathbf{i}$. *Then*

$$\mathbb{S}_{\mathbf{i}^*} = \mathbf{R}^\vee(\mathcal{T}_{\mathbf{i}})^{\mathrm{pol}}.$$

Remark 3.2. Rhombic tilings are dual to objects called pseudoline arrangements which are also associated to reduced words (see e.g. [BFZ96] for the definition) such that strips in a tiling correspond to pseudolines (see [DKK10, Section 2]). Using pseudoline arrangements the notion of Reineke crossings translates into the notion of rigorous paths which appear in the work [GP00] of Gleizer and Postnikov. Furthermore, Gleizer and Postnikov associate vectors to rigorous paths. The set of such vectors associated to all rigorous path in the pseudoline arrangement corresponding to the reduced word $\mathbf{i}$ coincides with the set of Reineke vectors $\mathbf{R}(\mathcal{T}_{\mathbf{i}^*})$ with coordinates labeled with respect to the total ordering $\leq_{\mathbf{i}}$ of $\mathcal{T}_{\mathbf{i}^*}$ (as usual using the identification of tiles with positive roots as in (2)).

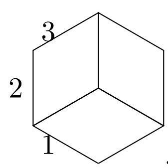
Gleizer and Postnikov show in loc. cit. that the set of vectors associated to rigorous paths of the pseudline arrangement corresponding to $\mathbf{i}$ is polar to the string cones $\mathbb{S}_{\mathbf{i}}$. Therefore Theorem 3.1 recovers the inequalities of $\mathbb{S}_{\mathbf{i}}$ found in [GP00].

Remark 3.3. For the special case of reduced words adapted to quivers of type A Zelikson proved the equality

$$\mathbb{S}_{\mathbf{i}^*} = \mathbf{R}^\vee(\mathcal{T}_{\mathbf{i}})^{\mathrm{pol}}$$

in [Z13] using the results of [R97, GP00]. Theorem 3.1 was inspired by Zelikson's result.

Example 3.4. We give an example for $n = 3$. Consider the following tiling corresponding to the reduced word $\mathbf{i} = (2, 1, 2)$.



We have $\mathbf{i}^* = (1, 2, 1)$. The ordering of the tiles with respect to the order $\leq_{\mathbf{i}^*}$ is:

$$[1,2] \leq_{\mathbf{i}^*} [1,3] \leq_{\mathbf{i}^*} [2,3].$$

We label the coordinates with respect to that order. The set of Reineke vectors $\mathbf{R}^\vee(\mathcal{T}_{\mathbf{i}})$ with relabeled coordinates is given as follows

$$\mathcal{R}(\mathcal{T}) = \{(1,0,0), (0,1,-1), (0,0,1)\}.$$

We obtain by Theorem 3.1 the inequalities for the string cone for this reduced word (see e.g. [L98]):

$$v_{1,2} \geq 0 \qquad v_{1,3} \geq v_{2,3} \geq 0.$$

□

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