Combinatorics of canonical bases revisited

Bea Schumann (Volker Genz and Gleb Koshevoy)

Hannover
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Today I will talk about recent work with Volker Genz and Gleb Koshevoy.
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Outline of the talk:
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1. Crystal structures on PBW-type bases
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2. Rhombic tilings
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3. Crystal structures via Rhombic tilings
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1. Crystal structures on PBW-type bases
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3. Crystal structures via Rhombic tilings
4. A duality between Lusztig’s and Kashiwara’s parametrization of canonical bases
Outline of the talk:

1. Crystal structures on PBW-type bases
2. Rhombic tilings
3. Crystal structures via Rhombic tilings
4. A duality between Lusztig’s and Kashiwara’s parametrization of canonical bases
5. An interpretation in the setup of cluster varieties
Setup

\[ g = \mathfrak{sl}_n = \{ A \in M_n(\mathbb{C}) \mid \text{trace} A = 0 \} \]

\[ h = \{ \text{diagonal matrices} \} \subset \mathfrak{sl}_n \]

Cartan subalgebra

\[ \Phi^+ = \{ \alpha_{k, \ell} - 1 = \epsilon_k - \epsilon_\ell \mid k, \ell \in \{1, \ldots, n\}; k < \ell \} \subset h^* \]

positive roots of \( g \), where \( \epsilon_k(\text{diag}(h_1, h_2, \ldots, h_n)) = h_k \).

A total ordering \( \leq \) on \( \Phi^+ \) is called convex if for each \( \beta_1, \beta_2, \beta_3 \in \Phi^+ \) with \( \beta_1 + \beta_3 = \beta_2 \)

\[ \beta_1 \leq \beta_2 \leq \beta_3 \text{ or } \beta_3 \leq \beta_2 \leq \beta_1. \]

If we replace \( \beta_1 \leq \beta_2 \leq \beta_3 \) by \( \beta_3 \leq \beta_2 \leq \beta_1 \), we call the convex ordering \( \leq \) a flip of \( \leq \).
\[ g = sl_n := \{ A \in M_n(\mathbb{C}) \mid \text{trace} A = 0 \} \]
Setup

- $\mathfrak{g} = sl_n := \{A \in M_n(\mathbb{C}) \mid \text{trace} A = 0\}$

- $\mathfrak{h} = \{\text{diagonal matrices}\} \subset sl_n$ Cartan subalgebra
Setup

- $\mathfrak{g} = sl_n := \{ A \in M_n(\mathbb{C}) \mid \text{trace} A = 0 \}$

- $\mathfrak{h} = \{$diagonal matrices$\} \subset sl_n$ Cartan subalgebra

- $\Phi^+ = \{ \alpha_{k,\ell-1} = \epsilon_k - \epsilon_\ell \mid k, \ell \in \{1, \ldots, n\}; k < \ell \} \subset \mathfrak{h}^*$
  positive roots of $\mathfrak{g}$, where $\epsilon_k (\text{diag}(h_1, h_2, \ldots, h_n)) = h_k$. 

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• $\mathfrak{g} = \mathfrak{sl}_n := \{ A \in M_n(\mathbb{C}) \mid \text{trace}A = 0 \}$

• $\mathfrak{h} = \{ \text{diagonal matrices} \} \subset \mathfrak{sl}_n$ Cartan subalgebra

• $\Phi^+ = \{ \alpha_{k,\ell-1} = \epsilon_k - \epsilon_{\ell} \mid k, \ell \in \{1, \ldots, n\}; k < \ell \} \subset \mathfrak{h}^*$ positive roots of $\mathfrak{g}$, where $\epsilon_k (\text{diag}(h_1, h_2, \ldots, h_n)) = h_k$.

• A total ordering $\leq$ on $\Phi^+$ is called convex if for each $\beta_1, \beta_2, \beta_3 \in \Phi^+$ with $\beta_1 + \beta_3 = \beta_2$

$$\beta_1 \leq \beta_2 \leq \beta_3 \quad \text{or} \quad \beta_3 \leq \beta_2 \leq \beta_1.$$
Setup

- \(\mathfrak{g} = \mathfrak{sl}_n := \{A \in M_n(\mathbb{C}) \mid \text{trace} A = 0\}\)

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- A total ordering \(\leq\) on \(\Phi^+\) is called convex if for each \(\beta_1, \beta_2, \beta_3 \in \Phi^+\) with \(\beta_1 + \beta_3 = \beta_2\)

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\beta_1 \leq \beta_2 \leq \beta_3 \quad \text{or} \quad \beta_3 \leq \beta_2 \leq \beta_1.
\]

If we replace \(\beta_1 \leq \beta_2 \leq \beta_3\) by \(\beta_3 \leq' \beta_2 \leq' \beta_1\), we call the convex ordering \(\leq'\) a flip of \(\leq\).
PBW-type bases

For each convex ordering $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_N$ on $\Phi^+ = \{\beta_1, \beta_2, \ldots, \beta_N\}$ Lusztig defined a PBW-type basis of $U^+_q$ ($=\text{positive part of the quantized enveloping algebra of } g$)

$B_{\leq}$ is in natural bijection with the canonical basis of $U^+_q$. We have a crystal structure (=a particular coloured graph structure) on each $B_{\leq}$. 
For each convex ordering $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_N$ on $\Phi^+ = \{\beta_1, \beta_2, \ldots, \beta_N\}$ Lusztig defined a PBW-type basis of $U_q^+$ (=positive part of the quantized enveloping algebra of $\mathfrak{g}$)

$$B_{\leq} = \{ F_{\beta_1}^{(x_{\beta_1})} F_{\beta_2}^{(x_{\beta_2})} \ldots F_{\beta_N}^{(x_{\beta_N})} \mid (x_{\beta_1}, x_{\beta_2}, \ldots, x_{\beta_N}) \in \mathbb{Z}_\geq^N \}. $$
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$B_\leq$ is in natural bijection with the canonical basis of $U_q^+$

$\leadsto$ we have a crystal structure (= a particular coloured graph structure) on each $B_\leq$. 

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Crystal structure

\[
\{ \text{Vertices of the crystal} \} = \{ \text{elements of } B_\leq \}
\]
Crystal structure

- \{\text{Vertices of the crystal}\} = \{\text{elements of } B_\leq\}\]

- For each \(a \in \{1, \ldots, n-1\}\) we have an operator \(f_a : B_\leq \rightarrow B_\leq\).
Crystal structure

- \{\text{Vertices of the crystal}\} = \{\text{elements of } B_{\leq}\}

- For each \(a \in \{1, \ldots, n - 1\}\) we have an operator \(f_a : B_{\leq} \to B_{\leq}\). For \(b, b' \in B_{\leq}\), there is an arrow in the crystal graph \(b \xrightarrow{a} b' \iff f_a b = b'\)
Crystal structure

- \{\text{Vertices of the crystal}\} = \{\text{elements of } B_{\leq}\}

- For each \( a \in \{1, \ldots, n - 1\} \) we have an operator \( f_a : B_{\leq} \to B_{\leq} \). For \( b, b' \in B_{\leq} \), there is an arrow in the crystal graph \( b \xrightarrow{a} b' \iff f_a b = b' \)

- \( \varepsilon_a(b) = \text{length of string of consecutive arrows of colour } a \) ending at \( b \).
Crystal structure

- \{\text{Vertices of the crystal}\} = \{\text{elements of } B_\leq\}\n
- For each \(a \in \{1, \ldots, n-1\}\) we have an operator \(f_a : B_\leq \to B_\leq\). For \(b, b' \in B_\leq\), there is an arrow in the crystal graph \(b \xrightarrow{a} b' \iff f_a b = b'\).

- \(\varepsilon_a(b) = \text{length of string of consecutive arrows of colour } a \text{ ending at } b\).

- \(f_a\) is given explicitly if \(\alpha_a\) is the \(\leq\)-minimal positive root.
Crystal structure

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- For each \(a \in \{1, \ldots, n - 1\}\) we have an operator \(f_a : B_{\leq} \rightarrow B_{\leq}\). For \(b, b' \in B_{\leq}\), there is an arrow in the crystal graph \(b \xrightarrow{a} b' \iff f_a b = b'\)

- \(\varepsilon_a(b) = \text{length of string of consecutive arrows of colour } a \text{ ending at } b\).

- \(f_a\) is given explicitly if \(\alpha_a\) is the \(\leq\)-minimal positive root. Otherwise \(f_a\) is given recursively via transition between different PBW-type bases obtained from each other by a sequence of flips of the corresponding convex orderings.
Crystal structure

- \{\text{Vertices of the crystal}\} = \{\text{elements of } B_\leq\}\n
- For each \(a \in \{1, \ldots, n-1\}\) we have an operator \(f_a : B_\leq \to B_\leq\). For \(b, b' \in B_\leq\), there is an arrow in the crystal graph \(b \xrightarrow{a} b' \iff f_a b = b'\)

- \(\varepsilon_a(b) = \text{length of string of consecutive arrows of colour } a \text{ ending at } b\).

- \(f_a\) is given explicitly if \(\alpha_a\) is the \(\leq\)-minimal positive root. Otherwise \(f_a\) is given recursively via transition between different PBW-type bases obtained from each other by a sequence of flips of the corresponding convex orderings (difficult to compute in general).
Aim:

Find nice description of $f_a b$ for $\leq$ arbitrary!
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Find nice description of $f_a b$ for $\leq$ arbitrary!

(nice=without using piecewise-linear maps)
Rhombic tilings

Elnitsky associated to each convex ordering on $\Phi^+$ a decomposition of the regular $2n$–gon into rhombi (Rhombic tilings).
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Example ($n = 5$)

$\alpha_2 < \alpha_{1,2} < \alpha_1 < \alpha_{1,3} < \alpha_{1,4} < \alpha_4 < \alpha_{2,4} < \alpha_{3,4} < \alpha_{2,3} < \alpha_3$
Elnitsky associated to each convex ordering on $\Phi^+$ a decomposition of the regular $2n-$gon into rhombi (*Rhombic tilings*).

**Example ($n = 5$)**

$$\alpha_2 < \alpha_{1,2} < \alpha_1 < \alpha_{1,3} < \alpha_{1,4} < \alpha_4 < \alpha_{2,4} < \alpha_{3,4} < \alpha_{2,3} < \alpha_3$$

$$\alpha_2 = \epsilon_2 - \epsilon_3$$
Elnitsky associated to each convex ordering on $\Phi^+$ a decomposition of the regular $2n$–gon into rhombi (*Rhombic tilings*).

**Example ($n = 5$)**

\[ \alpha_2 < \alpha_{1,2} < \alpha_1 < \alpha_{1,3} < \alpha_{1,4} < \alpha_4 < \alpha_{2,4} < \alpha_{3,4} < \alpha_{2,3} < \alpha_3 \]

\[ = \epsilon_2 - \epsilon_3 \]
Elnitsky associated to each convex ordering on $\Phi^+$ a decomposition of the regular $2n$–gon into rhombi (*Rhombic tilings*).

**Example ($n = 5$)**

\[ \alpha_2 < \alpha_{1,2} < \alpha_1 < \alpha_{1,3} < \alpha_{1,4} < \alpha_4 < \alpha_{2,4} < \alpha_{3,4} < \alpha_{2,3} < \alpha_3 \]

\[= \epsilon_1 - \epsilon_3\]
Elnitsky associated to each convex ordering on $\Phi^+$ a decomposition of the regular $2n$–gon into rhombi (*Rhombic tilings*).

**Example ($n = 5$)**

$$\alpha_2 < \alpha_{1,2} < \alpha_1 < \alpha_{1,3} < \alpha_{1,4} < \alpha_4 < \alpha_{2,4} < \alpha_{3,4} < \alpha_{2,3} < \alpha_3$$

![Diagram of Rhombic tiling example](image-url)
Elnitsky associated to each convex ordering on $\Phi^+$ a decomposition of the regular $2n$–gon into rhombi (*Rhombic tilings*).

**Example ($n = 5$)**

\[
\alpha_2 < \alpha_{1,2} < \alpha_1 = \epsilon_1 - \epsilon_2 < \alpha_{1,3} < \alpha_{1,4} < \alpha_4 < \alpha_{2,4} < \alpha_{3,4} < \alpha_{2,3} < \alpha_3
\]
Elnitsky associated to each convex ordering on $\Phi^+$ a decomposition of the regular $2n$–gon into rhombi (Rhombic tilings).

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$= \epsilon_1 - \epsilon_4$
Rhombic tilings

Elnitsky associated to each convex ordering on $\Phi^+$ a decomposition of the regular $2n$–gon into rhombi (*Rhombic tilings*).

**Example ($n = 5$)**

\[
\alpha_2 < \alpha_{1,2} < \alpha_1 < \alpha_{1,3} < \alpha_{1,4} < \alpha_4 < \alpha_{2,4} < \alpha_{3,4} < \alpha_{2,3} < \alpha_3 = \epsilon_1 - \epsilon_4
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Elnitsky associated to each convex ordering on $\Phi^+$ a decomposition of the regular $2n$–gon into rhombi (Rhombic tilings).

Example ($n = 5$)

$$\alpha_2 < \alpha_{1,2} < \alpha_1 < \alpha_{1,3} < \alpha_{1,4} < \alpha_4 < \alpha_{2,4} < \alpha_{3,4} < \alpha_{2,3} < \alpha_3$$

$$= \epsilon_1 - \epsilon_5$$
Elnitsky associated to each convex ordering on $\Phi^+$ a decomposition of the regular $2n$–gon into rhombi (Rhombic tilings).

Example ($n = 5$)

$$\alpha_2 < \alpha_{1,2} < \alpha_1 < \alpha_{1,3} < \alpha_{1,4} < \alpha_4 < \alpha_{2,4} < \alpha_{3,4} < \alpha_{2,3} < \alpha_3 = \epsilon_1 - \epsilon_5$$
Elnitsky associated to each convex ordering on $\Phi^+$ a decomposition of the regular $2n$–gon into rhombi (Rhombic tilings).

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Rhombic tilings

Elnitsky associated to each convex ordering on $\Phi^+$ a decomposition of the regular $2n$–gon into rhombi (\textit{Rhombic tilings}).

Example ($n = 5$)

\[
\alpha_2 < \alpha_{1,2} < \alpha_1 < \alpha_{1,3} < \alpha_{1,4} < \alpha_4 < \alpha_{2,4} < \alpha_{3,4} < \alpha_{2,3} < \alpha_3 = \epsilon_4 - \epsilon_5
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Rhombic tilings

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Rhombic tilings

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$= \epsilon_2 - \epsilon_5$
Elnitsky associated to each convex ordering on $\Phi^+$ a decomposition of the regular $2n$–gon into rhombi (Rhombic tilings).

Example ($n = 5$)

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Elnitsky associated to each convex ordering on $\Phi^+$ a decomposition of the regular $2n$–gon into rhombi (Rhombic tilings).

**Example ($n = 5$)**

\[ \alpha_2 < \alpha_{1,2} < \alpha_1 < \alpha_{1,3} < \alpha_{1,4} < \alpha_4 < \alpha_{2,4} < \alpha_{3,4} < \alpha_{2,3} < \alpha_3 = \epsilon_3 - \epsilon_5 \]
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$= \epsilon_2 - \epsilon_4$
Rhombic tilings

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Example ($n = 5$)

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Elnitsky associated to each convex ordering on $\Phi^+$ a decomposition of the regular $2n$–gon into rhombi (*Rhombic tilings*).

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![Diagram of a rhombic tiling](image-url)
Rhombic tilings

Elnitsky associated to each convex ordering on $\Phi^+$ a decomposition of the regular $2n$–gon into rhombi (*Rhombic tilings*).

Example $(n = 5)$

$$\alpha_2 < \alpha_{1,2} < \alpha_1 < \alpha_{1,3} < \alpha_{1,4} < \alpha_4 < \alpha_{2,4} < \alpha_{3,4} < \alpha_{2,3} < \alpha_3 = \epsilon_3 - \epsilon_4$$
Elnitsky associated to each convex ordering on $\Phi^+$ a decomposition of the regular $2n$–gon into rhombi (\textit{Rhombic tilings}).

\textbf{Example ($n = 5$)}

$\alpha_2 < \alpha_{1,2} < \alpha_1 < \alpha_{1,3} < \alpha_{1,4} < \alpha_4 < \alpha_{2,4} < \alpha_{3,4} < \alpha_{2,3} < \alpha_3$
Lusztig data

For a convex ordering $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_n$, let $\mathcal{T}_\leq$ be the corresponding tiling.
Lusztig data

For a convex ordering $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_n$, let $\mathcal{T}_\leq$ be the corresponding tiling.

We identify each positive root $\alpha_{k,\ell-1} = \epsilon_k - \epsilon_\ell$ with the tile

$$\langle k, \ell \rangle := \begin{array}{c}
\ell \\
k
\end{array}$$
Lusztig data

For a convex ordering $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_n$, let $\mathcal{T}_\leq$ be the corresponding tiling.

We identify each positive root $\alpha_{k,\ell-1} = \epsilon_k - \epsilon_\ell$ with the tile

$$\langle k, \ell \rangle := \begin{array}{c} \ell \\ \downarrow \\ k \end{array}$$

and an element

$$F_{\beta_1}^{(x_{\beta_1})} F_{\beta_2}^{(x_{\beta_2})} \ldots F_{\beta_n}^{(x_{\beta_n})} \in B_\leq$$
Lusztig data

For a convex ordering \( \beta_1 \leq \beta_2 \leq \ldots \leq \beta_n \), let \( \mathcal{T}_\leq \) be the corresponding tiling.

We identify each positive root \( \alpha_{k,\ell-1} = \epsilon_k - \epsilon_\ell \) with the tile

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\langle k, \ell \rangle := \begin{array}{c}
\ell \\
\downarrow \\
k
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and an element

\[
F_{\beta_1}^{(x_{\beta_1})} F_{\beta_2}^{(x_{\beta_2})} \ldots F_{\beta_N}^{(x_{\beta_N})} \in B_\leq
\]

with the vector

\[
x = (x_{\beta_j})_{\beta_j \in \mathcal{T}_\leq}
\]
For a convex ordering $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_n$, let $\mathcal{T}_\leq$ be the corresponding tiling.

We identify each positive root $\alpha_{k,\ell-1} = \epsilon_k - \epsilon_\ell$ with the tile

$$\langle k, \ell \rangle := \begin{array}{c}
\ell \\
\ell \\
\ell \\
k
\end{array}$$

and an element

$$F_{\beta_1}^{(x_{\beta_1})} F_{\beta_2}^{(x_{\beta_2})} \ldots F_{\beta_N}^{(x_{\beta_N})} \in B_\leq$$

with the vector

$$x = (x_{\beta_j})_{\beta_j \in \mathcal{T}_\leq}$$
called $\leq$-Lusztig-datum.
For $i \in \{1, \ldots, n\}$ we define the $i$-strip $S_i$ to be the sequence consisting of all rhombi parallel to the edge with label $i$. 
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Example (2-strip in $\mathcal{T}$)
For $i \in \{1, \ldots, n\}$ we define the $i$-strip $S_i$ to be the sequence consisting of all rhombi parallel to the edge with label $i$.

Example (2-strip in $\mathcal{T}$)
For \( i \in \{1, \ldots, n\} \) we define the \( i \)-strip \( S_i \) to be the sequence consisting of all rhombi parallel to the edge with label \( i \).

Example (2-strip in \( T \))
For $i \in \{1, \ldots, n\}$ we define the $i$-strip $S_i$ to be the sequence consisting of all rhombi parallel to the edge with label $i$.

Example (2-strip in $\mathcal{T}$)
For $i \in \{1, \ldots, n\}$ we define the $i$-strip $S_i$ to be the sequence consisting of all rhombi parallel to the edge with label $i$.

Example (2-strip in $T$)
For $a \in \{1, \ldots, n - 1\}$ we define the $a$-comb $\mathcal{W}_a$ to be the set of rhombi lying in the region starting at the left boundary cut out by the $a$-strip $S_a$ and the $(a + 1)$-strip $S_{a+1}$. 
For $a \in \{1, \ldots, n - 1\}$ we define the $a$-comb $\mathcal{W}_a$ to be the set of rhombi lying in the region starting at the left boundary cut out by the $a$-strip $S_a$ and the $(a + 1)$-strip $S_{a+1}$.

Example ($\mathcal{W}_1$ and $\mathcal{W}_3$ in $\mathcal{T}$)
For $a \in \{1, \ldots, n - 1\}$ we define the $a$-comb $\mathcal{W}_a$ to be the set of rhombi lying in the region starting at the left boundary cut out by the $a$-strip $S_a$ and the $(a + 1)$-strip $S_{a+1}$.

Example ($\mathcal{W}_1$ and $\mathcal{W}_3$ in $\mathcal{T}$)
An $a$-crossing $\gamma$ is a sequence of neighbour rhombi in $\mathcal{W}_a$ connecting the first rhombus of $S_a$ with the first rhombus of $S_{a+1}$ following strips oriented from $S_a$ to $S_{a+1}$.
An a-crossing $\gamma$ is a sequence of neighbour rhombi in $\mathcal{W}_a$ connecting the first rhombus of $S_a$ with the first rhombus of $S_{a+1}$ following strips oriented from $S_a$ to $S_{a+1}$.

Example (A 3-crossing in $\mathcal{T}$)
An a-crossing $\gamma$ is a sequence of neighbour rhombi in $\mathcal{W}_a$ connecting the first rhombus of $S_a$ with the first rhombus of $S_{a+1}$ following strips oriented from $S_a$ to $S_{a+1}$.

Example (A 3-crossing in $\mathcal{T}$)
An \textit{a-crossing} $\gamma$ is a sequence of neighbour rhombi in $\mathcal{W}_a$ connecting the first rhombus of $S_a$ with the first rhombus of $S_{a+1}$ following strips oriented from $S_a$ to $S_{a+1}$.

\textbf{Example (A 3-crossing in $\mathcal{T}$)}
An $a$-crossing $\gamma$ is a sequence of neighbour rhombi in $\mathcal{W}_a$ connecting the first rhombus of $S_a$ with the first rhombus of $S_{a+1}$ following strips oriented from $S_a$ to $S_{a+1}$.

Example (A 3-crossing in $\mathcal{T}$)
An \textit{a-crossing} $\gamma$ is a sequence of neighbour rhombi in $\mathcal{W}_a$ connecting the first rhombus of $S_a$ with the first rhombus of $S_{a+1}$ following strips oriented from $S_a$ to $S_{a+1}$.

\textbf{Example (A 3-crossing in $\mathcal{T}$)}
An $a$-crossing $\gamma$ is a sequence of neighbour rhombi in $\mathcal{W}_a$ connecting the first rhombus of $S_a$ with the first rhombus of $S_{a+1}$ following strips oriented from $S_a$ to $S_{a+1}$.

**Example (A 3-crossing in $\mathcal{T}$)**
An \( a \)-crossing \( \gamma \) is a sequence of neighbour rhombi in \( \mathcal{W}_a \) connecting the first rhombus of \( S_a \) with the first rhombus of \( S_{a+1} \) following strips oriented from \( S_a \) to \( S_{a+1} \).

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Example (Ordering of 3-crossings in \( \mathcal{T} \))
Vectors associated to crossings

To an $i$-crossing $\gamma$ we associate the vector $[\gamma] \in \mathbb{Z}^{\Phi^+}$ given by 

$$(1 \leq k < \ell \leq n)$$

$$[\gamma]_{\alpha_{k,\ell-1}} = \begin{cases} 
1 & \text{if } \gamma \text{ turns from the } k\text{-strip to the } \ell\text{-strip at } \langle k, \ell \rangle, \\
0 & \text{if } \gamma \text{ does not turn at } \langle k, \ell \rangle, \\
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Example

![Diagram of a 3D object with numbers 1 to 5 labeling vertices, showing a network of lines indicating crossings.]
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\[ [\gamma]_{\alpha_{k,\ell-1}} = 1 \]
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Example

![Diagram of a graph with labeled edges and a vector notation example]

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Example
The Crossing Formula

For a $\leq$-Lusztig datum $x$ we define for $1 \leq k < l \leq n$

$$F_a(x, \gamma) := \sum_{\langle k, \ell \rangle \in \gamma} x_{\alpha_{k, \ell-1}} - \sum_{\langle k, \ell \rangle \in \gamma, a \notin [k, \ell]} x_{\alpha_{k, \ell-1}}.$$
The Crossing Formula

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For a Lusztig datum \( x \in \mathbb{N}^T \), we have
The Crossing Formula

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F_a(x, \gamma) := \sum_{\langle k, \ell \rangle \in \gamma} x_{\alpha_k, \ell-1} - \sum_{\langle k, \ell \rangle \in \gamma} x_{\alpha_k, \ell-1}. 
\]

Example

Let \( \gamma \) be the following 3-crossing

For a Lusztig datum \( x \in \mathbb{N}^T \), we have

\[
F_3(x, \gamma) = -x_{[2,3]} + x_{[2,5]} + x_{[2,4]} - x_{[4,5]} + x_{[1,4]}.
\]
Crossing Formula (Genz, Koshevoy, S.)

Let $a \in \{1, \ldots, n - 1\}$ and let $x$ be a $\leq$-Lusztig datum. Then

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\varepsilon_a(x) = \max_{\gamma} F_a(x, \gamma)
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**Remark**

The above theorem was inspired by a result by Reineke for convex orderings adapted to quivers.
The set of Reineke crossings

Question:
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Do all vectors associated to crossings appear as actions of Kashiwara operators?
The set of Reineke crossings

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Do all vectors associated to crossings appear as actions of Kashiwara operators?

**Definition**
We say \( \gamma \in \Gamma_a \) is a *Reineke a-crossing* if it satisfies the following condition: For any \( \gamma_i = \langle s, t \rangle \in \gamma \) such that \( \gamma_{i-1}, \gamma_i \) and \( \gamma_{i+1} \) lie in the same strip sequence \( \mathcal{L}^s \) we have
The set of Reineke crossings

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We say $\gamma \in \Gamma_a$ is a Reineke $a$-crossing if it satisfies the following condition: For any $\gamma_i = \langle s, t \rangle \in \gamma$ such that $\gamma_{i-1}, \gamma_i$ and $\gamma_{i+1}$ lie in the same strip sequence $L^s$ we have

\[
\begin{align*}
    s &> t \quad \text{if } t \leq a \\
    s &< t \quad \text{if } a + 1 \leq t.
\end{align*}
\]
The set of Reineke crossings

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$$s > t \quad \text{if } t \leq a$$
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We denote the set of all Reineke $a$-crossings by $R_a$.
In our running example all 3-crossings are Reineke 3-crossings except:
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\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{example_diagram}
\end{figure}
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The Set of Reineke vectors

Theorem (Genz, Koshevoy, S.)

We have

\[ \{ f_a x - x \mid x \in \mathbb{N}^T \} = \{ [\gamma] \mid \gamma \in \mathcal{R}_a \}. \]
The Set of Reineke vectors

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Definition
The subset \( \mathcal{R}_a(T) := \{ f_a x - x \mid x \in \mathbb{N}^T \} \subset \mathbb{Z}^T \) is called the set of \( a \)-Reineke vectors.
Kashiwara’s parametrization

Each convex ordering \( \leq \) on \( \Phi^+ \) yields another parametrization \( S_\leq \subset \mathbb{Z}^N \) of the canonical basis by counting the lengths of certain canonical paths from the unique source to each element \( b \) in the crystal graph.
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By Berenstein-Zelevinsky and Littelmann the set \( S_{\leq} \) is a polyhedral cone in \( \mathbb{Z}^N \).
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The cone \( S_{\leq} \) is called the string cone (corresponding to \( \leq \)).
Opposite orderings

For a convex ordering

\[ \beta_1 \leq \beta_2 \leq \ldots \leq \beta_N \]

on \( \Phi^+ \)
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on \( \Phi^+ \) we define another convex ordering \( \leq^{\text{op}} \) by

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Recall that for a tiling $\mathcal{T}$

$$R_a(\mathcal{T}) := \{f_ax - x \mid x \in \mathbb{N}^\mathcal{T}\} \subset \mathbb{Z}^\mathcal{T}.$$
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Recall that for a tiling \( \mathcal{T} \)

\[ R_a(\mathcal{T}) := \{ f_a x - x \mid x \in \mathbb{N}^\mathcal{T} \} \subset \mathbb{Z}^\mathcal{T}. \]

We have the following duality between Lusztig’s and Kashiwara’s parametrization:
The duality

**Theorem (Genz, Koshevoy, S.)**

Let $\leq$ be a convex ordering on $\Phi^+$ and let $T$ be the tiling associated to the convex ordering $\leq^{\text{op}}$. 

Remark

The defining inequalities obtained for $S_{\leq}$ by the theorem above were already found by Gleizer-Postnikov. In the special case of convex orderings $\leq$ adapted to quivers of type $A$, the duality between Lusztig's and Kashiwara's parametrization above was discovered by Zelikson using work of Reineke.
The duality

Theorem (Genz, Koshevoy, S.)

Let $\leq$ be a convex ordering on $\Phi^+$ and let $\mathcal{T}$ be the tiling associated to the convex ordering $\leq^{\text{op}}$. We have

$$\mathbb{S}_{\leq} = \{ v \in \mathbb{Z}^N \mid (v, x) \geq 0 \quad \forall x \in R_a(\mathcal{T}), \quad \forall a \}.$$
The duality

**Theorem (Genz, Koshevoy, S.)**

Let \( \leq \) be a convex ordering on \( \Phi^+ \) and let \( \mathcal{T} \) be the tiling associated to the convex ordering \( \leq^{\text{op}} \). We have

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S_{\leq} = \{ v \in \mathbb{Z}^N \mid (v, x) \geq 0 \quad \forall x \in R_a(\mathcal{T}), \quad \forall a \}.
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Here \((, , )\) is the standard scalar product.
The duality

**Theorem (Genz, Koshevoy, S.)**

Let $\leq$ be a convex ordering on $\Phi^+$ and let $\mathcal{T}$ be the tiling associated to the convex ordering $\leq^{op}$. We have

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- In the special case of convex orderings $\leq$ adapted to quivers of type $A$, the duality between Lusztig’s and Kashiwara’s parametrization above was discovered by Zelikson using work of Reineke.
Let $N$ be the set of upper triangular matrices in $\text{SL}_n(\mathbb{C})$ with diagonal entries all equal to one.
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Let $N$ be the set of upper triangular matrices in $\text{SL}_n(\mathbb{C})$ with diagonal entries all equal to one. For

$$S = \{x_1, x_2, \ldots, x_k\} \subset \{1, 2, \ldots, n\},$$

let $\Delta_S \in \mathbb{C}[N]$ be the function associating to a matrix $A$ the (chamber) minor corresponding to the columns $x_1, x_2, \ldots, x_k$ and the first $k$ rows.
Reduced double Bruhat cells

Let $N$ be the set of upper triangular matrices in $SL_n(\mathbb{C})$ with diagonal entries all equal to one. For

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let $\Delta_S \in \mathbb{C}[N]$ be the function associating to a matrix $A$ the (chamber) minor corresponding to the columns $x_1, x_2, \ldots, x_k$ and the first $k$ rows. The variety

$$L^{e, w_0} = \{A \in N \mid \Delta_{\{n-k,n-k+1,\ldots,n\}} A \neq 0\}$$

is called the reduced double Bruhat cell (associated to $e, w_0$).
We identify the vertices of the tiling with a subset of the chamber minors by collecting the indices of the strips running below each vertex.
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Example

![Diagram of a tiling with vertices labeled 1, 2, 3, 4, 5.]
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The chamber minors corresponding to the vertices on the left boundary of any tiling $\mathcal{T}$ equal the principal minors. Thus $\Delta_S = 1$ for all $S$ on the left boundary of $\mathcal{T}$. 

Example:

$$\Delta_{\{1,2\}} = 3$$
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$$\Delta_{\{1\}} = 5$$
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The chamber minors corresponding to the vertices on the left boundary of any tiling $\mathcal{T}$ equal the principal minors. Thus $\Delta_S = 1$ for all $S$ on the left boundary of $\mathcal{T}$.

Example
The coordinate ring $\mathbb{C}[L^{e,w_0}]$ has the structure of a cluster algebra by Fomin-Zelevinsky.
Cluster associated to tilings

The coordinate ring $\mathbb{C}[L^e, w_0]$ has the structure of a cluster algebra by Fomin-Zelevinsky. This means that we have distinguished overlapping subsets of generators of $\mathbb{C}[L^e, w_0]$ (clusters).
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Let $V(\mathcal{T})$ be the set of vertices of $\mathcal{T}$ not lying on the left boundary.
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Let $V(\mathcal{T})$ be the set of vertices of $\mathcal{T}$ not lying on the left boundary. For each tiling $\mathcal{T}$ the set $V(\mathcal{T})$ of chamber minors forms a cluster in $\mathbb{C}[L^e,w_0]$. 
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To get from the cluster $V(\mathcal{T}_\leq)$ to the cluster $V(\mathcal{T}_{\leq'})$ we have to apply a sequence of maps (mutations) corresponding to a sequence of flips transforming $\leq$ to $\leq'$. 
Flips and mutations

A flip of a convex ordering $\leq$ corresponds to a flip of a subhexagon of $\mathcal{T}_{\leq}$. 
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Example
Recall that $N$ can be obtained from $L^{e,w_0}$ by allowing the chamber minors to vanish which correspond to the vertices on the right boundary of any tiling $\mathcal{T}$. 
Recall that $N$ can be obtained from $L^{e,w_0}$ by allowing the chamber minors to vanish which correspond to the vertices on the right boundary of any tiling $T$. Gross-Hacking-Keel-Kontsevich construct a canonical basis $\mathcal{B}$ for $\mathbb{C}[N]$. 
Recall that $N$ can be obtained from $L^{e,w_0}$ by allowing the chamber minors to vanish which correspond to the vertices on the right boundary of any tiling $\mathcal{T}$.

Gross-Hacking-Keel-Kontsevich construct a canonical basis $\mathcal{B}$ for $\mathbb{C}[N]$.

For each cluster $X$ we have a parametrization of $\mathcal{B}$ by a polyhedral cone $\mathcal{C}_X$ in $\mathbb{Z}^X$ defined via a tropical function $W$: 
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Gross-Hacking-Keel-Kontsevich construct a canonical basis $\mathbb{B}$ for $\mathbb{C}[N]$.

For each cluster $X$ we have a parametrization of $\mathbb{B}$ by a polyhedral cone $C_X$ in $\mathbb{Z}^X$ defined via a tropical function $W$:

$$C_X := \{v \in \mathbb{Z}^X \mid W|_{\mathbb{Z}^X}(v) \geq 0\}.$$
Here \( W = \sum_{a \in \{1, 2, \ldots, n-1\}} W_a \) and for a tiling \( T_a \) such that the tile \( \langle a, a + 1 \rangle \) intersects the right boundary with two edges
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W_a \mid_{\mathbb{Z}^V(T_a)}(v) := (v, e_{\Delta \{n-a, n-a+1, \ldots, n\}}).
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For two clusters $X$, $Y$, there is a natural piecewise-linear transformation between $Z^X$ and $Z^Y$ which can be used to compute $W|_{Z^Y}$ from $W|_{Z^X}$. 
To relate the tropical function $W$ (*potential function*) to the crystal operations on $\mathbb{N}^T$, we define a bijection...
The tropical Chamber Ansatz

To relate the tropical function $W$ (potential function) to the crystal operations on $\mathbb{N}^\mathcal{T}$, we define a bijection

$$CA_\mathcal{T} : \mathbb{Z}^{V(\mathcal{T})} \rightarrow \mathbb{Z}^\mathcal{T}$$

$$v = (v_S)_{S \in V(\mathcal{T})} \mapsto (CA_\mathcal{T}(v)_T)_{T \in \mathcal{T}},$$

given by the tropicalisation of Berenstein-Fomin-Zelevinsky’s Chamber Ansatz.
To relate the tropical function $W$ (*potential function*) to the crystal operations on $\mathbb{N}^T$, we define a bijection

$$CA_T : \mathbb{Z}^{V(T)} \rightarrow \mathbb{Z}^T$$

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given by the tropicalisation of Berenstein-Fomin-Zelevinsky’s Chamber Ansatz.

I.e. for a tile $T \in \mathcal{T}$, we have

$$CA_T(\nu)_T = \nu_{\ell(T)} + \nu_{r(T)} - \nu_{o(T)} - \nu_{u(T)}.$$

\[\documentclass{article}
\usepackage{amsmath,amsfonts}
\begin{document}

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\end{document} \]
Recall that for a tiling $\mathcal{T}$

$$R_a(\mathcal{T}) := \{ f_a x - x \mid x \in \mathbb{N}^\mathcal{T} \} \subset \mathbb{Z}^\mathcal{T}.$$
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We obtain the following relation of the crystal operations on Lusztig’s parametrizations and the parametrizations of the GHKK-canonical basis for $\mathbb{C}[\mathcal{N}]$. 
Theorem (Genz, Koshevoy, S.)

We have

\[ W_a \big|_{\mathbb{Z}^{\leq}} (v) = \min\{(CA_{\leq\text{op}}^{-1}(x), v) \mid x \in R_a(\mathcal{T}_{\leq\text{op}})\} \].
Theorem (Genz, Koshevoy, S.)

We have

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We have

\[ W_a|_{\mathbb{Z}^\leq} (v) = \min\{(CA_{\mathcal{T}_{\leq}^{\text{op}}}(x), v) \mid x \in R_a(\mathcal{T}_{\leq}^{\text{op}})\} \].

We obtain as a direct consequence.

Corollary

\[ C_{V}(\mathcal{T}_{\leq}) = CA_{\mathcal{T}_{\leq}}^{*} (S_{\mathcal{T}_{\leq}}) \].
Remark

A unimodular transformation from the weighted string cone for flag and Schubert varieties to the respective cone arising from the potential functions of the appropriate cluster varieties was recently obtained by Bossinger-Fourier.
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THANK YOU!!