SEMI–STABLE VECTOR BUNDLES ON ELLIPTIC CURVES AND THE ASSOCIATIVE YANG–BAXTER EQUATION

IGOR BURBAN AND THILO HENRICH

Abstract. In this paper we study unitary solutions of the associative Yang–Baxter equation (AYBE) with spectral parameters. We show that to each point τ from the upper half-plane and an invertible $(n \times n)$ matrix B with complex coefficients one can attach a solution of AYBE with values in $\mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n\times n}(\mathbb{C})$, depending holomorphically on τ and B. Moreover, we compute some of these solutions explicitly.

1. Introduction

Let $\tau \in \mathbb{C}$ be such that $\mathsf{Im}(\tau) > 0$, $q = \exp(\pi i \tau)$ and

$$\theta(z) = \theta_1(z|\tau) = 2q^{\frac{1}{4}} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \sin\big((2n+1)\pi z\big)$$

be the first theta-function of Jacobi. Consider the following meromorphic function $\sigma(u,x) = \dfrac{\theta'(0)\theta(u+x)}{\theta(u)\theta(x)}$ introduced by Kronecker, see for example [15, Chapter VIII]. It is well-known that $\sigma(u,x)$ satisfies the celebrated Fay's identity

$$\sigma(u,x)\sigma(u+v,y) = \sigma(u+v,x+y)\sigma(-v,x) + \sigma(v,y)\sigma(u,x+y). \tag{1}$$

Note that the Kronecker function also satisfies the equality $\sigma(-u,-x) = -\sigma(u,x)$. As it was shown by Polishchuk [12, Theorem 5], up to a certain equivalence relation the Kronecker function and its trigonometric and rational degenerations $\cot(u) + \cot(x)$ and $\frac{1}{u} + \frac{1}{x}$ are the only solutions of the functional equation (1).

In this paper we study solutions of the matrix-valued generalization of the Fay's identity (1). Namely, we are interested in meromorphic functions $r : (\mathbb{C}^2, 0) \to A \otimes A$, satisfying the following equality:

$$r^{12}(u,x)r^{23}(u+v,y) = r^{13}(u+v,x+y)r^{12}(-v,x) + r^{23}(v,y)r^{13}(u,x+y), \tag{2}$$

where $A = \mathsf{Mat}_{n\times n}(\mathbb{C})$. The upper indices in (2) indicate various embeddings of $A \otimes A$ into $A \otimes A \otimes A$. For example, the function r^{13} is defined as the composition

$$r^{13} : \mathbb{C}^2 \xrightarrow{r} A \otimes A \xrightarrow{\rho_{13}} A \otimes A \otimes A,$$

where $\rho_{13}(x \otimes y) = x \otimes \mathbb{1} \otimes y$. The two other maps r^{12} and r^{23} have a similar meaning. The equation (2), called *associative Yang–Baxter equation*, was introduced by Polishchuk [12]. Its theory was further developed by Burban and Kreußler in [6].

The first version of the associative Yang–Baxter equation (without spectral parameters) appeared in a paper of Fomin and Kirillov [7]. Later, it arose in a work of Aguiar in the framework of the deformation theory of Hopf algebras [1]. A special version of the equation (2) was also considered by Odesskii and Sokolov [11].

In what follows, we shall be interested in *unitary* solutions of the associative Yang–Baxter equation, i.e. in solutions of (2) satisfying an additional identity $r(-u,-x) = -\rho\bigl(r(u,x)\bigr)$, where $\rho : A\otimes A \to A\otimes A$ is the automorphism given by the rule $\rho(a\otimes b) = b\otimes a$ for all $a, b \in A$. In this case, the function $r(u,x)$ automatically satisfies the "dual equation"

$$r^{23}(v,y)r^{12}(u+v,x) = r^{12}(u,x)r^{13}(v,x+y) + r^{13}(u+v,x+y)r^{23}(-u,y), \tag{3}$$

see for example [6, Lemma 2.7]. The unitary solutions of (2) having the Laurent expansion with respect to the first variable of the form

$$r(u,x) = \frac{\mathbb{1}\otimes\mathbb{1}}{u} + r_0(x) + u r_1(x) + \dots \tag{4}$$

were studied by Polishchuk [12, 13] as well as by Burban and Kreußler [6]. Such solutions are closely related with the classical and quantum Yang–Baxter equations, see [12, 13, 6] for more details.

In this paper we construct non-degenerate unitary solutions of (2) *not* satisfying the residue condition (4). Moreover, we get solutions having *higher order* poles with respect to the first spectral parameter u. It turns our that they can be frequently expressed via the Kronecker function $\sigma(u,x)$ and its derivatives with respect to the first variable. For example, we show that the elliptic function

$$r(u,x) = \sigma(u,x)\bigl(e_{11}\otimes e_{11} + e_{22}\otimes e_{22} + e_{12}\otimes e_{21} + e_{21}\otimes e_{12}\bigr) + \sigma'(u,x)\bigl(e_{12}\otimes e_{11} - e_{12}\otimes e_{22} - e_{11}\otimes e_{12} + e_{22}\otimes e_{12}\bigr) - \sigma''(u,x) e_{12}\otimes e_{12}$$

is a unitary solution of (2) for $A = \mathsf{Mat}_{2\times 2}(\mathbb{C})$, where derivatives of $\sigma(u,x)$ are taken with respect to the first variable.

The study of solutions of the associative Yang–Baxter equation is also motivated by an observation of Kirillov [9] stating that any unitary solution of (2) determines a certain family of commuting first order differential operators and hence, a very interesting quantum integrable systems, see also [6, Proposition 2.9]. In particular, one can attach to any unitary solution of (2) a second-order differential operator of Calogero-Moser type, generalizing the construction of Buchstaber, Felder and Veselov [5]. We hope that our approach to the construction of these operators via the theory of vector bundles on genus one curves will be helpful to clarify their spectral properties. On the other hand, our explicit solutions of (2) provide new identities for the higher derivatives of the Kronecker function $\sigma(u,x)$.

The main result of our paper is the following. We fix a complex parameter $\tau \in \mathbb{C}$ such that $\mathsf{Im}(\tau) > 0$ and an invertible matrix $B \in \mathsf{GL}_n(\mathbb{C})$. Let $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ be the corresponding lattice in $\mathbb{C}$ and $\mathfrak{S}(B) = \{\lambda_1, \dots, \lambda_n\}$ the spectrum of B. We denote by $\Sigma = \Sigma_B$ the lattice $\bigl\{\lambda - \lambda' \bigm| \exp(2\pi i\lambda), \exp(2\pi i\lambda') \in \mathfrak{S}(B)\bigr\} + \Lambda \subset \mathbb{C}$. Then we attach to the pair (B, τ) a meromorphic tensor-valued function

$$r_B = r_B(v, y) : \mathbb{C} \times \mathbb{C} \longrightarrow \mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n\times n}(\mathbb{C})$$

having the following properties:

(1) The function r_B is a non-degenerate unitary solution of (2).

(2) Moreover, r_B depends *analytically* on the entries of the matrix B and is holomorphic on $(\mathbb{C} \setminus \Sigma) \times (\mathbb{C} \setminus \Lambda)$.

(3) Let $S \in \mathsf{GL}_n(\mathbb{C})$ and $A = S^{-1}BS$. Then we have:

$$r_A(v, y) = (S^{-1} \otimes S^{-1}) r_B(v, y)(S \otimes S)$$

i. e. r_A and r_B are *gauge equivalent* in the sense of [6, Definition 2.5].

(4) If $B = \mathsf{diag}\bigl(\exp(2\pi i\lambda_1), \dots, \exp(2\pi i\lambda_n)\bigr)$ for some $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ then the corresponding solution r_B is given by the following formula:

$$r_B(v, y) = \sum_{k,\, l=1}^{n} \sigma(v - \lambda_{kl}, y)\, e_{l,k} \otimes e_{k,l},$$

where $\lambda_{kl} = \lambda_k - \lambda_l$ and $\sigma(u, x)$ is the Kronecker function.

(5) If $B = J_n(1)$ is the Jordan block of size $n \times n$ with eigenvalue one then

$$r_B(v, y) = \sum_{\substack{0 \le k \le n-1 \\ 0 \le l \le n-1}} \nabla_{kl}\bigl(\sigma(v, y)\bigr) \sum_{\substack{1 \le i \le n-l \\ 1 \le j \le n-k}} e_{i,j+k} \otimes e_{j,i+l},$$

where ∇_{kl} are certain differential operators described in Definition 4.6.

The core of our method is the computation of certain *triple Massey products* in the derived category $D^b\bigl(\mathsf{Coh}(E)\bigr)$, where $E = \mathbb{C}/\Lambda$ is the complex torus corresponding to the lattice Λ.

Acknowledgement. This work was supported by the DFG grant Bu–1866/1–2.

2. Brief description of the main construction

In this section we present an algorithm attaching to a pair $(B, \tau) \in \mathsf{GL}_n(\mathbb{C}) \times \mathbb{H}$, where $\mathbb{H} \subset \mathbb{C}$ is the upper half–plane, a non-degenerate unitary solution of the associative Yang–Baxter equation (2) with values in $\mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n\times n}(\mathbb{C})$. The explanation of this method as well as proofs will be given in the next section.

In what follows, we denote $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$. Let $\mathfrak{S}(B) = \bigl\{\lambda_1, \dots, \lambda_n\bigr\}$ be the spectrum of B and $\Sigma = \Sigma_B \subset \mathbb{C}$ be the lattice $\bigl\{\lambda - \lambda' \bigm| \exp(2\pi i\lambda), \exp(2\pi i\lambda') \in \mathfrak{S}(B)\bigr\} + \Lambda$. We construct the tensor–valued function

$$r_B : (\mathbb{C} \setminus \Sigma) \times (\mathbb{C} \setminus \Lambda) \longrightarrow \mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n\times n}(\mathbb{C})$$

in the following way.

- For any $v \in \mathbb{C}$ consider the function
$$e(z) = e(z, v, \tau) := -\exp\bigl(-2\pi i(z + v + \tau)\bigr).$$
- Let $\mathsf{Sol} = \mathsf{Sol}_{B,\,v,\,\tau}$ be the following complex vector space:
$$\mathsf{Sol} = \left\{ \Phi : \mathbb{C} \longrightarrow \mathsf{Mat}_{n\times n}(\mathbb{C}) \;\middle|\; \begin{array}{l} \Phi \quad \text{is holomorphic} \\ \Phi(z+1) = \Phi(z) \\ \Phi(z+\tau)B = e(z)B\Phi(z) \end{array} \right\}.$$
- For any $y \in \mathbb{C} \setminus \Lambda$ consider the *evaluation map* $\mathsf{ev}_y : \mathsf{Sol} \to \mathsf{Mat}_{n\times n}(\mathbb{C})$ given by the formula $\mathsf{ev}_y(\Phi) = \frac{1}{\bar{\theta}(y+\frac{\tau+1}{2})}\Phi(y)$, where
$$\bar{\theta}(y) = \theta_3(y|\tau) = 1 + 2\sum_{n=0}^{\infty} q^{n^2}\cos\bigl(2\pi n y\bigr)$$
is the *third* Jacobian theta–function with $q = \exp(\pi i \tau)$. Next, consider the *residue map* $\mathsf{res}_0 : \mathsf{Sol} \to \mathsf{Mat}_{n\times n}(\mathbb{C})$ given by the formula $\mathsf{res}_0(\Phi) = \Phi(0)$.

Proposition 2.1. *For any $v \in \mathbb{C}$ and $B \in \mathsf{GL}_n(\mathbb{C})$ the vector space $\mathsf{Sol}_{B,\,v,\,\tau}$ has dimension n^2. Moreover, if $v \notin \Sigma$ then the linear map $\mathsf{res}_0 : \mathsf{Sol}_{B,\,v,\,\tau} \to \mathsf{Mat}_{n\times n}(\mathbb{C})$ is an isomorphism.*

For a proof of this Proposition, see Corollary 3.13, Theorem 3.14 and Remark 3.15.

Next, we continue the construction of the tensor valued function r_B.

- For any pair $(v, y) \in (\mathbb{C}\setminus\Sigma) \times (\mathbb{C}\setminus\Lambda)$ consider the linear map $\tilde{r}_B(v, y)$ given by the following commutative diagram:

$$\begin{tikzcd} \mathsf{Mat}_{n\times n}(\mathbb{C}) \arrow[rr, "\tilde{r}_B(v,y)"] & & \mathsf{Mat}_{n\times n}(\mathbb{C}) \\ & \mathsf{Sol}_{B,\,v,\,\tau} \arrow[ul, "\mathsf{res}_0"] \arrow[ur, "\mathsf{ev}_y"'] & \end{tikzcd} \tag{5}$$

In other words, $\tilde{r}_B(v, y) := \mathsf{ev}_y \circ \mathsf{res}_0^{-1}$.
- Let $r_B(v, y) \in \mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n\times n}(\mathbb{C})$ be the tensor corresponding to the linear map $\tilde{r}_B(v, y)$ via the canonical map of vector spaces
$$\mathsf{can} : \mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n\times n}(\mathbb{C}) \longrightarrow \mathsf{Hom}_{\mathbb{C}}\bigl(\mathsf{Mat}_{n\times n}(\mathbb{C}), \mathsf{Mat}_{n\times n}(\mathbb{C})\bigr)$$
sending a simple tensor $X \otimes Y$ to the linear map $Z \mapsto \mathrm{Tr}(XZ)Y$.

The following theorem is the main result of our article.

Theorem 2.2. *Let $(B, \tau) \in \mathsf{GL}_n(\mathbb{C}) \times \mathbb{H}$.*

(1) *The function $(\mathbb{C}\setminus\Sigma)\times(\mathbb{C}\setminus\Lambda) \to \mathsf{Mat}_{n\times n}(\mathbb{C})\otimes\mathsf{Mat}_{n\times n}(\mathbb{C})$, assigning to a pair (v, y) the tensor $r_B(v, y)$ constructed above, is a non–degenerate holomorphic unitary solution of the associative Yang–Baxter equation (2). Moreover, this function is meromorphic on $\mathbb{C} \times \mathbb{C}$.*

(2) *Let $S \in \mathsf{GL}_n(\mathbb{C})$ and $A = S^{-1}BS$. Then for any $(v, y) \in (\mathbb{C} \setminus \Sigma) \times (\mathbb{C} \setminus \Lambda)$ we have the following equality:*

$$r_A(v, y) = (S^{-1} \otimes S^{-1}) r_B(v, y)(S \otimes S).$$

In particular, the solutions r_A and r_B are gauge–equivalent in the sense of [6, Definition 2.5].

The key idea of the proof of this theorem is to interpret the linear morphism $\tilde{r}_B(v, y)$ from the diagram (5) as a certain triple Massey product in the derived category $D^b\big(\mathsf{Coh}(E)\big)$, where $E = \mathbb{C}/\Lambda$. The fact that $r_B(v, y)$ satisfies the equation (2) is a translation of the A_∞–constraint $m_3 \circ (m_3 \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes m_3 \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes m_3) = 0$. For further details, see Theorem 3.16, Proposition 3.18 and Proposition 3.20.

3. Proof of the main theorem

In this section we explain the algorithm of the construction of solutions of the associative Yang–Baxter equation (2) stated in Section 2 and prove Theorem 2.2.

3.1. **Vector bundles on a one-dimensional complex torus.** Let $\tau \in \mathbb{H}$, $\Lambda = \mathbb{Z} + \tau\mathbb{Z} \subset \mathbb{C}$ and $E = \mathbb{C}/\Lambda$ be the corresponding complex torus. In this subsection we recall the basic techniques for dealing with holomorphic vector bundles on E.

Definition 3.1. Let $A : \mathbb{C} \to \mathsf{GL}_n(\mathbb{C})$ be a holomorphic function satisfying the condition $A(z + 1) = A(z)$ for all $z \in \mathbb{C}$. Such a function A, called *automorphy factor*, defines the following topological space $\mathcal{E}(A) := \mathbb{C} \times \mathbb{C}^n / \sim$, where $(z, v) \sim (z + 1, v) \sim (z + \tau, A(z)v)$. Note that we have a Cartesian diagram of complex manifolds

$$\begin{tikzcd} \mathbb{C} \times \mathbb{C}^n \arrow[r] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{E}(A) \arrow[d] \\ \mathbb{C} \arrow[r, "\pi"] & E \end{tikzcd}$$

and $\mathcal{E}(A)$ is a vector bundle of rank n on the torus E.

Remark 3.2. Let $\pi : \mathbb{C} \to \mathbb{C}/\Lambda = E$ be the quotient map. Another way to define the locally free sheaf $\mathcal{E}(A)$ is the following.

The open subsets $U \subset E$ for which all connected components of $\pi^{-1}(U)$ map isomorphically to U, form a basis of the topology of E. For such U, we let U_0 be a connected component of $\pi^{-1}(U)$ and denote $U_\gamma = \gamma + U_0$ for all $\gamma \in \Lambda$. Then $\pi_* \mathcal{O}_{\mathbb{C}}^n\big(U\big) = \prod_{\gamma \in \Lambda} \mathcal{O}_{\mathbb{C}}^n(U_\gamma)$ and we define

$$\mathcal{E}(A)\big(U\big) := \left\{ (F_\gamma(z))_{\gamma \in \Lambda} \in \pi_*(\mathcal{O}_{\mathbb{C}}^n)\big(U\big) \;\middle|\; \begin{array}{l} F_{\gamma+1}(z + 1) = F_\gamma(z) \\ F_{\gamma+\tau}(z + \tau) = A(z) F_\gamma(z) \end{array} \right\}.$$

In this way we get an embedding $\mathsf{m}_A : \mathcal{E}(A) \subset \pi_* \mathcal{O}_{\mathbb{C}}^n$ as well as a trivialization γ_A of $\pi^*\big(\mathcal{E}(A)\big)$ given by the composition $\pi^* \mathcal{E}(A) \xrightarrow{\pi^*(\mathsf{m}_A)} \pi^* \pi_* \mathcal{O}_{\mathbb{C}}^n \xrightarrow{\mathrm{can}} \mathcal{O}_{\mathbb{C}}^n$.

The following classical result is due to A. Weil.

Theorem 3.3. *Let $E = \mathbb{C}/\Lambda$ be a one-dimensional complex torus.*

(1) *For any holomorphic rank n vector bundle $\mathcal{E}$ on the torus E there exists an automorphy factor $A : \mathbb{C} \to \mathsf{GL}_n(\mathbb{C})$ such that $\mathcal{E} \cong \mathcal{E}(A)$.*
(2) *For any automorphy factors $A : \mathbb{C} \to \mathsf{GL}_n(\mathbb{C})$ and $B : \mathbb{C} \to \mathsf{GL}_m(\mathbb{C})$ we have:*

$$\mathsf{Hom}\bigl(\mathcal{E}(A), \mathcal{E}(B)\bigr) \cong \mathsf{Sol}_{A,B} := \left\{ \Phi : \mathbb{C} \to \mathsf{Mat}_{m\times n}(\mathbb{C}) \;\middle|\; \begin{array}{l} \Phi \quad \text{is holomorphic} \\ \Phi(z+1) = \Phi(z) \\ \Phi(z+\tau)A(z) = B(z)\Phi(z) \end{array} \right\}$$

and $\mathcal{E}(A) \otimes \mathcal{E}(B) \cong \mathcal{E}(A \otimes B)$.

Proof. This result is a corollary of the monoidal equivalence of the category of Λ–equivariant holomorpic vector bundles on $\mathbb{C}$ and holomorphic vector bundles on the quotient torus $E = \mathbb{C}/\Lambda$. See [3] or [8] for a detailed proof. $\square$

Corollary 3.4. *For any pair of automorphy factors $A, S : \mathbb{C} \to \mathsf{GL}_n(\mathbb{C})$ we have an isomorphism of vector bundles $\mathcal{E}(A) \cong \mathcal{E}(B)$, where $B(z) = S(z+\tau)^{-1}A(z)S(z)$. In particular, we have an isomorphism $\mathcal{E}(A) \cong \mathcal{E}(\widehat{A})$, where $\widehat{A}(z) = \exp(2\pi i\tau)A(z)$.*

In the next step, we need an explicit description of the indecomposable semi-stable vector bundles on E of degree zero.

Theorem 3.5. *Let $E = \mathbb{C}/\Lambda$ be a complex torus.*

(1) *The map $\mathbb{C} \to \mathsf{Pic}(E)$ assigning to $\lambda \in \mathbb{C}$ the line bundle $\mathcal{L}_\lambda := \mathcal{E}\bigl(\exp(2\pi i\lambda)\bigr)$ yields a bijection between the points of E and the isomorphy classes of degree zero line bundles on E.*
(2) *For any $m \geq 1$ let*

$$J_m = J_m(1) = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \in \mathsf{GL}_m(\mathbb{C}).$$

Then $\mathcal{E}(J_m)$ is isomorphic to the Atiyah bundle $\mathcal{A}_m$ defined as follows. For $m = 1$ we set $\mathcal{A}_1 = \mathcal{O}$ and for $m \geq 2$ the vector bundle $\mathcal{A}_m$ is recursively defined by the following property: it is the unique (up to an isomorphism) vector bundle occurring as the middle term of a non-split short exact sequence

$$0 \longrightarrow \mathcal{A}_{m-1} \longrightarrow \mathcal{A}_m \longrightarrow \mathcal{O} \longrightarrow 0.$$

(3) *Let $B \in \mathsf{GL}_n(\mathbb{C})$ and $J = J_{m_1}(\mu_1) \oplus \dots \oplus J_{m_t}(\mu_t)$ be the Jordan normal form of B with $\mu_l = \exp(2\pi i\lambda_l)$ for some $\lambda_l \in \mathbb{C}$, $1 \leq l \leq t$. Then we have:*

$$\mathcal{E}(B) \cong (\mathcal{L}_{\lambda_1} \otimes \mathcal{A}_{m_1}) \oplus \dots \oplus (\mathcal{L}_{\lambda_t} \otimes \mathcal{A}_{m_t}).$$

In particular, $\mathcal{E}(B)$ is a semi–stable vector bundle of degree zero on the torus E, whose Jordan–Hölder quotients are $\mathcal{L}_{\lambda_1}, \dots, \mathcal{L}_{\lambda_t}$. Moreover, for any semi-stable vector bundle $\mathcal{E}$ of rank n and degree zero on the torus E there exists a matrix $B \in \mathsf{GL}_n(\mathbb{C})$ such that $\mathcal{E} \cong \mathcal{E}(B)$.

Proof. A proof of the first two statements can for instance be found in [6, Section 8.1] or in [8]. To show the third one observe that by Corollary 3.4 we have an isomorphism $\mathcal{E}(B) \cong \mathcal{E}(J)$. Since for any $\lambda \in \mathbb{C}$ and $m \in \mathbb{N}$ we have an isomorphism $\mathcal{E}\big(J_m(\lambda)\big) \cong \mathcal{L}_\lambda \otimes \mathcal{A}_m$, we have: $\mathcal{E}(J) \cong (\mathcal{L}_{\lambda_1} \otimes \mathcal{A}_{m_1}) \oplus \dots \oplus (\mathcal{L}_{\lambda_t} \otimes \mathcal{A}_{m_t})$. Hence, the result follows from Atiyah's classification of vector bundles on E [2]. □

Corollary 3.6. *Let $A \in \mathsf{GL}_n(\mathbb{C})$ and $\mathfrak{S}(A) = \big\{\exp(2\pi i \lambda_1), \dots, \exp(2\pi i \lambda_n)\big\}$ be its spectrum, $B \in \mathsf{GL}_m(\mathbb{C})$ and $\mathfrak{S}(B) = \big\{\exp(2\pi i \mu_1), \dots, \exp(2\pi i \mu_m)\big\}$ be its spectrum. Assume that $\lambda_k - \mu_l \notin \Lambda$ for all $1 \le k \le n$ and $1 \le l \le m$. Then we have:*

$$\mathsf{Hom}\big(\mathcal{E}(A), \mathcal{E}(B)\big) = 0 = \mathsf{Ext}^1\big(\mathcal{E}(A), \mathcal{E}(B)\big).$$

Proof. The assumption on the eigenvalues of A and B implies that the degree zero semi-stable vector bundles $\mathcal{E}(A)$ and $\mathcal{E}(B)$ have no common Jordan–Hölder quotients. From this fact it follows that

$$\mathsf{Hom}\big(\mathcal{E}(A), \mathcal{E}(B)\big) = 0 = \mathsf{Hom}\big(\mathcal{E}(B), \mathcal{E}(A)\big) \cong \mathsf{Ext}^1\big(\mathcal{E}(A), \mathcal{E}(B)\big)^*,$$

where the last isomorphism is given by the Serre duality. □

Lemma 3.7. *Let $\varphi(z) = \exp(-\pi i \tau - 2\pi i z)$, $x \in \mathbb{C}$ and $[x]$ be the corresponding divisor of degree one on E. Then we have an isomorphism:*

$$\mathcal{O}_E\big([x]\big) \cong \mathcal{E}\big(\varphi(z + \frac{\tau+1}{2} - x)\big).$$

Proof. A proof of this result can be for instance found in [6, Section 8.1]. □

3.2. **Residue and evaluation morphisms.** Let Ω_E denote the sheaf of regular differential one forms on the torus E. Then we have an isomorphism $\mathcal{O}_E \cong \Omega_E$ given by a nowhere vanishing differential form, e.g. by $\omega = dz$. For any $x \in E$ consider the canonical short exact sequence

$$0 \to \Omega_E \to \Omega_E(x) \xrightarrow{\mathsf{res}_x} \mathbb{C}_x \to 0. \tag{6}$$

Let $\mathcal{F}$ and $\mathcal{G}$ be a pair of vector bundles on E. We identify the line bundles Ω_E and $\mathcal{O}_E$ using the differential form ω, tensor the sequence (6) with $\mathcal{G}$ and then apply the functor $\mathsf{Hom}(\mathcal{F}, -)$. As a result, we obtain a long exact sequence

$$0 \longrightarrow \mathsf{Hom}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathsf{Hom}\big(\mathcal{F}, \mathcal{G}(x)\big) \longrightarrow \mathsf{Hom}\big(\mathcal{F}, \mathcal{G} \otimes \mathbb{C}_x\big) \longrightarrow \mathsf{Ext}^1(\mathcal{F}, \mathcal{G}). \tag{7}$$

Definition 3.8. The linear map $\mathsf{res}_x^{\mathcal{F},\mathcal{G}}(\omega) : \mathsf{Hom}\big(\mathcal{F}, \mathcal{G}(x)\big) \to \mathsf{Lin}\big(\mathcal{F}\big|_x, \mathcal{G}\big|_x\big)$ is the composition of the following canonical morphisms

$$\mathsf{Hom}\big(\mathcal{F}, \mathcal{G}(x)\big) \longrightarrow \mathsf{Hom}\big(\mathcal{F}, \mathcal{G} \otimes \mathbb{C}_x\big) \longrightarrow \mathsf{Lin}\big(\mathcal{F}\big|_x, \mathcal{G}\big|_x\big),$$

where the first map comes from the long exact sequence (7) and the second one is a canonical isomorphism. The morphism $\mathsf{res}_x^{\mathcal{F},\mathcal{G}}(\omega)$ is called *residue map*.

The following lemma is a straightforward corollary of the definition and the long exact sequence (7).

Lemma 3.9. *Let $\mathcal{F}$ and $\mathcal{G}$ be a pair of vector bundles on E such that $\mathsf{Hom}(\mathcal{F},\mathcal{G}) = 0 = \mathsf{Ext}^1(\mathcal{F},\mathcal{G})$. Then for any $x \in E$ the residue map $\mathsf{res}_x^{\mathcal{F},\mathcal{G}}(\omega)$ is an isomorphism.*

Definition 3.10. Let $\mathcal{F}$ and $\mathcal{G}$ be a pair of vector bundles on E and $x, y \in E$ be a pair of *distinct* points. Then the linear map $\mathsf{ev}_y^{\mathcal{F},\mathcal{G}(x)}$ defined as the composition of the following canonical morphisms

$$\mathsf{Hom}\big(\mathcal{F},\mathcal{G}(x)\big) \longrightarrow \mathsf{Hom}\big(\mathcal{F}\otimes\mathbb{C}_y, \mathcal{G}(x)\otimes\mathbb{C}_y\big) \xrightarrow{\cong} \mathsf{Lin}\big(\mathcal{F}\big|_y,\ \mathcal{G}\big|_y\big)$$

is called *evaluation map*.

Lemma 3.11. *Let $\mathcal{F}$ and $\mathcal{G}$ be a pair of vector bundles on E and $x, y \in E$ be a pair of distinct points such that $\mathsf{Hom}\big(\mathcal{F}(y),\mathcal{G}(x)\big) = 0 = \mathsf{Ext}^1\big(\mathcal{F}(y),\mathcal{G}(x)\big)$. Then the evaluation map $\mathsf{ev}_y^{\mathcal{F},\mathcal{G}(x)}$ is an isomorphism.*

Proof. Consider the short exact sequence

$$0 \longrightarrow \mathcal{O}(-y) \longrightarrow \mathcal{O} \xrightarrow{\mathsf{ev}_y} \mathbb{C}_y \longrightarrow 0. \tag{8}$$

It induces a short exact sequence of coherent sheaves

$$0 \longrightarrow \mathcal{G}(x-y) \longrightarrow \mathcal{G}(x) \xrightarrow{\mathbb{1}\otimes\mathsf{ev}_y} \mathcal{G}(x)\otimes\mathbb{C}_y \to 0.$$

Using the vanishing $\mathsf{Hom}\big(\mathcal{F},\mathcal{G}(x-y)\big) = 0 = \mathsf{Ext}\big(\mathcal{F},\mathcal{G}(x-y)\big)$, we get an isomorphism $\mathsf{Hom}\big(\mathcal{F},\mathcal{G}(x)\big) \to \mathsf{Hom}\big(\mathcal{F},\mathcal{G}(x)\otimes\mathbb{C}_y\big)$. It remains to observe that $\mathsf{ev}_y^{\mathcal{F},\mathcal{G}(x)}$ is the composition of the following canonical isomorphisms:

$$\mathsf{Hom}\big(\mathcal{F},\mathcal{G}(x)\big) \longrightarrow \mathsf{Hom}\big(\mathcal{F},\mathcal{G}(x)\otimes\mathbb{C}_y\big) \longrightarrow \mathsf{Hom}\big(\mathcal{F},\mathcal{G}\otimes\mathbb{C}_y\big) \longrightarrow \mathsf{Lin}\big(\mathcal{F}\big|_y,\ \mathcal{G}\big|_y\big).$$

$\square$

Lemma 3.12. *For $B \in \mathsf{GL}_n(\mathbb{C})$ and $v \in \mathbb{C}$ we set $\mathcal{F}_v = \mathcal{E}\big(\exp(2\pi i v)B\big) \cong \mathcal{E}(B)\otimes\mathcal{L}_v$. Then for any $v_1, v_2 \in \mathbb{C}$ and $y \in E$ we have:*

$$\dim_{\mathbb{C}}\Big(\mathsf{Hom}\big(\mathcal{F}_{v_1},\mathcal{F}_{v_2}(y)\big)\Big) = n^2.$$

Proof. The vector bundle $\mathcal{F}_{v_2}(y)$ is semi–stable of slope one. Hence, we have:

$$\mathsf{Ext}^1\big(\mathcal{F}_{v_1},\mathcal{F}_{v_2}(y)\big) \cong \mathsf{Hom}\big(\mathcal{F}_{v_2}(y),\mathcal{F}_{v_1}\big)^* = 0.$$

Thus, the statement of Lemma is a consequence of the Riemann–Roch formula. $\square$

Corollary 3.13. *For any $v, y \in \mathbb{C}$ the dimension of the complex vector space*

$$\mathsf{Sol} = \mathsf{Sol}_{B,\,v,\,y,\,\tau} := \left\{ \Phi : \mathbb{C} \longrightarrow \mathsf{Mat}_{n\times n}(\mathbb{C}) \;\middle|\; \begin{array}{l} \Phi \quad \text{is holomorphic} \\ \Phi(z+1) = \Phi(z) \\ \Phi(z+\tau)B = e(z)B\Phi(z) \end{array} \right\} \tag{9}$$

is n^2, where $e(z) = e(z, v, y, \tau) = -\exp\bigl(-2\pi i(z + v - y + \tau)\bigr)$.

Proof. By Theorem 3.3 and Lemma 3.7, we have an isomorphism of vector spaces $\mathsf{Sol} \cong \mathsf{Hom}\bigl(\mathcal{F}_{v_1}, \mathcal{F}_{v_2}(y)\bigr)$, where $v = v_1 - v_2$. Hence, by Lemma 3.12, the dimension of Sol is n^2. Taking $y = 0 \in E$, we also recover the first part of Proposition 2.1. □

Theorem 3.14. *Let $B \in \mathsf{GL}_n(\mathbb{C})$ and $\omega = \bar{\theta}'(\frac{1+\tau}{2})dz \in H^0(\Omega_E)$. Let $U \subset \mathbb{C}$ be a small neighborhood of 0. Using the projection map $\pi : \mathbb{C} \to E$, we identify U with a small neighborhood of $\pi(0) \in E$. Then for all $v_1, v_2; y_1, y_2 \in U$ such that $y_1 \neq y_2$ the following diagram of vector spaces is commutative:*

$$\begin{tikzcd}
\mathsf{Lin}\bigl(\mathcal{F}_{v_1}\big|_{y_1}, \mathcal{F}_{v_2}\big|_{y_1}\bigr) \arrow[d] & \mathsf{Hom}\bigl(\mathcal{F}_{v_1}, \mathcal{F}_{v_2}(y_1)\bigr) \arrow[l, "\mathsf{res}_{y_1}^{\mathcal{F}_{v_1}, \mathcal{F}_{v_2}}(\omega)"'] \arrow[r, "\mathsf{ev}_{y_2}^{\mathcal{F}_{v_1}, \mathcal{F}_{v_2}(y_1)}"] \arrow[d] & \mathsf{Lin}\bigl(\mathcal{F}_{v_1}\big|_{y_2}, \mathcal{F}_{v_2}\big|_{y_2}\bigr) \arrow[d] \\
\mathsf{Mat}_{n\times n}(\mathbb{C}) & \mathsf{Sol}_{B,\,v,\,y_1,\,\tau} \arrow[l, "\mathsf{res}_{y_1}"'] \arrow[r, "\mathsf{ev}_{y_2}"] & \mathsf{Mat}_{n\times n}(\mathbb{C}),
\end{tikzcd}$$

where $v = v_1 - v_2$, the middle vertical arrow is the isomorphism from Theorem 3.3, whereas the first and the last vertical arrows are isomorphisms induced by trivializations γ from Remark 3.2. The maps res_{y_1} and ev_{y_2} are given by the formulae:

$$\mathsf{res}_{y_1}\bigl(\Phi(z)\bigr) = \Phi(y_1) \quad \text{and} \quad \mathsf{ev}_{y_2}\bigl(\Phi(z)\bigr) = \frac{1}{\bar{\theta}(y_2 - y_1 + \frac{\tau+1}{2})}\Phi(y_2),$$

where $\bar{\theta}(y)$ is the third Jacobian theta–function.

Proof. The proof of this theorem is literally the same as the one given in [6, Section 8.2], see in particular [6, Corollary 8.10]. □

Remark 3.15. Let $v_1, v_2 \in \mathbb{C}$ be such that $v_1 - v_2$ does not belong to the lattice Σ. By Corollary 3.6 we get the vanishing $\mathsf{Hom}(\mathcal{F}_{v_1}, \mathcal{F}_{v_2}) = 0 = \mathsf{Ext}(\mathcal{F}_{v_1}, \mathcal{F}_{v_2})$. Next, Lemma 3.9 implies that the morphism $\mathsf{res}_{y_1}^{\mathcal{F}_{v_1}, \mathcal{F}_{v_2}}(\omega)$ is an isomorphism. The commutativity of the left square of the diagram from Theorem 3.14 implies that the linear map res_{y_1} is an isomorphism, too. Setting $y_1 = 0$, we obtain a proof of the second part of Proposition 2.1.

3.3. **Triple Massey products and the associative Yang–Baxter equation.** In this subsection, we give a proof of Theorem 2.2.

Theorem 3.16. *Let $B \in \mathsf{GL}_n(\mathbb{C})$, $v_1, v_2 \in \mathbb{C}$ such that $v = v_1 - v_2 \notin \Sigma$ and $y_1, y_2 \in \mathbb{C}$ such that $y_2 - y_1 \notin \Lambda$. Consider the linear map $\tilde{r}_B(v_1, v_2; y_1, y_2) : \mathsf{Mat}_{n\times n}(\mathbb{C}) \to$*

$\mathsf{Mat}_{n\times n}(\mathbb{C})$ *defined via the commutative diagram*

$$\begin{tikzcd}
\mathsf{Mat}_{n\times n}(\mathbb{C}) \arrow[rr, "\tilde{r}_B(v_1,v_2;\,y_1,y_2)"] & & \mathsf{Mat}_{n\times n}(\mathbb{C}) \\
& \mathsf{Sol}_{B,\,v,\,y_1,\,\tau} \arrow[ul, "\mathsf{res}_{y_1}"] \arrow[ur, "\mathsf{ev}_{y_2}"'] &
\end{tikzcd} \tag{10}$$

where res_{y_1} *and* ev_{y_2} *are as in Theorem 3.14. Let* $r_B(v_1, v_2; y_1, y_2) \in \mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n\times n}(\mathbb{C})$ *be the tensor corresponding to the linear map* $\tilde{r}_B(v_1, v_2; y_1, y_2)$ *via the canonical isomorphism of vector spaces*

$$\mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n\times n}(\mathbb{C}) \longrightarrow \mathsf{Lin}\big(\mathsf{Mat}_{n\times n}(\mathbb{C}),\ \mathsf{Mat}_{n\times n}(\mathbb{C})\big),$$

which sends a simple tensor $X \otimes Y$ *to the linear map* $Z \mapsto \mathrm{Tr}(XZ)Y$*. Then the obtained function of four variables*

$$r : \mathbb{C}^4_{(v_1,v_2;\,y_1,y_2)} \longrightarrow \mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n\times n}(\mathbb{C})$$

satisfies the following version of the associative Yang–Baxter equation:

$$r_B(v_1, v_2; y_1, y_2)^{12} r_B(v_1, v_3; y_2, y_3)^{23} = r_B(v_1, v_3; y_1, y_3)^{13} r_B(v_3, v_2; y_1, y_2)^{12} + \tag{11}$$

$$+ r_B(v_2, v_3; y_2, y_3)^{23} r_B(v_1, v_2; y_1, y_3)^{13}.$$

Moreover, the tensor-valued function r *is unitary, i.e. it satisfies the condition*

$$r_B(v_1, v_2; y_1, y_2)^{12} = -r_B(v_2, v_1; y_2, y_1)^{21}. \tag{12}$$

Proof. As above, for $v \in \mathbb{C}$ we set $\mathcal{F}_v := \mathcal{E}\big(\exp(2\pi i v)B\big) \cong \mathcal{E}(B) \otimes \mathcal{L}_v$. We split the proof into the following logical steps.

• By Corollary 3.6, for any $v_1, v_2 \in \mathbb{C}$ such that $v = v_1 - v_2 \notin \Sigma$ we have

$$\mathsf{Hom}(\mathcal{F}_{v_1}, \mathcal{F}_{v_2}) = 0 = \mathsf{Ext}^1(\mathcal{F}_{v_1}, \mathcal{F}_{v_2}).$$

For simplicity of notation we write $\mathcal{F}_i$ for $\mathcal{F}_{v_i}$, $i = 1, 2$. The following linear map

$$m_3 = m_{y_1,y_2}^{\mathcal{F}_1,\mathcal{F}_2} : \mathsf{Hom}(\mathcal{F}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Ext}^1(\mathbb{C}_{y_1}, \mathcal{F}_2) \otimes \mathsf{Hom}(\mathcal{F}_2, \mathbb{C}_{y_2}) \longrightarrow \mathsf{Hom}(\mathcal{F}_1, \mathbb{C}_{y_2}),$$

called *triple Massey product*, is defined as follows.

Let $a \in \mathsf{Ext}^1(\mathbb{C}_{y_1}, \mathcal{F}_2)$, $g \in \mathsf{Hom}(\mathcal{F}_1, \mathbb{C}_{y_1})$, $f \in \mathsf{Hom}(\mathcal{F}_2, \mathbb{C}_{y_2})$ and $0 \to \mathcal{F}_2 \xrightarrow{\alpha} \mathcal{A} \xrightarrow{\beta} \mathbb{C}_{y_1} \to 0$ be an extension representing the element a. The vanishing of $\mathsf{Hom}(\mathcal{F}_1, \mathcal{F}_2)$ and $\mathsf{Ext}^1(\mathcal{F}_1, \mathcal{F}_2)$ implies that we can uniquely lift the morphisms g and f to morphisms $\tilde{g} : \mathcal{F}_1 \to \mathcal{A}$ and $\tilde{f} : \mathcal{A} \to \mathbb{C}_{y_2}$ such that $\beta\tilde{g} = g$ and $\tilde{f}\alpha = f$. So, we obtain the following commutative diagram

$$\begin{tikzcd}
 & & & \mathcal{F}_1 \arrow[dl, "\tilde{g}"'] \arrow[d, "g"] & \\
a:\ 0 \arrow[r] & \mathcal{F}_2 \arrow[r, "\alpha"] \arrow[d, "f"'] & \mathcal{A} \arrow[r, "\beta"] \arrow[dl, "\tilde{f}"] & \mathbb{C}_{y_1} \arrow[r] & 0 \\
 & \mathbb{C}_{y_2} & & &
\end{tikzcd}$$

and the triple Massey product is defined as $m_3(g \otimes a \otimes f) = \tilde{f}\tilde{g}$.

• By the Serre duality, we have $\mathsf{Ext}^1(\mathbb{C}_{y_1}, \mathcal{F}_2)^* \cong \mathsf{Hom}(\mathcal{F}_2, \mathbb{C}_{y_1})$. Let

$$\widetilde{m}^{\mathcal{F}_1,\mathcal{F}_2}_{y_1,y_2} : \mathsf{Hom}(\mathcal{F}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}(\mathcal{F}_2, \mathbb{C}_{y_2}) \longrightarrow \mathsf{Hom}(\mathcal{F}_2, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}(\mathcal{F}_1, \mathbb{C}_{y_2}) \tag{13}$$

be the image of $m^{\mathcal{F}_1,\mathcal{F}_2}_{y_1,y_2}$ under the canonical isomorphism of vector spaces

$$\mathsf{Lin}\big(\mathsf{Hom}(\mathcal{F}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Ext}^1(\mathbb{C}_{y_1}, \mathcal{F}_2) \otimes \mathsf{Hom}(\mathcal{F}_2, \mathbb{C}_{y_2}), \mathsf{Hom}(\mathcal{F}_1, \mathbb{C}_{y_2})\big) \cong$$
$$\mathsf{Lin}\big(\mathsf{Hom}(\mathcal{F}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}(\mathcal{F}_2, \mathbb{C}_{y_2}), \mathsf{Hom}(\mathcal{F}_2, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}(\mathcal{F}_1, \mathbb{C}_{y_2})\big).$$

By [12, Theorem 1] the linear map $\widetilde{m}^{\mathcal{V}_1,\mathcal{F}_2}_{y_1,y_2}$ satisfies the following "triangle equation"

$$(\widetilde{m}^{\mathcal{F}_3,\mathcal{F}_2}_{y_1,y_2})^{12}(\widetilde{m}^{\mathcal{F}_1,\mathcal{F}_3}_{y_1,y_3})^{13} - (\widetilde{m}^{\mathcal{F}_1,\mathcal{F}_3}_{y_2,y_3})^{23}(\widetilde{m}^{\mathcal{F}_1,\mathcal{F}_2}_{y_1,y_2})^{12} + (\widetilde{m}^{\mathcal{F}_1,\mathcal{F}_2}_{y_1,y_3})^{13}(\widetilde{m}^{\mathcal{F}_2,\mathcal{F}_3}_{y_2,y_3})^{23} = 0. \tag{14}$$

Both sides of the equality (14) are viewed as linear maps

$$\mathsf{Hom}(\mathcal{F}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}(\mathcal{F}_2, \mathbb{C}_{y_2}) \otimes \mathsf{Hom}(\mathcal{F}_3, \mathbb{C}_{y_3}) \longrightarrow$$
$$\longrightarrow \mathsf{Hom}(\mathcal{F}_2, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}(\mathcal{F}_3, \mathbb{C}_{y_2}) \otimes \mathsf{Hom}(\mathcal{F}_1, \mathbb{C}_{y_3}).$$

Moreover, the tensor $\widetilde{m}^{\mathcal{F}_1,\mathcal{F}_2}_{y_1,y_2}$ is non-degenerate and skew-symmetric:

$$\rho(\widetilde{m}^{\mathcal{F}_1,\mathcal{F}_2}_{y_1,y_2}) = -\widetilde{m}^{\mathcal{F}_2,\mathcal{F}_1}_{y_2,y_1}, \tag{15}$$

where ρ is the isomorphism

$$\mathsf{Hom}(\mathcal{F}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}(\mathcal{F}_2, \mathbb{C}_{y_2}) \longrightarrow \mathsf{Hom}(\mathcal{F}_2, \mathbb{C}_{y_2}) \otimes \mathsf{Hom}(\mathcal{F}_1, \mathbb{C}_{y_1})$$

given by the rule $\rho(f \otimes g) = g \otimes f$.

• The *idea of the proof* of the relation (14) is the following. Since the derived category $D^b\big(\mathsf{Coh}(E)\big)$ has a structure of an A_∞–category, we have the equality:

$$m_3 \circ (m_3 \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes m_3 \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes m_3) = 0, \tag{16}$$

where both sides are viewed as linear operators mapping the tensor product

$$\mathsf{Hom}(\mathcal{F}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Ext}^1(\mathbb{C}_{y_1}, \mathcal{F}_2) \otimes \mathsf{Hom}(\mathcal{F}_2, \mathbb{C}_{y_2}) \otimes \mathsf{Ext}^1_E(\mathbb{C}_{y_2}, \mathcal{F}_3) \otimes \mathsf{Hom}(\mathcal{F}_3, \mathbb{C}_{y_3})$$

to the vector space $\mathsf{Hom}(\mathcal{F}_1, \mathbb{C}_{y_3})$. In other words, the Yang–Baxter relation (14) is just a translation of the A_∞–constraint (16). Similarly, the unitarity property (15) of $\widetilde{m}^{\mathcal{F}_1,\mathcal{F}_2}_{y_1,y_2}$ is a consequence of existence of a cyclic A_∞–structure on $D^b\big(\mathsf{Coh}(E)\big)$, see [12, Section 1] for more details.

• Consider the linear map

$$\tilde{r}^{\mathcal{F}_1,\mathcal{F}_2}_{y_1,y_2} : \mathsf{Lin}\big(\mathcal{F}_1\big|_{y_1}, \mathcal{F}_2\big|_{y_1}\big) \longrightarrow \mathsf{Lin}\big(\mathcal{F}_1\big|_{y_2}, \mathcal{F}_2\big|_{y_2}\big)$$

defined by the following commutative diagram of vector spaces:

$$\begin{tikzcd}[column sep=small] & \mathsf{Hom}\big(\mathcal{F}_1, \mathcal{F}_2(y_1)\big) \arrow[dl, "\mathsf{res}^{\mathcal{F}_1,\,\mathcal{F}_2}_{y_1}(\omega)"'] \arrow[dr, "\mathsf{ev}^{\mathcal{F}_1,\,\mathcal{F}_2(y_1)}_{y_2}"] & \\ \mathsf{Lin}\big(\mathcal{F}_1\big|_{y_1}, \mathcal{F}_2\big|_{y_1}\big) \arrow[rr, "\tilde{r}^{\mathcal{F}_1,\mathcal{F}_2}_{y_1,y_2}"] & & \mathsf{Lin}\big(\mathcal{F}_1\big|_{y_2}, \mathcal{F}_2\big|_{y_2}\big). \end{tikzcd} \tag{17}$$

Then $\tilde{r}^{\mathcal{F}_1,\mathcal{F}_2}_{y_1,y_2}$ is the image of $\widetilde{m}^{\mathcal{F}_1,\mathcal{F}_2}_{y_1,y_2}$ under the canonical isomorphism of vector spaces

$$\mathsf{Lin}\big(\mathsf{Hom}(\mathcal{F}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}(\mathcal{F}_2, \mathbb{C}_{y_2}), \mathsf{Hom}(\mathcal{F}_2, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}(\mathcal{F}_1, \mathbb{C}_{y_2})\big) \cong$$
$$\mathsf{Lin}\Big(\mathsf{Lin}\big(\mathcal{F}_1\big|_{y_1}, \mathcal{F}_2\big|_{y_1}\big),\ \mathsf{Lin}\big(\mathcal{F}_1\big|_{y_2}, \mathcal{F}_2\big|_{y_2}\big)\Big),$$

see [6, Theorem 4.17] and [12, Theorem 4].

• Let A be an arbitrary automorphy factor, $\mathcal{F} = \mathcal{E}(A)$ and $y \in E$. Then we have an isomorphism of vector spaces

$$\gamma(A, y) : \mathsf{Hom}(\mathcal{F}, \mathbb{C}_y) \longrightarrow \mathsf{Hom}(\mathcal{F} \otimes \mathbb{C}_y, \mathbb{C}_y) \longrightarrow \mathcal{F}\big|_y^* \longrightarrow \mathbb{C}^n,$$

induced by the trivialization γ_A from Remark 3.2. For any $v \in \mathbb{C}$ we denote by $\gamma(v, y)$ the isomorphism $\mathsf{Hom}(\mathcal{F}_v, \mathbb{C}_y) \to \mathbb{C}^n$. We obtain a linear map $\bar{r}_B(v_1, v_2; y_1, y_2)$, defined by the following commutative diagram of vector spaces:

$$\begin{tikzcd} \mathsf{Hom}(\mathcal{F}_{v_1}, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}(\mathcal{F}_{v_2}, \mathbb{C}_{y_2}) \arrow[r, "\widetilde{m}^{\mathcal{F}_1,\mathcal{F}_2}_{y_1,y_2}"] \arrow[d, "\gamma(v_1,y_1)\otimes\gamma(v_2,y_2)"'] & \mathsf{Hom}(\mathcal{F}_{v_2}, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}(\mathcal{F}_{v_1}, \mathbb{C}_{y_2}) \arrow[d, "\gamma(v_2,y_1)\otimes\gamma(v_1,y_2)"] \\ \mathbb{C}^n \otimes \mathbb{C}^n \arrow[r, "\bar{r}_B(v_1,v_2;y_1,y_2)"] & \mathbb{C}^n \otimes \mathbb{C}^n. \end{tikzcd}$$

• Using the canonical isomorphism $\mathsf{Lin}(\mathbb{C}^n \otimes \mathbb{C}^n,\ \mathbb{C}^n \otimes \mathbb{C}^n) \to \mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n\times n}(\mathbb{C})$, we end up with a tensor-valued meromorphic function

$$\mathbb{C}^2_{(v_1,v_2)} \times \mathbb{C}^2_{(y_1,y_2)} \xrightarrow{r_B} \mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n\times n}(\mathbb{C}),$$

satisfying the Yang–Baxter equation (11) and the unitarity condition (12). Moreover, for any $v_1, v_2; y_1, y_2 \in \mathbb{C}$ such that $v_1 - v_2 \notin \Sigma$ and $y_1 - y_2 \notin \Lambda$ the tensor $r_B(v_1, v_2; y_1, y_2)$ coincides with the image of $\tilde{r}^{\mathcal{F}_1,\mathcal{F}_2}_{y_1,y_2}$ under the composition of the canonical isomorphism of vector spaces

$$\mathsf{Lin}\Big(\mathsf{Lin}\big(\mathcal{F}_1\big|_{y_1}, \mathcal{F}_2\big|_{y_1}\big),\ \mathsf{Lin}\big(\mathcal{F}_1\big|_{y_2}, \mathcal{F}_2\big|_{y_2}\big)\Big) \longrightarrow \mathsf{Lin}\big(\mathcal{F}_2\big|_{y_1}, \mathcal{F}_1\big|_{y_1}\big) \otimes \mathsf{Lin}\big(\mathcal{F}_1\big|_{y_2}, \mathcal{F}_2\big|_{y_2}\big)\Big)$$

with the isomorphism $\mathsf{Lin}\big(\mathcal{F}_1\big|_{y_2}, \mathcal{F}_2\big|_{y_2}\big)\Big) \longrightarrow \mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n\times n}(\mathbb{C})$ induced by trivializations γ from Remark 3.2. $\square$

Remark 3.17. In order to prove that the function $r = r_B(v_1, v_2; y_1, y_2)$ is actually holomorphic with an analytic dependence on the entries of the matrix B, we need again the formalism of sheaves. This will be done in Subsection 3.5. The reader, interested in the actual solutions may go directly to Section 4.

3.4. **Remarks on the constructed solutions.** In the previous subsection we have seen how one can attach to a matrix $B \in \mathsf{GL}_n(\mathbb{C})$ a unitary solution

$$\mathbb{C}^2 \times \mathbb{C}^2 \xrightarrow{r_B} \mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n\times n}(\mathbb{C})$$

of the associative Yang–Baxter equation (11), see diagram (10) from Theorem 3.16.

Proposition 3.18. *For general $v_1, v_2, u; y_1, y_2, x \in \mathbb{C}$ we have the equality*

$$r_B(v_1 + u, v_2 + u;\ y_1 + x, y_2 + x) = r_B(v_1, v_2;\ y_1, y_2).$$

In other words, the function $r_B(v_1, v_2; y_1, y_2)$ depends only on the differences $v = v_1 - v_2$ and $y = y_2 - y_1$. In particular, the function $r_B(v, y) = r_B(v_1, v_2; y_1, y_2)$ satisfies the associative Yang–Baxter equation (2).

Proof. Since the vector space $\mathsf{Sol}_{B,v,y_1,\tau}$ from Theorem 3.14 only depends on the difference $v = v_2 - v_1$, whereas res_{y_1} and ev_{y_2} depend only on y_1 and y_2, we have the equality $r_B(v_1 + u, v_2 + u; y_1, y_2) = r_B(v_1, v_2; y_1, y_2)$. To show the translation invariance of the function r_B with respect to the second pair of spectral variables note that we have the following commutative diagram:

$$\begin{tikzcd}
 & \mathsf{Sol}_{B,v,y_1,\tau} \arrow[dl, "\mathsf{res}_{y_1}"'] \arrow[dr, "\mathsf{ev}_{y_2}"] \arrow[dd, "t_x"] & \\
\mathsf{Mat}_{n\times n}(\mathbb{C}) & & \mathsf{Mat}_{n\times n}(\mathbb{C}) \\
 & \mathsf{Sol}_{B,v,y_1+x,\tau} \arrow[ul, "\mathsf{res}_{y_1+x}"] \arrow[ur, "\mathsf{ev}_{y_2+x}"'] &
\end{tikzcd}$$

where $t_x\big(\Phi(z)\big) = \Phi(z - x)$. It proves that $r_B(v_1, v_2; y_1 + x, y_2 + x) = r_B(v_1, v_2; y_1, y_2)$. $\square$

Remark 3.19. Proposition 3.18 implies that in order to compute the linear map $r_B(v, y)$ we can take $y_1 = 0$ and $y_2 = y$ in the commutative diagram (10). In particular, the solution $r_B(v, y)$ can be computed using the diagram (5).

Proposition 3.20. *Let $B, S \in \mathsf{GL}_n(\mathbb{C})$ and $A := S^{-1}BS$. Then we have:*

$$r_A(v, y) = \big(S^{-1} \otimes S^{-1}\big)\ r_B(v, y)\ \big(S \otimes S\big).$$

Proof. For simplicity of notation we denote $\mathsf{Sol}_B = \mathsf{Sol}_{B,v,y,\tau}$ and $r_B = r_B(v, y)$. Observe that we have an isomorphism of vector spaces $\varphi_S : \mathsf{Sol}_B \to \mathsf{Sol}_A$ mapping a function $\Phi \in \mathsf{Sol}_B$ to $S^{-1}\,\Phi\, S \in \mathsf{Sol}_A$. We have a commutative diagram

$$\begin{tikzcd}
\mathsf{Mat}_{n\times n}(\mathbb{C}) \arrow[d, "c_S"'] & \mathsf{Sol}_B \arrow[l, "\mathsf{res}_0"'] \arrow[r, "\mathsf{ev}_y"] \arrow[d, "\varphi_S"] & \mathsf{Mat}_{n\times n}(\mathbb{C}) \arrow[d, "c_S"] \\
\mathsf{Mat}_{n\times n}(\mathbb{C}) & \mathsf{Sol}_A \arrow[l, "\mathsf{res}_0"'] \arrow[r, "\mathsf{ev}_y"] & \mathsf{Mat}_{n\times n}(\mathbb{C}),
\end{tikzcd} \tag{18}$$

where $c_S(X) = S^{-1}XS$. This implies that for any $X \in \mathsf{Mat}_{n\times n}(C)$ we have: $\tilde{r}_A(S^{-1}XS) = S^{-1}\tilde{r}_B(X)S$. The matrix S defines the following linear automorphism

$$\psi_S : \mathsf{Lin}\bigl(\mathsf{Mat}_{n\times n}(\mathbb{C}),\ \mathsf{Mat}_{n\times n}(\mathbb{C})\bigr) \longrightarrow \mathsf{Lin}\bigl(\mathsf{Mat}_{n\times n}(\mathbb{C}),\ \mathsf{Mat}_{n\times n}(\mathbb{C})\bigr)$$

sending $l \in \mathsf{Lin}\bigl(\mathsf{Mat}_{n\times n}(\mathbb{C}),\, \mathsf{Mat}_{n\times n}(\mathbb{C})\bigr)$ to the linear map $X \xrightarrow{\psi_S(l)} S^{-1}\, l(SXS^{-1})\, S$. Then we have: $\psi_S(\tilde{r}_B) = \tilde{r}_A$.

Finally, let $\mathsf{can} : \mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n\times n}(\mathbb{C}) \to \mathsf{Lin}\bigl(\mathsf{Mat}_{n\times n}(\mathbb{C}),\, \mathsf{Mat}_{n\times n}(\mathbb{C})\bigr)$ be the canonical isomorphism of vector spaces mapping a simple tensor $X \otimes Y$ to the linear map $Z \mapsto \mathrm{Tr}(XZ)Y$. Then the following diagram is commutative:

$$\begin{tikzcd}
\mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n\times n}(\mathbb{C}) \arrow[r, "\mathsf{can}"] \arrow[d, "c_S \otimes c_S"'] & \mathsf{Lin}\bigl(\mathsf{Mat}_{n\times n}(\mathbb{C}),\, \mathsf{Mat}_{n\times n}(\mathbb{C})\bigr) \arrow[d, "\psi_S"] \\
\mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n\times n}(\mathbb{C}) \arrow[r, "\mathsf{can}"] & \mathsf{Lin}\bigl(\mathsf{Mat}_{n\times n}(\mathbb{C}),\, \mathsf{Mat}_{n\times n}(\mathbb{C})\bigr).
\end{tikzcd}$$

But this implies that $r_A(v, y) = \bigl(S^{-1} \otimes S^{-1}\bigr)\, r_B(v, y)\, \bigl(S \otimes S\bigr)$. □

3.5. **Semi–universal family of degree zero semi-stable vector bundles on a complex torus and the associative Yang–Baxter equation.** In the previous subsections we have explained how one can attach to a matrix $B \in \mathsf{GL}_n(\mathbb{C})$ a unitary solution $r_B(v, y)$ of the associative Yang–Baxter equation (2). However, it still remains to be shown that r_B is a meromorphic function in v and y, holomorphic on $(\mathbb{C} \setminus \Sigma) \times (\mathbb{C} \setminus \Lambda)$ and with an analytic dependence on the matrix B. Although this fact can be verified by a direct computation, we prefer to give an abstract proof based on the technique of semi–universal families of semi–stable sheaves.

• Let $G = \mathsf{GL}_n(\mathbb{C})$ and $\mathcal{P} \in \mathsf{VB}(E \times G)$ be defined as follows

$$\mathcal{P} := \mathbb{C} \times G \times \mathbb{C}^n/\sim, \quad \text{where} \quad (z, g, v) \sim (z + 1, g, v) \sim (z + \tau, g, g \cdot v)$$

for all $(z, g, v) \in \mathbb{C} \times G \times \mathbb{C}^n$. Note that we have a Cartesian diagram

$$\begin{tikzcd}
(\mathbb{C} \times G) \times \mathbb{C}^n \arrow[r] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{P} \arrow[d] \\
\mathbb{C} \times G \arrow[r, "\pi \times \mathbb{1}"] & E \times G,
\end{tikzcd}$$

where $\pi : \mathbb{C} \to \mathbb{C}/\Lambda = E$ is the quotient map. Note that for any $g \in G$ we have an isomorphism $\mathcal{P}\big|_{E\times\{g\}} \cong \mathcal{E}(g)$, where $\mathcal{E}(g)$ is the semi–stable degree zero vector bundle on E determined by the automorphy factor $g \in \mathsf{GL}_n(\mathbb{C})$. Thus, the constructed vector bundle $\mathcal{P}$ is a *semi–universal* family of degree zero semi-stable vector bundle on the torus E.

• Let $J = \mathsf{Pic}^0(E)$ be the Jacobian of E. One can identify J with the torus E using the following construction. Consider the line bundle $\mathcal{L}$ on $E \times E = \mathbb{C}/\Lambda \times \mathbb{C}/\Lambda$ defined as the quotient space $\mathcal{L} := (\mathbb{C} \times \mathbb{C}) \times \mathbb{C}/\sim$, where

$$(z, w, v) \sim (z + 1, w, v) \sim (z, w + 1, v) \sim (z, w + \tau, v) \sim (z + \tau, w, \exp(2\pi i w)v)$$

for all $(z, w, v) \in (\mathbb{C} \times \mathbb{C}) \times \mathbb{C}$. The constructed line bundle $\mathcal{L}$ is a universal family of degree zero vector bundles on E.

• We denote $X = E \times J \times J \times E \times E \times G$, $T = J \times J \times E \times E \times G$ and set $q : X \to T$ and $p : X \to E \times G$ to be the canonical projection maps. Similarly, for $i = 1, 2$ we define $p_i : X \to E \times J$ and $h_i : T \to X$ to be given by the formulae $p_i(x, v_1, v_2, y_1, y_2, g) = (x, v_i)$ and $h_i(v_1, v_2, y_1, y_2, g) = (y_i, v_1, v_2, y_1, y_2, g)$. Note that h_1 and h_2 are sections of the canonical projection q.

• For $i = 1, 2$ we define $\mathcal{F}_i := p^*\mathcal{P} \otimes p_i^*\mathcal{L}$. Obviously, for any point $t = (v_1, v_2, y_1, y_2, g) \in T$ we have: $\mathcal{F}_i\big|_{q^{-1}(t)} \cong \mathcal{P}\big|_{E\times\{g\}} \otimes \mathcal{L}\big|_{E\times\{v_i\}} \cong \mathcal{E}\big(\exp(2\pi v_i) \cdot g\big)$.

Lemma 3.21. *The coherent sheaf $q_*\mathcal{H}om_X(\mathcal{F}_1, \mathcal{F}_2)$ is supported on a proper closed analytic subset of T.*

Proof. By Grauert's direct image theorem, the sheaf $q_*\mathcal{H}om_X(\mathcal{F}_1, \mathcal{F}_2)$ is coherent, hence it is supported on a closed analytic subset Δ of the base T. Since $\mathcal{H}om_X(\mathcal{F}_1, \mathcal{F}_2)$ is a vector bundle on X, it is flat over T and for any point $t = (v_1,\ v_2,\ y_1,\ y_2,\ g) \in T$ we have a base–change isomorphism

$$q_*\mathcal{H}om_X(\mathcal{F}_1, \mathcal{F}_2) \otimes \mathbb{C}_t \cong \mathsf{Hom}_E\big(\mathcal{E}(g) \otimes \mathcal{L}_{v_1}, \mathcal{E}(g) \otimes \mathcal{L}_{v_2}\big).$$

By Corollary 3.6, we have the vanishing $\mathsf{Hom}_E\big(\mathcal{E}(g) \otimes \mathcal{L}_{v_1}, \mathcal{E}(g) \otimes \mathcal{L}_{v_2}\big) = 0$ for generically chosen $v_1, v_2 \in J$ and $g \in G$. Hence, Δ is a proper subset of T. □

The following result can be proven along the same lines as Lemma 3.21.

Lemma 3.22. *Let $D_i := \mathsf{Im}(h_i) \subseteq X$. Then the sheaf $q_*\mathcal{H}om_X\big(\mathcal{F}_1(D_2), \mathcal{F}_2(D_1)\big)$ is supported on a proper closed analytic subset Δ' of T and $q_*\mathcal{H}om_X\big(\mathcal{F}_1, \mathcal{F}_2(D_1)\big)$ is a vector vector bundle of rank n^2.*

• Let $\breve{T} := T \setminus (\Delta \cup \Delta')$ and $\breve{X} := q^{-1}(\breve{T})$. For the sake of simplicity we denote the restrictions of $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\breve{X}$ by the same symbols. Let $\omega \in H^0\big(\Omega_{\breve{X}/\breve{T}}\big)$ be the pull-back of the differential form $dz \in H^0(\Omega_E)$. Note that we are in the situation of [6, Section 5.3]. In particular, we have the following commutative diagram in

$\mathsf{Coh}(\breve{T})$, where all arrows are isomorphisms of vector bundles on $\breve{T}$:

$$\begin{tikzcd}[column sep=small] & q_*\mathcal{H}om_{\breve{X}}\big(\mathcal{F}_1, \mathcal{F}_2(D_1)\big) \arrow[dl, "\mathsf{res}_{h_1}^{\mathcal{F}_1,\mathcal{F}_2}(\omega)"'] \arrow[dr, "\mathsf{ev}_{h_2}^{\mathcal{F}_1,\mathcal{F}_2(D_1)}"] & \\ \mathcal{H}om_{\breve{T}}\big(h_1^*\mathcal{F}_1, h_1^*\mathcal{F}_2\big) \arrow[rr, "\tilde{r}^{\mathcal{F}_1,\mathcal{F}_2}_{h_1,h_2}"] & & \mathcal{H}om_{\breve{T}}\big(h_2^*\mathcal{F}_1, h_2^*\mathcal{F}_2\big). \end{tikzcd} \tag{19}$$

The morphisms $\mathsf{res}_{h_1}^{\mathcal{F}_1,\mathcal{F}_2}(\omega)$ and $\mathsf{ev}_{h_2}^{\mathcal{F}_1,\mathcal{F}_2(D_1)}$ are induced by the short exact sequences

$$0 \to \Omega_{\breve{X}/\breve{T}} \to \Omega_{\breve{X}/\breve{T}}(D_1) \xrightarrow{\mathsf{res}_{D_1}} \mathcal{O}_{D_1} \to 0, \quad 0 \to \mathcal{O}_{\breve{X}}(-D_2) \to \mathcal{O}_{\breve{X}} \to \mathcal{O}_{D_1} \longrightarrow 0,$$

see [6, Section 5.3]. By [6, Theorem 5.17], after tensoring the diagram (19) with $\mathbb{C}_t$, where $t = (v_1, v_2, y_1, y_2, g) \in \breve{T}$, and applying base change isomorphisms, we get the commutative diagram (17). In particular, the function $\tilde{r}_B(v_1, v_2; y_1, y_2)$ from Theorem 3.16 is just the isomorphism of vector bundles $\tilde{r}^{\mathcal{F}_1,\mathcal{F}_2}_{h_1,h_2}$ written with respect of the trivialization γ, described in Remark 3.2. This implies that the tensor $r_B(v, y)$ is non-degenerate.

In a similar way, the isomorphism $\tilde{r}_B(v_1, v_2; y_1, y_2)$ determines a holomorphic section $r^{\mathcal{F}_1,\mathcal{F}_2}_{h_1,h_2} \in H^0\big(\breve{T}, \mathcal{H}om_{\breve{T}}\big(h_1^*\mathcal{F}_2, h_1^*\mathcal{F}_1\big) \otimes \mathcal{H}om_{\breve{T}}\big(h_2^*\mathcal{F}_1, h_2^*\mathcal{F}_2\big)\big)$. Trivializing $\mathcal{F}_1$ and $\mathcal{F}_2$ as in Remark 3.2, the section $r^{\mathcal{F}_1,\mathcal{F}_2}_{h_1,h_2}$ becomes the tensor-valued function $r_B(v, y)$ from Theorem 3.16. This proves that $r_B(v, y)$ is holomorphic on $(\mathbb{C} \setminus \Sigma_B) \times (\mathbb{C} \setminus \Lambda)$ and as a function of the input matrix B. To show that $r_B(v, y)$ is meromorphic on $\mathbb{C} \times \mathbb{C}$ note that $\mathsf{res}_{h_1}^{\mathcal{F}_1,\mathcal{F}_2}(\omega)$ and $\mathsf{ev}_{h_2}^{\mathcal{F}_1,\mathcal{F}_2(D_1)}$ are morphisms of vector bundles of rank n^2 on the whole base T and $\tilde{r}^{\mathcal{F}_1,\mathcal{F}_2}_{h_1,h_2} = \mathsf{ev}_{h_2}^{\mathcal{F}_1,\mathcal{F}_2(D_1)} \circ \big(\mathsf{res}_{h_1}^{\mathcal{F}_1,\mathcal{F}_2}(\omega)\big)^{-1}$ is a meromorphic isomorphism of $\mathcal{H}om_T\big(h_1^*\mathcal{F}_1, h_1^*\mathcal{F}_2\big)$ and $\mathcal{H}om_T\big(h_2^*\mathcal{F}_1, h_2^*\mathcal{F}_2\big)$.

4. Computations of solutions of AYBE

In this section we compute the solutions of the associative Yang–Baxter equation (2) attached to a diagonal matrix and to a Jordan block.

4.1. **Solution obtained from a diagonal matrix.** All or computations are based on the following standard fact.

Lemma 4.1. *Let $\varphi(z) = \exp(-\pi i\tau - 2\pi i z)$. Then the vector space*

$$\left\{ f : \mathbb{C} \longrightarrow \mathbb{C} \;\middle|\; \begin{array}{l} f \quad \text{is holomorphic} \\ f(z+1) = f(z) \\ f(z+\tau) = \varphi(z) f(z) \end{array} \right\} \tag{20}$$

is one–dimensional and generated by the third Jacobian theta–function

$$\bar{\theta}(z) = \theta_3(z|\tau) = \sum_{n \in \mathbb{Z}} \exp(\pi i n^2 \tau + 2\pi i n z).$$

Proof. A proof of this result can for instance be found in [10, Chapter 1]. $\square$

Theorem 4.2. *Let $B = \mathsf{diag}\big(\exp(2\pi i\lambda_1), \dots, \exp(2\pi i\lambda_n)\big)$ for some $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. Then the corresponding solution of the associative Yang–Baxter equation described in Theorem 3.16 is given by the following formula:*

$$r_B(v, y) = \sum_{k,\, l=1}^{n} \sigma(v - \lambda_{kl}, y)\, e_{l,k} \otimes e_{k,l}, \tag{21}$$

where $\lambda_{kl} = \lambda_k - \lambda_l$ for all $1 \le k, l \le n$ and $\sigma(u, x)$ is the Kronecker function.

Proof. Let $\Phi(z) = \big(a_{kl}(z)\big)$ be an element of $\mathsf{Sol} = \mathsf{Sol}_{B,\, v,\, 0,\, \tau}$, where $v = v_1 - v_2$. Then for all $1 \le k, l \le n$ we have:

$$\begin{cases} a_{kl}(z+1) &= a_{kl}(z) \\ a_{kl}(z+\tau) &= \exp\big(-\pi i\tau - 2\pi i(z + v + \frac{\tau+1}{2} - \lambda_{kl})\big)a_{kl}(z). \end{cases}$$

Hence, there exist $\beta_{kl} \in \mathbb{C}$ such that $a_{kl}(z) = \beta_{kl}\bar{\theta}(z + v + \frac{\tau+1}{2} - \lambda_{kl})$.

If $A = (\alpha_{kl}) \in \mathsf{Mat}_{n\times n}(\mathbb{C})$ is such that $\mathsf{res}_0\big(\Phi(z)\big) = A$ then $\beta_{kl} = \frac{1}{\bar{\theta}(v + \frac{\tau+1}{2} - \lambda_{kl})}\alpha_{kl}$. If $C = (\gamma_{kl}) := \mathsf{ev}_y\big(\Phi(z)\big)$ then for all $1 \le k, l \le n$ we have

$$\gamma_{kl} = \frac{\bar{\theta}(v + y - \lambda_{kl} + \frac{\tau+1}{2})}{\bar{\theta}(v - \lambda_{kl} + \frac{\tau+1}{2})\,\bar{\theta}(y + \frac{\tau+1}{2})}\alpha_{kl} = \frac{1}{i\exp(-\pi i\frac{\tau}{4})}\frac{\theta(v - \lambda_{kl} + y)}{\theta(v - \lambda_{kl})\theta(y)}\alpha_{kl},$$

where we have used the well–known relation between the first and the third Jacobian theta functions $\bar{\theta}(z + \frac{\tau+1}{2}) = i\exp(-\pi i(z + \frac{\tau}{4}))\theta(z)$. Hence, the linear map $\tilde{r}_B(v, y) : \mathsf{Mat}_{n\times n}(\mathbb{C}) \to \mathsf{Mat}_{n\times n}(\mathbb{C})$ sends the basis vector $e_{k,l}$ to $\frac{\exp(\pi i\frac{\tau}{4})}{i\theta'(0)}\sigma(v - \lambda_{kl}, y)e_{k,l}$. Neglecting the constant $\frac{\exp(\pi i\frac{\tau}{4})}{i\theta'(0)}$, we end up with the solution $r_B(v, y)$ given by (21). □

4.2. **Solution attached to a Jordan block.** In this subsection we compute the solution of the associative Yang–Baxter equation (2) attached to a Jordan block of size $n \times n$. Fist note the following easy fact.

Lemma 4.3. *For any $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}^*$ the solution $r_{J_n(\lambda)}(v, y)$ constructed in Theorem 3.16, is gauge equivalent to $r_J(v, y)$, where $J_n(\lambda)$ is the Jordan block of size $n \times n$ with eigenvalue λ and $J = J_n(1)$.*

Proof. Since the matrices $J_n(\lambda)$ and $\lambda \cdot J$ are conjugate, Proposition 3.20 implies that the corresponding solutions are gauge equivalent. From the algorithm of the construction of solutions of (2) presented in Theorem 3.16 it is clear that the matrices $\lambda \cdot J$ and J give the same solutions. □

Hence, it suffices to describe the solution of the associative Yang–Baxter equation (2) attached to the Jordan block J.

Definition 4.4. Let $n \in \mathbb{N}$ be fixed. For all $1 \le k \le n-1$ we set $a_k = \frac{(-1)^k}{k}$,

$$A_0 = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ a_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ a_{n-1} & \cdots & a_1 & 0 \end{pmatrix} \quad \text{and} \quad A_k = -a_k \cdot \mathbb{1}_{n\times n}.$$

Next, consider the following matrix N from $\mathsf{Mat}_{n^2\times n^2}(\mathbb{C})$:

$$N = \begin{pmatrix} A_0 & A_1 & \cdots & A_{n-1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_1 \\ 0 & \cdots & 0 & A_0 \end{pmatrix}. \tag{22}$$

Note the following easy fact.

Lemma 4.5. *The matrix N is nilpotent. More precisely, $N^{2n-1} = 0$.*

Definition 4.6. Consider the differential operator $\nabla = -\frac{1}{2\pi i}\cdot\frac{d}{dz}$ acting on the vector space $\mathcal{M}$ of meromorphic functions on $\mathbb{C}$. For all $0 \le k, l \le n-1$ we define the linear operator $\nabla_{k,l} : \mathcal{M} \to \mathcal{M}$ given by the following formula:

$$\nabla_{k,l} = e^t_{n(n-k-1)+l+1} \exp\left(\nabla\, N\right) e_{n(n-1)+1}. \tag{23}$$

Since the matrix N is nilpotent, the operators $\nabla_{k,l}$ are polynomials in ∇. Note that $\nabla_{0,0} = e^t_{n(n-1)+1} \exp\left(\nabla\, N\right) e_{n(n-1)+1}$ is the identity operator.

Now we can state the main result of this subsection.

Theorem 4.7. *Let J be the Jordan block of size $n\times n$ with eigenvalue one. Then the corresponding solution of the associative Yang-Baxter equation, described in Theorem 3.16, is given by the following formula:*

$$r_J(v,y) = \left(\sum_{\substack{0\le k\le n-1 \\ 0\le l\le n-1}} \nabla_{k,l}\big(\sigma\,(v,y)\big) \sum_{\substack{1\le i\le n-l \\ 1\le j\le n-k}} e_{i,j+k} \otimes e_{j,i+l} \right), \tag{24}$$

where $\sigma(v,y)$ is the Kronecker function and $\nabla_{k,l}$ acts on the first spectral variable.

Remark 4.8. Let $1 \le a, b, c, d \le n$. Then the coefficient of the tensor $e_{a,b} \otimes e_{c,d}$ in the expression for $r_J(v,y)$ from Equation (24) is zero unless $d \ge a$ and $b \ge c$. Moreover, this coefficient depends only on the differences $d-a$ and $b-c$.

Example 4.9. Let $n = 2$ and $J = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Note that

$$N = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right), \quad N^2 = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

and that all higher powers of N are zero. Hence,

$$\exp(\nabla N) = 1 + \nabla N + \frac{\nabla^2 N^2}{2} = \left(\begin{array}{cc|cc} \mathbb{1} & 0 & \nabla & 0 \\ -\nabla & \mathbb{1} & -\nabla^2 & \nabla \\ \hline 0 & 0 & \mathbb{1} & 0 \\ 0 & 0 & -\nabla & \mathbb{1} \end{array}\right)$$

and we derive that

$$\begin{aligned} r_J(v,y) &= \sigma(v,y)\big(e_{11}\otimes e_{11} + e_{22}\otimes e_{22} + e_{12}\otimes e_{21} + e_{21}\otimes e_{12}\big) + \\ &\quad \nabla\sigma(v,y)\big(e_{12}\otimes h - h\otimes e_{12}\big) - \nabla^2\sigma(v,y) e_{12}\otimes e_{12}, \end{aligned}$$

where $h = e_{11} - e_{22}$.

Remark 4.10. From the fact that the function $r_J(v,y)$ from Example 4.9 satisfies the associative Yang–Baxter equation (2) we obtain the following identity for derivatives of the Kronecker function with respect to the first spectral variable:

$$\begin{aligned} \sigma'(u,x+y)\sigma'(v,y) - \sigma'(u,x)\sigma'(u+v,y) - \sigma'(-v,x)\sigma'(u+v,x+y) \\ = \sigma(u,x)\sigma''(u+v,y) - \sigma(-v,x)\sigma''(u+v,x+y). \end{aligned}$$

Example 4.11. For $n = 3$ and $J = \left(\begin{smallmatrix} 1&1&0 \\ 0&1&1 \\ 0&0&1 \end{smallmatrix}\right)$ we have

$$N = \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 \\ \frac{1}{2} & -1 & 0 & 0 & 0 & 1 & 0 & 0 & -\frac{1}{2} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -1 & 0 \end{array}\right).$$

Note that

$$\exp(\nabla N)\, e_7 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ \frac{1}{2} \end{pmatrix} \nabla + \begin{pmatrix} 1 \\ 1 \\ -\frac{1}{2} \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \frac{\nabla^2}{2} + \begin{pmatrix} 0 \\ -3 \\ 0 \\ 0 \\ 0 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \frac{\nabla^3}{6} + \begin{pmatrix} 0 \\ 0 \\ 6 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \frac{\nabla^4}{24}$$

and that $\nabla_{k,l} = e^t_{3(2-k)+l+1}\big(\exp(\nabla\, N)\, e_7\big)$. Carrying out computations, we end up with the following solution of the associative Yang–Baxter equation:

$$
\begin{aligned}
r_J(v,y) =& \sigma \sum_{1\le i,j\le 3} e_{i,j}\otimes e_{j,i} + \nabla\sigma \sum_{\substack{1\le i\le 3\\ 1\le j\le 2}} \big(e_{i,j+1}\otimes e_{j,i} - e_{j,i}\otimes e_{i,j+1}\big)+\\
& \Big(-\frac{1}{2}\nabla + \frac{1}{2}\nabla^2\Big)\sigma \sum_{1\le i\le 3} e_{i,3}\otimes e_{1,i} + \Big(\frac{1}{2}\nabla + \frac{1}{2}\nabla^2\Big)\sigma \sum_{1\le i\le 3} e_{1,i}\otimes e_{i,3}\\
& \Big(\frac{1}{2}\nabla^2 - \frac{1}{2}\nabla^3\Big)\sigma \sum_{1\le i\le 2} e_{i,3}\otimes e_{1,i+1} + \Big(\frac{1}{2}\nabla^2 + \frac{1}{2}\nabla^3\Big)\sigma \sum_{1\le i\le 2} e_{1,i+1}\otimes e_{i,3}+\\
& -\nabla^2\sigma \sum_{1\le i,j\le 2} e_{i,j+1}\otimes e_{j,i+1} + \Big(-\frac{1}{4}\nabla^2 + \frac{1}{4}\nabla^4\Big)\sigma e_{1,3}\otimes e_{1,3},
\end{aligned}
$$

where $\sigma = \sigma(v,y)$ is the Kronecker function.

Proof of Theorem 4.7. We divide the proof into several steps.

Computation of a basis of Sol. First, we compute a basis of the vector space

$$
\mathsf{Sol}_{J,\,v,\,0,\,\tau} := \left\{ \Phi : \mathbb{C}\to \mathsf{Mat}_{m\times n}(\mathbb{C}) \;\middle|\; \begin{array}{l} \Phi \quad \text{is holomorphic}\\ \Phi(z+1) = \Phi(z)\\ \Phi(z+\tau)J = e(z)J\Phi(z) \end{array} \right\}, \tag{25}
$$

where $e(z) = e(z,v,\tau) = -\exp\big(-2\pi i(z+v+\tau)\big)$. The proof of the following result is straightforward.

Lemma 4.12. *Let $C : \mathbb{C}\to \mathsf{GL}_n(\mathbb{C})$ and $D :\to \mathsf{GL}_m(\mathbb{C})$ be a pair of automorphy factors. Let $\widetilde{D} := \big(D^{-1}\big)^t$ be the transpose of the inverse matrix of D. Next, we set*

$$
C\otimes \widetilde{D} = \begin{pmatrix} c_{11}\widetilde{D} & \dots & c_{1n}\widetilde{D}\\ \vdots & \ddots & \vdots\\ c_{n1}\widetilde{D} & \dots & c_{nn}\widetilde{D} \end{pmatrix}.
$$

(1) *In the notations of Theorem 3.3, we have isomorphisms*

$$
\mathsf{Hom}\big(\mathcal{E}(C),\,\mathcal{E}(D)\big) \xrightarrow{\cong} \mathsf{Sol}_{C,\,D} \xrightarrow{\alpha} \mathsf{Sol}_{(1),\,C\otimes\widetilde{D}} \xrightarrow{\cong} H^0\big(\mathcal{E}(C\otimes\widetilde{D})\big),
$$

where for $\Phi = \big(f_{ij}\big)_{1\le i,j\le n} \in \mathsf{Sol}_{C,\,D}$ we set $\alpha\big(\Phi\big) = \big(f_{n(i-1)+j}\big)_{1\le i,j\le n}$.

(2) *We have: $J\otimes\widetilde{J} = \exp(N)$, where N is the matrix from Definition 4.4.*

The following result is due to Polishchuk and Zaslow [14, Proposition 2].

Proposition 4.13. *As above, let $\nabla = -\frac{1}{2\pi i}\cdot\frac{d}{dz}$ and $e(z) = -\exp\big(-2\pi i(z+v+\tau)\big)$. Then we have an isomorphism of vector spaces:*

$$
\delta : H^0\Big(\mathcal{E}\big(e(z)\big)\Big)\otimes \mathbb{C}^{n^2} \longrightarrow H^0\Big(\mathcal{E}\big(e(z)\cdot\exp(N)\big)\Big)
$$

given by the rule $\delta(f \otimes u) = \big(\exp(\nabla N) f\big) u = \sum_{m=0}^{\infty} \frac{\nabla^m(f)}{m!} N^m(u)$ *for any* $f \in H^0\Big(\mathcal{E}\big(e(z)\big)\Big)$ *and* $u \in \mathbb{C}^{n^2}$*.*

Proposition 4.14. *Let* $\bar\theta_v(z) = \bar\theta(z + v + \frac{\tau+1}{2})$*. Then we have an isomorphism of vector spaces* $\Delta : \mathbb{C}^{n^2} \to \mathsf{Sol}_{J,\,v,\,0,\,\tau}$ *mapping a vector* $u \in \mathbb{C}^{n^2}$ *to the matrix-valued function* $\Delta(u)$*, where for any* $1 \le k, l \le n$ *we have:*

$$\big(\Delta(u)\big)_{k,\,l}(z) = e^t_{n(k-1)+l}\big(\exp(\nabla N)\bar\theta_v(z)\big)u.$$

Proof. By Lemma 4.1, the vector space $H^0\big(\mathcal{E}(e(z))\big)$ is one–dimensional and $\bar\theta_v(z) = \bar\theta(z+v+\frac{\tau+1}{2})$ is its basis element. Hence, Proposition 4.14 is a consequence of Lemma 4.12 and Proposition 4.13. □

Definition 4.15. In the notations of Proposition 4.14, let U be the element of $\mathsf{Sol} = \mathsf{Sol}_{J,\,v,\,0,\,\tau}$ corresponding to $u = e_{n(n-1)+1} \in \mathbb{C}^{n^2}$. Note that $\big(U(z)\big)_{n,1} = \bar\theta_v(z)$.

Proposition 4.16. *Let* $K = J_n(0)$ *be the Jordan block of size* $n \times n$ *with eigenvalue zero. For all* $1 \le i, j \le n$ *we set* $F_{ij} = K^{n-i} U K^{j-1}$*. Then we have:*

1. *All matrix–valued functions* $F_{ij} : \mathbb{C} \to \mathsf{Mat}_{n\times n}(\mathbb{C})$ *belong to* Sol*.*
2. *If* $1 \le p,\, q \le n$ *are such that* $i < p \le n$ *or* $1 \le q < j$ *then we have:* $\big(F_{ij}\big)_{p,\,q} = 0$*. Moreover,* $\big(F_{ij}\big)_{i,\,j} = \bar\theta_v$*. In other words, all non-zero entries of* F_{ij} *are located in the rectangle whose lower left corner is* (i, j)*.*
3. *Moreover,* $\big\{F_{ij}\big\}_{1\le i,j\le n}$ *is a basis of the vector space* Sol*.*

Proof. The statement that F_{ij} belongs to Sol is equivalent to the equality

$$K^{n-i}\, U(z+\tau)\, K^{j-1}\, J = e(z) J\, K^{n-i}\, U(z)\, K^{j-1}. \tag{26}$$

Since the matrices K and J commute, Equality (26) is equivalent to

$$K^{n-i}\big(U(z+\tau)J - e(z)JU(z)\big)K^{j-1} = 0,$$

which is true since U belongs to Sol. The second part of the proposition follows from the definition of the functions F_{ij}. From this part also follows that all elements of the set $\big\{F_{ij}\big\}_{1\le i,j\le n}$ are linearly independent. By Corollary 3.13, the dimension of Sol is n^2. Thus, $\big\{F_{ij}\big\}_{1\le i,j\le n}$ is a basis of Sol. □

Example 4.17. Let $n = 2$. Similarly to Example 4.9, we obtain:

$$F_{2,1} = U = \begin{pmatrix} \nabla\bar\theta_v & -\nabla^2\bar\theta_v \\ \bar\theta_v & -\nabla\bar\theta_v \end{pmatrix}.$$

Moreover, we have:

$$F_{1,1} = \begin{pmatrix} \bar\theta_v & -\nabla\bar\theta_v \\ 0 & 0 \end{pmatrix},\ F_{2,2} = \begin{pmatrix} 0 & \nabla\bar\theta_v \\ 0 & \bar\theta_v \end{pmatrix} \text{ and } F_{1,2} = \begin{pmatrix} 0 & \bar\theta_v \\ 0 & 0 \end{pmatrix}.$$

Computation of res_0^{-1}. As the next step, we compute the preimages of the elementary matrices $\big\{e_{a,b}\big\}_{1\le a,b\le n}$ under the isomorphism $\mathsf{res}_0 : \mathsf{Sol} \to \mathsf{Mat}_{n\times n}(\mathbb{C})$.

Let $X = (x_{p,q})_{1\le p,q\le n} \in \mathsf{Mat}_{n\times n}(\mathbb{C})$ be a given matrix and $A \in \mathsf{Sol}$ be such that $\mathsf{res}_0(A) = X$. By Proposition 4.16, we have an expansion $A = \sum_{1\le i,\,j\le n} \eta_{i,j} F_{i,j}$ for certain uniquely determined $\eta_{i,j} \in \mathbb{C}$. It is clear that for all $1 \le p, q \le n$ we get:

$$x_{p,q} = \sum_{\substack{p\le i\le n\\ 1\le j\le q}} \eta_{i,j}\big(F_{i,j}(0)\big)_{p,q}. \tag{27}$$

Next, for all $1 \le p, q \le n$ we have $\big(F_{p,q}(z)\big)_{p,q} = \bar{\theta}_v(z)$. This implies that

$$\eta_{p,q} = \frac{1}{\bar{\theta}_v(0)}\left[x_{p,q} - \sum_{\substack{p\le i\le n\\ 1\le j\le q\\ (i,j)\ne(p,q)}} \eta_{i,j}\,\big(F_{i,j}(0)\big)_{p,q}\right]. \tag{28}$$

Hence $\eta_{p,q}$ can be expressed as a linear combination of those $x_{i,j}$ for which $p \le i \le n$ and $1 \le i \le q$. Moreover, due to the recursive structure of the Equality (28), it is clear that $\eta_{p,\,q}$ can be written as a certain linear combination of $x_{i,j}$, whose structure is controlled by the set of paths starting at (p, q) and ending at (i, j). In order to make this more precise, let us make the following definition.

Definition 4.18. For any $\chi \in \mathbb{N}$ and $(i, j), (p, q) \in \mathbb{N}\times\mathbb{N}$ such that $i \ge p$ and $j \le q$ we denote

$$\mathcal{W}^{\chi}_{(p,q),(i,j)} = \left\{ (\alpha_s, \beta_s)_{0\le s\le\chi} \in (\mathbb{N}\times\mathbb{N})^{\chi+1} \;\middle|\; \substack{\alpha_s\le\alpha_{s+1},\,\beta_s\ge\beta_{s+1}\\ (\alpha_s,\beta_s)\ne(\alpha_{s+1},\beta_{s+1})\\ (\alpha_0,\beta_0)=(p,q)\\ (\alpha_\chi,\beta_\chi)=(i,j)} \right\}.$$

In other words, $\mathcal{W}^{\chi}_{(p,q),(i,j)}$ is the set of all paths of length χ on the square lattice $\mathbb{N}\times\mathbb{N}$ starting at (p, q), ending at (i, j) and going in the "south-west" direction.

Applying the recursive formula (28), we end up with the following result.

Lemma 4.19. *Let* $X = \big(x_{i,j}\big)_{1\le i,j\le n} \in \mathsf{Mat}_{n\times n}(\mathbb{C})$ *and* $\big\{\eta_{p,q}(X)\big\}_{1\le p,q\le n}$ *be such that the equality (27) is true. Then we have:*

$$\eta_{p,q}(X) = \sum_{\substack{p\le i\le n\\ 1\le j\le q}} x_{i,j}\left[\sum_{\chi=0}^{i-p+q-j}\ \sum_{\mathcal{W}^{\chi}_{(p,q),(i,j)}} \frac{(-1)^{\chi}}{\bar{\theta}_v(0)^{\chi+1}} \prod_{s=0}^{\chi-1} \big(F_{\alpha_{s+1},\beta_{s+1}}(0)\big)_{\alpha_s,\beta_s}\right], \tag{29}$$

where the third sum runs over all elements $\big(\alpha_s, \beta_s\big)_{0\le s\le\chi}$ *of* $\mathcal{W}^{\chi}_{(p,q),(i,j)}$.

Corollary 4.20. *Let $\{e_{a,b}\}_{1\le a,b\le n}$ be the standard basis of $\mathsf{Mat}_{n\times n}(\mathbb{C})$. If $a\ge p$ and $b\le q$ then we have:*

$$\eta_{p,q}\left(e_{a,b}\right)=\sum_{\chi=0}^{a-p+q-b}\;\sum_{(\alpha_s,\beta_s)_{0\le s\le\chi}\in\mathcal{W}^{\chi}_{(p,q),(a,b)}}\frac{(-1)^{\chi}}{\bar{\theta}_v(0)^{\chi+1}}\prod_{s=0}^{\chi-1}\left(F_{\alpha_{s+1},\beta_{s+1}}(0)\right)_{\alpha_s,\beta_s},$$

whereas in the remaining cases $\eta_{p,q}(e_{a,b})=0$. Hence, $\mathsf{res}_0^{-1}\left(e_{a,b}\right)=\gamma^{a,b}$, where

$$\gamma^{a,b}(z)=\sum_{\substack{1\le p\le a\\ b\le q\le n}}F_{p,q}(z)\left[\sum_{\chi=0}^{a-p+q-b}\;\sum_{(\alpha_s,\beta_s)_{0\le s\le\chi}\in\mathcal{W}^{\chi}_{(p,q),(a,b)}}\frac{(-1)^{\chi}}{\bar{\theta}_v(0)^{\chi+1}}\prod_{s=0}^{\chi-1}\left(F_{\alpha_{s+1},\beta_{s+1}}(0)\right)_{\alpha_s,\beta_s}\right].$$

Denote the matrix entries of $\left(\gamma^{a,b}(z)\right)_{c,d}$ by $\gamma^{a,b}_{c,d}(z)$. If $c<a$ or $d<b$ then $\gamma^{a,b}_{c,d}(z)=0$. On the other hand, if $a\ge c$ and $d\ge b$ then we get:

$$\gamma^{a,b}_{c,d}(z)=\sum_{\substack{c\le p\le a\\ b\le q\le d}}\left[\left(F_{p,q}(z)\right)_{c,d}\sum_{\chi=0}^{a-p+q-b}\;\sum_{\mathcal{W}^{\chi}_{(p,q),(a,b)}}\frac{(-1)^{\chi}}{\bar{\theta}_v(0)^{\chi+1}}\prod_{s=0}^{\chi-1}\left(F_{\alpha_{s+1},\beta_{s+1}}(0)\right)_{\alpha_s,\beta_s}\right].\tag{30}$$

Remark 4.21. In the formula (30) the function $\gamma^{a,b}_{c,d}$ depends on one variable z. However, from its definition it is clear that it also depends on the parameter $v\in\mathbb{C}\backslash\Lambda$. Hence, in what follows, we shall consider it as a function of two variables z and v.

Computation of $\tilde{r}_J(v,y):\mathsf{Mat}_{n\times n}(\mathbb{C})\to\mathsf{Mat}_{n\times n}(\mathbb{C})$. Recall that $\tilde{r}_B(v,y):\mathsf{Mat}_{n\times n}(\mathbb{C})\to\mathsf{Mat}_{n\times n}(\mathbb{C})$ is the composition $\mathsf{ev}_y\circ\mathsf{res}_0^{-1}$. The tensor $r_B(v,y)\in\mathsf{Mat}_{n\times n}(\mathbb{C})\otimes\mathsf{Mat}_{n\times n}(\mathbb{C})$ is the image of $\tilde{r}_B(v,y)$ under the canonical isomorphism

$$\mathsf{Lin}\big(\mathsf{Mat}_{n\times n}(\mathbb{C}),\mathsf{Mat}_{n\times n}(\mathbb{C})\big)\longrightarrow\mathsf{Mat}_{n\times n}(\mathbb{C})\otimes\mathsf{Mat}_{n\times n}(\mathbb{C}).$$

In the standard basis $\{e_{a,b}\}_{1\le a,b\le n}$ of $\mathsf{Mat}_{n\times n}(\mathbb{C})$ this map is given as follows:

$$\left(e_{a,b}\mapsto\sum_{1\le c,\,d\le n}\alpha^{a,b}_{c,d}\,e_{c,d}\right)\mapsto\sum_{1\le c,\,d\le n}\alpha^{a,b}_{c,d}\,e_{b,a}\otimes e_{c,d}.$$

Hence, the solution of the associative Yang–Baxter equation (2) attached to the Jordan block J is the following:

$$r_J(v,y)\;=\frac{1}{\bar{\theta}\left(\frac{1+\tau}{2}+y\right)}\sum_{\substack{0\le k\le n-1\\ 0\le l\le n-1}}\;\sum_{\substack{1\le i\le n-l\\ 1\le j\le n-k}}\gamma^{j+k,i}_{j,i+l}(v,y)\,e_{i,j+k}\otimes e_{j,i+l},\tag{31}$$

where $\gamma^{j+k,i}_{j,i+l}(v,y)$ are given by (30). Our next goal is to simplify this expression.

Definition 4.22. For any i,j,χ in $\mathbb{N}$, we set

$$\mathcal{W}^{\chi}_{(i,j)}=\left\{(\alpha_s,\beta_s)_{1\le s\le\chi}\in(\mathbb{N}\times\mathbb{N})^{\chi}\;\middle|\;(\alpha_s,\beta_s)\ne(0,0);\;\left(\sum_{s=1}^{\chi}\alpha_s,\sum_{s=1}^{\chi}\beta_s\right)=(i,j)\right\}.$$

We may think of $\mathcal{W}^{\chi}_{(i,j)}$ as the set paths of length χ in $\mathbb{N}\times\mathbb{N}$ starting at the point $(0,0)$, ending in (i,j) and going in the "nord-east" direction. Note that any path $\bigl\{(\alpha_s,\beta_s)\bigr\}_{0\le s\le\chi}$ from $\mathcal{W}^{\chi}_{(j+a,i+b),(j+k,i)}$ corresponds to the element $\bigl\{(\alpha_{s+1}-\alpha_s,\beta_s-\beta_{s+1})_s\bigr\}_{1\le s\le\chi}$ of $\mathcal{W}^{\chi}_{(k-a,b)}$. For the sake of simplicity, we shall use the same notation $\bigl\{(\alpha_s,\beta_s)\bigr\}_{0\le s\le\chi}$ for both elements.

In these notations, the formula (30) can be rewritten as follows:

$$\gamma^{j+k,i}_{j,i+l}(y)=\sum_{\substack{0\le a\le k\\0\le b\le l}}\left[\bigl(F_{j+a,i+b}(y)\bigr)_{j,i+l}\cdot\sum_{\chi=0}^{k-a+b}\sum_{\mathcal{W}^{\chi}_{(k-a,b)}}\frac{(-1)^{\chi}}{\bar{\theta}_v(0)^{\chi+1}}\prod_{s=1}^{\chi}\bigl(F_{\alpha_{s+1},\beta_{s+1}}(0)\bigr)_{\alpha_s,\beta_s}\right],$$

where the third sum is taken over all elements $\bigl(\alpha_s,\beta_s\bigr)_{1\le s\le\chi}$ of $\mathcal{W}^{\chi}_{(k-a,b)}$. Recall that $\bigl(F_{\alpha,\beta}\bigr)_{\gamma,\delta}=0$ if $\gamma>\alpha$ or $\beta>\delta$. For $\alpha\ge\gamma$ and $\delta\ge\beta$ we have: $\bigl(F_{\alpha,\beta}\bigr)_{\gamma,\delta}=U_{(n-(\alpha-\gamma),\delta-\beta+1)}$, where $\bigl(U(v,z)\bigr)_{\alpha,\beta}=e^t_{n(\alpha-1)+\beta}\exp(\nabla N)\bar{\theta}_v(z)\,e_{n(n-1)+1}$. Hence,

$$\gamma^{j+k,i}_{j,i+l}(v,y)=\sum_{\substack{0\le a\le k\\0\le b\le l}}\left[\bigl(U(v,y)\bigr)_{(n-a,l-b+1)}\cdot\sum_{\chi=0}^{k-a+b}\sum_{\mathcal{W}^{\chi}_{(k-a,b)}}\frac{(-1)^{\chi}}{\bar{\theta}_v(0)^{\chi+1}}\prod_{s=1}^{\chi}\bigl(U(0)\bigr)_{n-\alpha_s,\beta_s+1}\right]$$

Next, we need the following combinatorial lemma.

Lemma 4.23. *In the notations of Definition 4.4 we have the following results.*

(1) *For all $r\in\mathbb{N}$ we get*

$$N^r=\begin{pmatrix}N^{(r)}_0 & N^{(r)}_1 & \cdots & N^{(r)}_{n-1}\\ 0 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & N^{(r)}_1\\ 0 & \cdots & 0 & N^{(r)}_0\end{pmatrix},$$

where each $N^{(r)}_i$, $0\le i\le n-1$ is the matrix of size $n\times n$ given by

$$N^{(r)}_i=\sum_{k=0}^{r}\binom{r}{k}A_0^{r-k}\sum_{s\in S^k_i}\prod_{l=0}^{k-1}A_{s_l-s_{l+1}}$$

where for all $k,i\in\mathbb{N}$ we denote

$$S^k_i=\Bigl\{(s_j)_{0\le j\le k}\in\mathbb{N}^{k+1}\Bigm| s_0=i,\,s_k=0,\,s_{j+1}<s_j,\,0\le j\le k\Bigr\}.$$

(2) *For any $i,j,\chi\in\mathbb{N}$, $(u_s)_{1\le s\le\chi}\in[0,2n-1]^{\chi}$ and $u=\sum_{s=1}^{\chi}u_s$ holds:*

$$\sum_{(\alpha_s,\beta_s)_{1\le s\le\chi}\in\mathcal{W}^{\chi}_{(i,j)}}\prod_{s=1}^{\chi}e^t_{\beta_s+1}N^{(u_s)}_{\alpha_s}e_1=e^t_{j+1}N^{(u)}_i e_1.$$

The proof of this lemma is based on some elementary but tedious combinatorics, and is therefore omitted.

From Lemma 4.23(1) it follows that for all $0 \le \alpha < n$, $1 \le \beta \le n$ we have:

$$\bigl(U(v,z)\bigr)_{n-\alpha,\beta} = e^t_{n(n-\alpha-1)+\beta} \exp(\nabla_z N)\bar{\theta}_v(z)\, e_{n(n-1)+1} = \sum_{r=0}^{2n-1} e^t_\beta\, N^{(r)}_\alpha\, e_1 \,\frac{\nabla^r_z\bigl(\bar{\theta}_v(z)\bigr)}{r!}.$$

Recall that $\bar{\theta}_v(z) = \bar{\theta}(z+v+\frac{1+\tau}{2})$. Note that we have:

$$\nabla^r_z\bigl(\bar{\theta}_v(z)\bigr) = \nabla^r_v\bigl(\bar{\theta}_v(z)\bigr) = \left(\frac{\partial}{\partial v}\right)^r \bar{\theta}\left(z + \frac{\tau+1}{2} + v\right).$$

Therefore, we can rewrite the expression for $\gamma^{j+k,i}_{j,i+l}(v,y)$ as follows:

$$\gamma^{j+k,i}_{j,i+l}(v,y) = \sum_{\substack{0\le a\le k\\ 0\le b\le l}} \left(\left(\sum_{r=0}^{2n-1} e^t_{l-b+1}\, N^{(r)}_a\, e_1\, \frac{\nabla^r_v\bigl(\bar{\theta}_v(y)\bigr)}{r!}\right)\cdot\right.$$
$$\left.\cdot\sum_{\chi=0}^{k-a+b}\ \sum_{(\alpha_s,\beta_s)_{1\le s\le \chi}\in \mathcal{W}^{\chi}_{(k-a,b)}} \frac{(-1)^\chi}{\bar{\theta}_v(0)^{\chi+1}} \prod_{s=1}^{\chi}\left(\sum_{r_s=0}^{2n-1} e^t_{\beta_s+1}\, N^{(r_s)}_{\alpha_s}\, e_1\, \frac{\nabla^{r_s}_v\bigl(\bar{\theta}_v(0)\bigr)}{r_s!}\right)\right). \tag{32}$$

Next, we need the following generalization of the Leibniz formula.

Lemma 4.24. *Let f, g be any meromorphic functions on $\mathbb{C}$ and $\nabla = -\frac{1}{2\pi i}\frac{d}{dz}$. Then in the notations of Definition 4.6 the following formula is true:*

$$\nabla_{k,l}\Bigl(\frac{f}{g}\Bigr) = \sum_{\substack{0\le a\le k\\ 0\le b\le l}} \Bigl[\Bigl(\sum_{r=0}^{2n-1} e^t_{l-b+1}\, N^{(r)}_a\, e_1\, \frac{\nabla^r(f)}{r!}\Bigr)\cdot\Bigl(\sum_{\chi=0}^{k-a+b}\ \sum_{(\alpha_s,\beta_s)_{1\le s\le\chi}\in\mathcal{W}^{\chi}_{(k-a,b)}} \frac{(-1)^\chi}{g^{\chi+1}}\cdot$$
$$\cdot\prod_{s=1}^{\chi}\sum_{r_s=0}^{2n-1} e^t_{\beta_s+1}\, N^{(r_s)}_{\alpha_s}\, e_1\, \frac{\nabla^{r_s}(g)}{r_s!}\Bigr)\Bigr].$$

The proof of this lemma is based on some elementary combinatorics and the second part of Lemma 4.23. We leave it to an interested reader as an exercise. □

Applying Lemma 4.24 to Equality (32), we finally get:

$$\frac{1}{\bar{\theta}\left(\frac{1+\tau}{2}+y\right)}\gamma^{j+k,i}_{j,i+l}(v,y) = \bigl(\nabla_{k,l}\bigr)_v\left(\frac{\bar{\theta}(y+\frac{\tau+1}{2}+v)}{\bar{\theta}\left(\frac{1+\tau}{2}+y\right)\cdot\bar{\theta}(\frac{\tau+1}{2}+v)}\right).$$

Recall that the first and third theta functions θ and $\bar{\theta}$ are related by the equality $\bar{\theta}\left(z+\frac{1+\tau}{2}\right) = i\,q(z)\,\theta(z)$, where $q(z) = \exp(-\pi i(z+\frac{\tau}{4}))$. Thus, up to the constant $\frac{\exp(\pi i\frac{\tau}{4})}{i\theta'(0)}$, the coefficient of the tensor $e_{i,j+k}\otimes e_{j,i+l}$ in the expansion (31) is $(\nabla_{k,l})_v\bigl(\sigma(v,y)\bigr)$. This finishes the proof of Theorem 4.7.

Remark 4.25. The algorithm from Section 2 assigning to a matrix $B \in \mathsf{GL}_n(\mathbb{C})$ and a complex torus E a solution of the associative Yang–Baxter can be generalized to the case when E is a singular Weierstraß cubic curve. In this case, one can use a description of semi–stable vector bundles on E following the approach of [4], see also [6]. However, all solutions produced in this way turn out to be degenerations of the constructed elliptic solutions, where we replace the Kronecker function $\sigma(u, x)$ by its trigonometric or rational degenerations $\cot(u) + \cot(x)$ or $\dfrac{1}{u} + \dfrac{1}{x}$.

References

[1] M. Aguiar, *On the associative analog of Lie bialgebras*, J. Algebra **244** (2001), no. **2**, 492–532.

[2] M. Atiyah, *Vector bundles over an elliptic curve*, Proc. Lond. Math. Soc. (3) **7** (1957) 414–452.

[3] C. Birkenhake, H. Lange, *Complex abelian varieties*, Grundlehren der Mathematischen Wissenschaften **302** Springer, Berlin, 2004.

[4] L. Bodnarchuk, I. Burban, Yu. Drozd, G.-M. Greuel, *Vector bundles and torsion free sheaves on degenerations of elliptic curves*, Global aspects of complex geometry, 83–128, Springer, Berlin, 2006.

[5] V. Buchstaber, G. Felder, A. Veselov, *Elliptic Dunkl operators, root systems and functional equations*, Duke Math. J. **76** (1994), no. **3**, 885–911.

[6] I. Burban, B. Kreußler, *Vector bundles on degenerations of elliptic curves and Yang–Baxter equations*, arXiv:0708.1685.

[7] S. Fomin, A. Kirillov, *Quadratic algebras, Dunkl elements and Schubert calculus*, Advances in geometry, 147–182, Progr. Math., **172**, Birkhäuser Boston, 1999.

[8] O. Iena, *Vector bundles on elliptic curves and factors of automorphy*, arXiv:1009.3230.

[9] A. Kirillov, *Quadratic algebras-II*, preprint in preparation.

[10] D. Mumford, *Tata lectures on theta I*, Progress in Math. **28**, Birkhäuser Boston, 1983.

[11] A. Odesskii, V. Sokolov, *Pairs of compatible associative algebras, classical Yang–Baxter equation and quiver representations*, Comm. Math. Phys. **278** (2008), no. **1**, 83–99.

[12] A. Polishchuk, *Classical Yang–Baxter equation and the A_∞–constraint,* Adv. Math. **168** (2002), no. **1**, 56–95.

[13] A. Polishchuk, *Massey products on cycles of projective lines and trigonometric solutions of the Yang–Baxter equations*, in “Algebra, arithmetic, and geometry: in honor of Yu. I. Manin”. Vol. II, 573–617, Progr. Math. **270**, Birkhäuser, 2009.

[14] A. Polishchuk, E. Zaslow, *Categorical mirror symmetry: the elliptic curve*, Adv. Theor. Math. Phys. **2** (1998), no. **2**, 443-470.

[15] A. Weil, *Elliptic functions according to Eisenstein and Kronecker*, Ergebnisse der Mathematik und ihrer Grenzgebiete **88**, Springer, Berlin, 1976.

Mathematisches Institut, Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany

E-mail address: `burban@math.uni-bonn.de`

Mathematisches Institut, Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany

E-mail address: `henrich@math.uni-bonn.de`