COHEN–MACAULAY MODULES OVER SOME NON–REDUCED CURVE SINGULARITIES

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ABSTRACT. In this article, we study Cohen–Macaulay modules over non–reduced curve singularities. We prove that the rings $\mathbb{k}[\![x,y,z]\!]/(xy, y^q - z^2)$ have tame Cohen–Macaulay representation type. For the singularity $\mathbb{k}[\![x,y,z]\!]/(xy, z^2)$ we give an explicit description of all indecomposable Cohen–Macaulay modules and apply the obtained classification to construct families of indecomposable matrix factorizations of $x^2y^2 \in \mathbb{k}[\![x,y]\!]$.

INTRODUCTION

Cohen–Macaulay modules over Cohen–Macaulay rings have been intensively studied in recent years. They appear in the literature in various incarnations like matrix factorizations, objects of the triangulated category of singularities or lattices over orders.

Our interest in Cohen–Macaulay modules is representation theoretic. In the case of a *reduced* curve singularity, the behavior of the representation type of the category of Cohen–Macaulay modules $\mathsf{CM}(\mathsf{A})$ is completely understood. Assume, for simplicity, that A is an algebra over an algebraically closed field $\mathbb{k}$ of characteristic zero.

- According to Drozd and Roiter [17], Jacobinski [21] and Greuel and Knörrer [19], $\mathsf{CM}(\mathsf{A})$ is representation finite if and only if A dominates a simple curve singularity. See also the expositions in the monographs [25] and [28].
- Drozd and Greuel have proven in [15] that if $\mathsf{CM}(\mathsf{A})$ is tame then A dominates a singularity of type

$$\mathsf{T}_{pq}(\lambda) = \mathbb{k}[\![x,y]\!]/(x^{p-2} - y^2)(x^2 - \lambda y^{q-2}) \qquad \begin{cases} \frac{1}{p} + \frac{1}{q} = \frac{1}{2}, & \lambda \in \mathbb{k}\backslash\{0,1\}, \\ \frac{1}{p} + \frac{1}{q} < \frac{1}{2}, & \lambda = 1. \end{cases}$$

  In particular, they have shown that the singularities

$$\mathsf{P}_{pq} = \mathbb{k}[\![x,y,z]\!]/(xy, x^p + y^q - z^2), \qquad \text{where } p, q \in \mathbb{N}_{\geqslant 2},$$

  are Cohen–Macaulay tame.
- A reduced curve singularity which neither dominates a simple nor a $\mathsf{T}_{pq}(\lambda)$ singularity has wild Cohen–Macaulay representation type [14].
- There are also other approaches to establish tameness of $\mathsf{CM}\big(\mathsf{T}_{pq}(\lambda)\big)$: one using the generalized geometric McKay Correspondence [22, 16] and another via cluster–tilting theory [12].

The definition of Cohen–Macaulay representation type of a Cohen–Macaulay curve singularity (finite, discrete, tame or wild) is recalled in Section 4. Concerning the case of non–reduced curve singularities, the following results are known so far.

- By a theorem of Auslander [2], a non–reduced Cohen–Macaulay curve singularity always has infinite Cohen–Macaulay representation type.
- Buchweitz, Greuel and Schreyer have shown in [9] that the singularities $\mathsf{A}_\infty = \mathbb{k}[\![x,y]\!]/(y^2)$ and $\mathsf{D}_\infty = \mathbb{k}[\![x,y]\!]/(xy^2)$ have discrete Cohen–Macaulay representation type.

- Leuschke and Wiegand have proven in [26] that A_∞, D_∞ and $\Bbbk[\![x,y,z]\!]/(xy,yz,z^2)$ are the only Cohen–Macaulay curve singularities of bounded but infinite Cohen–Macaulay representation type. Here, "boundedness" means existence of an upper bound for the minimal number of generators of an indecomposable Cohen–Macaulay module.
- Burban and Drozd have proven in [11] that the singularities $\mathsf{T}_{\infty q} = \Bbbk[\![x,y]\!]/(x^2y^2 - y^q)$, where $q \in \mathbb{N}_{\geqslant 3}$, (respectively $\mathsf{T}_{\infty\infty} = \Bbbk[\![x,y]\!]/(x^2y^2)$) are Cohen–Macaulay tame (under the additional assumption that $\mathsf{char}(\Bbbk) = 0$, respectively $\mathsf{char}(\Bbbk) \neq 2$). However, an explicit description of the corresponding indecomposable matrix factorizations is still not known.

In this article, we obtain the following results.

1. First, we prove (see Theorem 2.1 and Remark 2.4) that the curve singularities

$$\mathsf{P}_{\infty q} = \Bbbk[\![x,y,z]\!]/(xy, y^q - z^2) \quad \text{and} \quad \mathsf{P}_{\infty\infty} = \Bbbk[\![x,y,z]\!]/(xy, z^2)$$

are Cohen–Macaulay tame for any algebraically closed field $\Bbbk$ of any characteristic (in the case $\mathsf{char}(\Bbbk) = 2$ the definition of $\mathsf{P}_{\infty q}$ has to be modified; see Remark 2.3). The method of the proof extends the approach of Drozd and Greuel [15] to the case of non–reduced curve singularities and is based on Bondarenko's work on representations of *bunches of semi–chains* [5]. Our approach can be summarized by the following diagram of categories and functors:

$$\mathsf{CM}(\mathsf{P}) \xleftarrow{\;\mathbb{I}\;} \mathsf{CM}(\mathsf{R}) \underset{\mathbb{G}}{\overset{\mathbb{F}}{\rightleftarrows}} \mathsf{Tri}(\mathsf{R}) \xrightarrow{\;\mathbb{P}\;} \mathsf{MP}(\mathsf{R}).$$

We start with a singularity $\mathsf{P} = \mathsf{P}_{\infty q}$ or $\mathsf{P}_{\infty\infty}$ and replace it by its *minimal overring* R. The forgetful functor $\mathbb{I}$ embeds $\mathsf{CM}(\mathsf{R})$ into $\mathsf{CM}(\mathsf{P})$ as a full subcategory. By a result of Bass [3], the "difference" between $\mathsf{CM}(\mathsf{R})$ and $\mathsf{CM}(\mathsf{P})$ is very small. The *category of triples* $\mathsf{Tri}(\mathsf{R})$ plays a key role in our approach. According to [11], the functors $\mathbb{F}$ and $\mathbb{G}$ are quasi–inverse equivalences of categories. Finally, $\mathsf{MP}(\mathsf{R})$ is a certain *bimodule category* in the sense of [13]. The functor $\mathbb{P}$ reflects isomorphism classes and indecomposability of objects. We prove that $\mathsf{MP}(\mathsf{R})$ is the category of representations of a certain *bunch of semi–chains*. According to a theorem of Bondarenko [5], $\mathsf{MP}(\mathsf{R})$ is representation tame. This implies representation tameness of $\mathsf{CM}(\mathsf{P})$.

2. Next, we show how to pass from canonical forms describing indecomposable objects of $\mathsf{MP}(\mathsf{R})$ to a concrete description of the corresponding indecomposable Cohen–Macaulay P–modules. We illustrate this technique by giving a full and explicit description of the indecomposable Cohen–Macaulay modules over $\mathsf{P}_{\infty\infty}$. They are described in terms of quite transparent combinatorial data: *bands* and *strings* (see Theorem 3.28). The obtained classification turns out to be perfectly adapted to separate those Cohen–Macaulay modules over $\mathsf{P}_{\infty\infty}$, which are *locally free on the punctured spectrum* from those which are not (see Remark 3.26).

3. At last, we construct explicit families of indecomposable matrix factorizations of $x^2y^2 \in \Bbbk[\![x,y]\!]$. In this context, there is the following diagram of categories and functors:

$$\mathsf{CM}(\mathsf{R}) \xhookrightarrow{\;\mathbb{J}\;} \mathsf{CM}(\mathsf{T}) \longrightarrow \underline{\mathsf{MF}}(x^2y^2).$$

Here, R is the minimal overring of $\mathsf{P}_{\infty\infty}$, the functor $\mathbb{J}$ is a fully faithful embedding, $\mathsf{T} = \Bbbk[\![x,y]\!]/(x^2y^2)$ and $\underline{\mathsf{MF}}(x^2y^2)$ is the homotopy category of matrix factorizations of x^2y^2 (which is equivalent to the stable category $\underline{\mathsf{CM}}(\mathsf{T})$ by a result of Eisenbud [18]).

Results of this article provide a partial classification of the indecomposable objects of $\underline{\mathsf{MF}}(x^2y^2)$ as well as an equivalent category $\underline{\mathsf{MF}}(x^2y^2+uv)$.

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1. Survey on Cohen–Macaulay modules over curve singularities

In this section we collect definitions and some known facts on Cohen–Macaulay modules over curve singularities. The proofs of the mentioned statements can be found in the monographs [7, 25, 28]; see also the survey article [10].

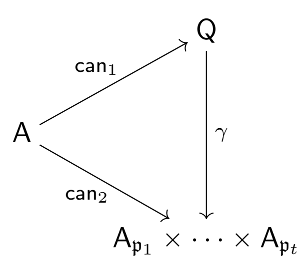
1.1. **Definitions and basic properties.** Let $(\mathsf{A}, \mathfrak{m})$ be a local Noetherian ring of Krull dimension one (a curve singularity), $\mathbb{k} = \mathsf{A}/\mathfrak{m}$ its residue field and $\mathsf{Q} = \mathsf{Q}(\mathsf{A})$ its total ring of fractions.

Definition 1.1. A curve singularity A is

- *Cohen–Macaulay* if and only if $\mathsf{Hom}_{\mathsf{A}}(\mathbb{k}, \mathsf{A}) = 0$ (equivalently, A contains a regular element).
- *Gorenstein* if and only if it is Cohen–Macaulay and $\mathsf{Ext}^1_A(\mathbb{k}, \mathsf{A}) \cong \mathbb{k}$ (equivalently, $\mathsf{inj.dim}_{\mathsf{A}}(\mathsf{A}) = 1$).

Note that a *reduced* curve singularity is automatically Cohen–Macaulay. However, in this article we mainly focus on non–reduced ones.

Lemma 1.2. *Let A be a Cohen–Macaulay curve singularity. Then Q is an Artinian ring. Moreover, if $\{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$ is the set of minimal prime ideals of A then there exists a ring isomorphism $\mathsf{Q} \xrightarrow{\gamma} \mathsf{A}_{\mathfrak{p}_1} \times \dots \times \mathsf{A}_{\mathfrak{p}_t}$ making the following diagram*



commutative, where can_1 and can_2 are canonical morphisms.

Proof. Since A is Cohen–Macaulay, the set of associated prime ideals of A coincides with $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Therefore, the set of zero divisors in A is $\mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_n$. By [6, Chapter IV, Proposition 2.5.10] Q is Artinian and its maximal ideals are $\mathfrak{p}_1\mathsf{Q}, \dots, \mathfrak{p}_n\mathsf{Q}$. Hence, $\mathsf{Q} \cong \mathsf{Q}_{\mathfrak{p}_1\mathsf{Q}} \times \dots \times \mathsf{Q}_{\mathfrak{p}_n\mathsf{Q}}$. Since $\mathsf{Q}_{\mathfrak{p}_i\mathsf{Q}} = \mathsf{A}_{\mathfrak{p}_i}$ for $1 \leqslant i \leqslant n$, the result follows. □

Definition 1.3. For an A–module M we set

$$\Gamma_{\mathfrak{m}}(M) := \left\{ x \in M \;\middle|\; \mathfrak{m}^t x = 0 \text{ for some } t \in \mathbb{N} \right\}.$$

The following result can be easily deduced from Lemma 1.2.

Lemma 1.4. *Let* A *be a Cohen-Macaulay curve singularity. For a Noetherian* A*–module* M *we have:*

$$\Gamma_{\mathfrak{m}}(M) = \mathsf{ker}\bigl(M \longrightarrow \mathsf{Q} \otimes_{\mathsf{A}} M\bigr) =: \mathsf{tor}(M).$$

Moreover, the following statements are equivalent:

- $\mathsf{Hom}_{\mathsf{A}}(\Bbbk, M) = 0$.
- M *is* torsion free*, i.e.* $\mathsf{tor}(M) = 0$.

Definition 1.5. A Noetherian module M satisfying the conditions of Lemma 1.4 is called *maximal Cohen–Macaulay*. In what follows we just say that M is Cohen–Macaulay. In this case, the Q–module $M_{\mathsf{Q}} = \mathsf{Q} \otimes_{\mathsf{A}} M$ is called the *rational envelope* of M. More generally, a Noetherian module N over a Noetherian ring S (say, of Krull dimension one) is (maximal) Cohen–Macaulay if for any maximal ideal $\mathfrak{n}$ in S the localization $N_{\mathfrak{n}}$ is Cohen–Macaulay over $\mathsf{S}_{\mathfrak{n}}$. In what follows, $\mathsf{CM}(\mathsf{S})$ denotes the category of Cohen–Macaulay S–modules.

Lemma 1.6. *Assume that a Cohen–Macaulay curve singularity* A *is Gorenstein in codimension zero (i.e.* Q *is self–injective). Then for a Noetherian* A*–module* M*, the following conditions are equivalent:*

- M *is Cohen–Macaulay.*
- M *embeds into a finitely generated free* A*–module.*

A proof of this Lemma can be found in [25, Appendix A, Corollary 15].

Remark 1.7. The statement of Lemma 1.6 is not true for an arbitrary Cohen–Macaulay curve singularity. For example, let $\mathsf{A} = \Bbbk[\![x, y, z]\!]/(x^2, xy, y^2)$ and K be a canonical A–module. Then K does not embed into a free A–module.

Definition 1.8. A ring R is an *overring* of A if $\mathsf{A} \subseteq \mathsf{R} \subset \mathsf{Q}$ and the ring extension $\mathsf{A} \subseteq \mathsf{R}$ is finite. We also say that R *birationally dominates* A.

Proposition 1.9. *Let* A *be a Cohen–Macaulay curve singularity and* R *an overring of* A*. Then the following results are true.*

- R *is Cohen–Macaulay.*
- *We have an adjoint pair* $\bigl(\mathsf{R} \boxtimes_{\mathsf{A}} -, \mathbb{I}(-)\bigr)$*, where* $\mathbb{I}\colon \mathsf{CM}(\mathsf{R}) \longrightarrow \mathsf{CM}(\mathsf{A})$ *is the restriction (or forgetful) functor and* $R \boxtimes_{\mathsf{A}} - : \mathsf{CM}(\mathsf{A}) \longrightarrow \mathsf{CM}(\mathsf{R})$ *sends a Cohen–Macaulay module* M *to* $\mathsf{R} \otimes_{\mathsf{A}} M/\mathsf{tor}(\mathsf{R} \otimes_{\mathsf{A}} M)$.
- $\mathbb{I}$ *is fully faithful.*
- *If* $M = \bigl\langle w_1, \dots, w_t \bigr\rangle_{\mathsf{A}} \subset \mathsf{Q}^n$*, then* $\mathsf{R} \boxtimes_{\mathsf{A}} M \cong \mathsf{R} \cdot M := \bigl\langle w_1, \dots, w_t \bigr\rangle_{\mathsf{R}} \subset \mathsf{Q}^n$.

Proof. The first statement follows from the fact that $\mathsf{depth}_{\mathsf{A}}(\mathsf{R}) = 1 = \mathsf{depth}_{\mathsf{R}}(\mathsf{R})$. The second result follows from the functorial isomorphisms

$$\mathsf{Hom}_{\mathsf{R}}\bigl(\mathsf{R} \boxtimes_{\mathsf{A}} M, N\bigr) \cong \mathsf{Hom}_{\mathsf{R}}\bigl(\mathsf{R} \otimes_{\mathsf{A}} M, N\bigr) \cong \mathsf{Hom}_{\mathsf{A}}\bigl(M, \mathbb{I}(N)\bigr).$$

For a proof of the third statement, see for example [25, Lemma 4.14]. The fourth result follows from the fact that the kernel of $\mathsf{R} \otimes_{\mathsf{A}} M \longrightarrow \mathsf{R} \cdot M$ is $\mathsf{tor}(\mathsf{R} \otimes_{\mathsf{A}} M)$. □

Corollary 1.10. *Let* A *be a Cohen–Macaulay curve singularity and* R *be an overring of* A*. Then the following statements are true.*

- *Let* N_1 *and* N_2 *be Cohen–Macaulay* R*–modules. Then* $N_1 \cong N_2$ *if and only if* $\mathbb{I}(N_1) \cong \mathbb{I}(N_2)$ *in* $\mathsf{CM}(\mathsf{A})$.
- *A Cohen–Macaulay* R*–module* N *is indecomposable if and only if* N *is indecomposable viewed as an* A*–module.*

The following result is due to Bass [3, Proposition 7.2] (see also [25, Lemma 4.9]).

Theorem 1.11. *Let* $(\mathsf{A}, \mathfrak{m})$ *be a Gorenstein curve singularity and let* $\mathsf{R} = \mathrm{End}_{\mathsf{A}}(\mathfrak{m})$*. Then the following results are true.*

- $\mathsf{R} \cong \{r \in \mathsf{Q} \mid r\,\mathfrak{m} \subseteq \mathfrak{m}\}$*. In particular,* R *is an overring of* A*.*
- *If* A *is not regular, we have an exact sequence of* A*–modules*

$$0 \longrightarrow \mathsf{A} \xrightarrow{\imath} \mathsf{R} \longrightarrow \mathbb{k} \longrightarrow 0,$$

  where $\imath$ *is the canonical inclusion. This short exact sequence defines a generator of the* A*–module* $\mathsf{Ext}^1_{\mathsf{A}}(\mathbb{k}, \mathsf{A}) \cong \mathbb{k}$*.*
- *In the latter case, let* S *be any other proper overring of* A*. Then* S *contains* R*. In other words,* R *is the* minimal overring *of the curve singularity* A*.*
- *Let* M *be a Cohen–Macaulay* A*–module without free direct summands. Then there exists a Cohen–Macaulay* R*–module* N *such that* $M = \mathbb{I}(N)$*.*

Remark 1.12. Theorem 1.11 gives a precise measure of the representation theoretic difference between the categories $\mathsf{CM}(\mathsf{A})$ and $\mathsf{CM}(\mathsf{R})$. Namely, an indecomposable Cohen–Macaulay A–module M is either regular or is the restriction of an indecomposable Cohen–Macaulay R–module. In more concrete terms, assume that $M = \langle w_1, \dots, w_t \rangle_{\mathsf{A}} \subset \mathsf{Q}^n$ contains no free direct summands (according to Lemma 1.6, any Cohen–Macaulay A–module admits such embedding). Then $M = \langle w_1, \dots, w_t \rangle_{\mathsf{R}}$.

Proposition 1.13. *In the situation of Theorem 1.11, assume that* $N = \langle w_1, \dots, w_t \rangle_{\mathsf{R}} \subset \mathsf{Q}^n$ *is an indecomposable Cohen–Macaulay* R*–module. Then either* $N \cong \mathsf{R}$ *or* $N = \langle w_1, \dots, w_t \rangle_{\mathsf{A}}$*.*

Proof. Put $M := \langle w_1, \dots, w_t \rangle_{\mathsf{A}}$. Obviously, we have: $N = \mathsf{R} \cdot M$. If M contains a free direct summand, i.e. $M \cong M' \oplus \mathsf{A}$ then $N = \mathsf{R} \cdot M \cong \mathsf{R} \cdot M' \oplus \mathsf{R}$. As N is assumed to be indecomposable, $N \cong \mathsf{R}$. If M has no free direct summands, then by Theorem 1.11 and Remark 1.12 we have: $\mathsf{R} \cdot M = M$. □

Definition 1.14. A Cohen–Macaulay A–module M is *locally free on the punctured spectrum of* A if for any minimal prime ideal $\mathfrak{p}$ in A the localization $M_{\mathfrak{p}}$ is free over $\mathsf{A}_{\mathfrak{p}}$.

Remark 1.15. According to Lemma 1.2, a Cohen–Macaulay A–module M is locally free on the punctured spectrum if and only if its rational envelope M_{Q} is projective over Q.

In what follows, $\mathsf{CM}^{\mathsf{lf}}(\mathsf{A})$ denotes the category of Cohen–Macaulay A–modules which are locally free on the punctured spectrum.

1.2. **Category of triples.** Let $(\mathsf{R}, \mathfrak{m})$ be a Cohen–Macaulay curve singularity, S an overring of R and $I = \mathsf{ann}_{\mathsf{R}}(\mathsf{S}/\mathsf{R})$ the corresponding *conductor ideal*. The next result is straightforward; see for example [11, Lemma 2.1].

Lemma 1.16. *The following statements are true.*

- $I = I\mathsf{R} = I\mathsf{S}$*. In other words,* I *is an ideal both in* R *and in* S*. Moreover,* I *is the biggest ideal having this property.*
- *The rings* $\overline{\mathsf{R}} = \mathsf{R}/I$ *and* $\overline{\mathsf{S}} = \mathsf{S}/I$ *are Artinian.*

For a Cohen–Macaulay R–module M we denote

- $\widetilde{M} := \mathsf{S} \boxtimes_{\mathsf{R}} M \in \mathsf{CM}(\mathsf{S})$.
- $\overline{M} := \overline{\mathsf{R}} \otimes_{\mathsf{R}} M \in \overline{\mathsf{R}}\text{–}\mathsf{mod}$.

- $\breve{M} := \overline{\mathsf{S}} \otimes_{\mathsf{S}} \widetilde{M} \in \overline{\mathsf{S}}$–mod.

Then the following result is true; see [11, Lemma 12.2].

Lemma 1.17. *The canonical map* $\theta_M : \overline{\mathsf{S}} \otimes_{\overline{\mathsf{R}}} \overline{M} \longrightarrow \breve{M}$ *is surjective and its adjoint map* $\widetilde{\theta}_M : \overline{M} \longrightarrow \breve{M}$ *is injective.*

Definition 1.18. Consider the following diagram of categories and functors:

$$\mathsf{CM}(\mathsf{S}) \xrightarrow{\overline{\mathsf{S}}\otimes_{\mathsf{S}} -} \overline{\mathsf{S}} - \mathsf{mod} \xleftarrow{\overline{\mathsf{S}}\otimes_{\overline{\mathsf{R}}} -} \overline{\mathsf{R}} - \mathsf{mod}.$$

According to [27, Section II.6], the corresponding *comma category* $\mathsf{Comma}(\mathsf{S}\,|\,\overline{\mathsf{R}})$ is defined as follows. Its objects are triples (N, V, θ), where

- N is a maximal Cohen–Macaulay S–module,
- V is a Noetherian $\overline{\mathsf{R}}$–module,
- $\theta : \overline{\mathsf{S}} \otimes_{\overline{\mathsf{R}}} V \to \overline{\mathsf{S}} \otimes_{\mathsf{S}} N$ is a morphism of $\overline{\mathsf{S}}$–modules (called *gluing map*).

A morphism between two triples (N, V, θ) and (N', V', θ') is given by a pair (ψ, φ), where

- $\psi : N \to N'$ is a morphism of S–modules and
- $\varphi : V \to V'$ is a morphism of $\overline{\mathsf{R}}$–modules

such that the following diagram of $\overline{\mathsf{S}}$–modules is commutative:

$$\begin{array}{ccc} \overline{\mathsf{S}} \otimes_{\overline{\mathsf{R}}} V & \xrightarrow{\theta} & \overline{\mathsf{S}} \otimes_{\mathsf{S}} N \\ \Big\downarrow{\scriptstyle \mathbb{1}\otimes\varphi} & & \Big\downarrow{\scriptstyle \mathbb{1}\otimes\psi} \\ \overline{\mathsf{S}} \otimes_{\overline{\mathsf{R}}} V' & \xrightarrow{\theta'} & \overline{\mathsf{S}} \otimes_{\mathsf{S}} N'. \end{array}$$

The *category of triples* $\mathsf{Tri}(\mathsf{R})$ is a full subcategory of $\mathsf{Comma}(\mathsf{S}\,|\,\overline{\mathsf{R}})$ consisting of those triples (N, V, θ) for which

- the gluing map θ is an epimorphism and
- the adjoint morphism of $\overline{\mathsf{S}}$–modules $\widetilde{\theta} : V \to \overline{\mathsf{S}} \otimes_{\mathsf{S}} N$ given as the composition $V \to \overline{\mathsf{S}} \otimes_{\overline{\mathsf{R}}} V \xrightarrow{\theta} \overline{\mathsf{S}} \otimes_{\mathsf{S}} N$ is a monomorphism.

Definition 1.18 is motivated by the following theorem; see [11, Theorem 2.5].

Theorem 1.19. *The functor* $\mathbb{F} : \mathsf{CM}(\mathsf{R}) \longrightarrow \mathsf{Tri}(\mathsf{R})$ *mapping a maximal Cohen–Macaulay module* M *to the triple* $\big(\widetilde{M}, \overline{M}, \theta_M\big)$*, is well–defined and is an equivalence of categories. A quasi–inverse functor* $\mathbb{G} : \mathsf{Tri}(\mathsf{R}) \longrightarrow \mathsf{CM}(\mathsf{R})$ *is defined as follows. Let* $T = (N, V, \theta)$ *be an object of* $\mathsf{Tri}(\mathsf{R})$*. Then* $M' = \mathbb{G}(T) := \pi^{-1}\big(\mathrm{Im}(\widetilde{\theta})\big) \subseteq N$*, where* $\pi : N \to \breve{N} := N/IN$ *is the canonical projection. In other words, we have the following commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & IN & \longrightarrow & M' & \longrightarrow & V & \longrightarrow & 0 \\ & & \Big\| & & \Big\downarrow & & \Big\downarrow{\scriptstyle \widetilde{\theta}} & & \\ 0 & \longrightarrow & IN & \longrightarrow & N & \xrightarrow{\pi} & \breve{N} & \longrightarrow & 0 \end{array} \tag{1.1}$$

in the category of R*–modules.*

In many cases, Theorem 1.19 provides an efficient tool to reduce the classification of indecomposable objects of $\mathsf{CM}(\mathsf{R})$ to a certain problem of linear algebra (a matrix problem).

Remark 1.20. There are several variations of the construction appearing in Theorem 1.19; see [11, Chapter 2] for an account of them.

1.3. **Cohen–Macaulay modules over simple curve singularities of type A.** Let $\Bbbk$ be an algebraically closed field. For simplicity, let us additionally assume that $\mathsf{char}(\Bbbk) \neq 2$ (see however Remark 1.22). For any $m \in \mathbb{N}$, denote

$$\mathsf{S} = \mathsf{A}_m := \Bbbk[\![x, u]\!]/(x^{m+1} - u^2) \tag{1.2}$$

the corresponding simple curve singularity of type A_m. The following is essentially due to Bass [3] (see also [25, 28]).

Theorem 1.21. *The indecomposable Cohen–Macaulay S–modules have the following description.*

- *Assume $m = 2n$, $n \in \mathbb{N}$. For any $1 \leqslant l \leqslant n$ consider the ideal $X_l := (x^l, u)$. Then $X_0 = (1) = \mathsf{S}, X_1, \ldots, X_n$ is the complete list of indecomposable objects of $\mathsf{CM}(\mathsf{S})$. Moreover, the Auslander–Reiten quiver of $\mathsf{CM}(\mathsf{S})$ has the form*

$$X_0 \underset{\iota}{\overset{\cdot x}{\rightleftarrows}} X_1 \underset{\iota}{\overset{\cdot x}{\rightleftarrows}} X_2 \underset{\iota}{\overset{\cdot x}{\rightleftarrows}} \cdots \underset{\iota}{\overset{\cdot x}{\rightleftarrows}} X_n \circlearrowright \pi \tag{1.3}$$

  Here, $\imath$ denotes the inclusion of ideals and $x\cdot$ is the multiplication by x. The endomorphism $\pi \in \mathrm{End}_{\mathsf{S}}(X_n)$ is defined as follows: $\pi(x^n) = u$ and $\pi(u) = x^{n+1}$.
- *Assume $m = 2n+1$, $n \in \mathbb{N}_0$. Again, for any $1 \leqslant l \leqslant n$ consider $X_l := (x^l, u) \subset \mathsf{S}$. Additionally, denote $X^{\pm}_{n+1} := (x^{n+1} \pm u)$. Then the indecomposable Cohen–Macaulay S–modules are $X_0 = (1) = \mathsf{S}, X_1, \ldots, X_n$ and $X^{\pm}_{n+1}$. Moreover,*

$$X_0 \underset{\iota}{\overset{\cdot x}{\rightleftarrows}} X_1 \underset{\iota}{\overset{\cdot x}{\rightleftarrows}} X_2 \underset{\iota}{\overset{\cdot x}{\rightleftarrows}} \cdots \underset{\iota}{\overset{\cdot x}{\rightleftarrows}} X_n \quad \begin{array}{l} \overset{\pi^+}{\nearrow}\!\!\underset{\iota^+}{\swarrow}\; X^{+}_{n+1} \\ \underset{\iota^-}{\nwarrow}\!\!\overset{\pi^-}{\searrow}\; X^{-}_{n+1} \end{array} \tag{1.4}$$

  is the Auslander–Reiten quiver of $\mathsf{CM}(\mathsf{S})$ in this case. Here, ι and $\iota^{\pm}$ denote inclusions of ideals, $x\cdot$ stands for multiplication by x. The maps $\pi^{\pm}\colon X_n \longrightarrow X^{\pm}_{n+1}$ are defined as follows: $\pi^{\pm}(x^n) = (x^{n+1} \pm u)$ and $\pi^{\pm}(u) = x(x^{n+1} \pm u)$.

Remark 1.22. In the case $\mathsf{char}(\Bbbk) = 2$ there are the following subtleties in defining simple curve singularities of type A_m.

- For $m = 2n+1$ one should take the ring $\mathsf{A}_{2n+1} = \Bbbk[\![x, u]\!]/(u(u - x^{n+1}))$ (the ring defined by (1.2) is no longer reduced!). Then the indecomposable Cohen–Macaulay modules are $X_0, \ldots, X_n, X^{\pm}_{n+1}$, where X_i has the same definition as in the case $\mathsf{char}(\Bbbk) \neq 2$ for $0 \leqslant i \leqslant n$, whereas $X^{+}_{n+1} := (u)$ and $X^{-}_{n+1} := (u - x^{n+1})$.
- For $m = 2n$ there are more simple singularities than in the case $\mathsf{char}(\Bbbk) \neq 2$. Namely, for $1 \leqslant s \leqslant n-1$ consider the ring $\mathsf{A}^s_{2n} = \Bbbk[\![x, u]\!]/(u^2 + x^{2n+1} + ux^{n+s})$. Then $\mathsf{A}^s_{2n} \ncong \mathsf{A}^t_{2n}$ for any $1 \leqslant s \neq t \leqslant n-1$. Moreover, $\mathsf{A}^s_{2n} \ncong \mathsf{A}_{2n}$ for any $1 \leqslant s \leqslant n-1$. However, the description of indecomposable Cohen–Macaulay modules over A^s_{2n} is essentially the same as over A_{2n}; see [3] and [23]. In particular, the Auslander–Reiten quivers of A^s_{2n} and A_{2n} coincide.

The following result is due to Buchweitz, Greuel and Schreyer [9, Section 4.1].

Theorem 1.23. *Let $\mathsf{S} = \mathsf{A}_\infty := \Bbbk[\![x,u]\!]/(u^2)$, where $\Bbbk$ is an algebraically closed field of arbitrary characteristic. Then the indecomposable Cohen–Macaulay S–modules are X_0, $X_1, \dots, X_\infty$, where $X_0 = \mathsf{S}, X_\infty = (u)$ and $X_l = (x^l, u)$ for $l \in \mathbb{N}$. In particular, X_∞ is the only indecomposable Cohen–Macaulay S–module which is not locally free on the punctured spectrum of S. The Auslander–Reiten quiver of the category $\mathsf{CM}^{\mathsf{lf}}(\mathsf{S})$ has the form*

$$X_0 \underset{\iota}{\overset{\cdot x}{\rightleftarrows}} X_1 \underset{\iota}{\overset{\cdot x}{\rightleftarrows}} \cdots \underset{\iota}{\overset{\cdot x}{\rightleftarrows}} X_i \underset{\iota}{\overset{\cdot x}{\rightleftarrows}} \cdots \tag{1.5}$$

Remark 1.24. It is natural to extend the quiver (1.5) with the remaining indecomposable Cohen–Macaulay S–module X_∞. Moreover, for any $l \in \mathbb{N}_0$ denote $\pi_l \colon X_l \longrightarrow X_\infty$ the map sending x^l to u and u to 0. Of course, $\pi_{l+1} = x \cdot \pi_l$ for any $l \in \mathbb{N}_0$. The entire structure of the category $\mathsf{CM}(\mathsf{S})$ can be visualized by the diagram:

$$X_0 \underset{\iota}{\overset{\cdot x}{\rightleftarrows}} X_1 \underset{\iota}{\overset{\cdot x}{\rightleftarrows}} \cdots \underset{\iota}{\overset{\cdot x}{\rightleftarrows}} X_i \underset{\iota}{\overset{\cdot x}{\rightleftarrows}} \cdots \underset{\iota}{\overset{\pi}{\rightleftarrows}} X_\infty \tag{1.6}$$

Definition 1.25. Let S be a curve singularity of type A_m for some $m \in \mathbb{N} \cup \{\infty\}$.

1. Consider the path algebra category $\overline{\overline{\mathsf{AR}}}(\mathsf{S})$ of the corresponding Auslander–Reiten quivers (1.3), (1.4) respectively (1.6) subject to the following zero relations:

- $(\cdot x) \circ \imath = \imath \circ (\cdot x) = 0$.
- The inclusion $\imath \colon X_1 \to X_0$ is zero in $\overline{\overline{\mathsf{AR}}}(\mathsf{S})$.
- $\begin{cases} \pi^2 = 0, & \text{if } m \text{ is even},\\ \pi^\pm \circ \imath^+ = \pi^\pm \circ \imath^- = \imath^+ \circ \pi^+ + \imath^- \circ \pi^- = 0, & \text{if } m \text{ is odd},\\ \pi \circ \iota = 0, & \text{if } m = \infty. \end{cases}$

In other words, the objects of $\overline{\overline{\mathsf{AR}}}(\mathsf{S})$ are vertices of the Auslander–Reiten quiver of $\mathsf{CM}(\mathsf{S})$ and morphisms are formal $\Bbbk$–linear combinations of equivalence classes of paths. Observe that all morphisms spaces in $\overline{\overline{\mathsf{AR}}}(\mathsf{S})$ are finite dimensional vector spaces over $\Bbbk$ and the endomorphism algebra of any object of $\overline{\overline{\mathsf{AR}}}(\mathsf{S})$ is local.

2. We define the category $\overline{\mathsf{AR}}(\mathsf{S})$ as the *additive closure* of $\overline{\overline{\mathsf{AR}}}(\mathsf{S})$. More concretely:

- The objects of $\overline{\mathsf{AR}}(\mathsf{S})$ are formal symbols $X_1 \oplus \cdots \oplus X_t$, where $t \in \mathbb{N}$ and $X_i \in \mathrm{Ob}\big(\overline{\mathsf{AR}}(S)\big)$ for all $1 \leqslant i \leqslant t$.
- Given two objects $X = X_1 \oplus \cdots \oplus X_t$ and $Y = Y_1 \oplus \cdots \oplus Y_s$ of $\overline{\mathsf{AR}}(\mathsf{S})$, we put

$$\mathsf{Hom}_{\overline{\mathsf{AR}}(\mathsf{S})}(Y, X) := \begin{pmatrix} \mathsf{Hom}(Y_1, X_1) & \dots & \mathsf{Hom}(Y_s, X_1) \\ \vdots & \ddots & \vdots \\ \mathsf{Hom}(Y_1, X_t) & \dots & \mathsf{Hom}(Y_s, X_t) \end{pmatrix},$$

  where $\mathsf{Hom}(Y_j, X_i)$ is taken in the category $\overline{\overline{\mathsf{AR}}}(\mathsf{S})$.
- The composition of morphisms in $\overline{\mathsf{AR}}(\mathsf{S})$ is given by the matrix product rule.

Definition 1.26. Let $(\mathsf{A}, \mathfrak{m})$ be a Cohen–Macaulay curve singularity with $\Bbbk = \mathsf{A}/\mathfrak{m}$. Consider the category $\overline{\mathsf{CM}}(\mathsf{A})$ defined as follows:

- $\mathsf{Ob}\big(\overline{\mathsf{CM}}(\mathsf{A})\big) = \mathsf{Ob}\big(\mathsf{CM}(\mathsf{A})\big)$.
- For $M, N \in \mathsf{Ob}\big(\overline{\mathsf{CM}}(\mathsf{A})\big)$ we set

$$\overline{\mathsf{Hom}}_{\mathsf{A}}(M, N) = \mathrm{Im}\Big(\mathsf{Hom}_{\mathsf{A}}(M, N) \longrightarrow \mathsf{Hom}_{\Bbbk}\big(M/\mathfrak{m}M, N/\mathfrak{m}N\big)\Big).$$

In other words, we have an isomorphism of $\Bbbk$–vector spaces

$$\overline{\mathsf{Hom}}_{\mathsf{A}}(M, N) \cong \mathsf{Hom}_{\mathsf{A}}(M, N)/\,\mathsf{Hom}^0_{\mathsf{A}}(M, N),$$

where a homomorphism of A–modules $M \xrightarrow{f} N$ belongs to $\mathsf{Hom}^0_{\mathsf{A}}(M, N)$ if any only if the induced map $M/\mathfrak{m}M \xrightarrow{\bar{f}} N/\mathfrak{m}N$ is zero.

- The composition of morphisms in $\overline{\mathsf{CM}}(\mathsf{A})$ is induced by the composition of morphisms in $\mathsf{CM}(\mathsf{A})$.

Lemma 1.27. *The canonical projection functor $\mathbb{T} : \mathsf{CM}(\mathsf{A}) \longrightarrow \overline{\mathsf{CM}}(\mathsf{A})$ is full and reflects isomorphism classes of objects (i.e. if $\mathbb{T}(M) \cong \mathbb{T}(N)$ in $\overline{\mathsf{CM}}(\mathsf{A})$ then $M \cong N$ in $\mathsf{CM}(\mathsf{A})$). Moreover, if S is a curve singularity of type A_m for some $m \in \mathbb{N} \cup \{\infty\}$, then the categories $\overline{\mathsf{CM}}(\mathsf{S})$ and $\overline{\mathsf{AR}}(\mathsf{S})$ are equivalent.*

Proof. Assume $M \xrightarrow{f} N$ and $N \xrightarrow{g} M$ are two morphisms in $\mathsf{CM}(\mathsf{A})$ such that the induced $\Bbbk$–linear maps $M/\mathfrak{m}M \xrightarrow{\bar{f}} N/\mathfrak{m}N$ and $N/\mathfrak{m}N \xrightarrow{\bar{g}} M/\mathfrak{m}M$ are mutually inverse isomorphisms in $\overline{\mathsf{CM}}(\mathsf{A})$. By Nakayama's lemma, $gf \in \mathrm{End}_{\mathsf{A}}(M)$ and $fg \in \mathrm{End}_{\mathsf{A}}(N)$ are epimorphisms. Since M and N are finitely generated A–modules, gf and fg are isomorphisms. Therefore, the functor $\mathbb{T}$ indeed reflects isomorphism classes of objects.

The second statement is a consequence of some basic Auslander–Reiten theory; see [25, Chapter 13] or [28, Chapter 5]. □

2. Tameness of $\mathsf{CM}(\mathsf{P}_{\infty q})$

Let $\Bbbk$ be an algebraically closed field such that $\mathsf{char}(\Bbbk) \neq 2$ and $p, q \in \mathbb{N}_{\geqslant 2}$. Consider the curve singularity

$$\mathsf{P}_{pq} := \Bbbk[\![x, y, z]\!]/(xy, x^p + y^q - z^2). \tag{2.1}$$

By a result of Drozd and Greuel [15, Section 3], the category $\mathsf{CM}(\mathsf{P}_{pq})$ is representation tame. For any $q \in \mathbb{N}_{\geqslant 2}$ consider the limiting singularity

$$\mathsf{P}_{\infty q} := \Bbbk[\![x, y, z]\!]/(xy, y^q - z^2). \tag{2.2}$$

Since $(xz)^2 = 0$ in $\mathsf{P}_{\infty q}$, the ring $\mathsf{P}_{\infty q}$ is non–reduced. Similarly, $\mathsf{P}_{\infty\infty} := \Bbbk[\![x, y, z]\!]/(xy, z^2)$ denotes the "largest degeneration" of the family (2.1).

The first major result of this article is the following.

Theorem 2.1. *The non–reduced curve singularities $\mathsf{P}_{\infty q}$ have tame Cohen–Macaulay representation type for any $q \in \mathbb{N}_{\geqslant 2} \cup \{\infty\}$.*

Proof. 1. Since $\mathsf{P} := \mathsf{P}_{\infty q}$ is a complete intersection, it is Gorenstein. Let $\mathsf{Q} = \mathsf{Q}(\mathsf{P})$ be the total ring of fractions of P and $\mathsf{R} = \mathrm{End}_{\mathsf{P}}(\mathfrak{m}) \cong \{a \in \mathsf{Q} | a\,\mathfrak{m} \subset \mathfrak{m}\} \subset \mathsf{Q}$ be the minimal overring of P. Note that $u := \dfrac{xz}{x+y}$ and $v = \dfrac{yz}{x+y}$ belong to R. Clearly, $u + v = z$. Moreover, by Theorem 1.11 we have: $\mathsf{R} = \mathsf{P} + \Bbbk u$. From this fact we get an isomorphism

$$\mathsf{R} \cong \Bbbk[\![x, y, u, v]\!]/(xy, yu, xv, uv, u^2, y^q - v^2). \tag{2.3}$$

The generator $y^q - v^2$ has to be replaced by v^2 for $q = \infty$. In these terms, the canonical inclusion $\imath \colon \mathsf{P} \longrightarrow \mathsf{R}$ maps x to x, y to y and z to $u + v$.

According to Theorem 1.11, any non–free indecomposable Cohen–Macaulay P–module is a restriction of some indecomposable Cohen–Macaulay R–module. Thus, it has to be shown that the category $\mathsf{CM}(\mathsf{R})$ has tame representation type.

2. Next, since $(x,u) \cap (y,v) = 0$ in R, we have an inclusion $\mathsf{R} \hookrightarrow \mathsf{S}_x \times \mathsf{S}_y$, where $\mathsf{S}_x := \mathsf{R}/(y,v) \cong \Bbbk[\![x,u]\!]/(u^2)$ and $\mathsf{S}_y = \mathsf{R}/(x,u) \cong \Bbbk[\![y,v]\!]/(y^q - v^2)$. Moreover, we have an inclusion $\mathsf{S} \subset \mathsf{Q}$ (thus S is an overring of R) and both idempotents of S can be expressed as follows:

$$e_x := (1,0) = \frac{x}{x+y} \quad \text{and} \quad e_y := (0,1) = \frac{y}{x+y}.$$

The reason to pass from P to its overring R is explained by the following observation: the conductor ideal $I := \mathsf{ann}_{\mathsf{R}}(\mathsf{S}/\mathsf{R})$ coincides with the maximal ideal $\mathfrak{m} = (x,y,u,v)_{\mathsf{R}}$. Moreover, $\mathfrak{m} = \mathfrak{m}_x \times \mathfrak{m}_y$, where $\mathfrak{m}_x$ and $\mathfrak{m}_y$ are the maximal ideals of S_x and S_y respectively. Hence, $\overline{\mathsf{R}} := \mathsf{R}/I \cong \Bbbk$ and $\overline{\mathsf{S}} := \mathsf{S}/I \cong \Bbbk_x \times \Bbbk_y = \Bbbk \times \Bbbk$. Under this identification, the canonical inclusion $\overline{\mathsf{R}} \to \overline{\mathsf{S}}$ is identified with the diagonal embedding.

According to Theorem 1.19, the category $\mathsf{CM}(\mathsf{R})$ is equivalent to the category of triples $\mathsf{Tri}(\mathsf{R})$. Thus, we have to show the representation tameness of $\mathsf{Tri}(\mathsf{R})$. Let $T = (N, V, \theta)$ be an object $\mathsf{Tri}(\mathsf{R})$. Then the following facts are true.

- Since V is just a module over $\overline{\mathsf{R}} \cong \Bbbk$, we have: $V \cong \Bbbk^t$ for some $t \in \mathbb{N}_0$.
- Since $\mathsf{S} = \mathsf{S}_x \times \mathsf{S}_y$, we have: $N \cong N_x \oplus N_y$, where $N_x \in \mathsf{CM}(\mathsf{S}_x)$ and $N_y \in \mathsf{CM}(\mathsf{S}_y)$. According to Theorem 1.21 and Theorem 1.23, the Cohen–Macaulay modules N_x and N_y split into a direct sum of ideals
  - $X_0 = \mathsf{S}_x$, $X_i = (x^i, u)$ for $i \in \mathbb{N}$ and $X_\infty = (u)$.
  - $Y_0 = \mathsf{S}_y$, $Y_j = (y^j, v)$ for $1 \leqslant j \leqslant s-1$ and $Y_s^{\pm} = (y^s \pm v)$ if $q = 2s$ is even, respectively $Y_s = (y^s, v)$ if $q = 2s+1$ is odd. If $q = \infty$ then N_y decomposes analogously to N_x.
- We have: $\overline{N}_x = N_x/\mathfrak{m}_x N_x \cong \Bbbk_x^m$ and $\overline{N}_y = N_y/\mathfrak{m}_y N_y \cong \Bbbk_y^n$ for some $m, n \in \mathbb{N}_0$. In what follows, we choose bases of $\overline{N}_x$ and $\overline{N}_y$ induced by the distinguished generators of the ideals which occur in a direct sum decomposition of N_x and N_y. Thus, the gluing map $\theta \colon \overline{\mathsf{S}} \otimes_{\overline{\mathsf{R}}} V \longrightarrow N/IN = \overline{N}_x \oplus \overline{N}_y$ is given by a pair of matrices $(\Theta_x, \Theta_y) \in \mathrm{Mat}_{m\times t}(\Bbbk) \times \mathrm{Mat}_{n\times t}(\Bbbk)$.
- The condition that the morphism of $\overline{\mathsf{S}}$–modules θ is surjective just means that both matrices Θ_x and Θ_y have full row rank. The condition that $\widetilde{\theta}$ is injective is equivalent to saying that the matrix $\widetilde{\Theta} := \begin{pmatrix} \Theta_x \\ \Theta_y \end{pmatrix}$ has full column rank.

3. Let us now proceed to the matrix problem underlying a description of the isomorphism classes of objects of $\mathsf{Tri}(\mathsf{R})$. If two triples $T = (N, V, \theta)$ and $T' = (N', V', \theta')$ are isomorphic, then $N \cong N'$ and $V \cong V'$. Hence, we may without loss of generality assume that:

- $N' = N = N_x \oplus N_y$, where $N_x = \bigoplus_{i=0}^{\infty} X_i^{\oplus m_i}$ for some $m_i \in \mathbb{N}_0$ and
$$N_y = \begin{cases} \oplus_{j=0}^{s-1} Y_j^{\oplus n_j} \oplus \left(Y_s^+\right)^{\oplus n_s^+} \oplus \left(Y_s^-\right)^{\oplus n_s^-} & \text{if } q = 2s \\ \oplus_{j=0}^{\infty} Y_j^{\oplus n_j} & \text{if } q = \infty \\ \oplus_{j=0}^{s} Y_j^{\oplus n_j} & \text{if } q = 2s+1 \end{cases}$$
  where $n_j, n_s^{\pm} \in \mathbb{N}_0$.
- $V' = V = \Bbbk^t$ for certain $t \in \mathbb{N}_0$.

Note that we have: $m = m_0 + 2(m_1 + \dots) + m_\infty$ and

$$n = \begin{cases} n_0 + 2(n_1 + \dots + n_{s-1}) + (n_s^+ + n_s^-) & \text{if } q = 2s \\ n_0 + 2(n_1 + \dots) + n_\infty & \text{if } q = \infty \\ n_0 + 2(n_1 + \dots + n_s) & \text{if } q = 2s+1. \end{cases}$$

According to Definition 1.18, we have an isomorphism $\big(N, V, (\Theta_x, \Theta_y)\big) \cong \big(N, V, (\Theta'_x, \Theta'_y)\big)$ in the category $\mathsf{Tri}(\mathsf{R})$ if any only if there exist automorphisms $\Psi_x \in \mathsf{Aut}_{\mathsf{S}_x}(N_x)$, $\Psi_y \in \mathsf{Aut}_{\mathsf{S}_y}(N_y)$ and $\Phi \in \mathsf{GL}_t(\Bbbk)$ such that

$$\Theta'_x = \overline{\Psi}_x \Theta_x \Phi^{-1} \quad \text{and} \quad \Theta'_y = \overline{\Psi}_y \Theta_y \Phi^{-1}, \tag{2.4}$$

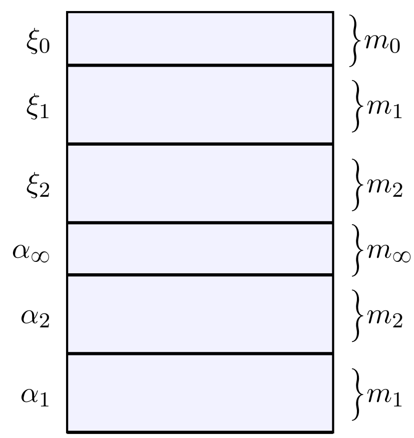
where $\overline{\Psi}_x \in \mathsf{GL}_m(\Bbbk)$ (respectively $\overline{\Psi}_y \in \mathsf{GL}_n(\Bbbk)$) is the induced automorphism of $\overline{N}_x \cong \Bbbk^m$ (respectively $\overline{N}_y \cong \Bbbk^n$).

Before proceeding to the matrix problem, consider the following special case.

Let $N_x = X_0^{\oplus m_0} \oplus X_1^{\oplus m_1} \oplus X_2^{\oplus m_2} \oplus X_\infty^{\oplus m_\infty}$ and $V = \Bbbk^t$. Then we have an isomorphism of vector spaces over $\Bbbk$:

$$\overline{N}_x = N_x/\mathfrak{m}_x N_x \xrightarrow{\gamma_x} \Bbbk^{m_0} \oplus \Bbbk^{m_1} \oplus \Bbbk^{m_2} \oplus \Bbbk^{m_\infty} \oplus \Bbbk^{m_2} \oplus \Bbbk^{m_1} = \Bbbk^m, \tag{2.5}$$

where $m = m_0 + 2(m_1 + m_2) + m_\infty$. The description of γ_x is specified by the following rule: the generator x of $X_1 = \langle x, u\rangle$ contributes to the second direct summand in the decomposition, whereas the generator u contributes to the last direct summand. Analogously, the generator x^2 of $X_2 = \langle x^2, u\rangle$ contributes to the third direct summand in the decomposition, whereas the generator u contributes to the second last direct summand. Therefore, the linear map $V = \Bbbk^t \xrightarrow{\Theta_x} \Bbbk^m \cong \overline{N}_x$ is represented by the following matrix.



Let us examine now the transformation rule (2.4). For any $\Psi_x \in \mathrm{End}_{\mathsf{S}_x}(N_x)$, the induced linear map $\overline{\Psi}_x \in \mathrm{End}_{\Bbbk}\big(\overline{N}_x\big)$ is represented by the following matrix:

$$\overline{\Psi}_x = \begin{array}{|c|c|c|c|c|c|}\hline \odot_0 & 0 & 0 & 0 & 0 & 0 \\ \hline \odot & \odot'_1 & 0 & 0 & 0 & 0 \\ \hline \odot & \odot & \odot'_2 & 0 & 0 & 0 \\ \hline \odot & \odot & \odot & \odot_\infty & 0 & 0 \\ \hline \odot & \odot & \odot & \odot & \odot''_2 & 0 \\ \hline \odot & \odot & \odot & \odot & \odot & \odot''_1 \\ \hline \end{array} \tag{2.6}$$

From the description of morphisms in the category $\overline{\mathsf{CM}}(\mathsf{S}_x) = \overline{\mathsf{AR}}(\mathsf{S}_x)$ given by the Auslander–Reiten quiver (1.6), we deduce the following results.

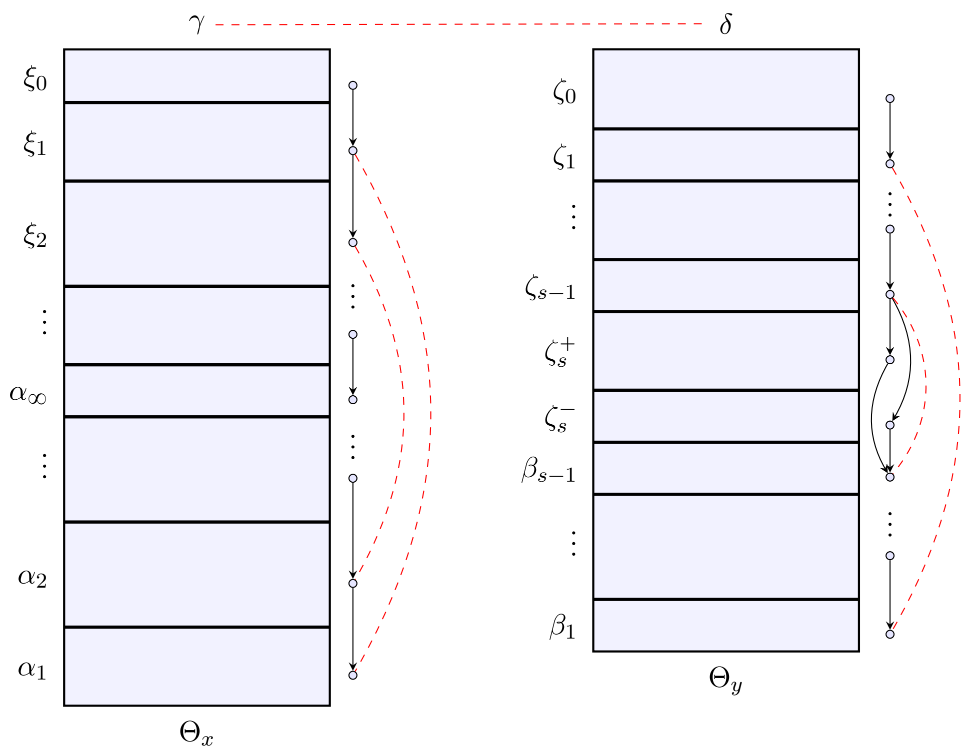
- We have the following equalities of diagonal blocks of $\overline{\Psi}_x$: $\odot_1' = \odot_1''$ and $\odot_2' = \odot_2''$.
- Any matrix $\overline{\Psi}_x \in \mathrm{Mat}_{m\times m}(\Bbbk)$ of the form (2.6) satisfying the constraints of the previous paragraph belongs to the image of the linear map $\mathrm{End}_{\mathsf{S}_x}(N_x) \longrightarrow \mathrm{Mat}_{m\times m}(\Bbbk)$.
- The morphism Ψ_x is an isomorphism if and only if all diagonal blocks of $\overline{\Psi}_x$ are invertible matrices.

4. Choose isomorphisms of vector spaces $\overline{N}_x \xrightarrow{\gamma_x} \Bbbk^m$ and $\overline{N}_y \xrightarrow{\gamma_y} \Bbbk^n$ analogously to (2.5). Then the transformation rule (2.4) leads to the following problem of linear algebra (a matrix problem).

- We have two matrices Θ_x and Θ_y over $\Bbbk$ with the same number of columns. The number of rows of Θ_x and Θ_y can be different. In particular, it can be zero for one of these matrices. Additionally, Θ_x and Θ_y have full row rank, and $\begin{pmatrix}\Theta_x\\ \Theta_y\end{pmatrix}$ has full column rank.
- Rows of Θ_x are divided into horizontal blocks indexed by elements of the linearly ordered set
$$\mathfrak{E}_x = \big\{\ \xi_0 < \xi_1 < \dots < \xi_i < \dots < \alpha_\infty < \dots < \alpha_i < \dots < \alpha_1\ \big\}.$$
The role of the ordering $<$ will be explained below.
- The block labeled by ξ_0 has m_0 rows, the block labeled by α_∞ has m_∞ rows. The blocks labeled by ξ_i and by α_i both have m_i rows. Thus, the matrix Θ_x has $m = m_0 + 2(m_1 + \dots + m_i + \dots) + m_\infty$ rows.
- The row division of Θ_y depends on the parity of the parameter q.
  - For $q = \infty$ the horizontal blocks of Θ_y are marked with the symbols of the linearly ordered set
  $$\mathfrak{E}_y = \mathfrak{E}_y^\infty = \big\{\ \zeta_0 < \zeta_1 < \dots < \zeta_j < \dots < \beta_\infty < \dots < \beta_j < \dots < \beta_1\ \big\},$$
  completely analogously as it is done for Θ_x.
  - For $q = 2s+1$, the labels are elements of the linearly ordered set
  $$\mathfrak{E}_y = \mathfrak{E}_y^q = \big\{\ \zeta_0 < \zeta_1 < \dots < \zeta_s < \beta_s < \dots < \beta_1\ \big\}.$$
  For any $1 \leqslant j \leqslant s$, the number of rows in blocks marked by ζ_j and β_j is the same (and equal to n_j).
  - For $q = 2s$, the labels are elements of an (only partially!) ordered set
  $$\mathfrak{E}_y = \mathfrak{E}_y^q = \big\{\ \zeta_0 < \zeta_1 < \dots < \zeta_{s-1} < \zeta_s^\pm < \beta_{s-1} < \dots < \beta_1\ \big\}$$
  as shown in (2.7) below. The elements ζ_s^+ and ζ_s^- are incomparable in $\mathfrak{E}_y$. Again, the number of rows in blocks ζ_j and β_j is the same for $1 \leqslant j \leqslant s-1$.
- We can perform any *simultaneous* elementary transformation of columns of Θ_x and Θ_y.
- Transformations of rows of Θ_x are of three types.
  - We can add any multiple of any row with lower weight to any row with higher weight.
  - For any $i \in \mathbb{N}$ we can perform any *simultaneous* elementary transformation of rows within the blocks marked by ξ_i and α_i.
  - We can make any elementary transformation of rows in block ξ_0 or α_∞.
- The transformation rules for rows of Θ_y *depend on the parity* of q.
  - Let us take the case q is even, which is the most complicated one; see (2.7).
    * We can add any multiple of any row with lower weight to any row with higher weight.

  * For any $1 \leqslant j \leqslant s-1$ we may perform any *simultaneous* elementary transformation of rows within blocks marked by ζ_j and β_j.
  * We can make any (independent) elementary transformation of rows in the block ζ_0 or $\zeta_s^{\pm}$.
- For $q = \infty$, the transformation rules for Θ_y are analogous to those listed above for Θ_x. The matrix problem for this case will be studied in detail in Subsection 3.1.
- For odd $q = 2s+1$, the transformation rules of Θ_y are the same as for $q = \infty$. The only difference between these cases lies in the absence of certain symbols in $\mathfrak{E}_y$.

(2.7)



5. As before, let $\mathsf{S}_x = \Bbbk[\![x,u]\!]/(u^2)$ and $\mathsf{S}_y = \Bbbk[\![y,v]\!]/(y^q - v^2)$. Next, define $\mathsf{MP}(\mathsf{R})$ to be the comma category of the following diagram of categories and functors:

$$\overline{\mathsf{AR}}(\mathsf{S}_x) \times \overline{\mathsf{AR}}(\mathsf{S}_y) \xrightarrow{\mathsf{For}\times\mathsf{For}} (\Bbbk \times \Bbbk) - \mathsf{mod} \xleftarrow{(\Bbbk\times\Bbbk)\otimes_\Bbbk -} \Bbbk - \mathsf{mod},$$

where $\overline{\mathsf{AR}}(\mathsf{S}_x)$ and $\overline{\mathsf{AR}}(\mathsf{S}_y)$ were introduced in Definition 1.26, and For is the forgetful functor. In other words, $\mathsf{MP}(\mathsf{R})$ is defined almost in the same way as the category of triples in Definition 1.18. An object of $\mathsf{MP}(\mathsf{R})$ is a triple $T = (\widetilde{N}_x \oplus \widetilde{N}_y, \Bbbk^t, \Theta)$, where $\widetilde{N}_x$ (respectively, $\widetilde{N}_x$) is an object of $\overline{\mathsf{AR}}(\mathsf{S}_x)$ (respectively, $\overline{\mathsf{AR}}(\mathsf{S}_y)$), $t \in \mathbb{N}_0$ and Θ is a $\Bbbk \times \Bbbk$–linear map. However, the isomorphism class of $\widetilde{N}_x$ (respectively, $\widetilde{N}_y$) is completely specified by the multiplicities attached to every vertex of the Auslander–Reiten quiver of $\mathsf{CM}(\mathsf{S}_x)$ (respectively, $\mathsf{CM}(\mathsf{S}_y)$). Therefore, we can represent (without loss of information) any object of $\mathsf{MP}(\mathsf{R})$ as a pair of partitioned matrices $(\Theta_x, \Theta_y) \in \mathrm{Mat}_{m\times t}(\Bbbk) \times \mathrm{Mat}_{n\times t}(\Bbbk)$, where $m = \mathsf{dim}_\Bbbk\big(N_x/\mathfrak{m}_x N_x\big)$, $n = \mathsf{dim}_\Bbbk\big(N_y/\mathfrak{m}_y N_y\big)$ and the row partitions of Θ_x and Θ_y arise from the direct sum decompositions of N_x and N_y, as well as by appropriately chosen isomorphisms $\overline{N}_x \xrightarrow{\gamma_x} \Bbbk^m$ and $\overline{N}_y \xrightarrow{\gamma_y} \Bbbk^n$, analogous to (2.5).

The matrix problem introduced in the previous paragraph corresponds precisely to the description of isomorphism classes of objects in the category $\mathsf{MP}(\mathsf{R})$. Moreover, the forgetful functor $\mathbb{P}\colon \mathsf{Tri}(\mathsf{R}) \longrightarrow \mathsf{MP}(\mathsf{R})$ assigning to a triple $\big(N, V, \theta\big)$ the pair of partitioned matrices (Θ_x, Θ_y) has the following properties.

- $\mathbb{P}$ is additive, full and reflects indecomposability and isomorphism classes of objects.
- The essential image of $\mathbb{P}$ consists of those $(\Theta_x, \Theta_y) \in \mathsf{Ob}\big(\mathsf{MP}(\mathsf{R})\big)$ for which Θ_x and Θ_y both have full row rank and $\begin{pmatrix}\Theta_x\\ \Theta_y\end{pmatrix}$ has full column rank.

According to a result of Bondarenko [5], the category $\mathsf{MP}(\mathsf{R})$ is *representation tame* for any $q \in \mathbb{N}_{\geqslant 2} \cup \{\infty\}$. More precisely,

- If $q \in 2\mathbb{N} \cup \{\infty\}$, then $\mathsf{MP}(\mathsf{R})$ is the category of representations of an appropriate *bunch of chains*. There are two types of indecomposable objects in $\mathsf{MP}(\mathsf{R})$: *bands* (continuous series) and *strings* (discrete series). In Section 3, the case $q = \infty$ will be discussed in detail and the terminology will be explained.
- For $q \in 2\mathbb{N} + 1$, the category $\mathsf{MP}(\mathsf{R})$ is the category of representations of an appropriate *bunch of semi–chains*. In this case, an indecomposable object either a band (continuous series) or a *bispecial, special, or usual* string (discrete series). The precise combinatorics of the discrete series is rather complicated.

6. Summing up, we have the following diagram of categories and functors:

$$\mathsf{CM}(\mathsf{P}) \xleftarrow{\;\mathbb{I}\;} \mathsf{CM}(\mathsf{R}) \underset{\mathbb{G}}{\overset{\mathbb{F}}{\rightleftarrows}} \mathsf{Tri}(\mathsf{R}) \xrightarrow{\;\mathbb{P}\;} \mathsf{MP}(\mathsf{R}). \tag{2.8}$$

Representation tameness of $\mathsf{MP}(\mathsf{R})$ implies that for any $q \in \mathbb{N}_{\geqslant 2} \cup \{\infty\}$, the curve singularity $\mathsf{P}_{\infty q}$ is Cohen–Macaulay representation tame in the "pragmatic sense": we have full control over the indecomposable objects of $\mathsf{CM}(\mathsf{R})$. See also Remark 4.6 concerning the proof of tameness in the formal sense. □

Remark 2.2. Following Bondarenko [5] and Drozd [13], there is another way to formalize the matrix problem, corresponding to the description of the isomorphism classes of the category $\mathsf{MP}(\mathsf{R})$. For any $q \in \mathbb{N}_{\geqslant 2} \cup \{\infty\}$, consider the following combinatorial data:

- The index set $\Omega = \{x, y\}$.
- Let $\mathfrak{F}_x = \{\gamma\}$, $\mathfrak{F}_y = \{\delta\}$, $\mathfrak{F} = \mathfrak{F}_x \cup \mathfrak{F}_y$.
- Let $\mathfrak{E}_x$ and $\mathfrak{E}_y = \mathfrak{E}_y^q$ be as in the proof of Theorem 2.1, $\mathfrak{E} = \mathfrak{E}_x \cup \mathfrak{E}_y$.
- In the set $\mathfrak{B} = \mathfrak{E} \cup \mathfrak{F}$ consider the symmetric (but not reflexive!) relation $\sim$ defined as follows:

$$\gamma \sim \delta, \qquad \xi_i \sim \alpha_i \quad \text{for } i \in \mathbb{N},$$
$$\zeta_j \sim \beta_j \quad \text{for } \begin{cases} 1 \leqslant j \leqslant s & \text{if } q = 2s \text{ or } q = 2s+1 \\ j \in \mathbb{N} & \text{if } q = \infty, \end{cases}$$
$$\zeta_s^{\pm} \sim \zeta_s^{\pm} \quad \text{for} \quad q = 2s.$$

The entire data $\mathfrak{B} = \big(\Omega, \mathfrak{E}, \mathfrak{F}, <, -, \sim\big)$ (which is an example of a bunch of (semi)–chains) defines a certain *bimodule category* $\mathsf{Rep}(\mathfrak{B})$; see [13] or [11, Section 7.2] for definitions and examples. This category *coincides* with $\mathsf{MP}(\mathsf{R})$. The combinatorics of Bondarenko of the indecomposable objects of $\mathsf{MP}(\mathsf{R})$ is given in terms of the data $\mathfrak{B}$; see the next Section 3 for the case $q = \infty$. The language of bimodule categories was introduced by Drozd [13] to formalize the notion of a *matrix problem*. Talking about representations of a bunch of (semi–)chains, bimodule categories allow to give a rigorous definition of the reduction procedure, which is crucial to prove the representation tameness as well as to establish the precise combinatorics of the indecomposable objects together with the Krull–Schmidt property (i.e. the uniqueness of a decomposition of an object into a direct sum of indecomposable ones). See also [11, Chapter 12] for a similar case. However, we do not need the language of bimodule categories in this article.

Remark 2.3. Let $\mathsf{char}(\mathbb{k}) = 2$. Then the simple curve singularities of type A have to be redefined according to Remark 1.22. It follows that the equation of $\mathsf{P}_{\infty,2s}$ should be $\mathbb{k}[\![x, y, z]\!]/(xy, z(y^s - z))$. Moreover, there are more singularities of type $\mathsf{P}_{\infty,2s+1}$, namely

$$\mathsf{P}^t_{\infty,2s+1} := \mathbb{k}[\![x, y, z]\!]/(xy, y^{2s+1} + y^{s+t}z - z^2), \qquad 1 \leqslant t \leqslant s-1.$$

Nevertheless, all rings $\mathsf{P}^t_{\infty,2s+1}$ are tame and the proof of Theorem 2.1 applies literally to this case as well.

Remark 2.4. For any $q \in \mathbb{N} \cup \{\infty\}$ consider the hypersurface singularity

$$\mathsf{T} = \mathsf{T}_{\infty,q+2} = \mathbb{k}[\![a, b]\!]/\big(b^2(a^2 - b^q)\big).$$

Observe that R is an overring of T via the embedding

$$\mathsf{T} \longrightarrow \mathsf{R}, \qquad a \longmapsto x + v, \quad b \longmapsto y + u$$

where R is the ring defined by (2.3). It was shown in [11, Theorem 11.1] that $\mathsf{CM}(\mathsf{T})$ has tame representation type (under the additional assumption $\mathsf{char}(\mathbb{k}) = 0$). This gives another argument that $\mathsf{CM}(\mathsf{R})$ (and hence $\mathsf{CM}(\mathsf{P})$) has either tame or discrete representation type. The latter case does also occur: if $q = 1$, then $\mathsf{T}_{\infty 3}$ is representation tame whereas

$$\mathsf{P}_{\infty 1} = \mathbb{k}[\![x, y, z]\!]/(xy, y - z^2) \cong \mathbb{k}[\![x, z]\!]/(xz^2) =: \mathsf{D}_\infty$$

is representation discrete [9].

3. Cohen–Macaulay modules over $\mathsf{P}_{\infty\infty}$ and $\mathsf{T}_{\infty\infty}$

In this section we shall explain that the technique of matrix problems, introduced in the course of the proof of Theorem 2.1, leads to a completely explicit description of indecomposable Cohen–Macaulay modules over $\mathsf{P} = \mathsf{P}_{\infty\infty} = \mathbb{k}[\![x, y, z]\!]/(xy, z^2)$. Although P is the "maximal degeneration" of the family (2.1), the combinatorics of the indecomposable objects in $\mathsf{CM}(\mathsf{P})$ are more transparent than for the less degenerate singularities $\mathsf{P}_{\infty,2s}$. The reason is that the underlying matrix problem has the type *representations of a bunch of chains* and not of *semi–chains* as for $\mathsf{P}_{\infty,2s}$. Another motivation to study Cohen–Macaulay modules over P is that it allows one to construct interesting examples of Cohen–Macaulay modules over the hypersurface singularity $\mathsf{T} = \mathsf{T}_{\infty\infty} = \mathbb{k}[\![a, b]\!]/(a^2b^2)$.

All results of this section concerning the study of indecomposable Cohen–Macaulay modules over P can be transferred in a straightforward way on the curve singularities of types $\mathsf{P}_{\infty,2s+1}$ or $\mathsf{P}_{2r+1,2s+1}$.

Until the end of this section we keep the following notation:

- $\mathsf{R} = \mathbb{k}[\![x, y, u, v]\!]/(xy, yu, uv, vx, u^2, v^2)$ is the minimal overring of P. The embedding $\mathsf{P} \longrightarrow \mathsf{R}$ sends z to $u + v$.
- Let $\mathsf{S} = \mathsf{S}_x \times \mathsf{S}_y = \mathbb{k}[\![x, u]\!]/(u^2) \times \mathbb{k}[\![y, v]\!]/(v^2)$.
- For any $l \in \mathbb{N}$ we denote $X_l = (u, x^l)_{\mathsf{S}_x}$ and $Y_l = (v, y^l)_{\mathsf{S}_y}$. Next, we put $X_0 = \mathsf{S}_x$, $Y_0 = \mathsf{S}_y$, $X_\infty = (u)_{\mathsf{S}_x}$ and $Y_\infty = (v)_{\mathsf{S}_y}$.

- Let $\mathsf{Q} = \Bbbk(\!(x)\!)[u]/(u^2) \times \Bbbk(\!(y)\!)[v]/(v^2)$.
- Denote $\mathfrak{m} = (x, y, u, v)_{\mathsf{R}}$. Recall that $\mathfrak{m} = \mathsf{ann}_{\mathsf{R}}(\mathsf{S}/\mathsf{R}) = \mathsf{rad}(\mathsf{S})$.

Observe that Q is the common total ring of fractions of P, R and S. In particular, we have the following equalities in Q:

$$u = \frac{xz}{x+y}, \quad v = \frac{yz}{x+y}, \quad e_x := (1,0) = \frac{x}{x+y} \quad \text{and} \quad e_y := (0,1) = \frac{y}{x+y}. \tag{3.1}$$

According to Remark 2.4, R is also an overring of T. Summing up, we have the following diagram of categories and functors:

$$\begin{array}{ccccc} \mathsf{CM}(\mathsf{P}) & & & & \\ \nwarrow{\scriptstyle \mathbb{I}} & & & & \\ & \mathsf{CM}(\mathsf{R}) & \underset{\mathbb{G}}{\overset{\mathbb{F}}{\rightleftarrows}}\,\sim & \mathsf{Tri}(\mathsf{R}) & \xrightarrow{\mathbb{P}} \mathsf{MP}(\mathsf{R}) \\ \swarrow{\scriptstyle \mathbb{J}} & & & & \\ \mathsf{CM}(\mathsf{T}) & & & & \end{array} \tag{3.2}$$

- $\mathbb{I}$ and $\mathbb{J}$ denote restriction functors. According to Proposition 1.9, they are both fully faithful. Moreover, by Theorem 1.11 $\mathsf{Ind}\big(\mathsf{CM}(\mathsf{P})\big) = \{\mathsf{P}\} \cup \mathsf{Ind}\big(\mathsf{CM}(\mathsf{R})\big)$.
- $\mathbb{F}$ and $\mathbb{G}$ are quasi–inverse equivalences of categories from Theorem 1.19.
- The functor $\mathbb{P}$ assigns to a triple $\big(N, V, \theta\big)$ a pair of matrices (Θ_x, Θ_y), whose rows are equipped with some additional "weights". $\mathbb{P}$ reflects isomorphism classes of objects as well as their indecomposability. However, $\mathbb{P}$ is not essentially surjective because Θ_x and Θ_y obey some additional constraints (see (3.4) below). Moreover, $\mathbb{P}$ is not faithful.

The goal of this section is to show how one can translate the combinatorics of indecomposable objects of MP(R) into an explicit description of indecomposable objects of CM(P) and CM(T).

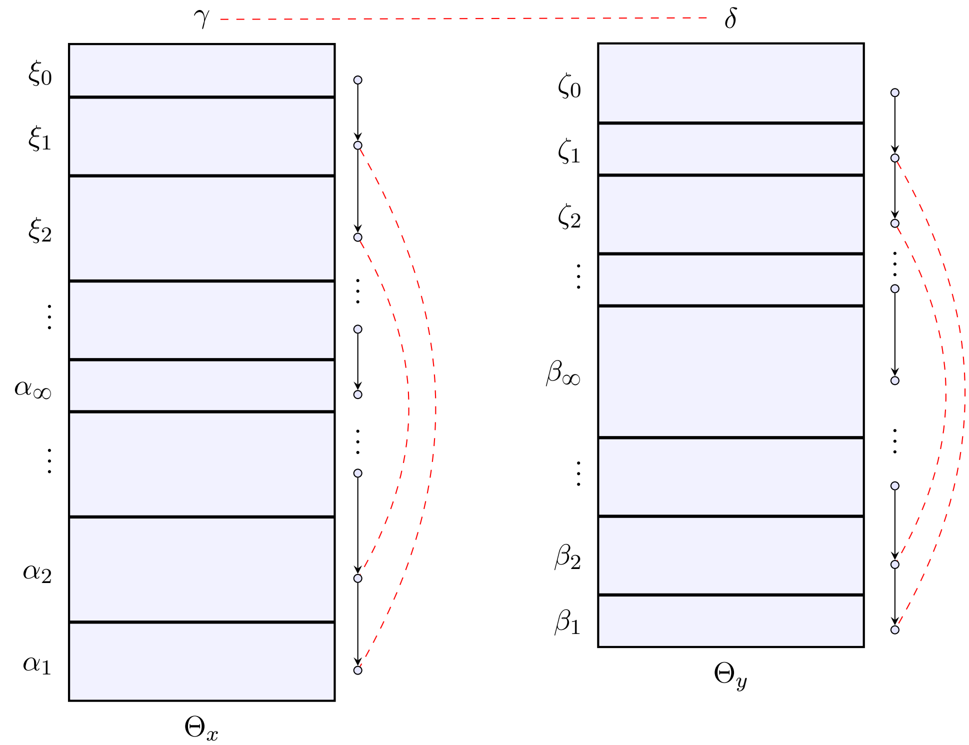
3.1. **Indecomposable objects of** MP(R)**.** According to the proof of Theorem 2.1, the matrix problem corresponding to a description of isomorphism classes of objects in MP(R) is as follows.

We are given two matrices Θ_x and Θ_y as depicted in (3.3) with entries from an algebraically closed field $\Bbbk$ and the same number of columns. The rows of Θ_x (respectively Θ_y) are both divided into horizontal blocks, labeled by symbols from the set $\big\{\xi_i \mid i \in \mathbb{N}_0\big\} \cup \big\{\alpha_j \mid j \in \mathbb{N} \cup \{\infty\}\big\}$ (respectively, set $\big\{\zeta_i \mid i \in \mathbb{N}_0\big\} \cup \big\{\beta_j \mid j \in \mathbb{N} \cup \{\infty\}\big\}$. Any two horizontal blocks in Θ_x (respectively Θ_y) connected by a dotted line have the same number of rows.

Transformation rules. The following transformations of columns and rows of Θ_x and Θ_y are admissible:

- any *simultaneous* elementary transformation of columns of Θ_x and Θ_y.
- addition of any multiple of any row of Θ_x (respectively, Θ_y) with lower weight to any row of Θ_x (respectively, Θ_y) with higher weight.
- any *simultaneous* elementary transformation of rows within horizontal blocks of Θ_x (respectively, Θ_y) connected by a dotted line.
- any elementary transformation of rows in the horizontal block of Θ_x (respectively, Θ_y) which is not connected to any other block by a dotted line. (These are the blocks labeled by $\xi_0, \alpha_\infty, \zeta_0$, or β_∞).

(3.3)



Additionally, there are the following *regularity constraints* on Θ_x and Θ_y:

$$\begin{gathered}\Theta_x \text{ and } \Theta_y \text{ both have full row rank,}\\ \text{the matrix } \begin{pmatrix}\Theta_x\\ \Theta_y\end{pmatrix} \text{ has full column rank.}\end{gathered} \tag{3.4}$$

Definition 3.1. Consider the following data $\mathfrak{B} = \big(\Omega, \mathfrak{E}, \mathfrak{F}, <, -, \sim\big)$, which is an example of a bunch of chains. It is given by the following ingredients.

- The index set $\Omega = \{x, y\}$.
- The set of column symbols $\mathfrak{F} = \mathfrak{F}_x \cup \mathfrak{F}_y$, where $\mathfrak{F}_x = \{\gamma\}$, $\mathfrak{F}_y = \{\delta\}$.
- The set of row symbols $\mathfrak{E} = \mathfrak{E}_x \cup \mathfrak{E}_y$, where $\mathfrak{E}_x$ and $\mathfrak{E}_y$ are "chains"

$$\begin{aligned}\mathfrak{E}_x &= \big\{\ \xi_0 < \xi_1 < \dots < \xi_i < \dots < \alpha_\infty < \dots < \alpha_i < \dots < \alpha_1\ \big\},\\ \mathfrak{E}_y &= \big\{\ \zeta_0 < \zeta_1 < \dots < \zeta_j < \dots < \beta_\infty < \dots < \beta_j < \dots < \beta_1\ \big\}.\end{aligned}$$

- The set $\mathfrak{B} = \mathfrak{E} \cup \mathfrak{F}$ is equipped with a symmetric (but not reflexive) relation $\sim$ defined as follows:

$$\gamma \sim \delta, \qquad \xi_l \sim \alpha_l \quad \text{and} \quad \zeta_l \sim \beta_l \quad \text{for } l \in \mathbb{N}.$$

Finally, we introduce another symmetric (but not reflexive) relation $-$ on $\mathfrak{B}$ as follows:

$$\gamma - \epsilon_x \text{ for any } \epsilon_x \in \mathfrak{E}_x \quad \text{ and } \quad \delta - \epsilon_y \text{ for any } \epsilon_y \in \mathfrak{E}_y.$$

Following Bondarenko [5], we define *strings* and *bands* of the bunch of chains $\mathfrak{B}$. They describe the invariants of the indecomposable objects of $\mathsf{MP}(\mathsf{R})$. As we already mentioned in Remark 2.2, one can describe $\mathsf{MP}(\mathsf{R})$ as the bimodule category $\mathsf{Rep}(\mathfrak{B})$.

Definition 3.2. Let $\mathfrak{B}$ be the bunch of chains from Definition 3.1.

(1) A *full word* w of $\mathfrak{B}$ is a sequence

$$w = \chi_1\rho_1\chi_2\rho_2\ldots\chi_{n-1}\rho_{n-1}\chi_n \tag{3.5}$$

of symbols $\chi_k \in \mathfrak{B}$ and relations $\rho_k \in \{\sim, -\}$ subject to the following conditions:
- the relation $\chi_k\rho_k\chi_{k+1}$ holds in $\mathfrak{B}$ for $1 \leqslant k \leqslant n-1$.
- the sequence of relations alternates, i.e. $\rho_k \neq \rho_{k+1}$ for $1 \leqslant k \leqslant n-2$.
- either $\chi_1 \in \{\xi_0, \alpha_\infty, \zeta_0, \beta_\infty\}$ or ρ_1 is $\sim$.
- either $\chi_n \in \{\xi_0, \alpha_\infty, \zeta_0, \beta_\infty\}$ or ρ_{n-1} is $\sim$.

The first and the last two conditions explain why w is called "full": if for some $1 \leqslant i \leqslant n$ the element χ_i has a "partner" in $\mathfrak{B}$, then this partner enters the word w as a neighbour of χ_i.

(2) The *opposite word* of a full word w given by (3.5) is defined by

$$w^o = \chi_n\rho_{n-1}\chi_{n-1}\rho_{n-2}\ldots\chi_2\rho_1\chi_1.$$

(3) A *string datum* of $\mathfrak{B}$ is given by any full word w.

(4) Two string data w and w' are *equivalent* if and only if $w' = w$ or $w' = w^o$.

Example 3.3. The word $\xi_1 - \gamma \sim \delta - \zeta_1 \sim \xi_1 - \delta \sim \gamma - \zeta_0$ is not full since the first element ξ_1 occurs without its partner ζ_1. On the other hand,

$$\zeta_1 \sim \xi_1 - \gamma \sim \delta - \zeta_1 \sim \xi_1 - \delta \sim \gamma - \zeta_0$$

is a full word. Its opposite word is $\zeta_0 - \gamma \sim \delta - \xi_1 \sim \zeta_1 - \delta \sim \gamma - \xi_1 \sim \zeta_1$

Definition 3.4. Let $\mathfrak{B}$ be the bunch of chains from Definition 3.1.

(1) A *cyclic word* $\widetilde{w}$ is the concatenation $\widetilde{w} = w-$, where $w = \chi_1\rho_1\chi_2\ldots\chi_{n-1}\rho_{n-1}\chi_n$ is a full word in the sense of Definition 3.2 such that the following additional conditions are satisfied.
- The relation $\chi_n - \chi_1$ holds in $\mathfrak{B}$.
- Both relations ρ_1 and ρ_{n-1} are equal to $\sim$.

Note that the length n of a cyclic word $\widetilde{w}$ is automatically divisible by eight.

(2) The opposite of a cyclic word $\widetilde{w} = w-$ is $\widetilde{w}^o = w^o-$.

(3) For any $k \in 2\mathbb{Z}$ the *k-th shift* of a cyclic word $\widetilde{w}$ is defined to be

$$\widetilde{w}^{[k]} = \chi_{k+1}\rho_{k+1}\chi_{k+2}\rho_{k+2}\ldots\chi_{k+n-1}\rho_{k+n-1}\chi_{k+n}\rho_{k+n}-$$

where the indices are taken modulo n.

(4) A cyclic word $\widetilde{w}$ is *periodic* if $\widetilde{w} = \widetilde{w}^{[k]}$ for some shift $k \notin n\mathbb{Z}$. Equivalently, $\widetilde{w}$ is periodic if there exists a smaller cyclic word $\widetilde{v}$ such that $\widetilde{w} = \widetilde{v}\ldots\widetilde{v}$.

(5) A *band datum* $(\widetilde{w}, m, \lambda)$ of $\mathfrak{B}$ consists of a non–periodic cyclic word $\widetilde{w}$, a "multiplicity" parameter $m \in \mathbb{N}$ and a "continuous" parameter $\lambda \in \mathbb{k}^*$.

(6) Two band data $(\widetilde{w}, m, \lambda)$ and $(\widetilde{w}', m', \lambda')$ are *equivalent* if and only if $(\widetilde{w}', m', \lambda')$ is equal to $(\widetilde{w}^o, m, \lambda)$, $(\widetilde{w}^{[4l]}, m, \lambda)$ or $(\widetilde{w}^{[4l+2]}, m, \lambda^{-1})$ for some $l \in \mathbb{Z}$.

Remark 3.5. The last equivalence relation explains why it is actually more natural to view a cyclic word $\widetilde{w}$ as a *cycle*.

Example 3.6. The words $\alpha_1 \sim \xi_1 - \gamma \sim \delta-$ and $\alpha_\infty - \gamma \sim \delta - \zeta_1 \sim \beta_1 - \delta \sim \gamma-$ are not cyclic. On the other hand, the word

$$\widetilde{w} = \alpha_1 \sim \xi_1 - \gamma \sim \delta - \zeta_1 \sim \beta_1 - \delta \sim \gamma-$$

is cyclic. We have:

$\widetilde{w}^{[2]} = \gamma \sim \delta - \zeta_1 \sim \beta_1 - \delta \sim \gamma - \alpha_1 \sim \xi_1 -$ and $\widetilde{w}^{[4]} = \zeta_1 \sim \beta_1 - \delta \sim \gamma - \alpha_1 \sim \xi_1 - \gamma \sim \delta -$.

Moreover, $\widetilde{w}^o = \gamma \sim \delta - \beta_1 \sim \zeta_1 - \delta \sim \gamma - \xi_1 \sim \alpha_1 -$.

The above definitions are motivated by the following result of Bondarenko [5].

Theorem 3.7. *There is a bijection between the equivalence classes of string and band data of the bunch of chains $\mathfrak{B}$ and the isomorphism classes of indecomposable objects in the category* $\mathsf{MP}(\mathsf{R})$.

Remark 3.8. There exists a more general class of tame matrix problems called representations of *decorated* bunches of (semi–)chains generalizing representations of bunches of chains; see [11, Chapter 12]. It arises in the study of Cohen–Macaulay modules over certain non–isolated surface singularities.

Following [5], we explain the construction of indecomposable objects in $\mathsf{MP}(\mathsf{R})$ corresponding to a string or a band datum.

1. Let w be a string datum of $\mathfrak{B}$. The corresponding object $\mathcal{S}(w)$ of $\mathsf{MP}(\mathsf{R})$ is given by a pair of matrices $\Theta_x(w)$ and $\Theta_y(w)$ defined as follows:

(1) Let t be the number of times the symbol γ occurs as a letter in w (since the word w is full, t is also the number of occurrences of δ). Then both matrices $\Theta_x(w)$ and $\Theta_y(w)$ have t columns.
(2) For each $\epsilon \in \mathfrak{E}$, let m_ϵ be the number of times the symbol ϵ occurs as a letter in w. Then the horizontal block labeled by ϵ in $\Theta_x(w)$ (respectively $\Theta_y(w)$) has m_ϵ rows.
(3) Next, we assign to every letter χ_k in w the number of times the letter χ_k occurred in the subsequence $\chi_1\rho_1 \dots \rho_{k-1}\chi_k$. In other words, we assign to every letter in w the number of times it occurred in w; see Example 3.9.
(4) Every appearance of the relation $-$ in w contributes to a non-zero entry of one of the matrices $\Theta_x(w), \Theta_y(w)$ in the following way. More concretely, we fill the entries of Θ_x and Θ_y according to the following rule.
    - Let $\epsilon - \nu$ or $\nu - \epsilon$ be a subsequence in w such that $\epsilon \in \mathfrak{E}_z$ and $\nu \in \mathfrak{F}_z$, where $z \in \Omega = \{x, y\}$. Let i be the occurrence number of ϵ and j be the occurrence number of ν. Then the (i, j)-th entry of the ϵ-th horizontal block of Θ_z is set to be 1. This rule is applied for every relation $-$ in w.
    - All remaining entries of $\Theta_x(w)$ and $\Theta_y(w)$ are set to be 0.

2. Let $(\widetilde{w}, m, \lambda)$ be a band datum. We assume that $\widetilde{w} = w-$ starts with an element of $\mathfrak{F} = \{\gamma, \delta\}$ (this can be achieved by applying a cyclic shift). The corresponding object $\mathcal{B}(\widetilde{w}, m, \lambda)$ of $\mathsf{MP}(\mathsf{R})$ is given by the pair of matrices $\big(\Theta_x(\widetilde{w}, m, \lambda), \Theta_y(\widetilde{w}, m, \lambda)\big)$ constructed in the following way.

(1) Using the recipe of the previous paragraph, construct first the matrices $\Theta_x(w)$ and $\Theta_y(w)$, viewing w as a *string* datum.
(2) Replace any zero entry in $\Theta_x(w)$ (respectively $\Theta_y(w)$) by the zero matrix and any identity entry in $\Theta_x(w)$ (respectively $\Theta_y(w)$) by the identity matrix I, both of size m.
(3) Finally, consider the relation $\chi_n - \chi_1$ in $\mathfrak{B}$. As we assumed that $\chi_1 \in \mathfrak{F}_z$ for $z \in \{x, y\}$, we have: $\chi_n \in \mathfrak{E}_z$. Replace the zero block of $\Theta_z(w)$ lying at the intersection of the *first* m columns of the vertical block labeled by χ_1 and the *last* m rows labeled by χ_n,

by the Jordan block

$$J_m(\lambda) = \begin{array}{|ccccc|} \hline \lambda & 1 & \dots & 0 & 0 \\ 0 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & \dots & 0 & \lambda \\ \hline \end{array} \in \mathsf{Mat}_{m\times m}(\Bbbk).$$

Here are some examples of canonical forms of band and string data.

Example 3.9. Consider the string datum given by the full word

$$w = \underset{\#1}{\alpha_i} \sim \underset{\#1}{\xi_i} - \underset{\#1}{\gamma} \sim \underset{\#1}{\delta} - \underset{\#1}{\zeta_j} \sim \underset{\#1}{\beta_j} - \underset{\#2}{\delta} \sim \underset{\#2}{\gamma} - \underset{\#2}{\xi_i} \sim \underset{\#2}{\alpha_i} - \underset{\#3}{\gamma} \sim \underset{\#3}{\delta}$$

The corresponding string object $\mathcal{S}(w)$ is given by the following pair of matrices $\big(\Theta_x(w), \Theta_y(w)\big)$

$$\begin{array}{c|ccc|} \multicolumn{1}{c}{} & \multicolumn{3}{c}{\gamma} \\ \cline{2-4} \multirow{2}{*}{ξ_i} & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ \cline{2-4} \multirow{2}{*}{α_i} & 0 & 0 & 0 \\ & 0 & 0 & 1 \\ \cline{2-4} \end{array} \qquad \begin{array}{|ccc|c} \multicolumn{3}{c}{\delta} & \\ \hline 1 & 0 & 0 & \zeta_j \\ \hline 0 & 1 & 0 & \beta_j \\ \hline \end{array}$$

Note that the first matrix $\Theta_x(w)$ does not have full row rank. Therefore, the regularity conditions (3.4) are not satisfied and the object $S(w)$ of $\mathsf{MP}(\mathsf{R})$ does not belong to the essential image of the functor $\mathbb{P}$.

Example 3.10. Consider a band datum $(\widetilde{w}, m, \lambda)$ where $\widetilde{w}$ is the following cyclic word:

$$\widetilde{w} = \underset{\#1}{\delta} \sim \underset{\#1}{\gamma} - \underset{\#1}{\xi_i} \sim \underset{\#1}{\alpha_i} - \underset{\#2}{\gamma} \sim \underset{\#2}{\delta} - \underset{\#1}{\zeta_{j_1}} \sim \underset{\#1}{\beta_{j_1}} - \underset{\#3}{\delta} \sim \underset{\#3}{\gamma} - \underset{\#2}{\alpha_i} \sim \underset{\#2}{\xi_i} - \underset{\#4}{\gamma} \sim \underset{\#4}{\delta} - \underset{\#1}{\zeta_{j_2}} \sim \underset{\#1}{\beta_{j_2}} -$$

Suppose that $\zeta_{j_1} < \zeta_{j_2}$. Actually, this assumption is not essential and only used to write the second matrix $\Theta_y(\widetilde{w}, m, \lambda)$ in the form conformal with the notation of this section. Then the corresponding canonical forms $\big(\Theta_x(\widetilde{w}, m, \lambda), \Theta_y(\widetilde{w}, m, \lambda)\big)$ are the following:

$$\begin{array}{c|cccc|} \multicolumn{1}{c}{} & \multicolumn{4}{c}{\gamma} \\ \cline{2-5} \multirow{2}{*}{ξ_i} & I & 0 & 0 & 0 \\ & 0 & 0 & 0 & I \\ \cline{2-5} \multirow{2}{*}{α_i} & 0 & I & 0 & 0 \\ & 0 & 0 & I & 0 \\ \cline{2-5} \end{array} \qquad \begin{array}{|cccc|c} \multicolumn{4}{c}{\delta} & \\ \hline 0 & I & 0 & 0 & \zeta_{j_1} \\ \hline 0 & 0 & 0 & I & \zeta_{j_2} \\ \hline J & 0 & 0 & 0 & \beta_{j_2} \\ \hline 0 & 0 & I & 0 & \beta_{j_1} \\ \hline \end{array}$$

Here, $I = I_m$ is the identity matrix and $J = J_m(\lambda)$ is the Jordan block with eigenvalue $\lambda \in \Bbbk^*$, both of size m.

Example 3.11. Consider the string datum given by the word

$$w = \delta \sim \gamma - \xi_i \sim \alpha_i - \gamma \sim \delta - \zeta_j \sim \beta_j - \delta \sim \gamma.$$

Then the corresponding canonical forms $\big(\Theta_x(w), \Theta_y(w)\big)$ are the following:

$$\begin{array}{c|ccc|} & & \gamma & \\ \hline \xi_i & 1 & 0 & 0 \\ \alpha_i & 0 & 1 & 0 \\ \hline \end{array} \qquad \begin{array}{|ccc|c} & \delta & & \\ \hline 0 & 1 & 0 & \zeta_j \\ 0 & 0 & 1 & \beta_j \\ \hline \end{array}$$

Example 3.12. Consider the string datum given by the word

$$w = \xi_0 - \gamma \sim \delta - \zeta_{j_1} \sim \beta_{j_1} - \delta \sim \gamma - \xi_i \sim \alpha_i - \gamma \sim \delta - \beta_{j_2} \sim \zeta_{j_2} - \delta \sim \gamma$$

Then the corresponding canonical forms $\big(\Theta_x(w), \Theta_y(w)\big)$ are the following:

$$\begin{array}{c|cccc|} & & \gamma & & \\ \hline \xi_0 & 1 & 0 & 0 & 0 \\ \xi_i & 0 & 1 & 0 & 0 \\ \alpha_i & 0 & 0 & 1 & 0 \\ \hline \end{array} \qquad \begin{array}{|cccc|c} & & \delta & & \\ \hline 1 & 0 & 0 & 0 & \zeta_{j_1} \\ 0 & 0 & 0 & 1 & \zeta_{j_2} \\ 0 & 0 & 1 & 0 & \beta_{j_2} \\ 0 & 1 & 0 & 0 & \beta_{j_1} \\ \hline \end{array}$$

Example 3.13. Consider the string datum given by the word

$$w = \alpha_\infty - \gamma \sim \delta - \beta_j \sim \zeta_j - \delta \sim \gamma - \xi_i \sim \alpha_i - \gamma \sim \delta - \beta_\infty.$$

Then the corresponding canonical forms $\big(\Theta_x(w), \Theta_y(w)\big)$ are the following:

$$\begin{array}{c|ccc|} & & \gamma & \\ \hline \xi_i & 0 & 1 & 0 \\ \alpha_\infty & 1 & 0 & 0 \\ \alpha_i & 0 & 0 & 1 \\ \hline \end{array} \qquad \begin{array}{|ccc|c} & \delta & & \\ \hline 0 & 1 & 0 & \zeta_j \\ 0 & 0 & 1 & \beta_\infty \\ 1 & 0 & 0 & \beta_j \\ \hline \end{array}$$

Remark 3.14. Let $(\widetilde{w}, m, \lambda)$ be a band datum and

$$\mathcal{B}(\widetilde{w}, m, \lambda) = \big(\Theta_x(\widetilde{w}, m, \lambda), \Theta_y(\widetilde{w}, m, \lambda)\big)$$

be the corresponding object of $\mathsf{MP}(\mathsf{R})$. Then both matrices $\Theta_x(\widetilde{w}, m, \lambda)$ and $\Theta_y(\widetilde{w}, m, \lambda)$ have the same size $mn \times mn$ and are invertible, where n is the length of $\widetilde{w}$. In particular, the regularity conditions (3.4) are automatically satisfied for bands. Moreover, we may replace $J_m(\lambda)$ by any conjugate matrix in $\mathsf{Mat}_{m\times m}(\mathbb{k})$, for example by the transposed matrix $J_m(\lambda)^{tr}$ or by the Frobenius block corresponding to the polynomial $(t-\lambda)^m \in \mathbb{k}[t]$. Any such substitution yields an isomorphic object in $\mathsf{MP}(\mathsf{R})$. In the case the base field $\mathbb{k}$ is not algebraically closed, essentially the same rules for the canonical forms remain true.

Remark 3.15. Let $\mathcal{S}(w) = \big(\Theta_x(w), \Theta_y(w)\big)$ be a string object of $\mathsf{MP}(\mathsf{R})$ associated to a full word w. The regularity constraints (3.4) are satisfied if and only if

- w begins and ends with symbols from the set $\mathfrak{E}_\circ \cup \mathfrak{F}$, where $\mathfrak{E}_\circ := \big\{\xi_0, \alpha_\infty, \zeta_0, \beta_\infty\big\}$.
- $w \neq \gamma \sim \delta, \delta \sim \gamma$ or e with $e \in \mathfrak{E}_\circ$.

3.2. **Indecomposable Cohen–Macaulay modules over $\mathsf{P}_{\infty\infty}$.** Our actual goal is to describe all indecomposable Cohen–Macaulay modules over $\mathsf{P} = \mathsf{P}_{\infty\infty}$ in an explicit way. To achieve this, recall the following logical steps.

- We view $\mathsf{CM}(\mathsf{P})$ as a full subcategory of $\mathsf{CM}(\mathsf{R})$, using the restriction functor $\mathbb{I}$.

- Next, we have an equivalence of categories $\mathbb{F} : \mathsf{CM}(\mathsf{R}) \longrightarrow \mathsf{Tri}(\mathsf{R})$. Its quasi–inverse functor $\mathbb{G}$ is also explicit.
- Next, we have a functor $\mathbb{P} : \mathsf{Tri}(\mathsf{R}) \longrightarrow \mathsf{MP}(\mathsf{R})$, which reflects the isomorphism classes and indecomposability of objects.
- Finally, the indecomposable objects of the category $\mathsf{MP}(\mathsf{R})$ are known: these are bands $\mathcal{B}(\widetilde{w}, m, \lambda)$ and strings $\mathcal{S}(w)$. Moreover, the essential image of the functor $\mathbb{P}$ is also completely known.

These steps were already summarized in the diagrams of categories and functors (2.8) and (3.2). The rest of this subsection is devoted to the the answer of the following question.

Question. How to go back from the category $\mathsf{MP}(\mathsf{R})$ to $\mathsf{CM}(\mathsf{P})$ in a constructive way? In other words, how to translate the combinatorics of strings and bands in $\mathfrak{B}$ and the corresponding canonical forms in $\mathsf{MP}(\mathsf{R})$ into an explicit description of Cohen–Macaulay modules over P?

Taking into account properties of categories and functors from diagram (3.2), this classification is essentially given by the combinatorics of strings and bands of the bunch of chains $\mathfrak{B}$. However, it is not the way we wish to state the final answer! The reason is that the combinatorial structure of $\mathfrak{B}$ has no intrinsic meaning for the category $\mathsf{CM}(\mathsf{P})$ and can be further compressed. The strategy is the following

- Consider a *new alphabet*

$$\mathfrak{G} := \mathfrak{G}_x \cup \mathfrak{G}_y = \bigl(\{\mathbf{x}_l^{\pm} \mid l \in \mathbb{N}\} \cup \{\mathbf{x}_0, \mathbf{x}_\infty\}\bigr) \bigcup \bigl(\{\mathbf{y}_l^{\pm} \mid l \in \mathbb{N}\} \cup \{\mathbf{y}_0, \mathbf{y}_\infty\}\bigr), \tag{3.6}$$

  which will replace the bunch of chains $\mathfrak{B}$ introduced in Definition 3.1.
- We shall *define* certain Cohen–Macaulay R–modules $S(\omega)$ and $B(\widetilde{\omega}, m, \lambda)$, where ω (respectively, $\widetilde{\omega}$) are certain words in $\mathfrak{G}$, whose precise shape will be specified below, whereas $m \in \mathbb{N}$ and $\lambda \in \Bbbk^*$.
- Then we shall show that $\mathbb{PF}\bigl(S(\omega)\bigr) \cong \mathcal{S}(w)$ and $\mathbb{PF}\bigl(B(\widetilde{\omega}, m, \lambda)\bigr) \cong \mathcal{B}(\widetilde{w}, m, \lambda)$ for some explicit translation rule $\mathfrak{G} \to \mathfrak{B}$ assigning to ω (respectively, $\widetilde{\omega}$) a full word (respectively, a non–periodic cyclic word) in the sense of Definitions 3.2 and 3.4. Because of that we shall say that $S(\omega)$ (respectively, $B(\widetilde{\omega}, m, \lambda)$) is a string (respectively, band) Cohen–Macaulay R–module.
- Finally, we shall specify the rules to determine the Cohen–Macaulay P–modules $\mathbb{I}\bigl(S(\omega)\bigr)$ and $\mathbb{I}\bigl(B(\widetilde{\omega}, m, \lambda)\bigr)$.

3.2.1. *Band and string Cohen–Macaulay* R*–modules.* We *define* certain Cohen–Macaulay modules over R, called *band* (respectively, *string*) modules. In the course of the proof of Theorem 3.25 below we shall see that their images under the functor $\mathbb{PF}$ are precisely the band (respectively, string) objects of $\mathsf{MP}(\mathsf{R})$.

Definition 3.16. A *band module* $B = B(\widetilde{\omega}, m, \lambda)$ is defined by the following parameters.

- A *band word* $\widetilde{\omega}$, which is a *non–periodic* sequence of the form

$$\widetilde{\omega} = \mathbf{x}_{i_1}^{\sigma_1}\mathbf{y}_{j_1}^{\tau_1}\mathbf{x}_{i_2}^{\sigma_2}\mathbf{y}_{j_2}^{\tau_2}\ldots\mathbf{x}_{i_n}^{\sigma_n}\mathbf{y}_{j_n}^{\tau_n} \tag{3.7}$$

  where $\sigma_k, \tau_k \in \{+, -\}$ and $i_k, j_k \in \mathbb{N}$ for $1 \leqslant k \leqslant n$. The condition for $\widetilde{\omega}$ to be non–periodic means that $\widetilde{\omega} \neq \widetilde{\gamma}\widetilde{\gamma}\ldots\widetilde{\gamma}$, where $\widetilde{\gamma}$ is a smaller word.
- $m \in \mathbb{N}$ and $\lambda \in \Bbbk^*$.

Consider the following Cohen–Macaulay S–module

$$N = N(\widetilde{\omega}, m) := X_{i_1}^{\oplus m} \oplus Y_{j_1}^{\oplus m} \oplus \cdots \oplus X_{i_n}^{\oplus m} \oplus Y_{j_n}^{\oplus m}$$

Note that $N \subseteq \mathsf{S}^{2mn}$. By definition, B is the following R–submodule of N:

$$B := \left\langle \begin{pmatrix} f''_{i_1} I \\ g'_{j_1} I \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ g''_{j_1} I \\ f'_{i_2} I \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ f''_{i_2} I \\ g'_{j_2} I \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ f''_{i_n} I \\ g'_{j_n} I \end{pmatrix}, \begin{pmatrix} f'_{i_1} J \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ g''_{j_n} I \end{pmatrix} \right\rangle_{\mathsf{R}} \tag{3.8}$$

where for any $1 \leqslant k \leqslant n$ the elements $f'_{i_k}, f''_{i_k} \in X_{i_k}$ and $g'_{j_k}, g''_{j_k} \in Y_{i_k}$ are determined by the values of $\sigma_k, \tau_k \in \{+,-\}$ according to the following tables:

(3.9)

σ_k	f'_{i_k}	f''_{i_k}
$+$	u	x^{i_k}
$-$	x^{i_k}	u

τ_k	g'_{j_k}	g''_{j_k}
$+$	v	y^{j_k}
$-$	y^{j_k}	v

Here as usual, $I = I_m$ is the identity matrix of size m and $J = J_m(\lambda)$ is a Jordan block of size m with eigenvalue λ. The number of columns in (3.8) is $2n$ (the length of $\widetilde{\omega}$). Recall that $X_l = \langle u, x^l \rangle$ and $Y_l = \langle v, y^l \rangle$ for any $l \in \mathbb{N}$. Therefore, $f'_{i_k}, f''_{i_k} \in X_{i_k}$ and $g'_{i_k}, g''_{i_k} \in Y_{i_k}$ for all $1 \leqslant k \leqslant n$ and B is indeed a submodule of N. Moreover, B is a submodule of R^{2mn}.

Definition 3.17. A *string module* $S = S(\omega)$ is defined by a *string* word ω in $\mathfrak{G}$ having the following structure: it has a beginning $\widehat{\mathbf{x}}\check{\mathbf{y}}$, an intermediate part and an end $\check{\mathbf{x}}\widehat{\mathbf{y}}$. The beginning as well as the end may consist of zero, one or two letters. The following table lists all possible beginnings and ends for ω (any beginning from the first two columns can match any ending from the last two columns):

(3.10)

<table>
<tr><th>$\widehat{\mathbf{x}}$</th><th>$\check{\mathbf{y}}$</th><th>intermediate part</th><th>$\check{\mathbf{x}}$</th><th>$\widehat{\mathbf{y}}$</th></tr>
<tr><td rowspan="4">void</td><td>void</td><td rowspan="6">$\mathbf{x}_{i_1}^{\sigma_1}\mathbf{y}_{j_1}^{\tau_1} \dots \mathbf{x}_{i_{n-1}}^{\sigma_{n-1}}\mathbf{y}_{j_{n-1}}^{\tau_{n-1}}$</td><td>void</td><td rowspan="4">void</td></tr>
<tr><td>$\mathbf{y}_0$</td><td>$\mathbf{x}_0$</td></tr>
<tr><td>$\mathbf{y}_\infty$</td><td>$\mathbf{x}_\infty$</td></tr>
<tr><td>$\mathbf{y}_{j_0}^{\tau_0}$</td><td>$\mathbf{x}_{i_n}^{\sigma_n}$</td></tr>
<tr><td>$\mathbf{x}_0$</td><td>$\mathbf{y}_{j_0}^{\tau_0}$</td><td>$\mathbf{x}_{i_n}^{\sigma_n}$</td><td>$\mathbf{y}_0$</td></tr>
<tr><td>$\mathbf{x}_\infty$</td><td>$\mathbf{y}_{j_0}^{\tau_0}$</td><td>$\mathbf{x}_{i_n}^{\sigma_n}$</td><td>$\mathbf{y}_\infty$</td></tr>
</table>

where $n \in \mathbb{N}$ and additionally

- For $n = 1$, the intermediate part of ω is void.
- For any $1 \leqslant k \leqslant n-1$ we have: $i_k, j_k \in \mathbb{N}$ and $\sigma_k, \tau_k \in \{+,-\}$.
- If $\check{\mathbf{y}} = \mathbf{y}_{j_0}^{\tau_0}$ then $j_0 \in \mathbb{N}$ and $\tau_0 \in \{+,-\}$. Analogously, if $\check{\mathbf{x}} = \mathbf{x}_{i_n}^{\sigma_n}$ then $i_n \in \mathbb{N}$ and $\sigma_n \in \{+,-\}$.

In other words, a string word ω is given by a sequence of letters from the alphabet $\mathfrak{G}$ such that the letters of $\mathfrak{G}_x$ and $\mathfrak{G}_y$ alternate and a letter from the set $\{\mathbf{x}_0, \mathbf{x}_\infty, \mathbf{y}_0, \mathbf{y}_\infty\}$ may only occur as the first or last letter of ω.

Consider the Cohen–Macaulay S–module

$$N = N(\omega) = \widehat{X} \oplus \check{Y} \oplus X_{i_1} \oplus Y_{j_1} \oplus \cdots \oplus X_{i_{n-1}} \oplus Y_{j_{n-1}} \oplus \check{X} \oplus \widehat{Y},$$

where for each $i \in \{i_0, i_n\}$ and each $j \in \{j_0, j_n\}$, we set

$$\widehat{X} = \begin{cases} X_0 & \text{if } \widehat{\mathbf{x}} = \mathbf{x}_0, \\ X_\infty & \text{if } \widehat{\mathbf{x}} = \mathbf{x}_\infty, \\ 0 & \text{if } \widehat{\mathbf{x}} \text{ is void} \end{cases} \qquad \check{X} = \begin{cases} X_0 & \text{if } \check{\mathbf{x}} = \mathbf{x}_0, \\ X_\infty & \text{if } \check{\mathbf{x}} = \mathbf{x}_\infty, \\ X_l & \text{if } \check{\mathbf{x}} = \mathbf{x}_l^{\pm}, \text{with } l \in \mathbb{N}, \\ 0 & \text{if } \check{\mathbf{x}} \text{ is void} \end{cases}$$

with analogous rules for $\widehat{Y}$ and $\check{Y}$.

By definition, $S = S(\omega)$ is the following R–submodule of N:

$$S := \left\langle \begin{pmatrix} f''_{i_0} \\ g'_{j_0} \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ g''_{j_0} \\ f'_{i_1} \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ g''_{j_{n-1}} \\ f'_{i_n} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ f''_{i_n} \\ g'_{j_n} \end{pmatrix} \right\rangle_{\mathsf{R}} \tag{3.11}$$

where for any $1 \leqslant k \leqslant n-1$, the elements $f'_{i_k}, f''_{i_k}, g'_{j_k}$ and g''_{j_k} are defined by the tables:

(3.12)

σ_k	f'_{i_k}	f''_{i_k}
$+$	ux	x^{i_k+1}
$-$	x^{i_k+1}	ux

τ_k	g'_{j_k}	g''_{j_k}
$+$	vy	y^{j_k+1}
$-$	y^{j_k+1}	vy

The remaining entries are defined as follows:

$\widehat{\mathbf{x}}$	f''_{i_0}
$\mathbf{x}_0$	x
$\mathbf{x}_\infty$	ux
void	0

$\widehat{\mathbf{y}}$	g'_{j_n}
$\mathbf{y}_0$	y
$\mathbf{y}_\infty$	vy
void	0

(3.13)

$\check{\mathbf{x}}$	σ_n	f'_{i_n}	f''_{i_n}
void		0	0
$i_n = 0$		x	0
$i_n = \infty$		ux	0
$i_n \in \mathbb{N}$	$+$	ux	x^{j_n+1}
$i_n \in \mathbb{N}$	$-$	x^{j_n+1}	ux

$\check{\mathbf{y}}$	τ_0	g'_{j_0}	g''_{j_0}
void		0	0
$j_0 = \infty$		0	vy
$j_0 = 0$		0	y
$j_0 \in \mathbb{N}$	$+$	vy	y^{j_0+1}
$j_0 \in \mathbb{N}$	$-$	y^{j_0+1}	vy

Remark 3.18. As in the case of bands, the number of generators of the R–module $S(\omega)$ defined by (3.11) is equal to the length of the word ω. Again, the rules (3.12) and (3.13) insure that we indeed have an inclusion $S(\omega) \subset N(\omega)$, as claimed. Comparing with the corresponding rule for band (3.9) there is a *deviation* in the definition of the entries

$f'_{i_k}, f''_{i_k}, g'_{i_k}$ and g''_{i_k}. The explanation is the following: since $x \in \mathsf{S}_x$ and $y \in \mathsf{S}_y$ are regular elements, we have: $X_l \cong xX_l$ and $Y_l \cong yY_l$ for all $l \in \mathbb{N}_0 \cup \{\infty\}$. Therefore,

$$N(\omega) \cong x\widehat{X} \oplus y\check{Y} \oplus xX_{i_1} \oplus yY_{j_1} \oplus \cdots \oplus xX_{i_{n-1}} \oplus yY_{j_{n-1}} \oplus x\check{X} \oplus y\widehat{Y} \subset \mathsf{R}^{2(n+1)}.$$

In this way we achieve that the R–module $S(\omega)$ is automatically defined as a submodule of $\mathsf{R}^{2(n+1)}$. This will be convenient in the light of Proposition 1.13, since we actually wish to describe $B(\widetilde{\omega}, m, \lambda)$ and $S(\omega)$ as modules over P. If the string parameter ω contains neither $\mathbf{x}_0$ nor $\mathbf{y}_0$, then there is a simpler presentation of the module $S(\omega)$, analogous to the case of bands. Namely, for $1 \leqslant k \leqslant n-1$, the elements $f'_{i_k}, f''_{i_k}, g'_{j_k}$ and g''_{j_k} are defined according to (3.9), whereas the rules for the remaining entries are given by the tables:

$\widehat{\mathbf{x}}$	f''_{i_0}
$\mathbf{x}_\infty$	u
void	0

$\widehat{\mathbf{y}}$	g'_{j_n}
$\mathbf{y}_\infty$	v
void	0

(3.14)

$\check{\mathbf{x}}$	σ_n	f'_{i_n}	f''_{i_n}
void		0	0
$i_n = \infty$		u	0
$i_n \in \mathbb{N}$	$+$	u	x^{j_n}
$i_n \in \mathbb{N}$	$-$	x^{j_n}	u

$\check{\mathbf{y}}$	τ_0	g'_{j_0}	g''_{j_0}
void		0	0
$j_0 = \infty$		0	v
$j_0 \in \mathbb{N}$	$+$	v	y^{j_0}
$j_0 \in \mathbb{N}$	$-$	y^{j_0}	v

These rules give an isomorphic R–module. For the sake of uniqueness, we shall use the convention of Definition 3.17 (in particular, in Remark 3.29 and Remark 3.32).

Remark 3.19. Any string module $S = S(\omega)$ defined by (3.11) has a more compact presentation by "merging" every odd row with its subsequent row:

$$S \cong \left\langle \begin{pmatrix} f''_{i_0} + g'_{j_0} \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} g''_{j_0} \\ f'_{i_1} \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ f''_{i_1} + g'_{j_1} \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ g''_{j_{n-1}} \\ f'_{i_n} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f''_{i_n} + g'_{j_n} \end{pmatrix} \right\rangle_{\mathsf{R}}$$

The same will be done with the horizontal stripes of any band module $B = B(\widetilde{\omega}, m, \lambda)$ defined by (3.8):

$$B \cong \left\langle \begin{pmatrix} (f''_{i_1} + g'_{j_1})I \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} g''_{j_1}I \\ f'_{i_2}I \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ (f''_{i_n} + g'_{j_n})I \end{pmatrix}, \begin{pmatrix} f'_{i_1}J \\ 0 \\ \vdots \\ g''_{j_n}I \end{pmatrix} \right\rangle_{\mathsf{R}}$$

The interpretation of this rule is the following: we use the embedding $X_l \oplus Y_k \longrightarrow \mathsf{S}$ for any $k, l \in \mathbb{N}_0 \cup \{\infty\}$. This allows one to embed a string module $S(\omega)$ (respectively, a band module $B(\widetilde{\omega}, m, \lambda)$ into a free R–module of a smaller rank.

Example 3.20. In the following we construct some indecomposable Cohen–Macaulay modules over R from strings and bands.

(1) Let $(\widetilde{\omega}, m, \lambda)$ be the band datum given by $\widetilde{\omega} = \mathbf{x}_i^- \mathbf{y}_{j_1}^- \mathbf{x}_i^+ \mathbf{y}_{j_2}^-$ with $i, j_1, j_2 \in \mathbb{N}$, $m \in \mathbb{N}$ and $\lambda \in \Bbbk^*$. Then the corresponding band module is given by

$$B(\widetilde{\omega}, m, \lambda) \cong \left\langle \begin{pmatrix} uI \\ y^{j_1} I \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ vI \\ uI \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x^i I \\ y^{j_2} I \end{pmatrix}, \begin{pmatrix} x^i J \\ 0 \\ 0 \\ vI \end{pmatrix} \right\rangle_{\mathsf{R}}$$
$$\cong \left\langle \begin{pmatrix} (u + y^{j_1}) I \\ 0 \end{pmatrix}, \begin{pmatrix} vI \\ uI \end{pmatrix}, \begin{pmatrix} 0 \\ (x^i + y^{j_2}) I \end{pmatrix}, \begin{pmatrix} x^i J \\ vI \end{pmatrix} \right\rangle_{\mathsf{R}}$$

Here, J denotes the Jordan block with eigenvalue λ and I the identity matrix, both of size m.

(2) Let ω be the string $\omega = \mathbf{x}_i^- \mathbf{y}_j^-$, where $i, j \in \mathbb{N}$. The corresponding string module $S(\omega)$ is given by

$$S(\omega) \cong \left\langle \begin{pmatrix} x^{i+1} \\ 0 \end{pmatrix}, \begin{pmatrix} ux \\ y^{j+1} \end{pmatrix}, \begin{pmatrix} 0 \\ vy \end{pmatrix} \right\rangle_{\mathsf{R}} \cong (x^i, u + y^j, v)_{\mathsf{R}}$$

(3) Let ω be the string $\omega = \mathbf{x}_0 \mathbf{y}_{j_1}^- \mathbf{x}_i^- \mathbf{y}_{j_2}^+$, where $i, j_1, j_2 \in \mathbb{N}$. The corresponding string module is given by

$$S(\omega) \cong \left\langle \begin{pmatrix} x \\ y^{j_1+1} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ vy \\ x^{i+1} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ ux \\ vy \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ y^{j_2+1} \end{pmatrix} \right\rangle_{\mathsf{R}}$$
$$\cong \left\langle \begin{pmatrix} x + y^{j_1+1} \\ 0 \end{pmatrix}, \begin{pmatrix} vy \\ x^{i+1} \end{pmatrix}, \begin{pmatrix} 0 \\ ux + vy \end{pmatrix}, \begin{pmatrix} 0 \\ y^{j_2+1} \end{pmatrix} \right\rangle_{\mathsf{R}}$$

(4) Let ω be the string $\omega = \mathbf{x}_\infty \mathbf{y}_j^+ \mathbf{x}_i^- \mathbf{y}_\infty$, where $i, j \in \mathbb{N}$. The corresponding string module is given by

$$S(\omega) \cong \left\langle \begin{pmatrix} ux \\ vy \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y^{j+1} \\ x^{i+1} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ ux \\ vy \end{pmatrix} \right\rangle_{\mathsf{R}} \cong \left\langle \begin{pmatrix} u + v \\ 0 \end{pmatrix}, \begin{pmatrix} y^j \\ x^i \end{pmatrix}, \begin{pmatrix} 0 \\ u + v \end{pmatrix} \right\rangle_{\mathsf{R}}$$

Definition 3.21. Let $\widetilde{\omega}$ be a band word as in (3.7) and ω be a string word as in (3.10).

- For any $k \in \mathbb{Z}$, we define the shifted word $\widetilde{\omega}^{[k]}$ by
$$\widetilde{\omega}^{[k]} := \mathbf{x}_{i_{k+1}}^{\sigma_{k+1}} \mathbf{y}_{j_{k+1}}^{\tau_{k+1}} \mathbf{x}_{i_{k+2}}^{\sigma_{k+2}} \mathbf{y}_{j_{k+2}}^{\tau_{k+2}} \dots \mathbf{x}_{i_{k+n+1}}^{\sigma_{k+n+1}} \mathbf{y}_{j_{k+n+1}}^{\tau_{k+n+1}},$$
where all indices are taken modulo n.
- The opposite word of $\widetilde{\omega}$ is by definition
$$\widetilde{\omega}^o := \mathbf{x}_{i_1}^{\bar{\sigma}_1} \mathbf{y}_{j_n}^{\bar{\tau}_n} \mathbf{x}_{i_n}^{\bar{\sigma}_n} \mathbf{y}_{j_{n-1}}^{\bar{\tau}_{n-1}} \dots \mathbf{x}_{i_2}^{\bar{\sigma}_2} \mathbf{y}_{j_1}^{\bar{\tau}_1},$$
where $\bar{\sigma}_j$ and $\bar{\tau}_j$ denote the opposite signs of σ_j respectively τ_j for each $1 \leqslant j \leqslant n$.

- The opposite word ω^o of ω is given by reversing the letters and taking opposite signs in the word ω. For example, in the notations of Table 3.10, if

  $\omega = \;\widehat{\mathbf{x}}\;\mathbf{y}_{j_0}^{\tau_0}\mathbf{x}_{i_1}^{\sigma_1}\mathbf{y}_{j_1}^{\tau_1}\dots\mathbf{x}_{i_{n-1}}^{\sigma_{n-1}}\mathbf{y}_{j_{n-1}}^{\tau_{n-1}}\mathbf{x}_{i_n}^{\sigma_n}\;\widehat{\mathbf{y}}$ then $\omega^o = \widehat{\mathbf{y}}\;\mathbf{x}_{i_n}^{\bar{\sigma}_n}\mathbf{y}_{j_{n-1}}^{\bar{\tau}_{n-1}}\mathbf{x}_{i_{n-1}}^{\bar{\sigma}_{n-1}}\dots\mathbf{y}_{j_1}^{\bar{\tau}_1}\mathbf{x}_{i_1}^{\bar{\sigma}_1}\mathbf{y}_{j_0}^{\bar{\tau}_0}\;\widehat{\mathbf{x}}.$

  Note that ω^o need not have the form (3.10): if for example $\omega = \mathbf{y}_0\mathbf{x}_\infty$ then $\omega^o = \mathbf{x}_\infty\mathbf{y}_0$ does not occur in (3.10).
- We say that two bands $(\widetilde{\omega}, m, \lambda)$ and $(\widetilde{v}, n, \mu)$ are equivalent if and only if there is some $k \in \mathbb{Z}$ such that $(\widetilde{\omega}^{[k]}, m, \lambda) = (\widetilde{v}, n, \mu)$ or $(\widetilde{v}^o, n, \mu^{-1})$.

Remark 3.22. If the beginning and the ending of ω are void then ω is also a band word. This is different from the case of full and cyclic words over the alphabet $\mathfrak{B}$.

3.2.2. *From triples to Cohen–Macaulay modules.* Let us look more closely at the functor $\mathbb{G} : \mathsf{Tri}(\mathsf{R}) \longrightarrow \mathsf{CM}(\mathsf{P})$. Consider an object $T = \bigl(N, V, (\Theta_x, \Theta_y)\bigr)$ of the category $\mathsf{Tri}(\mathsf{R})$. Then we may assume that

- $N = X_{i_1} \oplus \dots \oplus X_{i_k} \oplus Y_{j_1} \oplus \dots \oplus Y_{j_l}$ for some $i_1, \dots, i_k, j_1, \dots, j_l \in \mathbb{N}_0 \cup \{\infty\}$.
- $V = \mathbb{k}^t$ for some $t \in \mathbb{N}_0$.

According to Theorem 1.19, the corresponding Cohen–Macaulay R–module $M = \mathbb{G}(T)$ is determined by the following commutative diagram in the category of R–modules:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{m}N & \longrightarrow & M & \xrightarrow{\sigma} & \mathbb{k}^t & \longrightarrow & 0 \\ & & \Vert & & \downarrow & & \downarrow{\scriptstyle \widetilde{\theta}} & & \\ 0 & \longrightarrow & \mathfrak{m}N & \longrightarrow & N & \xrightarrow{\pi} & N/\mathfrak{m}N & \longrightarrow & 0. \end{array} \tag{3.15}$$

Lemma 3.23. *Let $\{e_1, \dots, e_t\}$ be the standard basis of $\mathbb{k}^t$. For any $1 \leqslant i \leqslant t$ choose $w_i \in N$ such that $\pi(w_i) = \widetilde{\theta}(e_i)$. Then we have:*

$$M = \bigl\langle w_1, \dots, w_t \bigr\rangle_{\mathsf{R}} \subseteq \mathsf{S}^{k+l} \tag{3.16}$$

and t is the minimal number of generators of M.

Proof. By definition of M, for any $1 \leqslant i \leqslant t$ we have: $w_i \in M$. Moreover, $M = \bigl\langle w_1, \dots, w_t \bigr\rangle_{\mathsf{R}} + \mathfrak{m}N$. Next, $\mathfrak{m}M = \mathfrak{m}N$, the induced map $\overline{\sigma} : M/\mathfrak{m}M \longrightarrow \mathbb{k}^t$ is an isomorphism (see the proof of [11, Theorem 2.5]) and $\overline{\sigma}(w_i) = e_i$. Hence, $\{\overline{w}_1, \dots, \overline{w}_t\}$ is a basis of $M/\mathfrak{m}M$ and (3.16) follows from Nakayama's Lemma. □

Lemma 3.24. *Let $T = \bigl(N, \mathbb{k}^t, (\Theta_x, \Theta_y)\bigr)$ be an indecomposable object of $\mathsf{Tri}(\mathsf{R})$ as above and $M = \mathbb{G}(T) = \bigl\langle w_1, \dots, w_t \bigr\rangle_{\mathsf{R}} \subseteq \mathsf{S}^{k+l}$. Then the following results are true.*

(1) We have either
- $T \cong \Bigl(\mathsf{S}, \mathbb{k}, \bigl((1), (1)\bigr)\Bigr)$ *(in this case $M \cong \mathsf{R}$), or*
- $\mathbb{I}(M) = \bigl\langle w_1, \dots, w_t \bigr\rangle_{\mathsf{P}} \subseteq \mathsf{S}^{k+l}$.

In both cases we have: $\mathbb{J}(M) = \bigl\langle w_1, \dots, w_t, vw_1, \dots, vw_t, uw_1, \dots, uw_t \bigr\rangle_{\mathsf{T}} \subseteq \mathsf{S}^{k+l}$.

(2) The Cohen–Macaulay module M, respectively $\mathbb{I}(M)$ and $\mathbb{J}(M)$, is locally free on the punctured spectrum of R, respectively P and T, if and only if N contains no direct summands isomorphic to X_∞ or Y_∞.

Proof. (1) The statement about $\mathbb{I}(M)$ is a corollary of Proposition 1.13. The statement about $\mathbb{J}(M)$ follows from the fact that $\mathbb{J}(\mathsf{T}) = \langle 1, u, v\rangle_{\mathsf{R}} \subset \mathsf{Q}$ in R–mod.

(2) The Cohen–Macaulay module M is locally free on the punctured spectrum of R if and only $M \cong \mathsf{S} \boxtimes_{\mathsf{R}} M$ is locally free on the punctured spectrum of S; see Remark 1.15. The

latter is equivalent to the condition that N contains no direct summands isomorphic to X_∞ or Y_∞. Since the rational envelopes of M, $\mathbb{I}(M)$ and $\mathbb{J}(M)$ are the same, the result follows. □

3.2.3. *Classification of indecomposable Cohen–Macaulay* R*–modules.* Recall that

$$\mathsf{R} = \Bbbk[\![x, y, u, v]\!]/(xy, xv, yu, uv, u^2, v^2).$$

Theorem 3.25. *The following results are true.*

- *The Cohen–Macaulay* R*–modules* $B(\widetilde{\omega}, m, \lambda)$ *and* $S(\omega)$ *are indecomposable. Moreover, any indecomposable Cohen–Macaulay* R*–module is isomorphic to some band module* $B(\widetilde{\omega}, m, \lambda)$ *or some string module* $S(\omega)$*.*
- $B(\widetilde{\omega}, m, \lambda) \not\cong S(\omega')$ *for any choice of parameters* $\widetilde{\omega}, \omega', m$ *and* λ*.*
- $S(\omega) \cong S(\omega')$ *if and only if* $\omega' = \omega$ *or* $\omega' = \omega^o$*, where* ω^o *is the opposite word.*
- $B(\widetilde{\omega}, m, \lambda) \cong B(\widetilde{\omega}', m', \lambda')$ *if and only if the bands* $(\widetilde{\omega}, m, \lambda)$ *and* $(\widetilde{\omega}', m', \lambda')$ *are equivalent in the sense of Definition 3.21.*

Proof. According to Theorem 2.1, the classification problem of indecomposable objects of $\mathsf{CM}(\mathsf{R})$ is equivalent to the matrix problem defined by the bunch of chains $\mathfrak{B}$ from Definition 3.1. More precisely, we had a diagram of categories and functors

$$\mathsf{CM}(\mathsf{R}) \underset{\mathbb{G}}{\overset{\mathbb{F}}{\rightleftarrows}} \mathsf{Tri}(\mathsf{R}) \xrightarrow{\mathbb{P}} \mathsf{MP}(\mathsf{R})$$

where $\mathbb{F}$ and $\mathbb{G}$ are mutually inverse equivalences of categories and $\mathbb{P}$ is a full functor reflecting isomorphism classes and indecomposability of objects.

1. The indecomposable objects of the category $\mathsf{MP}(\mathsf{R})$ are classified by string and band data according to Theorem 3.7. Moreover, the indecomposable objects of $\mathsf{MP}(\mathsf{R})$ lying in the essential image of $\mathbb{P}$ are described by Remark 3.15.

2. Let w (respectively, $\widetilde{w}$) be a full (respectively, cyclic) word of string (respectively, band) datum in $\mathfrak{B}$; see Definition 3.2 (respectively, Definition 3.4). It follows that we may delete from w and $\widetilde{w}$ all subsequences of the form $\gamma \sim \delta$ or $\delta \sim \gamma$ and relations $-$ *without any loss of information.* Now we can translate the remaining subsequences as follows:

ξ_0	$\alpha_i \sim \xi_i$	$\xi_i \sim \alpha_i$	α_∞	ζ_0	$\beta_j \sim \zeta_j$	$\zeta_j \sim \beta_j$	β_∞
$\mathbf{x}_0$	$\mathbf{x}_i^+$	$\mathbf{x}_i^-$	$\mathbf{x}_\infty$	$\mathbf{y}_0$	$\mathbf{y}_j^+$	$\mathbf{y}_j^-$	$\mathbf{y}_\infty$

This table yields the translation rule $\mathfrak{B} \longrightarrow \mathfrak{G}$, which will allow to pass from the indecomposable objects of $\mathsf{MP}(\mathsf{R})$ to the underlying indecomposable Cohen–Macaulay R–modules.

3. Consider a string datum (w) (obeying the constraint from Remark 3.15), respectively a band datum $(\widetilde{w}, m, \lambda)$. In Subsection 3.1 we explained the construction of the corresponding indecomposable object $\Theta = \big(\Theta_x, \Theta_y\big)$ of $\mathsf{MP}(\mathsf{R})$. Now we give the construction of a triple $T = (N, V, \theta)$ in $\mathsf{Tri}(\mathsf{R})$ such that $\mathbb{P}(T) = \Theta$. Let m_0, $m_l^\pm$, m_∞, n_0, $n_k^\pm$ respectively n_∞ be the number of times the letter $\mathbf{x}_0$, $\mathbf{x}_l^\pm$, $\mathbf{x}_\infty$, $\mathbf{y}_0$, $\mathbf{y}_k^\pm$ respectively $\mathbf{y}_\infty$ occurs in w, respectively in $\widetilde{w}$. For any $l \in \mathbb{N}$, we put $m_l = m_l^+ + m_l^-$ and $n_l = n_l^+ + n_l^-$. Let t be the number of times γ (or δ) occurs in w. Then $T = (N, V, \theta)$, where

- $N = \big(\oplus_{l=0}^\infty X_l^{\oplus m_l}\big) \oplus \big(\oplus_{l=0}^\infty Y_l^{\oplus n_l}\big)$,
- $V = \Bbbk^t$,
- θ is the $\Bbbk \times \Bbbk$–linear map given by $\big(\Theta_x, \Theta_y\big)$.

4. Now recall the construction of the indecomposable Cohen–Macaulay R–module $M = \mathbb{G}(T)$. Consider a basis of a $\Bbbk$–vector space $N/\mathfrak{m}N$ given by the images of the distinguished generators of the indecomposable direct summands of N. Let $\pi : N \to N/\mathfrak{m}N$ be the canonical projection and $\widetilde{\Theta} := \begin{pmatrix} \Theta_x \\ \Theta_y \end{pmatrix} : V \to N/\mathfrak{m}N$. By Theorem 1.19 we have:

$$M := \mathbb{G}(T) = \pi^{-1}\bigl(\mathsf{im}(\widetilde{\Theta})\bigr) \subseteq N.$$

Moreover, the description of M as a submodule of a free R–module is provided by Lemma 3.23. Namely, M is generated by the columns of the matrix obtained from $\widetilde{\Theta}$ by the following procedure.

(1) At the first step, we (formally!) multiply each entry of $\widetilde{\Theta}$ with its horizontal weight.
(2) At the next step, we construct from it a matrix with entries from S using the following table:

ξ_0	ξ_k	α_k	α_∞	ζ_0	ζ_k	β_k	β_∞
e_x	x^k	u	u	e_y	y^k	v	v

$k \in \mathbb{N}.$

Here, e_x, e_y are the primitive idempotents of the ring $\mathsf{S} = \mathsf{S}_x \times \mathsf{S}_y$.

- If the matrix $\widetilde{\Theta}$ was constructed from a band datum $(\widetilde{w}, m, \lambda)$ then we get precisely the module $B(\widetilde{\omega}, m, \lambda)$ from Definition 3.16.
- In the case the matrix $\widetilde{\Theta}$ was constructed from a string datum (w), we use an isomorphism of R–modules $M \cong (x + y)M$. Again, we get precisely the R–module from Definition 3.17.

5. The statement about the isomorphism classes of string modules in $\mathsf{CM}(\mathsf{R})$ is a direct translation of the corresponding result for the category $\mathsf{MP}(\mathsf{R})$ stated in Theorem 3.7. Considering all pairwise non–equivalent band data $(\widetilde{w}, m, \lambda)$, we may assume that the first letter of w is δ by the equivalence conditions in Definition 3.4. Then Theorem 3.7 yields the statement about the isomorphism classes of band modules $B(\widetilde{\omega}, m, \lambda)$ as stated in the theorem.

6. Summing up, the key point of the proof of Theorem 3.25 is that *by construction* we have the following isomorphisms in the category $\mathsf{MP}(\mathsf{R})$:

$$\mathbb{PF}\bigl(B(\widetilde{\omega}, m, \lambda)\bigr) \cong \mathcal{B}(w, m, \lambda) = \bigl(\Theta_x(w, m, \lambda), \Theta_x(w, m, \lambda)\bigr)$$

for a band module $B(\widetilde{\omega}, m, \lambda)$ from Definition 3.16 and

$$\mathbb{PF}\bigl(S(\omega)\bigr) \cong \mathcal{S}(w) = \bigl(\Theta_x(w), \Theta_x(w)\bigr).$$

for a string module $S(\omega)$ from Definition 3.17. □

Remark 3.26. According to Lemma 3.24, any band module $B(\omega, m, \lambda)$ is locally free on the punctured spectrum. A string module $S(\omega)$ is *not* locally free on the punctured spectrum if and only if ω contains a letter $\mathbf{x}_\infty$ or $\mathbf{y}_\infty$.

Remark 3.27. The canonical forms of Examples 3.10, 3.11, 3.12 and 3.13 of the preceding subsection correspond exactly to the string and band modules of Example 3.20 via the translation in the proof above.

3.2.4. *Classification of indecomposable Cohen–Macaulay* P*–modules.* Our original motivation was to describe indecomposable Cohen–Macaulay modules over $\mathsf{P} = \mathbb{k}[\![x, y, z]\!]/(xy, z^2)$. Theorem 1.11, Lemma 3.24 and Theorem 3.25 yield the following result.

Theorem 3.28. *An indecomposable Cohen–Macaulay* P*–module is either* P*, or one of the band modules (3.8) respectively string modules (3.11). Moreover, in the formulae (3.8) and (3.11), the generation over* R *can be replaced by the generation over* P *(with the only exception of* $S(\mathbf{y}_0\mathbf{x}_0) \cong \mathsf{R}$*).*

Remark 3.29. Any string or band module M over R can be translated into a Cohen–Macaulay module $\mathbb{I}(M)$ over P as follows.

(1) Assume that $M = B(\widetilde{\omega}, m, \lambda)$ is a band module given by (3.8). We compute $\mathbb{I}(M)$ replacing the elements of R by elements of P according to the following table:

x^i	u	y^j	v
x^{i+1}	xz	y^{j+1}	yz

(2) If $M = S(\omega) \ncong S(\mathbf{y}_0\mathbf{x}_0)$ is a string module given by (3.11), (3.12) and (3.13), then the translation rule is the following:

x^i	ux	y^j	vy
x^i	xz	y^j	yz

This statement follows directly from Lemma 3.24, taking into account that the embedding $\mathsf{P} \to \mathsf{R}$ is given by the formulae (3.1). The reason for a deviation in the recipes for bands and strings is explained in Remark 3.18.

Remark 3.30. An analogue of Theorem 3.28 remains valid for any curve singularity of type $\mathsf{P}_{2r+1,2s+1}$, where $r, s \in \mathbb{N}_0 \cup \{\infty\}$, but string and band modules have to be redefined in the following way:

(1) The band and string modules over $\mathsf{P}_{2r+1,\infty}$ are given by the Definitions 3.16 and 3.17, but their string and band words ω may only contain letters $\mathbf{x}_i$ such that $0 \leqslant i \leqslant r$ or $\mathbf{y}_j$, where $j \in \mathbb{N}_0 \cup \{\infty\}$.
(2) Band and string data over $\mathsf{P}_{2r+1,2s+1}$, where $r, s \in \mathbb{N}_0$, may only contain the letters $\mathbf{x}_i$ such that $0 \leqslant i \leqslant r$ or $\mathbf{y}_j$ such that $0 \leqslant j \leqslant s$.

The method of this section can also be generalized using Bondarenko's work on representations of bunches of semi–chains [5] to obtain an explicit classification of indecomposable Cohen–Macaulay modules over the remaining curve singularities $\mathsf{P}_{2r,q}$, where $r \in \mathbb{N}$ and $q \in \mathbb{N} \cup \{\infty\}$. This generalization is straightforward to carry out, but the explicit combinatorics is too complicated to be stated in the present article.

Example 3.31. In the following, we apply Remark 3.29 to rewrite the string and band modules over R from Example 3.20 as indecomposable Cohen–Macaulay modules over P.

(1) Let $(\widetilde{\omega}, m, \lambda)$ be a band datum with $\widetilde{\omega} = \mathbf{x}_i^- \mathbf{y}_{j_1}^- \mathbf{x}_i^+ \mathbf{y}_{j_2}^-$, where $i, j_1, j_2 \in \mathbb{N}$. Then the image of the corresponding band module $B = B(\widetilde{\omega}, m, \lambda)$ under $\mathbb{I}$ is

$$\mathbb{I}(B) \cong \left\langle \begin{pmatrix} (xz + y^{j_1+1})\, I \\ 0 \end{pmatrix}, \begin{pmatrix} yz\, I \\ xz\, I \end{pmatrix}, \begin{pmatrix} 0 \\ (x^{i+1} + y^{j_2+1})\, I \end{pmatrix}, \begin{pmatrix} x^{i+1}\, J \\ yz\, I \end{pmatrix} \right\rangle_{\mathsf{P}}$$

(2) Let $\omega = \mathbf{x}_i^- \mathbf{y}_j^-$ with $i, j \in \mathbb{N}$. Then $\mathbb{I}\big(S(\omega)\big)$ is given by

$$\mathbb{I}\big(S(\omega)\big) \cong \left\langle \begin{pmatrix} x^{i+1} \\ 0 \end{pmatrix}, \begin{pmatrix} xz \\ y^{j+1} \end{pmatrix}, \begin{pmatrix} 0 \\ yz \end{pmatrix} \right\rangle_{\mathsf{P}} \cong \big(x^{i+1}, xz + y^{j+1}, yz\big)_{\mathsf{P}}$$

(3) Let $\omega = \mathbf{x}_0 \mathbf{y}_{j_1}^- \mathbf{x}_i^- \mathbf{y}_{j_2}^+$, where $i, j_1, j_2 \in \mathbb{N}$. Then $\mathbb{I}\big(S(\omega)\big)$ is given by

$$\mathbb{I}\big(S(\omega)\big) \cong \left\langle \begin{pmatrix} x + y^{j_1+1} \\ 0 \end{pmatrix}, \begin{pmatrix} yz \\ x^{i+1} \end{pmatrix}, \begin{pmatrix} 0 \\ (x+y)\, z \end{pmatrix}, \begin{pmatrix} 0 \\ y^{j_2+1} \end{pmatrix} \right\rangle_{\mathsf{P}}$$

(4) Let $\omega = \mathbf{x}_\infty \mathbf{y}_j^+ \mathbf{x}_i^- \mathbf{y}_\infty$ with $i, j \in \mathbb{N}$. Then $\mathbb{I}\big(S(\omega)\big)$ is given by

$$\mathbb{I}\big(S(\omega)\big) \cong \left\langle \begin{pmatrix} xz \\ yz \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y^{j+1} \\ x^{i+1} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ xz \\ yz \end{pmatrix} \right\rangle_{\mathsf{P}} \cong \left\langle \begin{pmatrix} z \\ 0 \end{pmatrix}, \begin{pmatrix} y^j \\ x^i \end{pmatrix}, \begin{pmatrix} 0 \\ z \end{pmatrix} \right\rangle_{\mathsf{P}}$$

3.3. **Cohen–Macaulay modules over $\mathsf{T}_{\infty\infty}$ and their matrix factorizations.** Our next motivation was to study Cohen–Macaulay modules over the hypersurface singularity $\mathsf{T} = \Bbbk[\![a, b]\!]/(a^2b^2)$. At the beginning of Section 3 we have constructed a fully faithful functor $\mathbb{J}\colon \mathsf{CM}(\mathsf{R}) \hookrightarrow \mathsf{CM}(\mathsf{T})$. Its explicit description, adapted to the combinatorics of bands and strings, was explained in Lemma 3.24.

Remark 3.32. The image under the functor $\mathbb{J}$ of any band (respectively, string) Cohen–Macaulay R–module M defined by (3.8) (respectively, (3.11)) can be computed as follows.

- Let M be a band module $B(\widetilde{\omega}, m, \lambda)$. To compute $\mathbb{J}(M)$, we use Lemma 3.24 and the table

x^i	u	y^j	v
a^{i+2}	a^2b	b^{j+2}	ab^2

- Let $M = S(\omega)$ be a string module. In this case, the translation rule is the following:

x^i	ux	y^j	vy
a^{i+1}	a^2b	b^{j+1}	ab^2

Example 3.33. Now we translate the string and band modules over R from Example 3.20 into indecomposable Cohen-Macaulay modules over T using Remark 3.32.

(1) Let $(\widetilde{\omega}, m, \lambda)$ be a band datum with $\widetilde{\omega} = \mathbf{x}_i^- \mathbf{y}_{j_1}^- \mathbf{x}_i^+ \mathbf{y}_{j_2}^-$, where $i, j_1, j_2 \in \mathbb{N}$. Then the image of the corresponding band module $B(\widetilde{\omega}, m, \lambda)$ under $\mathbb{J}$ is

$$\left\langle \begin{pmatrix} (a^2b + b^{j_1+2})\, I \\ 0 \end{pmatrix}, \begin{pmatrix} ab^2\, I \\ a^2b\, I \end{pmatrix}, \begin{pmatrix} 0 \\ (a^{i+2} + b^{j_2+2})\, I \end{pmatrix} \begin{pmatrix} a^{i+2}\, J \\ ab^2\, I \end{pmatrix} \right\rangle_{\mathsf{T}}$$

(2) Let $\omega = \mathbf{x}_i^- \mathbf{y}_j^-$, where $i, j \in \mathbb{N}$. Then $\mathbb{J}\big(S(\omega)\big)$ is given by

$$\mathbb{J}(S) \cong \big(a^{i+2}, a^2b + y^{j+2}, ab^2\big)_{\mathsf{T}}$$

(3) Let $\omega = \mathbf{x}_0\mathbf{y}_{j_1}^-\mathbf{x}_i^-\mathbf{y}_{j_2}^+$, where $i, j_1, j_2 \in \mathbb{N}$. Then $\mathbb{J}\big(S(\omega)\big)$ is given by

$$\mathbb{J}\big(S(\omega)\big) \cong \left\langle \begin{pmatrix} a^2 + b^{j_1+2} \\ 0 \end{pmatrix}, \begin{pmatrix} ab^2 \\ a^{i+2} \end{pmatrix}, \begin{pmatrix} 0 \\ a^2b + ab^2 \end{pmatrix}, \begin{pmatrix} 0 \\ b^{j_2+2} \end{pmatrix} \right\rangle_{\mathsf{T}}$$

(4) Let $\omega = \mathbf{x}_\infty\mathbf{y}_j^+\mathbf{x}_i^-\mathbf{y}_\infty$, where $i, j \in \mathbb{N}$. Then $\mathbb{J}\big(S(\omega)\big)$ is given by

$$\mathbb{J}\big(S(\omega)\big) \cong \left\langle \begin{pmatrix} a^2b + ab^2 \\ 0 \end{pmatrix}, \begin{pmatrix} b^{j+2} \\ a^{i+2} \end{pmatrix}, \begin{pmatrix} 0 \\ a^2b + ab^2 \end{pmatrix} \right\rangle_{\mathsf{T}}$$

Remark 3.34. There is an involution τ on $\mathsf{R} = \Bbbk[\![x, y, u, v]\!]/(xy, xv, yu, uv, u^2, v^2)$ which interchanges x and y, u and v. Restricted to $\mathsf{P} = \Bbbk[\![x, y, z]\!]/(xy, z^2) \subset \mathsf{R}$, τ is still an involution such that $\tau(z) = z$. The restriction of τ to $\mathsf{T} = \Bbbk[\![a, b]\!]/(a^2b^2) \subset A$ interchanges a and b. Overall, τ induces an involution on the category of Cohen–Macaulay modules over R, P or T. The corresponding action of τ on words ω of string or band data of $\mathsf{CM}(\mathsf{R})$ is given by interchanging $\mathbf{x}$ and $\mathbf{y}$ in ω.

In the following table, we give a list of all indecomposable Cohen–Macaulay ideals of R and P (except P itself) and the corresponding ideals of T up to isomorphism and involution τ. Let $i, j \in \mathbb{N}$ and $\lambda \in \Bbbk^*$. For all band data the multiplicity parameter m is set to 1.

word of $\mathfrak{G}$	ideal in R	ideal in P	ideal in T
$\mathbf{x}_0$	(x)	(x)	(a^2)
$\mathbf{x}_\infty$	(u)	(xz)	(a^2b)
$\mathbf{x}_i^-$	(x^i, u)	(x^{i+1}, xz)	(a^{i+2}, a^2b)
$\mathbf{y}_0\mathbf{x}_0$	(1)	$(x + y, xz)$	$(a^2 + b^2, a^2b, ab^2)$
$\mathbf{x}_i^+\mathbf{y}_0$	$(ux, x^{i+1} + y)$	$(xz, x^{i+1} + y)$	$(a^2b, a^{i+2} + b^2, ab^2)$
$\mathbf{x}_i^+\mathbf{y}_j^-$	$(u, x^i + y^j, v)$	$(xz, x^{i+1} + y^{j+1}, yz)$	$(a^2b, a^{i+2} + b^{j+2}, ab^2)$
$\mathbf{x}_i^-\mathbf{y}_0$	$(x^{i+1}, ux + y)$	$(x^{i+1}, xz + y)$	$i = 1 : (a^3, a^2b + b^2)$ $i \geqslant 2 : (a^{i+2}, a^2b + b^2, ab^2)$
$\mathbf{x}_i^-\mathbf{y}_j^-$	$(x^i, u + y^j, v)$	$(x^{i+1}, xz + y^{j+1}, yz)$	$(a^{i+2}, a^2b + b^{j+2}, ab^2)$
$\mathbf{x}_i^-\mathbf{y}_j^+$	$(x^i, u + v, y^j)$	(x^i, z, y^j)	$(a^{i+2}, a^2b + ab^2, b^{j+2})$
$\mathbf{y}_0\mathbf{x}_\infty$	$(ux + y)$	$(xz + y)$	$(a^2b + b^2, ab^2)$
$\mathbf{x}_\infty\mathbf{y}_j^-$	$(u + y^j, v)$	$(xz + y^{j+1}, yz)$	$(a^2b + b^{j+2}, ab^2)$
$\mathbf{x}_\infty\mathbf{y}_j^+$	$(u + v, y^j)$	(z, y^j)	$(a^2b + ab^2, b^{j+2})$
$\mathbf{y}_\infty\mathbf{x}_\infty$	$(u + v)$	(z)	(ab)
$(\mathbf{x}_i^-\mathbf{y}_j^-, 1, \lambda)$	$(x^i + \lambda v, y^j + u)$	$(x^{i+1} + \lambda yz, y^{j+1} + xz)$	$(a^{i+2} + \lambda ab^2, b^{j+2} + a^2b)$ unless $i = j = 1, \lambda = 1$
$(\mathbf{x}_1^-\mathbf{y}_1^-, 1, 1)$	$(x + v, y + u)$	$(x^2 + yz, y^2 + xz)$	(a, b)
$(\mathbf{x}_i^-\mathbf{y}_j^+, 1, \lambda)$	$(x^i + \lambda y^j, u + v)$	$(x^i + \lambda y^j, z)$	$(a^{i+2} + \lambda b^{j+2}, a^2b + ab^2)$

Remark 3.35. The above list does not contain all indecomposable ideals of T. For example, the ideal $(a^2+\lambda b^2, a^2b+ab^2)$ is not a restriction of an ideal in R for any $\lambda \in \Bbbk^*$.

Let $\underline{\mathsf{MF}}(a^2b^2)$ be the homotopy category of matrix factorizations of a^2b^2. By a result of Eisenbud [18] there is an equivalence of triangulated categories $\underline{\mathsf{CM}}(\mathsf{T}) \xrightarrow{\sim} \underline{\mathsf{MF}}(a^2b^2)$. In the following table, we list the matrix factorizations of a^2b^2 which originate from an indecomposable ideal in R (up to isomorphism and involution). Let $i, j \in \mathbb{N}$ and $\lambda \in \Bbbk^*$.

ideal in $\Bbbk[\![a,b]\!]/(a^2b^2)$	matrix factorization (ϕ,ψ) of a^2b^2
(a^2)	$(b^2)\,(a^2)$
(a^2b)	$(b)\,(a^2b)$
(a^{i+2}, a^2b)	$\begin{pmatrix} b & 0 \\ -a^i & b \end{pmatrix} \begin{pmatrix} a^2b & 0 \\ a^{i+2} & a^2b \end{pmatrix}$
$(a^{i+1}+b^{j+1}, a^2b, ab^2)$	$\begin{pmatrix} ab & 0 & 0 \\ -a^i & b & 0 \\ -b^j & 0 & a \end{pmatrix} \begin{pmatrix} ab & 0 & 0 \\ a^{i+1} & a^2b & 0 \\ b^{j+1} & 0 & ab^2 \end{pmatrix}$
(a^3, a^2b+b^2)	$\begin{pmatrix} ab & 0 \\ -a^2 & a^2b \end{pmatrix} \begin{pmatrix} ab & 0 \\ -a & b \end{pmatrix}$
$(a^{i+3}, a^2b+b^2, ab^2)$	$\begin{pmatrix} b & 0 & 0 \\ -a^{i+1} & ab & 0 \\ 0 & -b & a \end{pmatrix} \begin{pmatrix} a^2b & 0 & 0 \\ a^{i+2} & ab & 0 \\ a^{i+1}b & b^2 & ab^2 \end{pmatrix}$
$(a^{i+2}, a^2b+b^{j+2}, ab^2)$	$\begin{pmatrix} b & 0 & 0 \\ -a^i & ab & 0 \\ a^{i-1}b^j & -b^{j+1} & a \end{pmatrix} \begin{pmatrix} a^2b & 0 & 0 \\ a^{i+1} & ab & 0 \\ 0 & b^{j+2} & ab^2 \end{pmatrix}$
$(a^{i+2}, b^{j+2}, a^2b+ab^2)$	$\begin{pmatrix} b & 0 & 0 \\ 0 & a & 0 \\ -a^i & -b^j & ab \end{pmatrix} \begin{pmatrix} a^2b & 0 & 0 \\ 0 & ab^2 & 0 \\ a^{i+1} & b^{j+1} & ab \end{pmatrix}$
(a^2b+b^{j+1}, ab^2)	$\begin{pmatrix} ab & 0 \\ -b^j & a \end{pmatrix} \begin{pmatrix} ab & 0 \\ b^{j+1} & ab^2 \end{pmatrix}$
(b^{j+2}, a^2b+ab^2)	$\begin{pmatrix} a & 0 \\ -b^j & ab \end{pmatrix} \begin{pmatrix} ab^2 & 0 \\ b^{j+1} & ab \end{pmatrix}$
(ab)	$(ab)\,(ab)$
$(a^{i+2}+\lambda ab^2, a^2b+b^{j+2})$ where i or j or $\lambda \neq 1$	$\begin{pmatrix} ab & -b^{j+1} \\ -a^{i+1} & \lambda ab \end{pmatrix} \begin{pmatrix} u\,ab & \lambda^{-1}\,u\,b^{j+1} \\ \lambda^{-1}\,u\,a^{i+1} & \lambda^{-1}\,u\,ab \end{pmatrix}$ where u is the unit $\left(1-\lambda^{-1}a^{i-1}b^{j-1}\right)^{-1}$
(a,b)	$\begin{pmatrix} ab^2 & 0 \\ 0 & a^2b \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$

ideal in $\mathbb{k}[\![a,b]\!]/(a^2b^2)$	matrix factorization (ϕ,ψ) of a^2b^2
$(a^3+ab^2, a^2b+b^3, ab^3)$	$\begin{pmatrix} b & 0 & 0 \\ -a & ab & 0 \\ 0 & -b & a \end{pmatrix} \begin{pmatrix} a^2b & 0 & 0 \\ a^2 & ab & 0 \\ ab & b^2 & ab^2 \end{pmatrix}$
$(a^{i+2}+\lambda b^{j+2}, a^2b+ab^2)$	$\begin{pmatrix} ab & 0 \\ -\lambda b^{j+1}-a^{i+1} & ab \end{pmatrix} \begin{pmatrix} ab & 0 \\ \lambda b^{j+1}+a^{i+1} & ab \end{pmatrix}$

Remark 3.36. By Knörrer's periodicity [24] the functor

$$\begin{array}{ccc} \underline{\mathsf{MF}}(a^2b^2) & \xrightarrow{\sim} & \underline{\mathsf{MF}}(a^2b^2+uv) \\ \bigl(\phi,\psi\bigr) & \longmapsto & \begin{pmatrix} \phi & -u\cdot I \\ v\cdot I & \psi \end{pmatrix} \begin{pmatrix} \psi & v\cdot I \\ -u\cdot I & \phi \end{pmatrix} \end{array}$$

is an equivalence of triangulated categories. It allows to get explicit families of matrix factorizations of any potential of type

$$a^2b^2 + u_1v_1 + \ldots + u_dv_d \in \mathbb{k}[\![a,b,u_1,\ldots,u_d,v_1,\ldots,v_d]\!].$$

Remark 3.37. Let $\mathsf{char}(\mathbb{k}) \neq 2$. Then there is a ring isomorphism

$$\mathbb{k}[\![a,b,c]\!]/(a^2b^2-c^2) \cong \mathbb{k}[\![x,y,z]\!]/(z^2-xyz) =: \mathsf{T}_{\infty\infty 2}.$$

The indecomposable Cohen–Macaulay modules over the surface singularity $\mathsf{T}_{\infty\infty 2}$ have been classified in [11]. On the other hand, Knörrer's correspondence [24] relates $\mathsf{T}_{\infty\infty 2}$ to $\mathsf{T}_{\infty\infty}$ by a restriction functor

$$\underline{\mathsf{MF}}(a^2b^2-c^2) \longrightarrow \underline{\mathsf{MF}}(a^2b^2),$$

such that every indecomposable matrix factorization of a^2b^2 appears as a *direct summand* of the restriction of some indecomposable matrix factorization of $a^2b^2-c^2$. With some efforts, one can compute the matrix factorizations of a^2b^2 corresponding to Cohen–Macaulay $\mathsf{T}_{\infty\infty 2}$–modules of small rank. However, the derivation of all indecomposable matrix factorizations of a^2b^2 is not straightforward by this approach.

Remark 3.38. The approach to classify indecomposable Cohen–Macaulay modules using the technique of tame matrix problems is close in spirit to the study of torsion free sheaves on degenerations of elliptic curves. See [4] for a survey of the corresponding results and methods.

3.4. **Some remarks on the stable category of Cohen–Macaulay modules.** Let $(\mathsf{A},\mathfrak{m})$ be a Gorenstein singularity (of any Krull dimension d). By a result of Buchweitz [8], the natural functor

$$\underline{\mathsf{CM}}(\mathsf{A}) \longrightarrow D_{sg}(\mathsf{A}) := \frac{D^b\bigl(\mathsf{A}-\mathsf{mod}\bigr)}{\mathsf{Perf}(\mathsf{A})}$$

is an equivalence of triangulated categories. If the singularity A is not isolated, then $\underline{\mathsf{CM}}(\mathsf{A})$ is Hom–infinite [1]. On the other hand, the stable category of Cohen–Macaulay modules $\underline{\mathsf{CM}}^{\mathsf{lf}}(\mathsf{A})$ is always a Hom–finite triangulated subcategory of $\underline{\mathsf{CM}}(A)$. By a result of Auslander [1], the category $\underline{\mathsf{CM}}^{\mathsf{lf}}(A)$ is $(d-1)$–Calabi–Yau. This means that for any objects M_1 and M_2 of $\underline{\mathsf{CM}}^{\mathsf{lf}}(\mathsf{A})$ we have an isomorphism

$$\underline{\mathsf{Hom}}_{\mathsf{A}}(M_1, M_2) \cong \mathbb{D}\big(\underline{\mathsf{Hom}}_{\mathsf{A}}(M_2, \Sigma^{d-1}(M_1))\big),$$

functorial in both arguments M_1 and M_2, where $\mathbb{D}$ is the Matlis duality functor and $\Sigma = \Omega^{-1}$ is the suspension functor. In particular, if A is a Gorenstein curve singularity, then for any $M \in \underline{\mathsf{CM}}^{\mathsf{lf}}(\mathsf{A})$ the algebra $\underline{\mathrm{End}}_{\mathsf{A}}(M)$ is Frobenius. Thus, Theorem 2.1 gives a family of examples of representation tame 0–Calabi–Yau triangulated categories and Theorem 3.28 provides a complete and explicit description of indecomposable objects in one of such categories $\underline{\mathsf{CM}}^{\mathsf{lf}}(\mathsf{P})$ for $\mathsf{P} = \Bbbk[\![x, y, z]\!]/(xy, z^2)$.

4. On the definition of tame representation type of a curve singularity

Let $\Bbbk$ be an *uncountable* algebraically closed field (the field of complex numbers $\mathbb{C}$ is of major interest) and $(\mathsf{A}, \mathfrak{m})$ a Cohen–Macaulay $\Bbbk$–algebra of Krull dimension one (a Cohen–Macaulay curve singularity).

Definition 4.1. Let $\mathsf{Ind}\big(\mathsf{CM}(\mathsf{A})\big)$ be the set of the isomorphism classes of indecomposable Cohen–Macaulay A–modules. The Cohen–Macaulay representation type of A is

- *finite* if the set $\mathsf{Ind}\big(\mathsf{CM}(\mathsf{A})\big)$ is finite.
- *discrete* if the set $\mathsf{Ind}\big(\mathsf{CM}(\mathsf{A})\big)$ is infinite but *countable*.
- *wild* if for any finitely generated commutative $\Bbbk$–algebra Λ there exists an exact functor $\Lambda - \mathsf{fdmod} \xrightarrow{\mathbb{E}} \mathsf{CM}(\mathsf{A})$ preserving indecomposability and isomorphism classes of objects.

The definition of tame representation type of a Cohen–Macaulay curve singularity is more involved. Let $\mathsf{Q} = \mathsf{Q}(\mathsf{A})$ be the total ring of fractions of A. According to Lemma 1.2, Q is an Artinian ring. For a Cohen–Macaulay A–module M we denote by $M_{\mathsf{Q}} := \mathsf{Q} \otimes_{\mathsf{A}} M$ its rational envelope, also called the *rational type* of M in what follows. For any finite length Q–module F we denote by $\mathsf{Ind}_F\big(\mathsf{CM}(\mathsf{A})\big)$ the set of the isomorphism classes of indecomposable Cohen–Macaulay A–modules such that $M_{\mathsf{Q}} \cong F$.

Let $X = \mathsf{Spec}(\mathsf{A})$ be the spectrum of A. As usual, we shall identify a coherent sheaf on X with the corresponding Noetherian A–module of global sections.

Definition 4.2. Let F be a finite length Q–module and T a scheme of finite type over $\Bbbk$. A coherent sheaf $\mathcal{F}$ on $X \times T$ is a *family* of Cohen–Macaulay modules of *rational type* F over the *base* T if the following conditions are satisfied:

- $\mathcal{F}$ is flat over T.
- For any point $\lambda \in T$ we have: $\mathcal{F}_\lambda := \mathcal{F}\big|_{X \times \{\lambda\}}$ is a Cohen–Macaulay A–module.
- Moreover, $\big(\mathcal{F}_\lambda\big)_{\mathsf{Q}} \cong F$ for all $\lambda \in T$.

Definition 4.3. A Cohen–Macaulay curve singularity A has *tame* Cohen–Macaulay representation type if the following conditions are satisfied.

- The set $\mathsf{Ind}\big(\mathsf{CM}(\mathsf{A})\big)$ is uncountable.
- The set $\mathsf{Ind}\big(\mathsf{Q}\text{–}\mathsf{mod}\big)$ is finite, i.e. the Artinian ring Q has finite representation type.
- For any finite length Q–module F there exists an at most countable (but possibly finite or empty) set $\Omega(F)$ with the following properties.
  - For any $i \in \Omega(F)$ there exists a quasi–projective curve E_i and a family of maximal Cohen–Macaulay A–modules $\mathcal{F}_i$ of rational type F such that for any $\lambda \in E_i$ the Cohen–Macaulay A–module $\mathcal{F}_{i,\lambda} := \mathcal{F}_i\big|_{X \times \{\lambda\}}$ is indecomposable.

– For any indecomposable Cohen–Macaulay A–module M of rational type F there exist $i \in \Omega(F)$ and $\lambda \in E_i$ such that $M \cong \mathcal{F}_{i,\lambda}$.

In other words, Cohen–Macaulay representation tameness of A means that $\mathsf{Ind}\big(\mathsf{CM}(\mathsf{A})\big)$ is uncountable and can be written as a union of countably many families, whose bases are quasi–projective curves.

Remark 4.4. Let A be a Cohen–Macaulay curve singularity over $\mathbb{k}$.

- If A is reduced then Q is semi–simple, hence of finite representation type. The rational type of a Cohen–Macaulay A–module is given by its multi–rank. In this case, our definition of tameness coincides with the one of Drozd and Greuel [15, Section 1]. Note that for the reduced Cohen–Macaulay tame singularities, all sets $\Omega(F)$ are actually *finite* [15].
- Our definition of tameness for curve singularities is consistent with the corresponding definition for surface singularities [11, Definition 8.13] as well as with the notion of tameness for finite dimensional $\mathbb{k}$–algebras [11, Proposition 8.20].
- Conjecturally, any Cohen–Macaulay curve singularity has either finite, countable, tame or wild Cohen–Macaulay representation type. This result is known to be true for the reduced curve singularities [14]. In fact, the Cohen–Macaulay representation type of all reduced curve singularities was determined in [15].
- In all non–wild cases of Cohen–Macaulay curve singularities known to us, the total ring of fractions Q is a product of several copies of $\mathbb{k}(\!(t)\!)$ or $\mathbb{k}(\!(t)\!)[\varepsilon]/(\varepsilon^2)$.
- A rigorous definition of Cohen–Macaulay representation types in the case the base field $\mathbb{k}$ is *countable* (e.g. $\overline{\mathbb{Q}}$) requires further elaboration.

Theorem 4.5. *Let* $\mathsf{P} = \mathbb{k}[\![x, y, z]\!]/(xy, z^2)$ *and* R *be its minimal overring. Then* P *and* R *have tame Cohen–Macaulay representation type in the sense of Definition 4.3.*

Proof. First of all, note that $\mathsf{Q} := \mathsf{Q}(\mathsf{P}) = \mathsf{Q}(\mathsf{R}) \cong \mathbb{k}(\!(t)\!)[\varepsilon]/(\varepsilon^2) \times \mathbb{k}(\!(t)\!)[\varepsilon]/(\varepsilon^2)$. Therefore, Q has finite representation type, as required in Definition 4.3.

Using the morphism of schemes $\mathsf{Spec}(\mathsf{R}) \to \mathsf{Spec}(\mathsf{P})$, we can push–forward any family of indecomposable Cohen–Macaulay R–modules to a family of indecomposable P–modules. Because of Theorem 1.11, it is sufficient to prove tameness of R.

Let M be a Cohen–Macaulay R–module and $\widetilde{M} := \mathsf{S} \otimes_{\mathsf{R}} M$, where $\mathsf{S} = \mathbb{k}[\![x, u]\!]/(u^2) \times \mathbb{k}[\![y, v]\!]/(v^2)$. Clearly, the rational types of M and $\widetilde{M}$ are the same. For any indecomposable Cohen–Macaulay module M of string type, we attach a constant family of Cohen–Macaulay modules. Clearly, for any finite length Q–module F, there are at most countably many indecomposable Cohen–Macaulay R–modules of rational type F, which are strings. Therefore, we only need to care about families for band modules.

Let $(\widetilde{w}, m, \lambda)$ be a band datum of the bunch of chains $\mathfrak{B}$ from Definition 3.4. We denote by $\widetilde{\omega}$ the corresponding non–periodic word over the alphabet $\mathfrak{G}$, as in Definition 3.16. Let $N = N(\widetilde{\omega})$ be the Cohen–Macaulay S–module from Definition 3.16 and $B = B(\widetilde{\omega}, m, \lambda)$ the indecomposable Cohen–Macaulay R–module given by (3.8). Note that $B_{\mathsf{Q}} \cong N_{\mathsf{Q}} \cong \mathsf{Q}^r$ for some $r \in \mathbb{N}$.

Put $\mathsf{C} = \mathbb{k}[t, t^{-1}]$ and $T = \mathsf{Spec}(\mathsf{C}) = \mathbb{A}^1_{\mathbb{k}} \backslash \{0\}$. For any R–module L we denote $L[t^{\pm}] := L \otimes_{\mathbb{k}} \mathsf{C}$. Note that $L[t^{\pm}]$ is free and hence *flat* as a module over C. Now we define the following $\mathsf{R}[t^{\pm}]$–module $M = M(\widetilde{\omega}, m)$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (IN)[t^{\pm}] & \longrightarrow & M & \longrightarrow & \mathsf{C}^r & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow {\scriptstyle \widetilde{\Theta}[t^{\pm}]} & & \\ 0 & \longrightarrow & (IN)[t^{\pm}] & \longrightarrow & N[t^{\pm}] & \xrightarrow{\pi} & \check{N}[t^{\pm}] & \longrightarrow & 0, \end{array} \tag{4.1}$$

where the matrix $\widetilde{\Theta}[t^{\pm}] := \begin{pmatrix} \Theta_x(\widetilde{w}, m) \\ \Theta_y(\widetilde{w}, m) \end{pmatrix}$ is determined by the same recipe as in the discussion of Theorem 3.7. Observe that both modules $(IN)[t^{\pm}]$ and C^r are free over C. Thus, $M(\widetilde{\omega}, m)$ is free (hence, flat) over C as well. According to Theorem 1.19 and Theorem 3.25 we have: $\big(M(\widetilde{\omega}, m)\big)\big|_{X \times \{\lambda\}} \cong B(\widetilde{\omega}, m, \lambda)$ for any $\lambda \in T$. Therefore, $M(\widetilde{\omega}, m)$ is a family of indecomposable Cohen–Macaulay R–modules of rational type Q^r over the base T. This gives us a recipe to construct the set $\Omega\big(\mathsf{Q}^r\big)$ from the Definition 4.3 as well as the corresponding families of indecomposable Cohen–Macaulay R–modules. □

Remark 4.6. In the same way, based on the proof of Theorem 2.1 and Bondarenko's result [5], we can complete the formal proof of tameness of the curve singularities $\mathsf{P}_{\infty,q} = \Bbbk[\![x, y, z]\!]/(xy, y^q - z^2)$ for $q \in \mathbb{N}_{\geqslant 2}$.

References

[1] M. Auslander, *Functors and morphisms determined by objects*, Representation theory of algebras, Lecture Notes in Pure Appl. Math. **37**, Dekker, New York, 1978, 1–244.
[2] M. Auslander, *Isolated singularities and existence of almost split sequences*, Representation theory II, Proc. ICRA IV, Springer Lecture Notes in Math. **1178**, Springer, Berlin, 1986, 194–242.
[3] H. Bass, *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1963), 8–28.
[4] L. Bodnarchuk, I. Burban, Yu. Drozd and G.-M. Greuel, *Vector bundles and torsion free sheaves on degenerations of elliptic curves*, Global aspects of complex geometry, Springer, Berlin, 2006, 83–128.
[5] V. M. Bondarenko, *Representations of bundles of semichained sets and their applications*, Algebra i Analiz **3** (1991), no. **5**, 38–61.
[6] N. Bourbaki, *Commutative algebra. Chapters 1–7*, Elements of Mathematics, Springer, Berlin, 1998.
[7] W. Bruns and J. Herzog, *Cohen–Macaulay Rings*, Cambridge Studies in Advanced Mathematics **39**, Cambridge Univ. Press, 1993.
[8] R.-O. Buchweitz, *Maximal Cohen–Macaulay modules and Tate–Cohomology over Gorenstein rings*, Preprint 1987.
[9] R.-O. Buchweitz, G.-M. Greuel and F.-O. Schreyer, *Cohen–Macaulay modules on hypersurface singularities II*, Invent. Math. **88** (1987), no. **1**, 165–182.
[10] I. Burban and Yu. Drozd, *Maximal Cohen–Macaulay modules over surface singularities*, Trends in representation theory of algebras and related topics, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2008, 101–166.
[11] I. Burban and Yu. Drozd, *Maximal Cohen–Macaulay modules over non–isolated surface singularities*, arXiv:1002.3042, to appear in the Memoirs of the AMS.
[12] I. Burban, O. Iyama, B. Keller and I. Reiten, *Cluster tilting for one–dimensional hypersurface singularities*, Adv. Math. **217** (2008), no. **6**, 2443–2484.
[13] Yu. Drozd, *Matrix problems and categories of matrices*, Investigations on the theory of representations, Zap. Nauchn. Semin. Leningr. Otd. Mat. Inst. Steklova **28**, 1972, 144–153.
[14] Yu. Drozd and G.-M. Greuel, *Tame–wild dichotomy for Cohen–Macaulay modules*, Math. Ann. **294** (1992), no. **3**, 387–394.
[15] Yu. Drozd and G.-M. Greuel, *Cohen–Macaulay module type*, Compositio Math. **89** (1993), no. **3**, 315–338.
[16] Yu. Drozd, G.-M. Greuel and I. Kashuba, *On Cohen–Macaulay modules on surface singularities*, Mosc. Math. J. **3** (2003), no. **2**, 397–418.
[17] Yu. Drozd and A. Roiter, *Commutative rings with a finite number of indecomposable integral representations*, Izv. Akad. Nauk SSSR Ser. Mat. **31** (1967), 783–798.

[18] D. Eisenbud, *Homological algebra on a complete intersection, with an application to group representations*, Trans. Amer. Math. Soc. **260** (1980), no. **1**, 35–64.
[19] G.-M. Greuel and H. Knörrer, *Einfache Kurvensingularitäten und torsionsfreie Moduln*, Math. Ann. **270** (1985), no. **3**, 417–425.
[20] G.-M. Greuel and H. Kröning, *Simple singularities in positive characteristic*, Math. Z. **203** (1990), no. **2**, 339–354.
[21] H. Jacobinski, *Sur les ordres commutatifs avec un nombre fini de réseaux indécomposables*, Acta Math. **118** (1967), 1–31.
[22] C. Kahn, *Reflexive modules on minimally elliptic singularities*, Math. Ann. **285** (1989), no. **1**, 141–160.
[23] K. Kiyek and G. Steinke, *Einfache Kurvensingularitäten in beliebiger Charakteristik*, Arch. Math. (Basel) **45** (1985), no. **6**, 565–573.
[24] H. Knörrer, *Cohen–Macaulay modules on hypersurface singularities I*, Invent. Math **88** (1987), no. **1**, 153–164.
[25] G. Leuschke and R. Wiegand, *Cohen–Macaulay representations*, Mathematical Surveys and Monographs **181**, American Mathematical Society, Providence, RI, 2012.
[26] G. Leuschke and R. Wiegand, *Local rings of bounded Cohen–Macaulay type*, Algebr. Represent. Theory **8** (2005), no. **2**, 225–238.
[27] S. Mac Lane, *Categories for the working mathematician*, Graduate Texts in Mathematics **5**, Springer, New York, 1998.
[28] Y. Yoshino, *Cohen–Macaulay modules over Cohen–Macaulay rings*, London Mathematical Society Lecture Note Series **146**, Cambridge Univ. Press, 1990.

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