Crystal structures on PBW-type bases via Rhombic tilings

Bea Schumann

22. August 2016

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4. Connection to MV-polytopes

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1. Crystal structures on PBW-type bases
2. Rhombic tilings
3. Crystal structures via Rhombic tilings
4. Connection to MV-polytopes
5. A proof of the Anderson-Mirkovic conjecture via Rhombic tilings

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  If we replace $\beta_1 \leq \beta_2 \leq \beta_3$ by $\beta_3 \leq' \beta_2 \leq' \beta_1$, we call the convex ordering $\leq'$ a *flip* of $\leq$.

PBW-type bases

For each convex ordering $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_N$ on $\Phi^+ = \{\beta_1, \beta_2, \ldots, \beta_N\}$ Lusztig defined a PBW-type basis of U_q^+ (=positive part of the quantized enveloping algebra of $\mathfrak{g}$)

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$$B_{\leq} = \{F_{\beta_1}^{(m_{\beta_1})} F_{\beta_2}^{(m_{\beta_2})} \cdots F_{\beta_N}^{(m_{\beta_N})} \mid (m_{\beta_1}, m_{\beta_2}, \ldots, m_{\beta_N}) \in \mathbb{Z}_{\geq 0}^N\}.$$

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$\rightsquigarrow$ we have a crystal structure (= a particular coloured graph structure) on each $B_{\leq}$.

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(nice=withouth using piecewise-linear maps)

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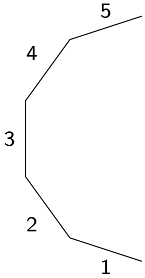

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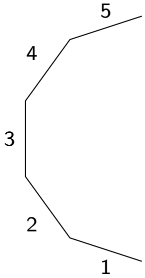

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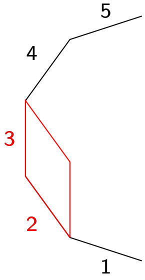

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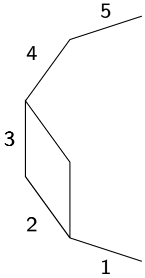

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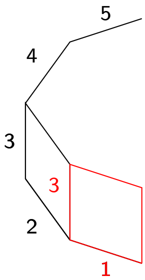

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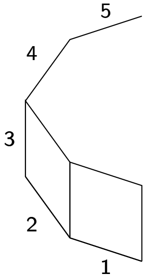

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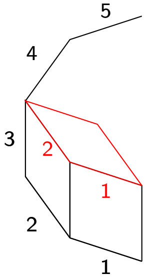

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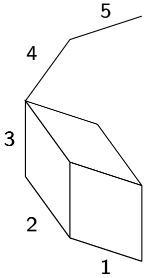

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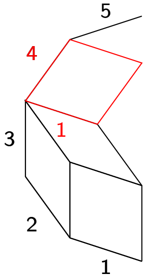

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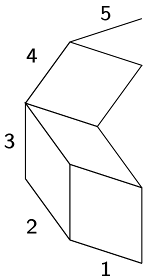

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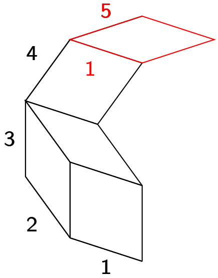

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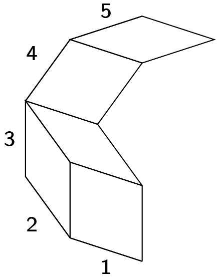

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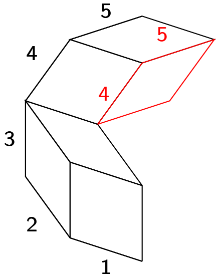

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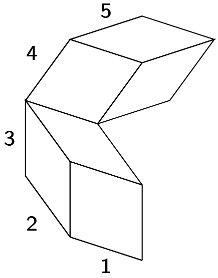

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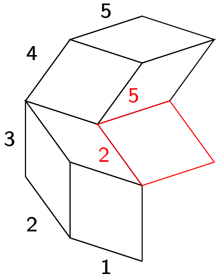

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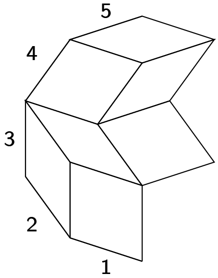

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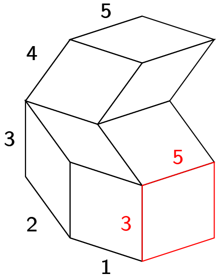

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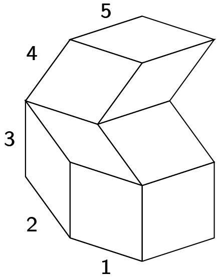

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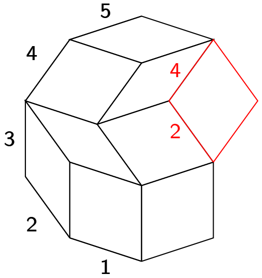

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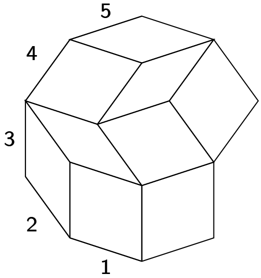

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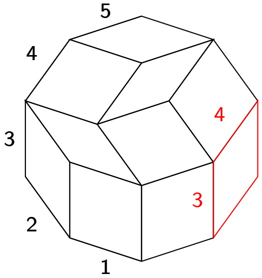

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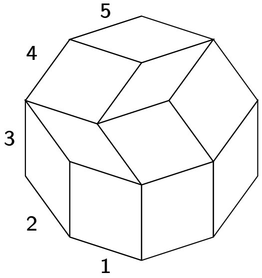

Lusztig data

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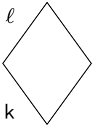
$\langle k, \ell \rangle :=$ 

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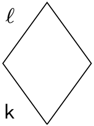

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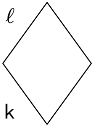

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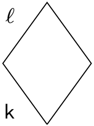

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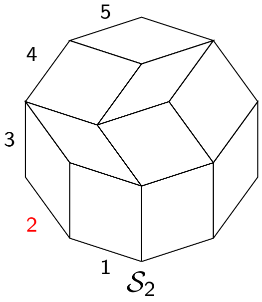

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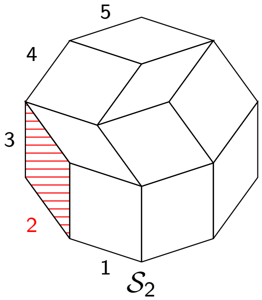

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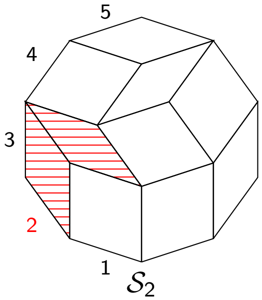

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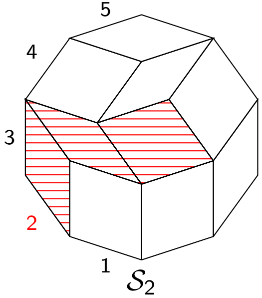

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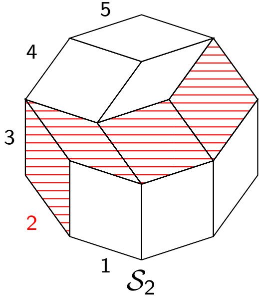

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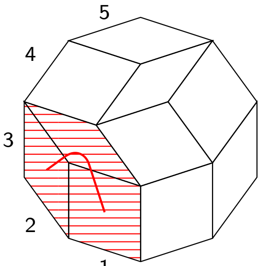

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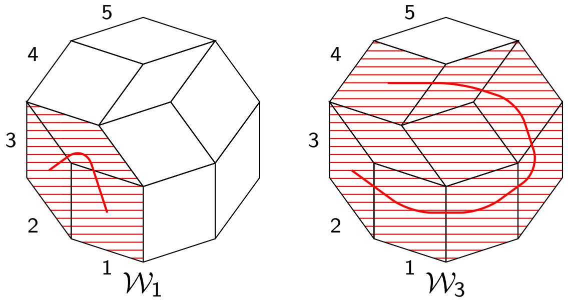

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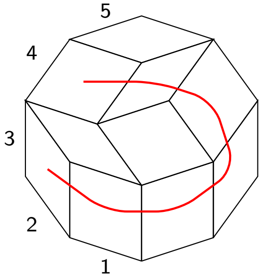

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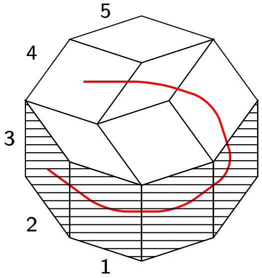

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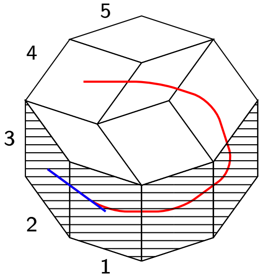

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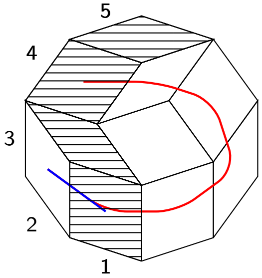

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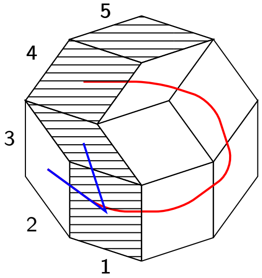

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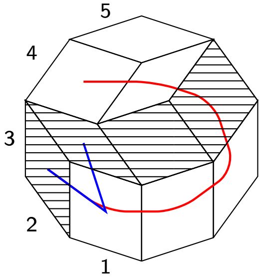

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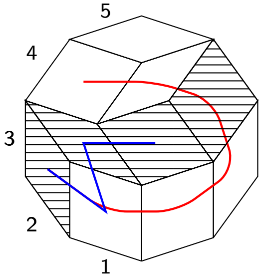

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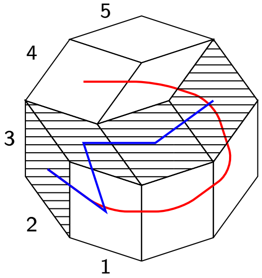

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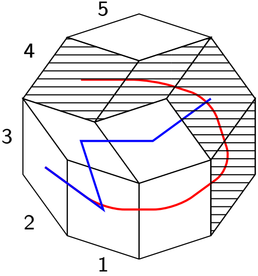

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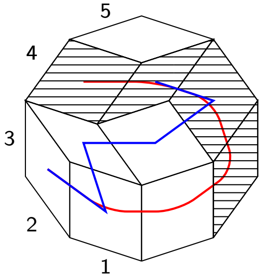

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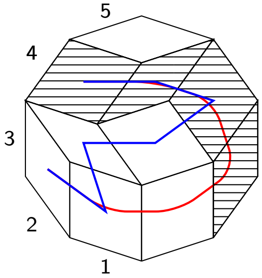

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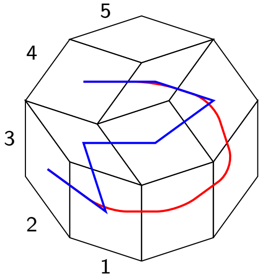

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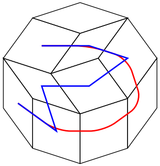

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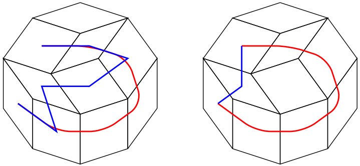

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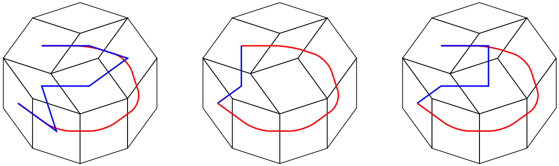

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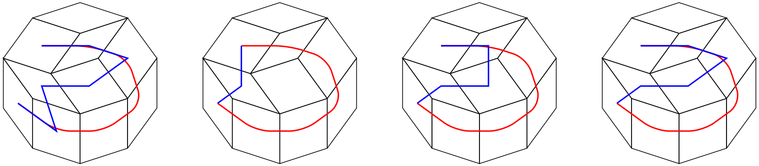

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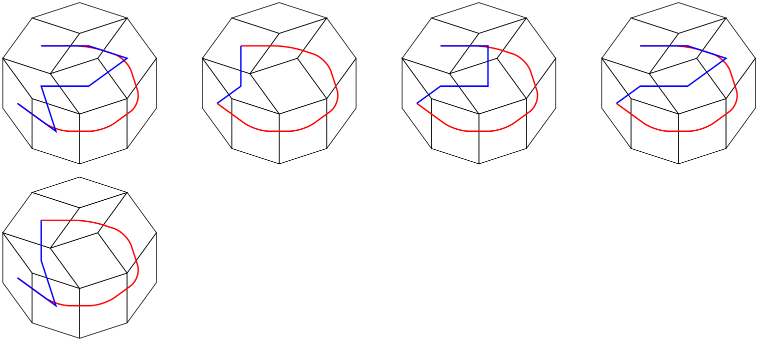

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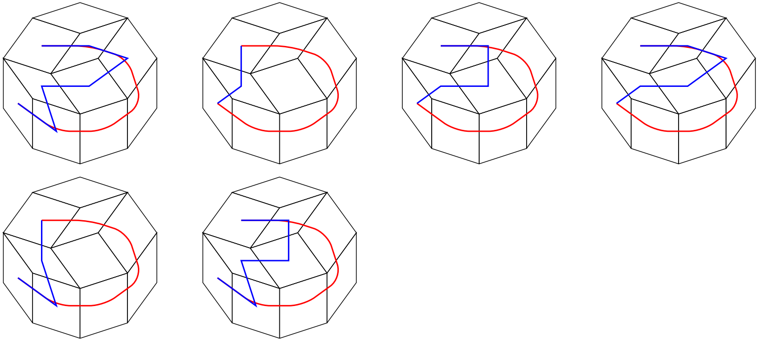

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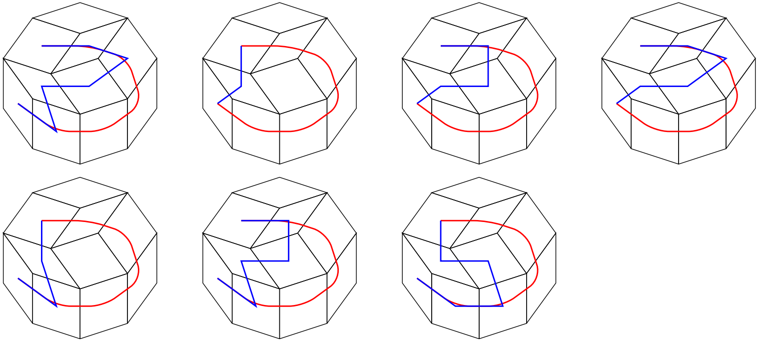

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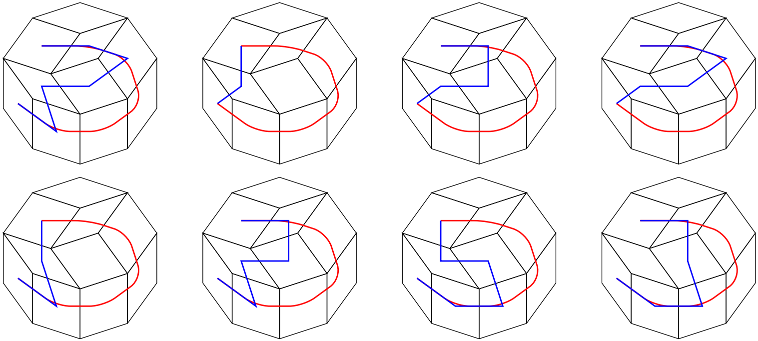

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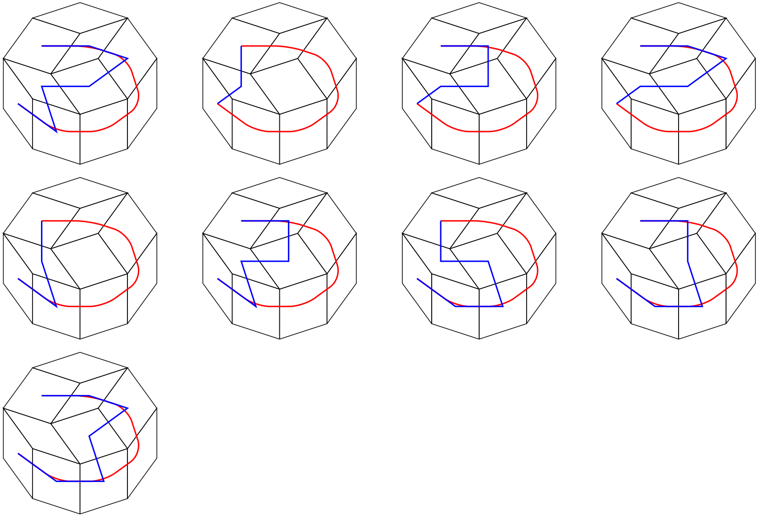

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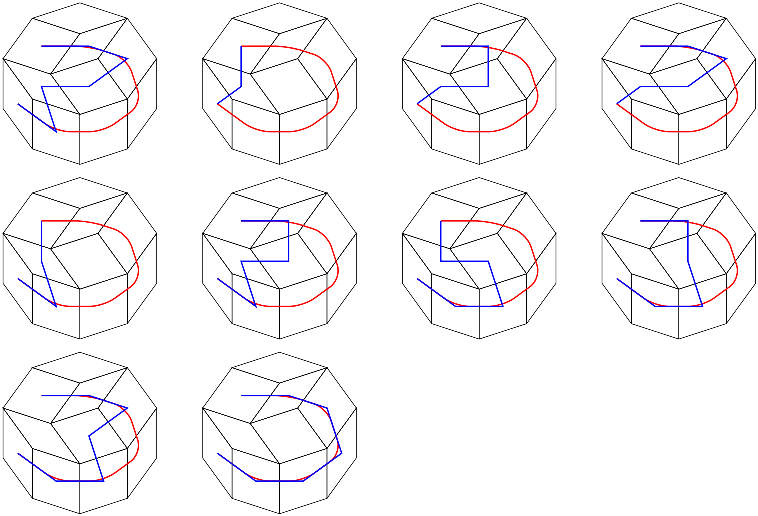

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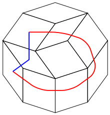

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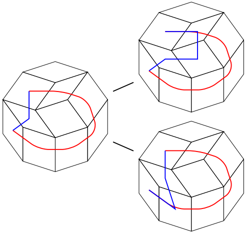

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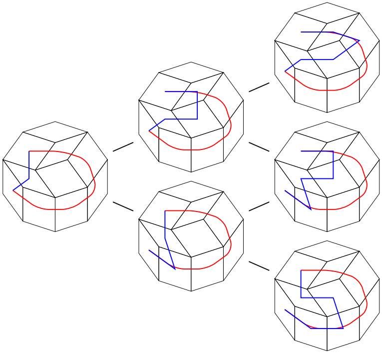

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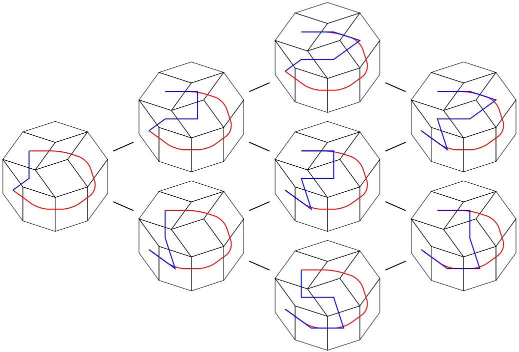

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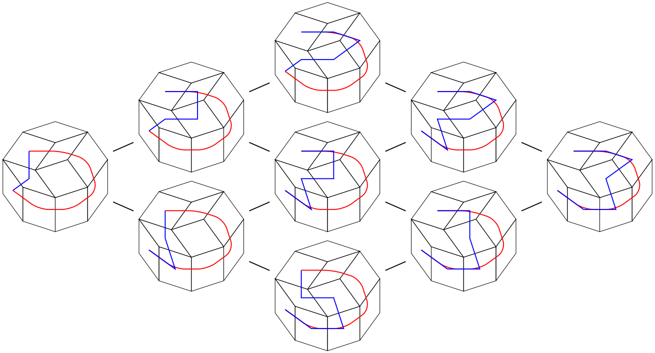

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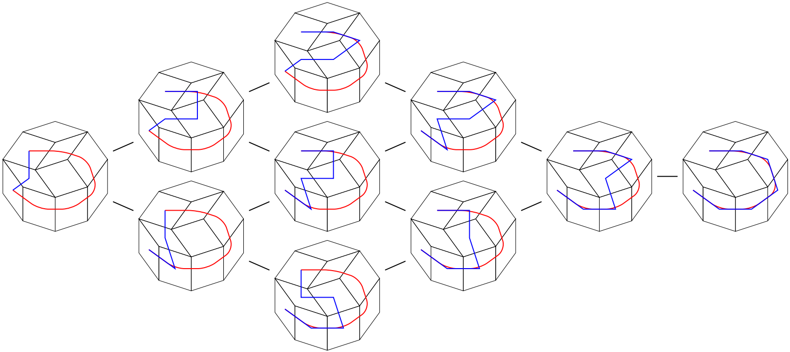

Vectors associated to crossings

To an i-crossing γ we associate the vector $[\gamma] \in \mathbb{Z}^{\Phi^+}$ given by $(1 \leq k < \ell \leq n)$

$$[\gamma]_{\alpha_{k,\ell-1}} = \begin{cases} 1 & \text{if } \gamma \text{ turns from the } k\text{-strip to the } \ell\text{-strip at } \langle k, \ell \rangle, \\ 0 & \text{if } \gamma \text{ does not turn at } \langle k, \ell \rangle, \\ -1 & \text{if } \gamma \text{ turns from the } \ell\text{-strip to the } k\text{-strip at } \langle k, \ell \rangle. \end{cases}$$

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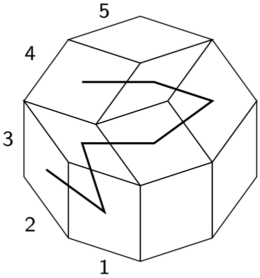

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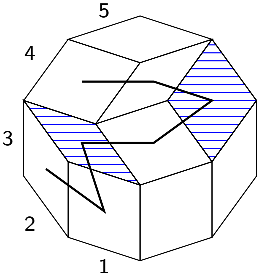

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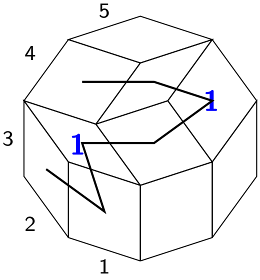

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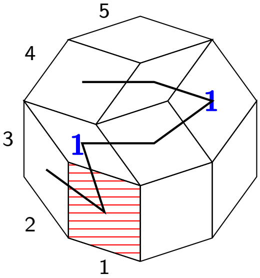

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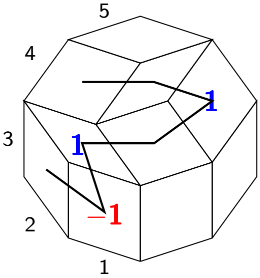

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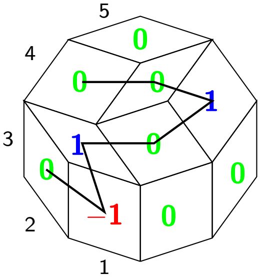

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Description of Kashiwara operators

For a $\leq$-Lusztig datum m we define for $1 \leq k < l \leq n$

$$F_i(m,\gamma) := \sum_{\substack{\langle k,\ell\rangle \in \gamma \\ i \in [k,\ell]}} m_{\alpha_{k,\ell-1}} - \sum_{\substack{\langle k,\ell\rangle \in \gamma \\ i \notin [k,\ell] \\ [\gamma]_{\alpha_{k,\ell-1}}=0}} m_{\alpha_{k,\ell-1}}.$$

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Theorem 1 (Genz, Koshevoy, S.)

Let $i \in \{1,\ldots,n-1\}$ and let m be a $\leq$-Lusztig datum. Then

$$\varepsilon_i(m) = \max_{\gamma} F_i(m,\gamma)$$

and

$$\tilde{f}_i(m) - m = [\gamma^m],$$

where γ^m is the maximal i-crossing with $F_i(m,\gamma^m) = \varepsilon_i(m)$.

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- The set of $\leq$-Lusztig data is in bijection with the set of (normalized) BZ data as illustrated in the following example:

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We identify the vertices of the tiling with a subset of Γ_n by collecting the indices of the strips running below each vertex.

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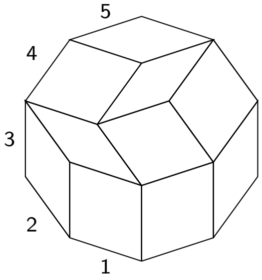

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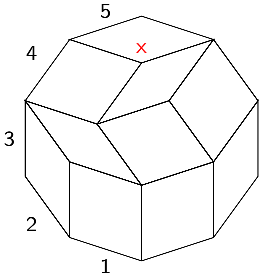

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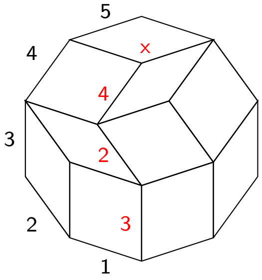

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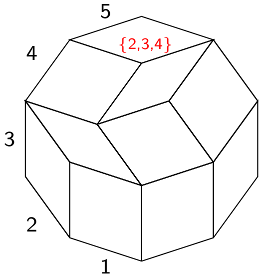

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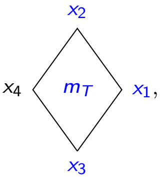

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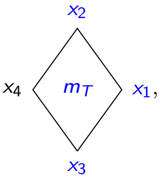

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- The values of the chamber weights not lying in the vertex set of $\mathcal{T}$ are determined by the relations of BZ-data.

Crystal structure on MV-polytopes

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Theorem 2 (Genz, Koshevoy, S.)

Let M be a BZ-datum and $M' = \tilde{f}_i M$. Then

$$M'_x = \begin{cases} M_x - 1 & \text{if } i \in x, i+1 \notin x \text{ and } M_x - M_{s_i x} \geq M_{\{1,\dots,i\}} - M_{s_i\{1,\dots,i\}} \\ M_x & \text{else.} \end{cases}$$

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Remark

Since in the first case in Theorem 2 the equality

$$M_x - M_{s_i x} = M_{\{1,\ldots,i\}} - M_{s_i\{1,\ldots,i\}}$$

follows directly from Theorem 1, we obtain the Anderson-Mirkovic conjecture (originally proved by Kamnitzer) as a corollary.

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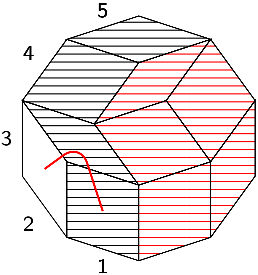

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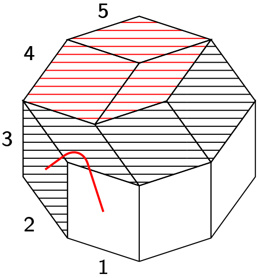

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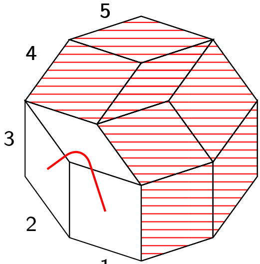

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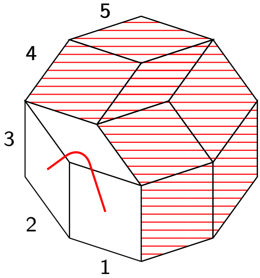

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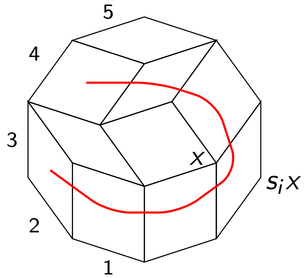

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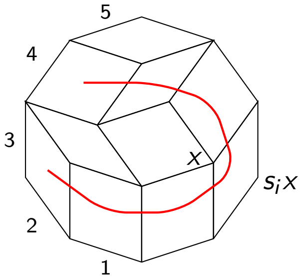

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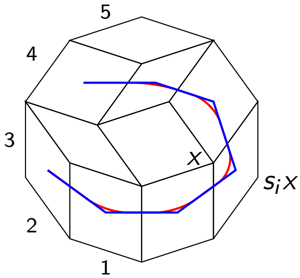

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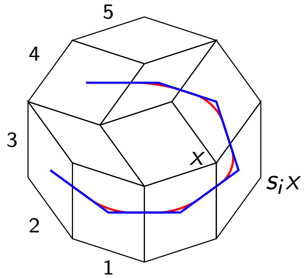

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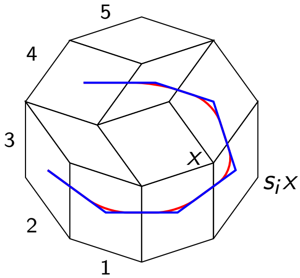

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- Translating $\varepsilon_i(m)$ and $F_i(m, \gamma)$ into the language of BZ-data yields the statement.

Thank you!