Combinatorics of canonical bases revisited

Bea Schumann (Volker Genz and Gleb Koshevoy)

Hannover
05. Dezember, 2016

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5. An interpretation in the setup of cluster varieties

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If we replace $\beta_1 \leq \beta_2 \leq \beta_3$ by $\beta_3 \leq' \beta_2 \leq' \beta_1$, we call the convex ordering $\leq'$ a *flip* of $\leq$.

PBW-type bases

For each convex ordering $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_N$ on $\Phi^+ = \{\beta_1, \beta_2, \ldots, \beta_N\}$ Lusztig defined a PBW-type basis of U_q^+ (=positive part of the quantized enveloping algebra of $\mathfrak{g}$)

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$$B_{\leq} = \{F_{\beta_1}^{(x_{\beta_1})} F_{\beta_2}^{(x_{\beta_2})} \cdots F_{\beta_N}^{(x_{\beta_N})} \mid (x_{\beta_1}, x_{\beta_2}, \ldots, x_{\beta_N}) \in \mathbb{Z}_{\geq 0}^N\}.$$

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$\rightsquigarrow$ we have a crystal structure (= a particular coloured graph structure) on each $B_{\leq}$.

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- f_a is given explicitly if α_a is the $\leq$-minimal positive root. Otherwise f_a is given recursively via transition between different PBW-type bases obtained from each other by a sequence of flips of the corresponding convex orderings (difficult to compute in general).

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(nice=withouth using piecewise-linear maps)

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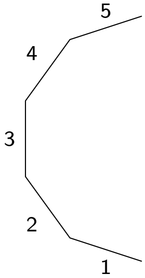

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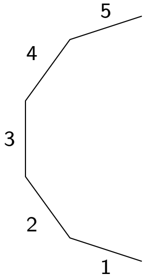

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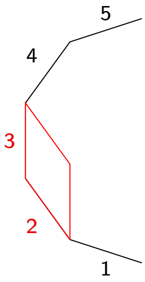

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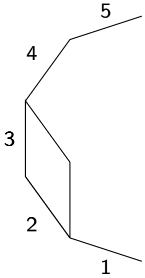

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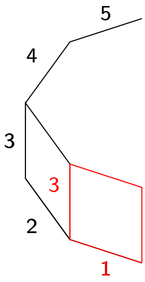

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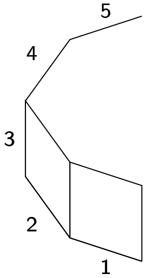

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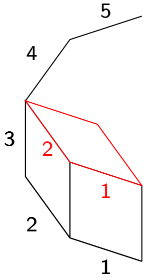

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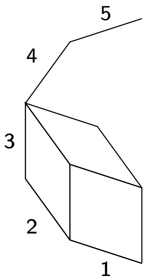

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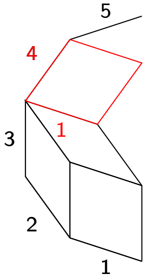

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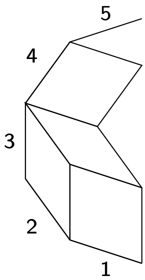

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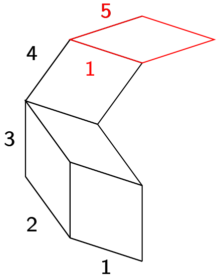

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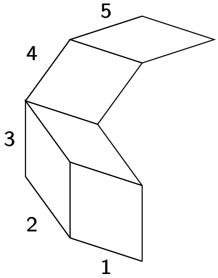

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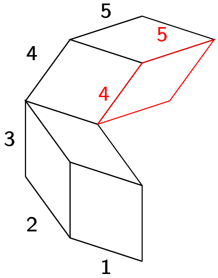

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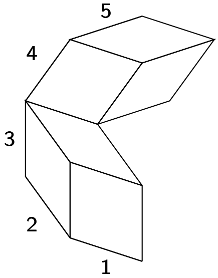

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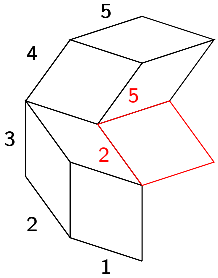

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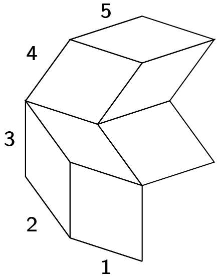

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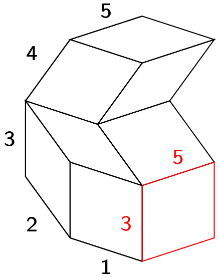

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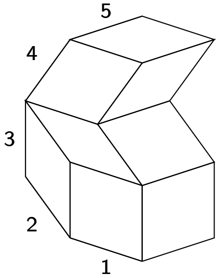

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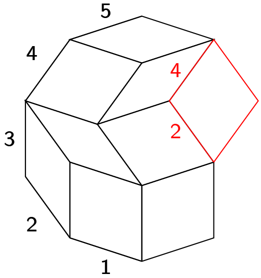

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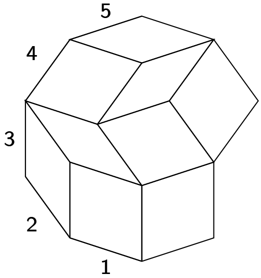

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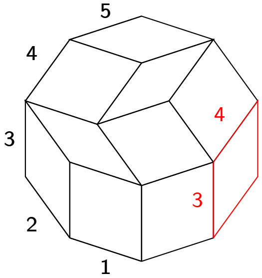

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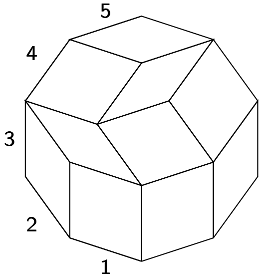

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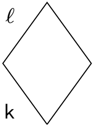
$$\langle k, \ell \rangle :=$$



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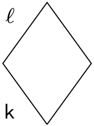
$\langle k, \ell \rangle :=$ 

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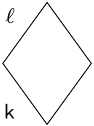

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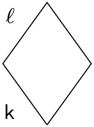

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For $i \in \{1, \ldots, n\}$ we define the *i-strip* $\mathcal{S}_i$ to be the sequence consisting of all rhombi parallel to the edge with label i.

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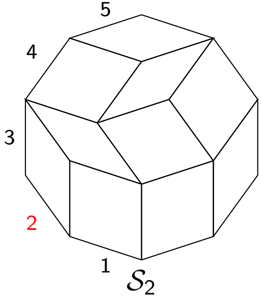

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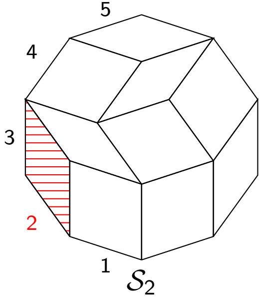

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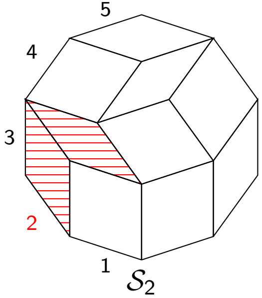

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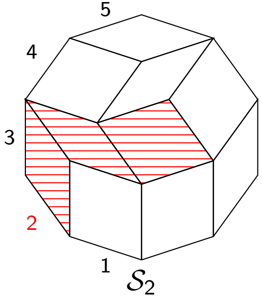

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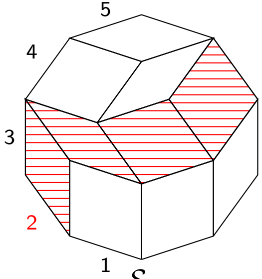

$\mathcal{S}_2$

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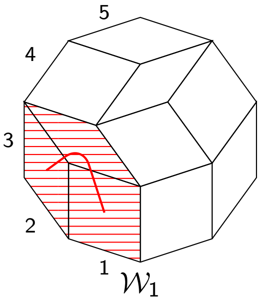

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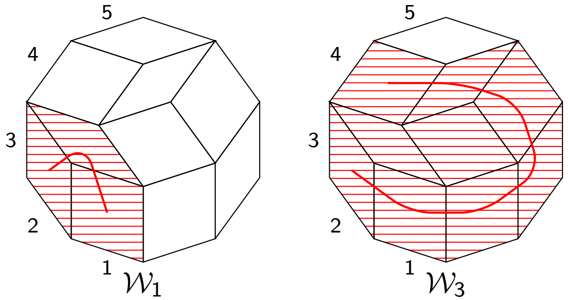

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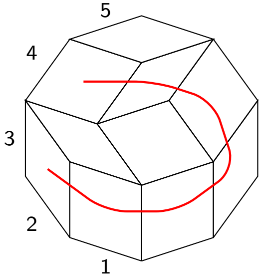

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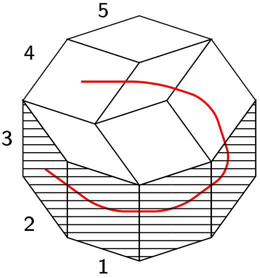

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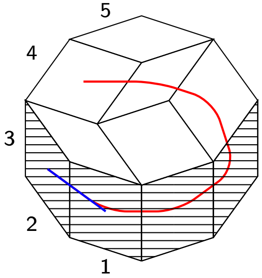

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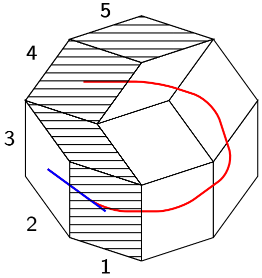

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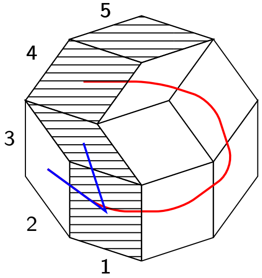

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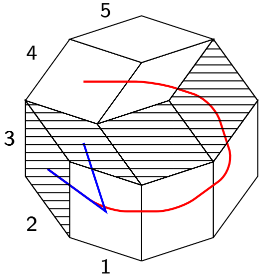

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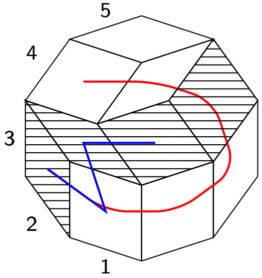

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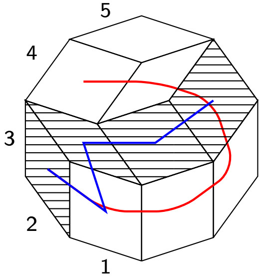

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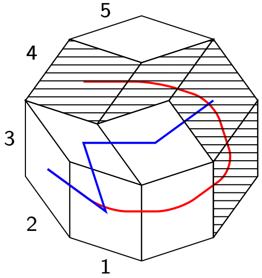

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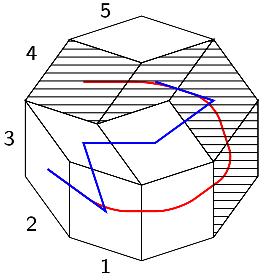

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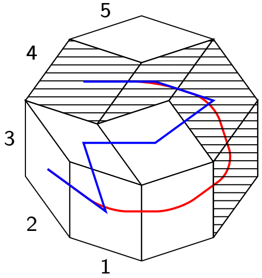

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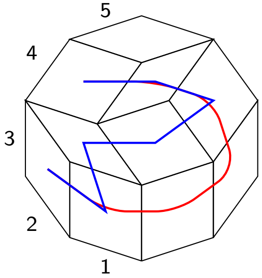

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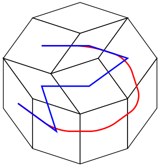

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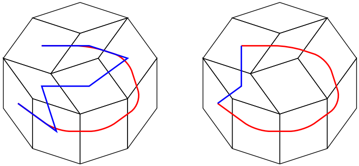

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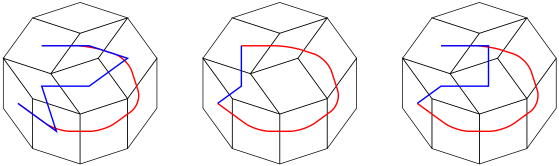

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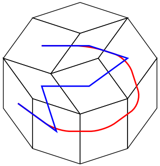

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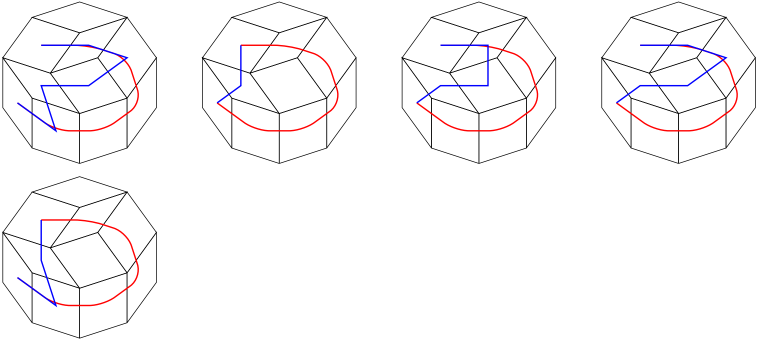

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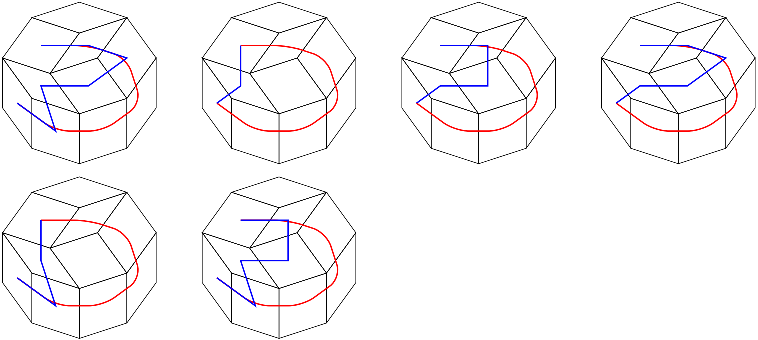

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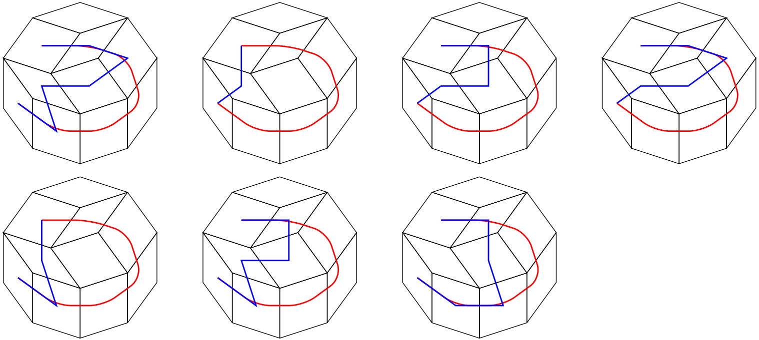

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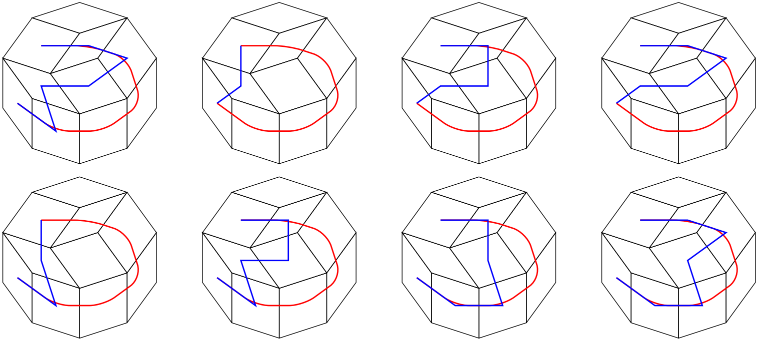

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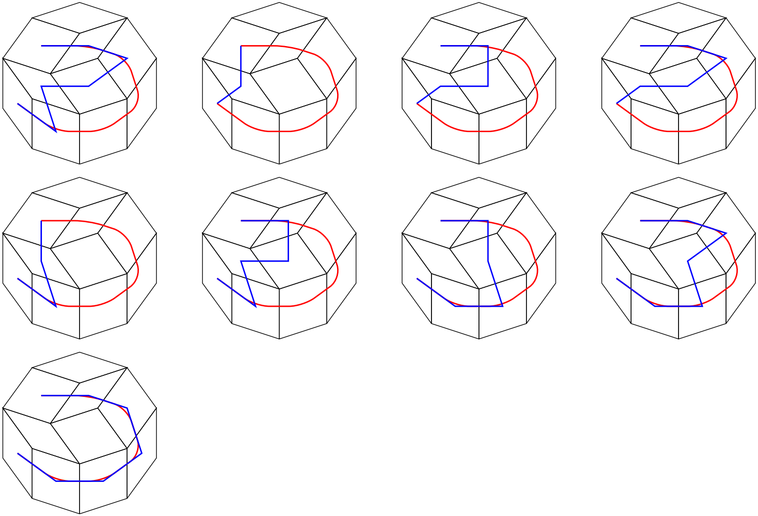

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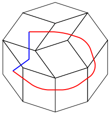

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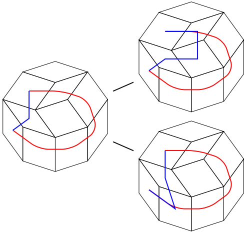

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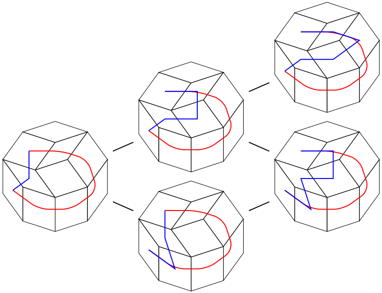

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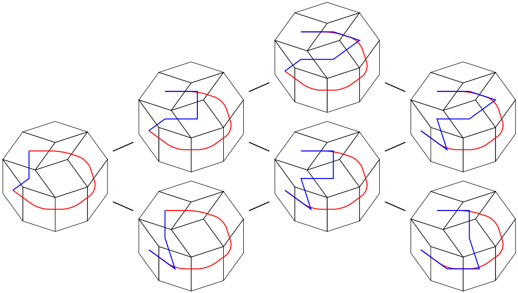

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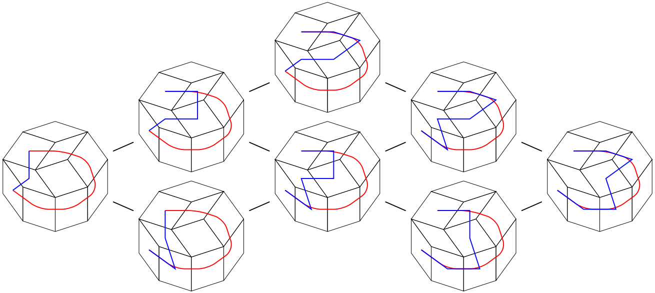

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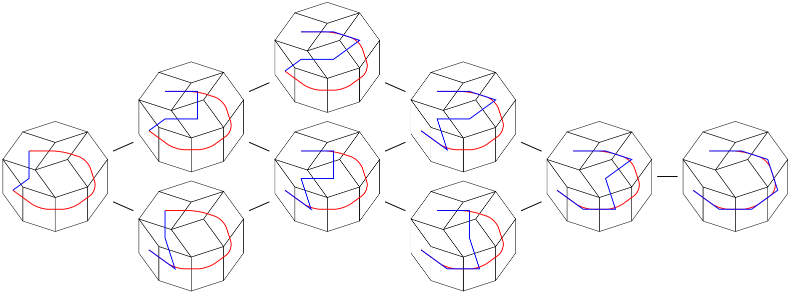

Vectors associated to crossings

To an i-crossing γ we associate the vector $[\gamma] \in \mathbb{Z}^{\Phi^+}$ given by $(1 \leq k < \ell \leq n)$

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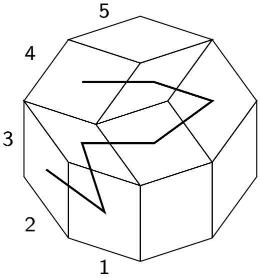

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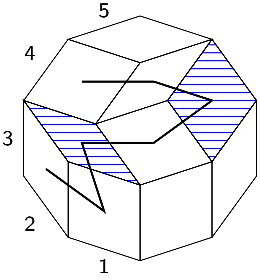

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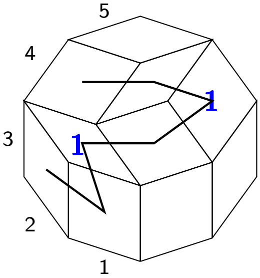

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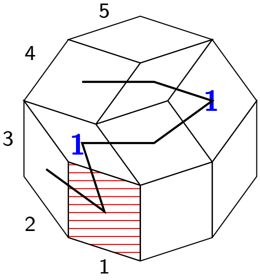

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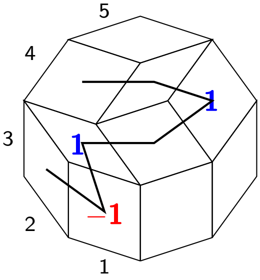

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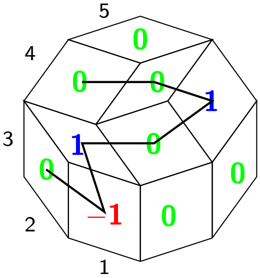

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For a $\leq$-Lusztig datum x we define for $1 \leq k < l \leq n$

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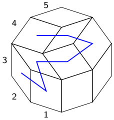

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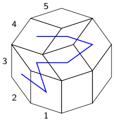

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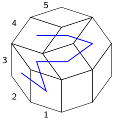

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$$F_3(x,\gamma) = -x_{[2,3]} + x_{[2,5]} + x_{[2,4]} - x_{[4,5]} + x_{[1,4]}.$$

Crossing Formula (Genz, Koshevoy, S.)

Let $a \in \{1, \ldots, n-1\}$ and let x be a $\leq$-Lusztig datum. Then

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Remark

The above theorem was inspired by a result by Reineke for convex orderings adapted to quivers.

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We denote the set of all Reineke a-crossings by $\mathcal{R}_a$

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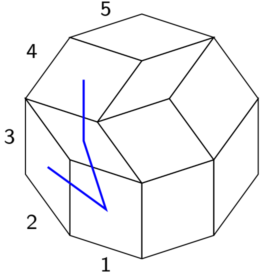

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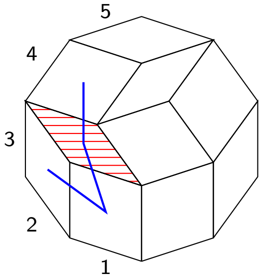

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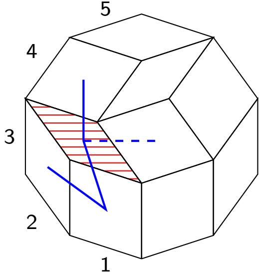

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$$\{f_a x - x \mid x \in \mathbb{N}^{\mathcal{T}}\} = \{[\gamma] \mid \gamma \in \mathcal{R}_a\}.$$

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The subset $\mathbf{R}_a(\mathcal{T}) := \{f_a x - x \mid x \in \mathbb{N}^{\mathcal{T}}\} \subset \mathbb{Z}^{\mathcal{T}}$ is called the set of *a-Reineke vectors*.

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Each convex ordering $\leq$ on Φ^+ yields another parametrization $\mathbb{S}_{\leq} \subset \mathbb{Z}^N$ of the canonical basis by counting the lengths of certain canonical paths from the unique source to each element b in the crystal graph.

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The cone $\mathbb{S}_\leq$ is called the *string cone* (corresponding to $\leq$).

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We have the following duality between Lusztig's and Kashiwara's parametrization:

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- In the special case of convex orderings $\leq$ adapted to quivers of type A, the duality between Lusztig's and Kashiwara's parametrization above was discovered by Zelikson using work of Reineke.

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Let N be the set of upper triangular matrices in $\mathrm{SL}_n(\mathbb{C})$ with diagonal entries all equal to one.

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is called the *reduced double Bruhat cell* (associated to e, w_0).

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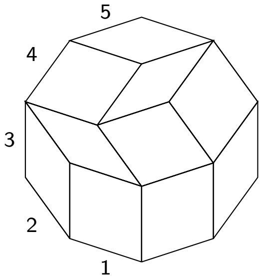

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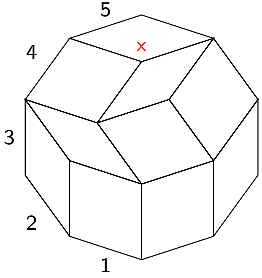

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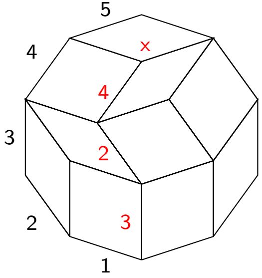

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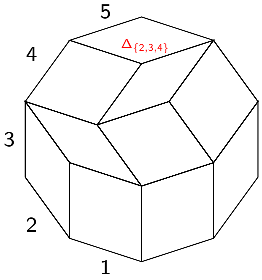

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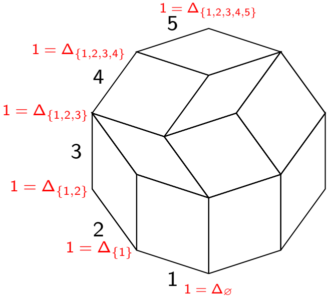

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To get from the cluster $V(\mathcal{T}_{\leq})$ to the cluster $V(\mathcal{T}_{\leq'})$ we have to apply a sequence of maps (*mutations*) corresponding to a sequence of flips transforming $\leq$ to $\leq'$.

Flips and mutations

A flip of a convex ordering $\leq$ corresponds to a flip of a subhexagon of $\mathcal{T}_{\leq}$.

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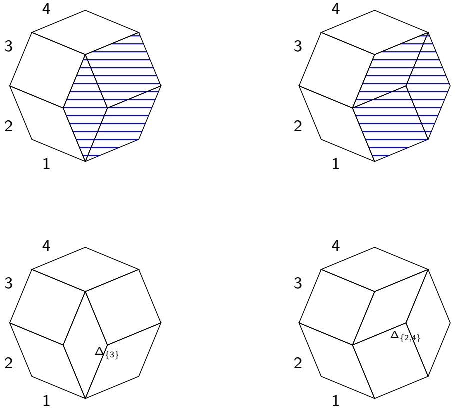

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$$\mathcal{C}_X := \{v \in \mathbb{Z}^X \mid W|_{\mathbb{Z}^X}(v) \geq 0\}.$$

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For two clusters X, Y, there is a natural piecewise-linear transformation between $\mathbb{Z}^X$ and $\mathbb{Z}^Y$ which can be used to compute $W|_{\mathbb{Z}^Y}$ from $W|_{\mathbb{Z}^X}$.

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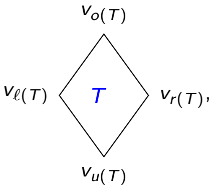
I.e. for a tile $T \in \mathcal{T}$, we have



$$\mathrm{CA}_{\mathcal{T}}(v)_T = v_{\ell(T)} + v_{r(T)} - v_{o(T)} - v_{u(T)}.$$

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Crystal operations and potential functions

Recall that for a tiling $\mathcal{T}$

$$\mathbf{R}_a(\mathcal{T}) := \{f_a x - x \mid x \in \mathbb{N}^{\mathcal{T}}\} \subset \mathbb{Z}^{\mathcal{T}}.$$

We obtain the following relation of the crystal operations on Lusztig's parametrizations and the parametrizations of the GHKK-canonical basis for $\mathbb{C}[N]$.

Theorem (Genz,Koshevoy,S.)

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Corollary

$$\mathcal{C}_{V(\mathcal{T}_\leq)} = \mathsf{CA}^*_{\mathcal{T}_\leq}\left(\mathbb{S}_{\mathcal{T}_\leq}\right).$$

Remark

A unimodular transformation from the weighted string cone for flag and Schubert varieties to the respective cone arising from the potential functions of the appropriate cluster varieties was recently obtained by Bossinger-Fourier.

THANK YOU!!