VECTOR BUNDLES ON PLANE CUBIC CURVES AND THE CLASSICAL YANG–BAXTER EQUATION

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Abstract. In this article, we develop a geometric method to construct solutions of the classical Yang–Baxter equation, attaching to the Weierstrass family of plane cubic curves and a pair of coprime positive integers, a family of classical r–matrices. It turns out that all elliptic r–matrices arise in this way from smooth cubic curves. For the cuspidal cubic curve, we prove that the obtained solutions are rational and compute them explicitly. We also describe them in terms of Stolin's classification and prove that they are degenerations of the corresponding elliptic solutions.

1. Introduction

Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ and $U = U(\mathfrak{g})$ be its universal enveloping algebra. The classical Yang–Baxter equation (CYBE) is

$$[r^{12}(x_1,x_2), r^{13}(x_1,x_3)] + [r^{13}(x_1,x_3), r^{23}(x_2,x_3)] + [r^{12}(x_1,x_2), r^{23}(x_2,x_3)] = 0, \tag{1}$$

where $r : (\mathbb{C}^2, 0) \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ is the germ of a meromorphic function. The upper indices in this equation indicate various embeddings of $\mathfrak{g} \otimes \mathfrak{g}$ into $U \otimes U \otimes U$. For example, the function r^{13} is defined as

$$r^{13} : \mathbb{C}^2 \xrightarrow{\ r\ } \mathfrak{g} \otimes \mathfrak{g} \xrightarrow{\rho_{13}} U \otimes U \otimes U,$$

where $\rho_{13}(x \otimes y) = x \otimes 1 \otimes y$. Two other maps r^{12} and r^{23} have a similar meaning.

A solution of (1) (also called *r–matrix* in the physical literature) is *unitary* if $r(x_1,x_2) = -\rho\big(r(x_2,x_1)\big)$, where ρ is the automorphism of $\mathfrak{g} \otimes \mathfrak{g}$ permuting both factors. A solution of (1) is *non-degenerate* if its image under the isomorphism

$$\mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathsf{End}(\mathfrak{g}), \quad a \otimes b \mapsto \big(c \mapsto \mathsf{tr}(ac) \cdot b\big)$$

is an invertible operator for some (and hence, for a generic) value of the spectral parameters (x_1, x_2). On the set of solutions of (1) there exists a natural action of the group of holomorphic function germs $\phi : (\mathbb{C}, 0) \longrightarrow \mathrm{Aut}(\mathfrak{g})$ given by the rule

$$r(x_1,x_2) \mapsto \tilde{r}(x_1,x_2) := \big(\phi(x_1) \otimes \phi(x_2)\big) r(x_1,x_2). \tag{2}$$

It is easy to see that $\tilde{r}(x_1,x_2)$ is again a solution of (1). Moreover, $\tilde{r}(x_1,x_2)$ is unitary (respectively non-degenerate) provided $r(x_1,x_2)$ is unitary (respectively non-degenerate). The solutions $r(x_1,x_2)$ and $\tilde{r}(x_1,x_2)$ related by the formula (2) for some ϕ are called *gauge equivalent.*

According to Belavin and Drinfeld [4], any non-degenerate unitary solution of the equation (1) is gauge-equivalent to a solution $r(x_1,x_2) = r(x_2 - x_1)$ depending just on the difference (or the quotient) of spectral parameters. This means that (1) is essentially equivalent to the equation

$$\big[r^{12}(x), r^{13}(x+y)\big] + \big[r^{13}(x+y), r^{23}(y)\big] + \big[r^{12}(x), r^{23}(y)\big] = 0. \tag{3}$$

By a result of Belavin and Drinfeld [3], a non-degenerate solution of (3) is automatically unitary, has a simple pole at 0 with the residue equal to a multiple of the Casimir element, and is either *elliptic* or *trigonometric*, or *rational*. In [3], Belavin and Drinfeld also gave a complete classification of all elliptic and trigonometric solutions of (3). A classification of rational solutions of (3) was achieved by Stolin in [33, 34].

In this paper we study a connection between the theory of vector bundles on curves of genus one and solutions of the classical Yang–Baxter equation (1). Let $E = V(wv^2 - 4u^3 - g_2uw^2 - g_3w^3) \subset \mathbb{P}^2$ be a Weiestraß curve over $\mathbb{C}$, $o \in E$ some fixed smooth point and $0 < d < n$ two coprime integers. Consider the sheaf of Lie algebras $\mathcal{A} := \mathsf{Ad}(\mathcal{P})$, where $\mathcal{P}$ is a simple vector bundle $\mathcal{P}$ of rank n and degree d on E (note that up to an automorphism $\mathcal{A}$ does not depend on a particular choice of $\mathcal{P}$). For any pair of distinct smooth points x, y of E, consider the linear map $\mathcal{A}\big|_x \longrightarrow \mathcal{A}\big|_y$ defined as the composition:

$$\mathcal{A}\big|_x \xrightarrow{\mathsf{res}_x^{-1}} H^0\big(\mathcal{A}(x)\big) \xrightarrow{\mathsf{ev}_y} \mathcal{A}\big|_y, \tag{4}$$

where res_x is the *residue map* and ev_y is the *evaluation map*. Choosing a trivialization $\mathcal{A}(U) \xrightarrow{\xi} \mathfrak{sl}_n\big(\mathcal{O}(U)\big)$ of the sheaf of Lie algebras $\mathcal{A}$ for some small neighborhood U of o, we get the tensor $r^{\xi}_{(E,(n,d))}(x,y) \in \mathfrak{g} \otimes \mathfrak{g}$. The first main result of this paper is the following.

Theorem A. In the above notations we have:

- The tensor-valued function $r^{\xi}_{(E,(n,d))} : U \times U \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ is meromorphic. Moreover, it is a non-degenerate unitary solution of the classical Yang–Baxter equation (1).
- The function $r^{\xi}_{(E,(n,d))}$ is *analytic* with respect to the parameters g_2 and g_3.
- A different choice of trivialization $\mathcal{A}(U) \xrightarrow{\zeta} \mathfrak{sl}_n\big(\mathcal{O}(U)\big)$ gives a gauge equivalent solution $r^{\zeta}_{(E,(n,d))}$.

Our next aim is to describe all solutions of (3) corresponding to elliptic curves. Let $\varepsilon = \exp\big(\frac{2\pi i d}{n}\big)$ and $I := \big\{(p,q) \in \mathbb{Z}^2 \big| 0 \le p \le n-1, 0 \le q \le n-1, (p,q) \ne (0,0)\big\}$. Consider the following matrices

$$X = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \varepsilon & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \varepsilon^{n-1} \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \end{pmatrix}. \tag{5}$$

For any $(k,l) \in I$ denote $Z_{k,l} = Y^k X^{-l}$ and $Z^{\vee}_{k,l} = \frac{1}{n} X^l Y^{-k}$.

Theorem B. Let $\tau \in \mathbb{C}$ be such that $\mathrm{Im}(\tau) > 0$ and $E = \mathbb{C}/\langle 1, \tau\rangle$ be the corresponding complex torus. Let $0 < d < n$ be two coprime integers. Then we have:

$$r_{(E,(n,d))}(x,y) = \sum_{(k,l)\in I} \exp\Big(-\frac{2\pi i d}{n} kz\Big)\sigma\Big(\frac{d}{n}\big(l - k\tau\big), z\Big) Z^{\vee}_{k,l} \otimes Z_{k,l}, \tag{6}$$

where $z = y - x$ and

$$\sigma(a,z) = 2\pi i \sum_{n\in\mathbb{Z}} \frac{\exp(-2\pi i n z)}{1-\exp\bigl(-2\pi i(a - 2\pi i n\tau)\bigr)} \tag{7}$$

is the Kronecker elliptic function. Hence, $r_{(E,(n,d))}$ is the elliptic r–matrix of Belavin [2], see also [3, Proposition 5.1].

Our next goal is to describe the solutions of (1) corresponding to the data $(E,(n,d))$ for the cuspidal cubic curve $E = V(wv^2 - u^3)$. Using the classification of simple vector bundles on E due to Bodnarchuk and Drozd [7] as well as methods developed by Burban and Kreußler [14], we derive an explicit recipe to compute the tensor $r^{\xi}_{(E,(n,d))}(x,y)$ from Theorem A. It turns out that the obtained solutions of (1) are always rational. By Stolin's classification [33, 34], the rational solutions of (1) are parameterized by certain Lie algebraic objects, which we shall call *Stolin triples*. Such a triple $(\mathfrak{l}, k, \omega)$ consists of

- a Lie subalgebra $\mathfrak{l} \subseteq \mathfrak{g}$,
- an integer k such that $0 \le k \le n$,
- a skew symmetric bilinear form $\omega : \mathfrak{l}\times\mathfrak{l} \to \mathbb{C}$ which is a 2-cocycle, i.e.
$$\omega\bigl([a,b],c\bigr) + \omega\bigl([b,c],a\bigr) + \omega\bigl([c,a],b\bigr) = 0$$
for all $a,b,c \in \mathfrak{l}$,

such that for the k-th parabolic Lie subalgebra $\mathfrak{p}_k$ of $\mathfrak{g}$ (with $\mathfrak{p}_0 = \mathfrak{p}_n = \mathfrak{g}$) the following two conditions are fulfilled:

- $\mathfrak{l} + \mathfrak{p}_k = \mathfrak{g}$,
- ω is non-degenerate on $(\mathfrak{l}\cap\mathfrak{p}_k)\times(\mathfrak{l}\cap\mathfrak{p}_k)$.

Let $0 < d < n$ be two coprime integers, $e = n - d$. We construct a certain matrix $J \in \mathsf{Mat}_{n\times n}(\mathbb{C})$ whose entries are equal to 0 or 1 (and their positions are uniquely determined by n and d) such that the pairing

$$\omega_J : \mathfrak{p}_e \times \mathfrak{p}_e \longrightarrow \mathbb{C}, \quad (a,b) \mapsto \mathsf{tr}\bigl(J^t \cdot [a,b]\bigr)$$

is non-degenerate. The following result was conjectured by Stolin.

Theorem C. Let E be the cuspidal cubic curve and $0 < d < n$ a pair of coprime integers. Then the solution $r_{(E,(n,d))}$ of the classical Yang–Baxter equation (1), described in Theorem A, is gauge equivalent to the solution $r_{(\mathfrak{g},e,\omega_J)}$ attached to the Stolin triple $(\mathfrak{g}, e, \omega_J)$.

Moreover, we derive an algorithm to compute the solution $r_{(E,(n,d))}$ explicitly. In particular, for $d = 1$ this leads to the following closed formula (see Example 9.7):

$$r_{(E,(n,1))} \sim r_{(\mathfrak{g},n-1,\omega)} = \frac{c}{y-x} +$$
$$+x\left[e_{1,2}\otimes \check{h}_1 - \sum_{j=3}^{n} e_{1,j}\otimes\left(\sum_{k=1}^{n-j+1} e_{j+k-1,k+1}\right)\right] - y\left[\check{h}_1\otimes e_{1,2} - \sum_{j=3}^{n}\left(\sum_{k=1}^{n-j+1} e_{j+k-1,k+1}\right)\otimes e_{1,j}\right]$$
$$+\sum_{j=2}^{n-1} e_{1,j}\otimes\left(\sum_{k=1}^{n-j} e_{j+k,k+1}\right) + \sum_{i=2}^{n-1} e_{i,i+1}\otimes\check{h}_i - \sum_{j=2}^{n-1}\left(\sum_{k=1}^{n-j} e_{j+k,k+1}\right)\otimes e_{1,j} - \sum_{i=2}^{n-1}\check{h}_i\otimes e_{i,i+1}$$
$$+\sum_{i=2}^{n-2}\left(\sum_{k=2}^{n-i}\left(\sum_{l=1}^{n-i-k+1} e_{i+k+l-1,l+i}\right)\otimes e_{i,i+k}\right) - \sum_{i=2}^{n-2}\left(\sum_{k=2}^{n-i} e_{i,i+k}\otimes\left(\sum_{l=1}^{n-i-k+1} e_{i+k+l-1,l+i}\right)\right),$$

where c is the Casimir element in $\mathfrak{g} \otimes \mathfrak{g}$ with respect to the trace form, $e_{i,j}$ are the matrix units for $1 \le i, j \le n$, and $\check{h}_l$ is the dual of $h_l = e_{l,l} - e_{l+1,l+1}$, $1 \le l \le n-1$. Theorem A implies, that up to a certain (not explicitly known) gauge transformation and a change of variables, this rational solution is a degeneration of the Belavin's elliptic r–matrix (6) for $\varepsilon = \exp\bigl(\frac{2\pi i}{n}\bigr)$. It seems that it is rather difficult to prove this result using just direct analytic methods.

Finally, we show that the solutions $r_{(E,(n,d))}$ and $r_{(E,(n,e))}$ are gauge equivalent.

Notations and terminology. In this article we shall use the following notations.

– $\Bbbk$ denotes an algebraically closed field of characteristic zero.

– Given an algebraic variety X, $\mathsf{Coh}(X)$ respectively $\mathsf{VB}(X)$ denotes the category of coherent sheaves respectively vector bundles on X. We denote $\mathcal{O}$ the structure sheaf of X. Of course, the theory of Yang–Baxter equations is mainly interesting in the case $\Bbbk = \mathbb{C}$. In that case, one can (and probably should) work in the *complex analytic category*. However, all relevant results and proofs of this article remain valid in that case, too.

– We denote by $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X)$ the triangulated category of bounded complexes of $\mathcal{O}$–modules with coherent cohomology, whereas $\mathsf{Perf}(X)$ stands for the triangulated category of perfect complexes, i.e. the full subcategory of $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X)$ admitting a bounded locally free resolution.

– We always write Hom, End and Ext when working with global morphisms and extensions between coherent sheaves whereas Lin is used when we work with vector spaces. If not explicitly otherwise stated, Ext always stands for Ext^1.

– For a vector bundle $\mathcal{F}$ on X and $x \in X$, we denote by $\mathcal{F}\big|_x$ the fiber of $\mathcal{F}$ over x, whereas $\Bbbk_x$ denotes the skyscraper sheaf of length one supported at x.

– A *Weierstraß curve* is a plane projective cubic curve given in homogeneous coordinates by an equation $wv^2 = 4u^3 + g_2uw^2 + g_3w^3$, where $g_1, g_2 \in \Bbbk$. Such a curve is always irreducible. It is singular if and only if $\Delta(g_2, g_3) = g_2^3 + 27g_3^2 = 0$. Unless $g_2 = g_3 = 0$, the singularity is a *node*, whereas in the case $g_2 = g_3 = 0$ the singularity is a *cusp*.

– A *Calabi–Yau curve* E is a reduced projective Gorenstein curve with trivial dualizing sheaf. Note that the complete list of such curves is actually known, see for example [32, Section 3]: E is either

- an elliptic curve,
- a Kodaira cycle of $n \ge 1$ projective lines (for $n = 1$ it is a nodal Weierstraß curve), also called Kodaira fiber of type I_n,
- a cuspidal plane cubic curve (Kodaira fiber II), a tachnode cubic curve (Kodaira fiber III) or a generic configuration of n concurrent lines in $\mathbb{P}^{n-1}$ for $n \ge 3$.

The irreducible Calabi–Yau curves are precisely the Weierstraß curves. For a Calabi–Yau curve E we denote by $\breve{E}$ the regular part of E.

– We denote by Ω the sheaf of regular differential one–forms on a Calabi–Yau curve E, which we always view as a dualizing sheaf. Taking a non-zero section $w \in H^0(\Omega)$, we get an isomorphism of $\mathcal{O}$–modules $\mathcal{O} \xrightarrow{w} \Omega$.

– Next, $\mathcal{P}$ will always denote a simple vector bundle on a Calabi–Yau curve E, i.e. a locally free coherent sheaf satisfying $\mathsf{End}(\mathcal{P}) = \mathbb{k}$. Note that we automatically have: $\mathsf{Ext}(\mathcal{P}, \mathcal{P}) \cong \mathbb{k}$.

– Finally, for $n \ge 2$ we denote $\mathfrak{a} = \mathfrak{gl}_n(\mathbb{k})$ and $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{k})$. For $1 \le k \le n-1$ we denote by $\mathfrak{p}_k$ the k-th parabolic subalgebra of $\mathfrak{g}$.

Plan of the paper and overview of methods and results.

The main message of this article is the following: to any triple $(E, (n, d))$, where

- E is a Weierstraß curve,
- $0 < d < n$ is a pair of coprime integers,

one can *canonically* attach a solution $r_{(E,(n,d))}$ of the classical Yang–Baxter equation (1) for the Lie algebra $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, see Section 4. The construction goes as follows.

Let $\mathcal{P}$ be a simple vector bundle of rank n and degree d on E and $\mathcal{A} = \mathsf{Ad}(\mathcal{P})$ be the sheaf of traceless endomorphisms of $\mathcal{P}$. Obviously, $\mathcal{A}$ is a sheaf of Lie algebras on E satisfying $H^0(\mathcal{A}) = 0 = H^1(\mathcal{A})$. In can be shown that $\mathcal{A}$ does not depend on the particular choice of $\mathcal{P}$ and up to an isomorphism determined by n and d, see Proposition 2.14.

Let x, y be a pair of smooth points of E. Since the triangulated category $\mathsf{Perf}(E)$ has a (non-canonical) structure of an A_∞–category, we have the following linear map

$$\mathsf{m}_3 : \mathsf{Hom}(\mathcal{P}, \mathbb{k}_x) \otimes \mathsf{Ext}(\mathbb{k}_x, \mathcal{P}) \otimes \mathsf{Hom}(\mathcal{P}, \mathbb{k}_y) \longrightarrow \mathsf{Hom}(\mathcal{P}, \mathbb{k}_x).$$

Using Serre duality, we get from here the induced linear map

$$\overline{\mathsf{m}}_{x,y} : \mathfrak{sl}\big(\mathsf{Hom}(\mathcal{P}, \mathbb{k}_x)\big) \longrightarrow \mathfrak{pgl}\big(\mathsf{Hom}(\mathcal{P}, \mathbb{k}_y)\big)$$

and the corresponding tensor $\mathsf{m}_{x,y} \in \mathfrak{pgl}\big(\mathsf{Hom}(\mathcal{P}, \mathbb{k}_x)\big) \otimes \mathfrak{pgl}\big(\mathsf{Hom}(\mathcal{P}, \mathbb{k}_y)\big)$. It turns out that this element $m_{x,y}$ is a *triangulated invariant* of $\mathsf{Perf}(E)$, i.e. it does not depend on a (non-canonical) choice of an A_∞–structure on the category $\mathsf{Perf}(E)$.
Let E be an elliptic curve. According to Polishchuk, [31, Theorem 2], the tensor $\mathsf{m}_{x,y}$ is unitary and satisfies the classical Yang–Baxter equation

$$\big[\mathsf{m}^{12}_{x_1,x_2}, \mathsf{m}^{13}_{x_1,x_3}\big] + \big[\mathsf{m}^{12}_{x_1,x_2}, \mathsf{m}^{23}_{x_2,x_3}\big] + \big[\mathsf{m}^{12}_{x_1,x_2}, \mathsf{m}^{13}_{x_1,x_3}\big] = 0. \tag{8}$$

Relation (8) follows from the following two ingredients.

- The A_∞–constraint
$$\mathsf{m}_3 \circ \big(\mathsf{m}_3 \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes \mathsf{m}_3 \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes \mathsf{m}_3\big) + \dots = 0$$
on the triple product m_3.
- Existence of a cyclic A_∞–structure with respect of the canonical Serre–pairing on the triangulated category $\mathsf{Perf}(E)$.

The unitarity of $\mathsf{m}_{x,y}$ follows from existence of a cyclic A_∞ structure as well. To generalize the relation (8) on singular Weierstraß curves as well as on the relative situation of genus one fibrations, we need the following result (Theorem 3.7): the diagram

$$
\begin{array}{ccc}
\mathcal{A}\big|_x & \xrightarrow{\;Y_1\;} & \mathfrak{sl}\big(\mathsf{Hom}(\mathcal{P}, \Bbbk_x)\big) \\
\uparrow{\scriptstyle \mathsf{res}_x} & & \\
H^0\big(\mathcal{A}(x)\big) & & \Big\downarrow{\scriptstyle \overline{\mathsf{m}}_{x,y}} \\
\downarrow{\scriptstyle \mathsf{ev}_y} & & \\
\mathcal{A}\big|_y & \xrightarrow{\;Y_2\;} & \mathfrak{pgl}\big(\mathsf{Hom}(\mathcal{P}, \Bbbk_y)\big)
\end{array}
\tag{9}
$$

is commutative, where Y_1 and Y_2 are certain canonical anti-isomorphisms of Lie algebras. A version of this important fact has been stated in [31, Theorem 4(b)].

Using the commutative diagram (9), we prove Theorem A. As a consequence, we obtain the continuity property of the solution $r_{(E,(n,d))}$ with respect to the Weierstraß parameters g_2 and g_3 of the curve E. This actually leads to certain unexpected analytic consequences about classical r–matrices, see Corollary 9.10.

The above construction can be rephrased in the following way. Let E be an arbitrary Weierstraß cubic curve. Then there exists a canonical meromorphic section

$$r \in \Gamma\big(\breve{E} \times \breve{E}, p_1^*\mathcal{A} \otimes p_2^*\mathcal{A}\big),$$

where $p_1, p_2 : \breve{E} \times \breve{E} \to E$ are canonical projections, satisfying the equation

$$\big[r^{12}, r^{13}\big] + \big[r^{13}, r^{23}\big] + \big[r^{12}, r^{23}\big] = 0,$$

see Theorem 4.4. It seems that in the case of elliptic curves, similar ideas have been suggested already in 1983 by Cherednik [16]. For an elliptic curve E with a marked point $o \in E$, the Lie algebra $\mathfrak{sel}_{(E,(n,d))} := \Gamma\big(E \setminus \{o\}, \mathcal{A}\big)$ was studied by Ginzburg, Kapranov and Vasserot [21], who constructed its realization using "correspondences" in the spirit of the geometric representation theory.

Talking about the proposed method of constructing of solutions of the classical Yang–Baxter equation, one may pose following natural question: to what extent is this method *constructive*? It turns out, that one can end up with explicit solutions in the case of all types of the Weierstraß curves. See also [14], where the similar technique in the case of solutions of the associative Yang–Baxter equation has been developed.

We first show that for an elliptic curve E, the corresponding solution $r_{(E,(n,d))}$ is the *elliptic* r–matrix of Belavin given by the formula (6), see Theorem 5.5. This result can be also deduced from [31, Formula (2.5)]. However, Polishchuk's proof, providing on one side a spectacular and impressive application of methods of mirror symmetry, is on the other hand rather undirect, as it requires the strong A_∞–version of the homological mirror symmetry for elliptic curves, explicit formulae for higher products in the Fukaya category of a torus and finally leads to a more complicated expression than (6).

Next, we focus on solutions of (1) originating from the cuspidal cubic curve $E = V(uv^2 - w^3)$. The motivation to deal with this problem comes from the fact that all obtained solutions turn out to be *rational*, which is the most complicated class of solutions from the point of view of the Belavin–Drinfeld classification [3]. Our approach is based on the general methods of study of vector bundles on the singular curves of genus one developed in

[18, 10, 6] and especially on Bodnarchuk–Drozd classification [7] of simple vector bundles on E. The above abstract way to construct solutions of (1) can be reduced to a very explicit recipe (see Algorithm 6.7), summarized as follows.

- To any pair of positive coprime integers d, e such that $e+d=n$ we attach a certain matrix $J = J_{(e,\,d)} \in \mathsf{Mat}_{n\times n}(\mathbb{C})$, whose entries are either 0 or 1.
- For any $x \in \mathbb{C}$, the matrix J defines a certain linear subspace $\mathsf{Sol}\big((e,d),x\big)$ in the Lie algebra of currents $\mathfrak{g}[z]$. For any $x \in \mathbb{C}$, we denote the evaluation map by $\phi_x : \mathfrak{g}[z] \to \mathfrak{g}$.
- Let $\overline{\mathsf{res}}_x := \phi_x$ and $\overline{\mathsf{ev}}_y := \dfrac{1}{y-x}\phi_y$. It turns out that $\overline{\mathsf{res}}_x$ and $\overline{\mathsf{ev}}_y$ yield isomorphisms between $\mathsf{Sol}\big((e,d),x\big)$ and $\mathfrak{g}$. Moreover, these maps are just the coordinate versions of the sheaf-theoretic morphisms $\mathsf{res}_x : H^0\big(\mathcal{A}(x)\big) \to \mathcal{A}\big|_x$ and $\mathsf{ev}_y : H^0\big(\mathcal{A}(x)\big) \to \mathcal{A}\big|_y$ appearing in the diagram (9).

The constructed matrix J turns out to be useful in a completely different situation. Let $\mathfrak{p} = \mathfrak{p}_e$ denote the e-th parabolic subalgebra of $\mathfrak{g}$. This Lie algebra is known to be *Frobenius*, see for example [19] and [33]. We prove (see Theorem 7.2) that the pairing

$$\omega_J : \mathfrak{p} \times \mathfrak{p} \longrightarrow \mathbb{C}, \quad (a,b) \mapsto \mathsf{tr}\big(J^t \cdot [a,b]\big)$$

is non-degenerate, making the Frobenius structure on $\mathfrak{p}$ explicit. This result will later be used to get explicit formulae for the solutions $r_{(E,(n,d))}$.

The study of rational solutions of the classical Yang–Baxter equation (1) was a subject of Stolin's investigation [33, 34, 35]. The first basic fact of his theory states that the gauge equivalence classes of rational solutions of (1) with values in $\mathfrak{g}$, which satisfy a certain additional Ansatz on the residue, stand in bijection with the conjugacy classes of certain Lagrangian Lie subalgebras $\mathfrak{w} \subset \mathfrak{g}((z^{-1}))$ called *orders*. The second basic result of Stolin's theory states that Lagrangian orders are parameterized (although not in a unique way) by certain triples $(\mathfrak{l}, k, \omega)$ mentioned in the Introduction.

The problem of description of all Stolin triples $(\mathfrak{l}, k, \omega)$ is known to be *representation-wild*, as it contains as a subproblem [3, 33] the wild problem of classification of all abelian Lie subalgebras of $\mathfrak{g}$ [17]. Thus, it is natural to ask what Stolin triples $(\mathfrak{l}, k, \omega)$ correspond to the "geometric" rational solutions $r_{(E,(n,d))}$, since the latter ones have discrete combinatorics and obviously form a "distinguished" class of rational solutions. This problem is completely solved in Theorem C, what is the third main result of this article.

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2. Some algebraic and geometric preliminaries

In this section we collect some known basic facts from linear algebra, homological algebra, and the theory of vector bundles on Calabi–Yau curves, which are crucial for further applications.

2.1. **Preliminaries from linear algebra.** For a finite dimensional vector space V over $\mathbb{k}$ we denote by $\mathfrak{sl}(V)$ the Lie subalgebra of $\mathsf{End}(V)$ consisting of endomorphisms with zero trace and $\mathfrak{pgl}(V) := \mathsf{End}(V)/\langle \mathbb{1}_V \rangle$. Since the proofs of all statements from this subsection are completely elementary, we left them to the reader as an exercise.

Lemma 2.1. *The non-degenerate bilinear pairing* $\mathsf{tr} : \mathsf{End}(V) \times \mathsf{End}(V) \longrightarrow \mathbb{k}$, $(f, g) \mapsto \mathsf{tr}(fg)$ *induces another non-degenerate pairing* $\mathsf{tr} : \mathfrak{sl}(V) \times \mathfrak{pgl}(V) \longrightarrow \mathbb{k}$, $(f, \bar{g}) \mapsto \mathsf{tr}(fg)$. *In particular, for any finite dimensional vector space U we get a canonical isomorphism of vector spaces*

$$\mathfrak{pgl}(U) \otimes \mathfrak{pgl}(V) \longrightarrow \mathsf{Lin}\big(\mathfrak{sl}(U), \mathfrak{pgl}(V)\big).$$

Lemma 2.2. *The* Yoneda map $Y : \mathsf{End}(V) \longrightarrow \mathsf{End}(V^*)$, *assigning to an endomorphism f its adjoint f^*, induces an* anti-isomorphisms of Lie algebras

- $Y_1 : \mathfrak{sl}(V) \longrightarrow \mathfrak{sl}(V^*)$ *and*
- $Y_2 : \mathfrak{sl}(V) \longrightarrow \mathfrak{pgl}(V^*)$, $f \mapsto \bar{f}^*$, *where $\bar{f}^*$ is the equivalence class of f^*.*
- *The following diagram*

$$\begin{array}{ccc} \mathfrak{sl}(V) \times \mathfrak{sl}(V) & \xrightarrow{\;Y_1 \times Y_2\;} & \mathfrak{sl}(V^*) \times \mathfrak{pgl}(V^*) \\ & \searrow^{\mathsf{tr}} \quad {}^{\mathsf{tr}}\swarrow & \\ & \mathbb{k} & \end{array}$$

  is commutative.

Note that the fist part of the statement is valid for any field $\mathbb{k}$, whereas the second one is only true if $\mathsf{dim}_{\mathbb{k}}(V)$ is invertible in $\mathbb{k}$.

Lemma 2.3. *Let $H \subseteq V$ be a linear subspace. Then we have the canonical linear map* $r_H : \mathsf{End}(V) \longrightarrow \mathsf{Lin}(H, V/H)$ *sending an endomorphism f to the composition* $H \longrightarrow V \xrightarrow{f} V \longrightarrow V/H$. *Moreover, the following results are true.*

- *We have: $r_H(\mathbb{1}_V) = 0$. In particular, there is an induced canonical map* $\bar{r}_H : \mathfrak{pgl}(V) \longrightarrow \mathsf{Lin}(H, V/H)$.
- *Let $f \in \mathsf{End}(V)$ be such that for any one-dimensional subspace $H \subseteq V$ we have: $r_H(f) = 0$. Then $\bar{f} = 0$ in $\mathfrak{pgl}(V)$.*
- *Let U be a finite dimensional vector space and $g_1, g_2 : U \longrightarrow \mathfrak{pgl}(V)$ be two linear maps such that for any one-dimensional subspace $H \subseteq V$ we have: $\bar{r}_H \circ g_1 = \bar{r}_H \circ g_2$. Then $g_1 = g_2$.*

2.2. **Triple Massey products.** In this article, we use the notion of triple Massey products in the following special situation.

Definition 2.4. Let D be a $\mathbb{k}$–linear triangulated category, $\mathcal{P}$, $\mathcal{X}$ and $\mathcal{Y}$ some objects of D satisfying the following conditions:

$$\mathsf{End}(\mathcal{P}) = \mathbb{k} \quad \text{and} \quad \mathsf{Hom}(\mathcal{X}, \mathcal{Y}) = 0 = \mathsf{Ext}(\mathcal{X}, \mathcal{Y}). \tag{10}$$

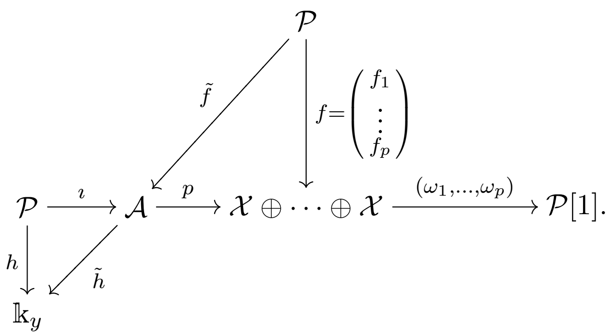
Consider the linear subspace

$$K := \mathsf{Ker}\big(\mathsf{Hom}(\mathcal{P}, \mathcal{X}) \otimes \mathsf{Ext}(\mathcal{X}, \mathcal{P}) \xrightarrow{\;\circ\;} \mathsf{Ext}(\mathcal{P}, \mathcal{P})\big). \tag{11}$$

and a linear subspace $H \subseteq \mathsf{Hom}(\mathcal{P}, \mathcal{Y})$. The triple Massey product is the map

$$M_H : K \longrightarrow \mathsf{Lin}\big(H, \mathsf{Hom}(\mathcal{P}, \mathcal{Y})/H\big) \tag{12}$$

defined as follows. Let $t = \sum_{i=1}^{p} f_i \otimes \omega_i \in K$ and $h \in H$. Consider the following commutative diagram in the triangulated category D:



The horizontal sequence is a distinguished triangle in D determined by the morphism $(\omega_1, \dots, \omega_p)$. Since $\sum_{i=1}^{p} \omega_i f_i = 0$ in $\mathsf{Ext}(\mathcal{P}, \mathcal{P})$, there exists a morphism $\tilde{f} : \mathcal{P} \longrightarrow \mathcal{A}$ such that $p\tilde{f} = f$. Note that such a morphism is only defined up to a translation $\tilde{f} \mapsto \tilde{f} + \lambda\imath$ for some $\lambda \in \Bbbk$. Since $\mathsf{Hom}(\mathcal{X}, \mathcal{Y}) = 0 = \mathsf{Ext}(\mathcal{X}, \mathcal{Y})$, there exists a unique morphism $\tilde{h} : \mathcal{A} \longrightarrow \mathcal{Y}$ such that $\tilde{h}\imath = h$. We set: $\big(M_H(t)\big)(h) := \overline{\tilde{h}\tilde{f}}$. □

The following result is well-known, see for instance [20, Exercise IV.2.3].

Proposition 2.5. *The map M_H is well-defined, i.e. it is independent of a presentation of $t \in K$ as a sum of simple tensors and a choice of the horizontal distinguished triangle. Moreover, M_H is $\Bbbk$–linear.*

2.3. **A_∞–structures and triple Massey products.** Let B be a $\Bbbk$–linear Grothendieck abelian category, A be its full subcategory of Noetherian objects and E the full subcategory of injective objects. For simplicity, we assume A to be Ext–finite. The derived category $D^+(\mathsf{B})$ is equivalent to the homotopy category $\mathsf{Hot}^{+,\mathsf{b}}_{\mathsf{coh}}(\mathsf{E})$. This identifies the triangulated category $\mathsf{D} = D^b_{\mathsf{A}}(\mathsf{B})$ of complexes with cohomology from A with the corresponding full subcategory of $\mathsf{Hot}^{+,\mathsf{b}}_{\mathsf{coh}}(\mathsf{E})$. Since $\mathsf{Hot}^{\mathsf{b}}_{\mathsf{coh}}(\mathsf{E})$ is the homotopy category of the dg–category $\mathsf{Com}^{\mathsf{b}}_{\mathsf{coh}}(\mathsf{E})$, by the homological perturbation lemma of Kadeishvili [24], the triangulated category D inherits a structure of an A_∞–category. This means that for any $n \geq 2$, $i_1, i_2, \dots, i_n \in \mathbb{Z}$ and objects $\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_n$ of D, we have linear maps

$$\mathsf{m}_n : \mathsf{Ext}^{i_1}(\mathcal{F}_0, \mathcal{F}_1) \otimes \mathsf{Ext}^{i_2}(\mathcal{F}_1, \mathcal{F}_2) \otimes \cdots \otimes \mathsf{Ext}^{i_n}(\mathcal{F}_{n-1}, \mathcal{F}_n) \longrightarrow \mathsf{Ext}^{i_1+\cdots+i_n+(2-n)}(\mathcal{F}_0, \mathcal{F}_n)$$

satisfying the identities

$$\sum_{\substack{r,s,t\geq 0\\ r+s+t=n}} (-1)^{r+st} \mathsf{m}_{r+1+t}\big(\underbrace{\mathbb{1} \cdots \otimes \mathbb{1}}_{r\ times} \otimes \mathsf{m}_s \otimes \underbrace{\mathbb{1} \cdots \otimes \mathbb{1}}_{s\ times}\big) = 0, \tag{13}$$

where m_2 is the composition of morphisms in D. The higher operations $\{\mathsf{m}_n\}_{n\geq 3}$ are unique up to an A_∞–automorphism of D. On the other hand, they are *not* determined by the triangulated structure of D, although they turn out to be compatible with the Massey products. Throughout this subsection, we fix some A_∞–structure $\{\mathsf{m}_n\}_{n\geq 3}$ on D.

Assume we have object $\mathcal{P}$, $\mathcal{X}$ and $\mathcal{Y}$ of D satisfying the conditions of Definition 2.4. Consider the linear map

$$\mathsf{m} = \mathsf{m}_3^\infty \ : \ \mathsf{Hom}(\mathcal{P}, \mathcal{X}) \otimes \mathsf{Ext}(\mathcal{X}, \mathcal{P}) \otimes \mathsf{Hom}(\mathcal{P}, \mathcal{Y}) \longrightarrow \mathsf{Hom}(\mathcal{P}, \mathcal{Y}).$$

It induces another linear map $K \longrightarrow \mathsf{End}\big(\mathsf{Hom}(\mathcal{P}, \Bbbk_y)\big)$ assigning to an element $t \in K$ the functional $g \mapsto \mathsf{m}(t \otimes g)$. Composing this map with the canonical projection $\mathsf{End}\big(\mathsf{Hom}(\mathcal{P}, \mathcal{Y})\big) \longrightarrow \mathfrak{pgl}\big(\mathsf{Hom}(\mathcal{P}, \mathcal{Y})\big)$, we obtain the linear map

$$\mathsf{m}_{\mathcal{X},\mathcal{Y}}^{\mathcal{P}} : \ K \longrightarrow \mathfrak{pgl}\big(\mathsf{Hom}(\mathcal{P}, \mathcal{Y})\big). \tag{14}$$

Lemma 2.6. *The map $\mathsf{m}_{\mathcal{X},\mathcal{Y}}^{\mathcal{P}}$ does not depend on the choice of an A_∞–structure on D.*

Proof. Of coarse, we may without loss of generality assume that $\mathsf{Hom}(\mathcal{P}, \mathcal{Y}) \neq 0$. First note that for any choice of an A_∞–structure on D and any one-dimensional linear subspace $H \subseteq \mathsf{Hom}(\mathcal{P}, \mathcal{Y})$, the following diagram

$$\begin{array}{ccc} K & \xrightarrow{\ \mathsf{m}_{\mathcal{X},\mathcal{Y}}^{\mathcal{P}}\ } & \mathfrak{pgl}\big(\mathsf{Hom}(\mathcal{P}, \mathcal{Y})\big) \\ & {\scriptstyle M_H} \searrow \quad \swarrow {\scriptstyle \bar{r}_H} & \\ & \mathsf{Lin}\big(H, \mathsf{Hom}(\mathcal{P}, \mathcal{Y})/H\big) & \end{array} \tag{15}$$

is commutative. Here, M_H is the triple Massey product (12) and $\bar{r}_H$ is the canonical linear map from Lemma 2.3. This compatibility between the triangulated Massey products and higher A_∞–products is well-known, see for example [26] a proof of a much more general statement. Let $\big\{\underline{\mathsf{m}}_n\big\}_{n \geq 3}$ be another A_∞–structure on D. From the last part of Lemma 2.3 it follows that $\mathsf{m}_{\mathcal{X},\mathcal{Y}}^{\mathcal{P}} = \underline{\mathsf{m}}_{\mathcal{X},\mathcal{Y}}^{\mathcal{P}}$. This proves the claim. □

2.4. **On the sheaf of Lie algebras $\mathsf{Ad}(\mathcal{F})$.** Let X be an algebraic variety over $\Bbbk$ and $\mathcal{F}$ a vector bundle on X.

Definition 2.7. The locally free sheaf $\mathsf{Ad}(\mathcal{F})$ of the traceless endomorphisms of $\mathcal{F}$ is defined by via the following short exact sequence

$$0 \longrightarrow \mathsf{Ad}(\mathcal{F}) \longrightarrow \mathcal{E}nd(\mathcal{F}) \xrightarrow{\mathsf{Tr}_{\mathcal{F}}} \mathcal{O} \longrightarrow 0, \tag{16}$$

where $\mathsf{Tr}_{\mathcal{F}} : \mathcal{E}nd(\mathcal{F}) \longrightarrow \mathcal{O}$ is the canonical trace map.

In the proposition below we collect some basic facts on the vector bundle $\mathsf{Ad}(\mathcal{F})$.

Proposition 2.8. *In the above notation the following statements are true.*

- *The vector bundle $\mathsf{Ad}(\mathcal{F})$ is a sheaf of Lie algebras on X.*
- *Next, we have: $H^0\big(\mathsf{Ad}(\mathcal{F})\big) = 0$.*
- *For any $\mathcal{L} \in \mathsf{Pic}(X)$ we have the natural isomorphism of sheaves of Lie algebras $\mathsf{Ad}(\mathcal{F}) \longrightarrow \mathsf{Ad}(\mathcal{F} \otimes \mathcal{L})$ which is induced by the natural isomorphism of sheaves of algebras $\mathcal{E}nd(\mathcal{F}) \longrightarrow \mathcal{E}nd(\mathcal{F} \otimes \mathcal{L})$.*
- *We have a symmetric bilinear pairing $\mathsf{Ad}(\mathcal{F}) \times \mathsf{Ad}(\mathcal{F}) \longrightarrow \mathcal{O}$ given on the level of local sections by the rule $(f, g) \mapsto \mathsf{tr}(fg)$. This pairing induces an isomorphism of $\mathcal{O}$–modules $\mathsf{Ad}(\mathcal{F}) \longrightarrow \mathsf{Ad}(\mathcal{F})^\vee$.*

2.5. **Serre duality pairing on a Calabi–Yau curve.** Let E be a Calabi–Yau curve and $w \in H^0(\Omega)$ a no-where vanishing regular differential form. For any pair of objects $\mathcal{F}, \mathcal{G} \in \mathsf{Perf}(E)$ we have the bilinear form

$$\langle - \,,\, - \rangle = \langle - \,,\, - \rangle^{w}_{\mathcal{F},\mathcal{G}} \;:\; \mathsf{Hom}(\mathcal{F},\mathcal{G}) \times \mathsf{Ext}(\mathcal{G},\mathcal{F}) \longrightarrow \Bbbk \tag{17}$$

defined as the composition

$$\mathsf{Hom}(\mathcal{F},\mathcal{G}) \times \mathsf{Ext}(\mathcal{G},\mathcal{F}) \xrightarrow{\;\circ\;} \mathsf{Ext}(\mathcal{F},\mathcal{F}) \xrightarrow{\mathsf{Tr}_{\mathcal{F}}} H^1(\mathcal{O}) \xrightarrow{\;w\;} H^1(\Omega) \xrightarrow{\;t\;} \Bbbk,$$

where $\circ$ denotes the composition operation, $\mathsf{Tr}_{\mathcal{F}}$ is the trace map and t is the canonical morphism described in [14, Subsection 4.3]. The following result is well-known, see for example [14, Corollary 3.3] for a proof.

Theorem 2.9. *For any $\mathcal{F}, \mathcal{G} \in \mathsf{Perf}(E)$ the pairing $\langle - \,,\, - \rangle^{w}_{\mathcal{F},\,\mathcal{G}}$ is non-degenerate. In particular, we have an isomorphism of vector spaces*

$$\mathbb{S} = \mathbb{S}_{\mathcal{F},\,\mathcal{G}} : \quad \mathsf{Ext}(\mathcal{G},\mathcal{F}) \longrightarrow \mathsf{Hom}(\mathcal{F},\mathcal{G})^*, \tag{18}$$

which is functorial in both arguments.

Let $\mathcal{P}$ be a simple vector bundle on E and $x, y \in \breve{E}$ a pair of points from the same irreducible components. Note that we are in the situation of Definition 2.4 for $\mathsf{D} = \mathsf{Perf}(E)$, $\mathcal{X} = \Bbbk_x$ and $\mathcal{Y} = \Bbbk_y$. Note the following easy fact.

Lemma 2.10. *Let K be as in (11). Then the linear isomorphism*

$$\overline{\mathbb{S}} : \mathsf{Hom}(\mathcal{P},\Bbbk_x) \otimes \mathsf{Ext}(\Bbbk_x,\mathcal{P}) \xrightarrow{\mathbb{1}\otimes\mathbb{S}} \mathsf{Hom}(\mathcal{P},\Bbbk_x) \otimes \mathsf{Hom}(\mathcal{P},\Bbbk_x)^* \xrightarrow{\mathsf{ev}} \mathsf{End}\big(\mathsf{Hom}(\mathcal{P},\Bbbk_x)\big)$$

identifies the vector space K with $\mathfrak{sl}\big(\mathsf{Hom}(\mathcal{P},\Bbbk_x)\big)$.

2.6. **Simple vector bundles on Calabi–Yau curves.** In this subsection, we collect some basic results on the classification of vector bundles on (possibly reducible) Calabi–Yau curves.

Definition 2.11. Let $\big\{E^{(1)}, \dots, E^{(p)}\big\}$ be the set of the irreducible components of a Calabi–Yau curve E. For a vector bundle $\mathcal{F}$ on E we denote by

$$\underline{\deg}(\mathcal{F}) = (d_1, \dots, d_p) \in \mathbb{Z}^p$$

its *multi-degree*, where $d_i = \deg\big(\mathcal{F}\big|_{E^{(i)}}\big)$ for $1 \le i \le p$.

For $\mathbb{d} \in \mathbb{Z}^p$ we denote $\mathsf{Pic}^{\mathbb{d}}(E) := \big\{\mathcal{L} \in \mathsf{Pic}(E) \,\big|\, \underline{\deg}(\mathcal{L}) = \mathbb{d}\big\}$. In particular, for $\mathbb{o} = (0, \dots, 0)$ we set: $J(E) = \mathsf{Pic}^{\mathbb{o}}(E)$. Then $J(E)$ is an algebraic group called *Jacobian* of E.

Proposition 2.12. *For $\Bbbk = \mathbb{C}$ we have the following isomorphisms of Lie groups:*

$$J(E) \cong \begin{cases} \mathbb{C}/\Lambda & \text{if } E \text{ is elliptic,} \\ \mathbb{C}^* & \text{if } E \text{ is a Kodaira cycle,} \\ \mathbb{C} & \text{in the remaining cases.} \end{cases}$$

Moreover, for any multi-degree $\mathbb{d}$ we have a (non-canonical) isomorphism of algebraic varieties $J(E) \longrightarrow \mathsf{Pic}^{\mathbb{d}}(E)$.

A proof of this result follows from [22, Exercise II.6.9] or [6, Theorem 16].

Next, recall the description of simple vector bundles on Calabi–Yau curves.

Theorem 2.13. *Let E be a reduced plane cubic curve with p irreducible components and $\mathcal{P}$ be a simple vector bundle on E. Then the following statements are true.*

- *Let $n = \mathrm{rk}(\mathcal{P})$ be the rank of $\mathcal{P}$ and $d = d_1(\mathcal{P}) + \cdots + d_p(\mathcal{P}) = \chi(\mathcal{P})$ its degree. Then n and d are mutually prime.*
- *If E is irreducible then $\mathcal{P}$ is stable.*
- *Let $n \in \mathbb{N}$ and $\mathbb{d} = (d_1, \dots, d_p) \in \mathbb{Z}^p$ be such that $\gcd(n, d_1 + \cdots + d_p) = 1$. Denote by $M_E(n, \mathbb{d})$ the set of simple vector bundles on E of rank n and multi-degree $\mathbb{d}$. Then the map $\det : M_E(n, \mathbb{d}) \longrightarrow \mathsf{Pic}^{\mathbb{d}}(E)$ is a bijection. Moreover, for any $\mathcal{P} \not\cong \mathcal{P}' \in M_E(n, \mathbb{d})$ we have: $\mathsf{Hom}(\mathcal{P}, \mathcal{P}') = 0 = \mathsf{Ext}(\mathcal{P}, \mathcal{P}')$.*
- *The group $J(E)$ acts transitively on $M_E(n, \mathbb{d})$. Moreover, given $\mathcal{P} \in M_E(n, \mathbb{d})$ and $\mathcal{L} \in J(E)$, we have: $\mathcal{P} \cong \mathcal{P} \otimes \mathcal{L} \iff \mathcal{L}^{\otimes n} \cong \mathcal{O}$.*

Comment on the proof. In the case of elliptic curves all these statements are due to Atiyah [1]. The case of a nodal Weierstraß curve has been treated by the first-named author in [9], the corresponding result for a cuspidal cubic curve is due to Bodnarchuk and Drozd [7]. The remaining cases (Kodaira fibers of type I_2, I_3, III and IV) are due to Bodnarchuk, Drozd and Greuel [8]. Their method actually allows to prove this theorem for arbitrary Kodaira cycles of projective lines. In that case, one can also deduce this result from another description of simple vector bundles obtained in [11, Theorem 5.3]. On the other hand, this result is still open for n concurrent lines in $\mathbb{P}^{n-1}$ if $n \geq 4$.

Proposition 2.14. *Let E be a reduced plane cubic curve and $\mathcal{P}$ be a simple vector bundle on E of rank n and multi-degree $\mathbb{d}$. Then the following results are true.*

- *The sheaf of Lie algebras $\mathcal{A} = \mathcal{A}_{n,\mathbb{d}} := \mathsf{Ad}(\mathcal{P})$ does not depend on the choice of $\mathcal{P} \in M_E(n, \mathbb{d})$.*
- *We have: $H^0(\mathcal{A}) = 0 = H^1(\mathcal{A})$. Moreover, this result remains true for an arbitrary Calabi–Yau curve.*
- *For $\mathcal{L} \in J(E) \setminus \{\mathcal{O}\}$ we have: $H^0(\mathcal{A} \otimes \mathcal{L}) \neq 0$ if and only if $\mathcal{L}^{\otimes n} \cong \mathcal{O}$. Moreover, in this case we have: $H^0(\mathcal{A} \otimes \mathcal{L}) \cong \mathbb{k} \cong H^1(\mathcal{A} \otimes \mathcal{L})$.*

Proof. The first part follows from the transitivity of the action of $J(E)$ on $M_E(n, \mathbb{d})$ (see Theorem 2.13) and the fact that $\mathsf{Ad}(\mathcal{P}) \cong \mathsf{Ad}(\mathcal{P} \otimes \mathcal{L})$ for any line bundle $\mathcal{L}$ (see Proposition 2.8). The second statement follows from the long exact sequence

$$0 \to H^0(\mathcal{A}) \longrightarrow \mathsf{End}(\mathcal{P}) \xrightarrow{H^0(\mathsf{Tr}_{\mathcal{P}})} H^0(\mathcal{O}) \longrightarrow H^1(\mathcal{A}) \longrightarrow \mathsf{Ext}(\mathcal{P}, \mathcal{P}) \longrightarrow H^1(\mathcal{O}) \to 0,$$

the isomorphisms $\mathsf{End}(\mathcal{P}) \cong \mathbb{k} \cong \mathsf{Ext}(\mathcal{P}, \mathcal{P})$, $H^0(\mathcal{O}) \cong \mathbb{k} \cong H^1(\mathcal{O})$ and the fact that $H^0(\mathsf{Tr}_{\mathcal{P}})(\mathbb{1}_{\mathcal{P}}) = \mathrm{rk}(\mathcal{P})$.

To show the last statement, note that we have the exact sequence

$$0 \longrightarrow H^0(\mathcal{A} \otimes \mathcal{L}) \longrightarrow \mathsf{Hom}(\mathcal{P}, \mathcal{P} \otimes \mathcal{L}) \longrightarrow H^0(\mathcal{L})$$

and $H^0(\mathcal{L}) = 0$. By Theorem 2.13 we know that $\mathsf{Hom}(\mathcal{P}, \mathcal{P} \otimes \mathcal{L}) = 0$ unless $\mathcal{L}^{\otimes n} \cong \mathcal{O}$. In the latter case, $H^0(\mathcal{A} \otimes \mathcal{L}) \cong \mathsf{End}(\mathcal{P}) \cong \mathbb{k}$. Since $\mathcal{A} \otimes \mathcal{L}$ is a vector bundle of degree zero, by the Riemann-Roch formula we obtain that $H^1(\mathcal{A} \otimes \mathcal{L}) \cong \mathbb{k}$. □

3. Triple products on Calabi–Yau curves and the classical Yang–Baxter equation

In this section we shall explain an interplay between the theory of vector bundles on Calabi–Yau curves, triple Massey products, A_∞–structures and the classical Yang–Baxter equation. Let E be a Calabi–Yau curve, $x, y \in \breve{E}$ a pair of points from the same irreducible component of E and $\mathcal{P}$ a simple vector bundle on E. By (14) and Lemma 2.10, we have the canonical linear map

$$\overline{\mathsf{m}}_{x,\,y} := \mathsf{m}^{\mathcal{P}}_{\Bbbk_x, \Bbbk_y} : \ \mathfrak{sl}\bigl(\mathsf{Hom}(\mathcal{P}, \Bbbk_x)\bigr) \longrightarrow \mathfrak{pgl}\bigl(\mathsf{Hom}(\mathcal{P}, \Bbbk_y)\bigr). \tag{19}$$

By Lemma 2.1, this map corresponds to a certain (canonical) tensor

$$\mathsf{m}_{x,y} \in \mathfrak{pgl}\bigl(\mathsf{Hom}(\mathcal{P}, \Bbbk_x)\bigr) \otimes \mathfrak{pgl}\bigl(\mathsf{Hom}(\mathcal{P}, \Bbbk_y)\bigr). \tag{20}$$

3.1. **The case of an elliptic curve.** The following result is due to Polishchuk, see [31, Theorem 2].

Theorem 3.1. *Let E be an elliptic curve, $\mathcal{P}$ be a simple vector bundle on E and $x_1, x_2, x_3 \in E$ be pairwise distinct. Then we have the following equality*

$$\bigl[\mathsf{m}^{12}_{x_1,x_2}, \mathsf{m}^{13}_{x_1,x_3}\bigr] + \bigl[\mathsf{m}^{12}_{x_1,x_2}, \mathsf{m}^{23}_{x_2,x_3}\bigr] + \bigl[\mathsf{m}^{12}_{x_1,x_2}, \mathsf{m}^{13}_{x_1,x_3}\bigr] = 0, \tag{21}$$

where both sides are viewed as elements of $\mathfrak{g}_1 \otimes \mathfrak{g}_2 \otimes \mathfrak{g}_3$. Here, $\mathfrak{g}_i = \mathfrak{pgl}\bigl(\mathsf{Hom}(\mathcal{P}, \Bbbk_{x_i})\bigr)$ for $i = 1, 2, 3$. Moreover, the tensor m_{x_1,x_2} is unitary:

$$\mathsf{m}_{x_2,x_1} = -\tau\bigl(\mathsf{m}_{x_1,x_2}\bigr) \tag{22}$$

where $\tau : \mathfrak{g}_1 \otimes \mathfrak{g}_2 \longrightarrow \mathfrak{g}_2 \otimes \mathfrak{g}_1$ is the map permuting both factors.

Idea of the proof. The equality (22) follows from existence of an A_∞–structure on $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(E)$ which is *cyclic* with respect to the pairing (17). In particular, this means that for any objects $\mathcal{F}_1, \mathcal{F}_2, \mathcal{G}_1, \mathcal{G}_2$ in $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(E)$ and morphisms $a_1 \in \mathsf{Hom}(\mathcal{F}_1, \mathcal{G}_1), a_2 \in \mathsf{Hom}(\mathcal{F}_2, \mathcal{G}_2), \omega_1 \in \mathsf{Ext}(\mathcal{G}_1, \mathcal{F}_2)$ and $\omega_2 \in \mathsf{Ext}(\mathcal{F}_2, \mathcal{G}_1)$ we have:

$$\bigl\langle \mathsf{m}(a_1 \otimes \omega_1 \otimes a_2), \omega_2 \bigr\rangle = -\bigl\langle a_1, \mathsf{m}(\omega_1 \otimes a_2 \otimes \omega_2)\bigr\rangle = -\bigl\langle \mathsf{m}(a_2 \otimes \omega_2 \otimes a_1), \omega_1 \bigr\rangle, \tag{23}$$

where $\mathsf{m} = \mathsf{m}_3^\infty$ is the triple product this A_∞–structure. A proof of the existence of such an A_∞–structure has been outlined by Polishchuk in [30, Theorem 1.1], see also [25, Theorem 10.2.2] for a different approach using non-commutative symplectic geometry. The identity (23) applied to $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{P}$ and $\mathcal{G}_i = \Bbbk_{x_i}$ leads to the equality (22). The fact that m_{x_1,x_2} satisfies the classical Yang–Baxter equation (21) follows from (22) and the equality

$$\mathsf{m} \circ (\mathsf{m} \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes m \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes \mathsf{m}) + \text{other terms} = 0$$

(which is one of the equalities (13)) viewed as a linear map

$$\mathsf{Hom}(\mathcal{P}, \Bbbk_{x_1}) \otimes \mathsf{Ext}(\Bbbk_{x_1}, \mathcal{P}) \otimes \mathsf{Hom}(\mathcal{P}, \Bbbk_{x_2}) \otimes \mathsf{Ext}(\Bbbk_{x_2}, \mathcal{P}) \otimes \mathsf{Hom}(\mathcal{P}, \Bbbk_{x_3}) \to \mathsf{Hom}(\mathcal{P}, \Bbbk_{x_3}).$$

□

Remark 3.2. Up to now, we are not aware of a complete proof of existence of an A_∞–structure on the triangulated category $\mathsf{Perf}(E)$ for a singular Calabi–Yau curve E, which is cyclic with respect to the pairing (17). Hence, in order to derive the identities (21) and (22) for a singular Weierstraß curve E, we use a different approach which is similar in spirit to the work [14]. Following [31], we give another description of the tensor $\mathsf{m}_{x,y}$ and show some kind of its continuity with respect to the degeneration of the complex structure on E. This approach also provides a constructive way of computing of the tensor $\mathsf{m}_{x,y}$.

3.2. **Residues and traces.** Let Ω be the sheaf of regular differential one–forms on a (possibly reducible) Calabi–Yau curve E, $w \in H^0(\Omega)$ some no-where vanishing regular differential form and $x, y \in \breve{E}$ a pair points from the same irreducible component of E. First recall that we have the following canonical short exact sequence

$$0 \longrightarrow \Omega \longrightarrow \Omega(x) \xrightarrow{\mathsf{res}_x} \Bbbk_x \longrightarrow 0. \tag{24}$$

The section w induces the short exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(x) \longrightarrow \Bbbk_x \longrightarrow 0. \tag{25}$$

Hence, for any vector bundle $\mathcal{F}$ we get a short exact sequence of coherent sheaves

$$0 \longrightarrow \mathcal{F} \xrightarrow{\imath} \mathcal{F}(x) \xrightarrow{\mathsf{res}_x^{\mathcal{F}}} \mathcal{F} \otimes \Bbbk_x \longrightarrow 0. \tag{26}$$

Next, recall the following result relating categorical traces with the usual trace of an endomorphism of a finite dimensional vector space.

Proposition 3.3. *In the above notation, the following results are true.*

- *There is an isomorphism of functors* $\delta_x : \mathsf{Hom}(\Bbbk_x, - \otimes \Bbbk_x) \longrightarrow \mathsf{Ext}(\Bbbk_x, -)$ *from the category of vector bundles on* E *to the category of vector spaces over* $\Bbbk$, *given by the boundary map induced by the short exact sequence (26).*
- *For any vector bundle* $\mathcal{F}$ *on the curve* E *and a pair of morphisms* $b : \mathcal{F} \longrightarrow \Bbbk_x, a : \Bbbk_x \longrightarrow \mathcal{F} \otimes \Bbbk_x$, *we have the equality:*

$$t^w\bigl(\mathsf{Tr}_{\mathcal{F}}(\delta_x(a) \circ b)\bigr) = \mathsf{tr}(a \circ b_x), \tag{27}$$

where $\mathsf{Tr}_{\mathcal{F}} : \mathsf{Ext}(\mathcal{F}, \mathcal{F}) \longrightarrow H^1(\mathcal{O})$ *is the trace map and* t^w *is the composition* $H^0(\mathcal{O}) \xrightarrow{w} H^0(\Omega) \xrightarrow{t} \Bbbk$ *of the isomorphism induced by* w *and the canonical map* t *described in* [14, Subsection 4.3].

Comment on the proof. The first part of the statement is just [14, Lemma 4.18]. The content of the second part is explained by the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{Q} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle b} & & \\
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{R} & \longrightarrow & \Bbbk_x & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle a} & & \\
0 & \longrightarrow & \mathcal{F} & \xrightarrow{\imath} & \mathcal{F}(x) & \xrightarrow{\mathsf{res}_x^{\mathcal{F}}} & \mathcal{F} \otimes \Bbbk_x & \longrightarrow & 0.
\end{array}$$

The lowest horizontal sequence of this diagram is (26). The middle sequence corresponds to the element $\delta_x(a) \in \mathsf{Ext}(\Bbbk_x, \mathcal{F})$ and the top one corresponds to $\delta_x(a) \circ b \in \mathsf{Ext}(\mathcal{F}, \mathcal{F})$. The endomorphism $a \circ b_x \in \mathsf{End}(\mathcal{F}\big|_x)$ is the induced map in the fiber of $\mathcal{F}$ over x. The equality (27) follows from [14, Lemma 4.20]. $\square$

Proposition 3.4. *The following diagram of vector spaces is commutative.*

$$\begin{array}{ccc}
\mathsf{Hom}(\mathcal{F},\Bbbk_x)\otimes\mathsf{Ext}(\Bbbk_x,\mathcal{F}) & \xrightarrow{\ \mathbb{1}\otimes\mathbb{S}\ } & \mathsf{Hom}(\mathcal{F},\Bbbk_x)\otimes\mathsf{Hom}(\mathcal{F},\Bbbk_x)^{*} \\
{\scriptstyle \mathbb{1}\otimes\delta_x^{\mathcal{F}}}\big\uparrow & & \big\uparrow{\scriptstyle \mathbb{1}\otimes\mathsf{can}} \\
\mathsf{Hom}(\mathcal{F},\Bbbk_x)\otimes\mathsf{Hom}(\Bbbk_x,\mathcal{F}\otimes\Bbbk_x) & \xrightarrow{\ \mathbb{1}\otimes\mathsf{tr}\ } & \mathsf{Hom}(\mathcal{F},\Bbbk_x)\otimes\mathsf{Hom}(\mathcal{F}\otimes\Bbbk_x,\Bbbk_x)^{*} \\
{\scriptstyle \circ}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{ev}} \\
\mathsf{Lin}(\mathcal{F}\big|_x,\mathcal{F}\big|_x) & \xrightarrow{\ Y_1\ } & \mathsf{End}\big(\mathsf{Lin}(\mathcal{F}\big|_x,\Bbbk)\big).
\end{array}$$

Here, $\mathbb{S}$ *is given by (18),* $\delta_x^{\mathcal{F}}$ *is the isomorphism from Proposition 3.3,* $\circ$ *is* m_2 *composed with the induced map in the fiber over* x, Y_1 *is the canonical isomorphism of vector spaces from Lemma 2.1,* ev *and* tr *are canonical isomorphisms of vector spaces and* can *is the isomorphism induced by* $\underline{\mathsf{res}}_x^{\mathcal{F}}$.

Proof. The commutativity of the top square is given by [14, Lemma 4.21]. The commutativity of the lower square can be easily verified by diagram chasing. ☐

Lemma 3.5. *The following diagram of vector spaces is commutative.*

$$\begin{tikzcd}
\mathsf{Hom}(\mathcal{F},\Bbbk_x)\otimes\mathsf{Ext}(\Bbbk_x,\mathcal{F}) \arrow[rrr, "\overline{\mathbb{S}}"] \arrow[ddd, "T"'] & & & \mathsf{End}\big(\mathsf{Hom}(\mathcal{F},\Bbbk_x)\big) \arrow[ddd] \\
& K \arrow[r, "\overline{\mathbb{S}}"] \arrow[d, "\overline{T}"'] \arrow[ul, hook'] & \mathfrak{sl}\big(\mathsf{Hom}(\mathcal{F},\Bbbk_x)\big) \arrow[d] \arrow[ur, hook] & \\
& \mathfrak{sl}(\mathcal{F}\big|_x) \arrow[r, "Y_1"] \arrow[dl, hook'] & \mathfrak{sl}\big(\mathsf{Lin}(\mathcal{F}\big|_x,\Bbbk)\big) \arrow[dr, hook] & \\
\mathsf{End}(\mathcal{F}\big|_x) \arrow[rrr, "Y"] & & & \mathsf{End}\big(\mathsf{Lin}(\mathcal{F}\big|_x,\Bbbk)\big)
\end{tikzcd}$$

In this diagram, $\overline{\mathbb{S}}$ *is the isomorphism induced by the Serre duality (18),* Y *and* Y_1 *are canonical isomorphisms from Lemma 2.2,* K *is the subspace of* $\mathsf{Hom}(\mathcal{F},\otimes\Bbbk_x)\otimes\mathsf{Ext}(\Bbbk_x,\mathcal{F})$ *defined in (11),* T *is the composition of* $\mathbb{1}\otimes(\delta_x^{\mathcal{F}})^{-1}$ *from Proposition 3.4 and* $\circ$, *whereas* $\overline{T}$ *is the restriction of* T. *The remaining arrows are canonical morphisms of vector spaces.*

Proof. Commutativity of the big square is given by Proposition 3.4. For the left small square it follows from the equality (27) whereas the commutativity of the remaining parts of this diagram is obvious. ☐

3.3. **Geometric description of the triple Massey products.** Let E, $\mathcal{P}$, x and y be as at the beginning of this section. In what follows, we shall frequently use the notation $\mathcal{A} := \mathsf{Ad}(\mathcal{P})$ and $\mathcal{E} := \mathcal{E}nd(\mathcal{P})$.

Lemma 3.6. *We have a canonical isomorphism of vector spaces*

$$\mathsf{res}_x := H^0(\underline{\mathsf{res}}_x^{\mathcal{A}}) : \quad H^0\big(\mathcal{A}(x)\big) \longrightarrow \mathcal{A}\big|_x \tag{28}$$

induced by the short exact sequence (26). Moreover, we have the canonical morphism

$$\mathsf{ev}_y := H^0(\underline{\mathsf{ev}}_y^{\mathcal{A}}) : \quad H^0\bigl(\mathcal{A}(x)\bigr) \longrightarrow \mathcal{A}\big|_y \tag{29}$$

obtained by composing the induced map in the fibers with the canonical isomorphism $\mathcal{A}(x)\big|_y \longrightarrow \mathcal{A}\big|_y$. *When E is a reduced plane cubic curve,* ev_y *is an isomorphism if and only if* $n \cdot \bigl([x]-[y]\bigr) \neq 0$ *in* $J(E)$*, where* $n = \mathrm{rk}(\mathcal{P})$.

Proof. The short exact sequence

$$0 \longrightarrow \mathcal{A} \xrightarrow{\imath} \mathcal{A}(x) \xrightarrow{\mathsf{res}_x^{\mathcal{A}}} \mathcal{A} \otimes \Bbbk_x \longrightarrow 0$$

yields the long exact sequence

$$0 \longrightarrow H^0(\mathcal{A}) \longrightarrow H^0\bigl(\mathcal{A}(x)\bigr) \xrightarrow{\mathsf{res}_x} \mathcal{A}\big|_x \longrightarrow H^1(\mathcal{A}).$$

Thus, the first part of the statement follows from the vanishing $H^0(\mathcal{A}) = 0 = H^1(\mathcal{A})$ given by Proposition 2.14.

In order to show the second part note that we have the canonical short exact sequence

$$0 \longrightarrow \mathcal{O}(-y) \longrightarrow \mathcal{O} \xrightarrow{\mathsf{ev}_y} \Bbbk_y \longrightarrow 0$$

yielding the short exact sequence

$$0 \longrightarrow \mathcal{A}(x-y) \longrightarrow \mathcal{A}(x) \longrightarrow \mathcal{A}(x) \otimes \Bbbk_y \longrightarrow 0.$$

Hence, we get the long exact sequence

$$0 \longrightarrow H^0\bigl(\mathcal{A}(x-y)\bigr) \longrightarrow H^0\bigl(\mathcal{A}(x)\bigr) \xrightarrow{\mathsf{ev}_y} \mathcal{A}\big|_y \longrightarrow H^1\bigl(\mathcal{A}(x-y)\bigr).$$

Since the dimensions of $H^0\bigl(\mathcal{A}(x)\bigr)$ and $\mathcal{A}\big|_y$ are the same, ev_y is an isomorphism if and only if $H^0\bigl(\mathcal{A}(x-y)\bigr) = 0$. By Proposition 2.14, this vanishing is equivalent to the condition $n \cdot \bigl([x]-[y]\bigr) \neq 0$ in $J(E)$. □

The following key result was stated for the first time in [31, Theorem 4].

Theorem 3.7. *In the notation as at the beginning of this section, the following diagram of vector spaces is commutative:*

$$\begin{array}{ccc} \mathcal{A}\big|_x & \xrightarrow{\quad Y_1 \quad} & \mathfrak{sl}\bigl(\mathsf{Hom}(\mathcal{P}, \Bbbk_x)\bigr) \\ \uparrow{\scriptstyle \mathsf{res}_x} & & \\ H^0\bigl(\mathcal{A}(x)\bigr) & & \downarrow{\scriptstyle \overline{\mathsf{m}}_{x,y}} \\ \downarrow{\scriptstyle \mathsf{ev}_y} & & \\ \mathcal{A}\big|_y & \xrightarrow{\quad Y_2 \quad} & \mathfrak{pgl}\bigl(\mathsf{Hom}(\mathcal{P}, \Bbbk_y)\bigr). \end{array} \tag{30}$$

In this diagram, $\overline{\mathsf{m}}_{x,y}$ *is the linear map (19) induced by the triple* A_∞*–product in* $\mathsf{Perf}(E)$, res_x *and* ev_y *are the linear maps (28) and (29), whereas* $\overline{Y}_1$ *and* $\overline{Y_2}$ *are obtained by composing the canonical isomorphisms* Y_1 *and* Y_2 *from Lemma 2.2 with the canonical isomorphisms induced by* $\mathsf{Hom}(\mathcal{P}, \Bbbk_z) \longrightarrow \mathsf{Lin}\bigl(\mathcal{P}\big|_z, \Bbbk\bigr)$ *for* $z \in \{x, y\}$.

3.4. **A proof of the Comparison Theorem.** We split the proof of Theorem 3.7 into three smaller logical steps.

Step 1. First note that we have a well-defined linear map

$$\imath^! : \mathsf{Hom}\big(\mathcal{P}, \mathcal{P}(x)\big) \longrightarrow \mathsf{End}\big(\mathsf{Hom}(\mathcal{P}, \Bbbk_y)\big)$$

defined as follows. Let $g \in \mathsf{Hom}\big(\mathcal{P}, \mathcal{P}(x)\big)$ and $h \in \mathsf{Hom}(\mathcal{P}, \Bbbk_y)$ be arbitrary morphisms. Then there exists a unique morphism $\tilde{h} \in \mathsf{Hom}(\mathcal{P}, \Bbbk_y)$ such that $\imath \circ \tilde{h} = h$, where $\imath : \mathcal{P} \longrightarrow \mathcal{P}(x)$ is the canonical inclusion. Then we set: $\imath^!(g)(h) = \tilde{h} \circ g$. It follows from the definition that $\imath^!(\imath) = \mathbb{1}_{\mathsf{Hom}(\mathcal{P}, \Bbbk_y)}$. This yields the following result.

Lemma 3.8. *We have a well-defined linear map*

$$\bar{\imath}^! : \frac{\mathsf{Hom}\big(\mathcal{P}, \mathcal{P}(x)\big)}{\langle \imath \rangle} \longrightarrow \mathfrak{pgl}\big(\mathsf{Hom}(\mathcal{P}, \Bbbk_y)\big)$$

given by the rule: $\bar{\imath}^!(\bar{g}) = \overline{h \mapsto g \circ \tilde{h}}$.

Lemma 3.9. *The canonical morphism of vector spaces*

$$\jmath : H^0\big(\mathcal{A}(x)\big) \longrightarrow \frac{\mathsf{Hom}\big(\mathcal{P}, \mathcal{P}(x)\big)}{\langle \imath \rangle} \tag{31}$$

given by the composition

$$H^0\big(\mathsf{Ad}(\mathcal{P})(x)\big) \hookrightarrow H^0\big(\mathcal{E}nd(\mathcal{P})(x)\big) \longrightarrow \mathsf{Hom}\big(\mathcal{P}, \mathcal{P}(x)\big) \longrightarrow \frac{\mathsf{Hom}\big(\mathcal{P}, \mathcal{P}(x)\big)}{\langle \imath \rangle}$$

is an isomorphism.

Proof. The short exact sequences (16) and (25) together with the vanishing $H^0(\mathcal{A}) = 0 = H^1(\mathcal{A})$ imply that we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(\mathcal{O}) & \longrightarrow & H^0\big(\mathcal{O}(x)\big) & \xrightarrow{0} & \Bbbk & \longrightarrow & H^1(\mathcal{O}) \\
 & & \uparrow & & \uparrow & & \uparrow {\scriptstyle \mathsf{tr}} & & \uparrow \\
0 & \longrightarrow & H^0(\mathcal{E}) & \longrightarrow & H^0\big(\mathcal{E}(x)\big) & \longrightarrow & \mathcal{E}\big|_x & \longrightarrow & H^1(\mathcal{E}) \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & 0 & \longrightarrow & H^0\big(\mathcal{A}(x)\big) & \xrightarrow{\mathsf{res}_x} & \mathcal{A}\big|_x & \longrightarrow & 0.
\end{array}$$

The fact that $\jmath$ is an isomorphism follows from a straightforward diagram chase. □

Lemma 3.10. *The following diagram is commutative.*

$$\begin{array}{ccc}
\mathsf{Hom}\big(\mathcal{P}, \mathcal{P}(x)\big) & \xrightarrow{\mathsf{ev}_y} & \mathsf{End}\big(\mathcal{P}\big|_y\big) \\
{\scriptstyle \imath^!}\downarrow & & \downarrow{\scriptstyle Y} \\
\mathsf{End}\big(\mathsf{Hom}(\mathcal{P}, \Bbbk_y)\big) & \xrightarrow{\mathsf{can}} & \mathsf{End}\big(\mathsf{Lin}(\mathcal{P}\big|_y, \Bbbk)\big).
\end{array}$$

Proof. The result follows from a straightforward diagram chase. □

Proposition 3.11. *The following diagram is commutative.*

$$
\begin{CD}
H^0\bigl(\mathcal{A}(x)\bigr) @>{\mathsf{ev}_y}>> @. \mathfrak{sl}\bigl(\mathcal{P}\big|_y\bigr) \\
@V{\jmath}VV @. @VV{Y_2}V \\
\dfrac{\mathsf{Hom}\bigl(\mathcal{P},\mathcal{P}(x)\bigr)}{\langle \imath \rangle} @>{\bar{\imath}^!}>> \mathfrak{pgl}\bigl(\mathsf{Hom}(\mathcal{P},\Bbbk_y)\bigr) @>>> \mathfrak{pgl}\bigl(\mathsf{Lin}(\mathcal{P}\big|_y,\Bbbk)\bigr).
\end{CD}
\tag{32}
$$

In particular, if E is a reduced plane cubic curve then $\bar{\imath}^!$ is an isomorphism if and only if $n\cdot\bigl([x]-[y]\bigr)\neq 0$ in $J(E)$.

Proof. Note that the following diagram is commutative:

$$
\begin{array}{ccccccc}
\mathfrak{sl}\bigl(\mathcal{P}\big|_y\bigr) & \hookrightarrow & \mathsf{End}\bigl(\mathcal{P}\big|_y\bigr) & & \xrightarrow{\quad Y \quad} & & \\
\uparrow{\scriptstyle \mathsf{ev}_y} & & \uparrow{\scriptstyle \mathsf{ev}_y} & & & & \\
H^0\bigl(\mathcal{A}(x)\bigr) & \hookrightarrow & \mathsf{Hom}\bigl(\mathcal{P},\mathcal{P}(x)\bigr) & \xrightarrow{\imath^!} & \mathsf{End}\bigl(\mathsf{Hom}(\mathcal{P},\Bbbk_y)\bigr) & \longrightarrow & \mathsf{End}\bigl(\mathsf{Lin}(\mathcal{P}\big|_y,\Bbbk)\bigr) \\
& \searrow{\scriptstyle \jmath} & \downarrow & & \downarrow & & \downarrow \\
& & \dfrac{\mathsf{Hom}\bigl(\mathcal{P},\mathcal{P}(x)\bigr)}{\langle \imath \rangle} & \xrightarrow{\bar{\imath}^!} & \mathfrak{pgl}\bigl(\mathsf{Hom}(\mathcal{P},\Bbbk_y)\bigr) & \longrightarrow & \mathfrak{pgl}\bigl(\mathsf{Lin}(\mathcal{P}\big|_y,\Bbbk)\bigr).
\end{array}
$$

Indeed, the right top square is commutative by Lemma 3.10, the commutativity of the remaining parts is straightforward. Hence, the diagram (32) is commutative, too.

Next, observe that all maps in the diagram (32) but $\bar{\imath}^!$ and ev_y are isomorphisms. By Lemma 3.6, the map ev_y is an isomorphism if and only if $n\cdot\bigl([x]-[y]\bigr)\neq 0$ in $J(E)$. This proves the second part of this Proposition. □

Note that from the exact sequence (26) we get the induced map

$$
R := H^0\bigl(\underline{\mathsf{res}}_x^{\mathcal{E}nd(\mathcal{P})}\bigr): \quad \mathsf{Hom}\bigl(\mathcal{P},\mathcal{P}(x)\bigr) \longrightarrow \mathsf{End}\bigl(\mathcal{P}\big|_x\bigr)
$$

sending an element $g\in\mathsf{Hom}\bigl(\mathcal{P},\mathcal{P}(x)\bigr)$ to $\bigl(\underline{\mathsf{res}}_x^{\mathcal{P}}\circ g\bigr)_x\in\mathsf{End}\bigl(\mathcal{P}\big|_x\bigr)$. Clearly, $R(\imath)=0$, thus we obtain the induced map

$$
\overline{R}: \quad \frac{\mathsf{Hom}\bigl(\mathcal{P},\mathcal{P}(x)\bigr)}{\langle \imath \rangle} \longrightarrow \mathsf{End}\bigl(\mathcal{P}\big|_x\bigr). \tag{33}
$$

Lemma 3.12. *In the above notation, the following statements are true.*

(1) $\mathsf{Im}(\overline{R})=\mathfrak{sl}\bigl(\mathcal{P}\big|_x\bigr)$.

(2) *Moreover, the map* $\overline{R}: \dfrac{\mathsf{Hom}\bigl(\mathcal{P},\mathcal{P}(x)\bigr)}{\langle \imath \rangle} \longrightarrow \mathfrak{sl}\bigl(\mathcal{P}\big|_x\bigr)$ *is an isomorphism.*

Proof. The result follows from the commutativity of the diagram

$$\begin{array}{ccc} H^0\big(\mathsf{Ad}(\mathcal{P})(x)\big) & \xrightarrow{\mathsf{res}_x} & \mathfrak{sl}\big(\mathcal{P}\big|_x\big) \\ \jmath\downarrow & & \downarrow \\ \dfrac{\mathsf{Hom}\big(\mathcal{P},\mathcal{P}(x)\big)}{\langle\imath\rangle} & \xrightarrow{\overline{R}} & \mathsf{End}\big(\mathcal{P}\big|_x\big) \end{array}$$

and the fact that the morphisms res_x and $\jmath$ are isomorphisms. □

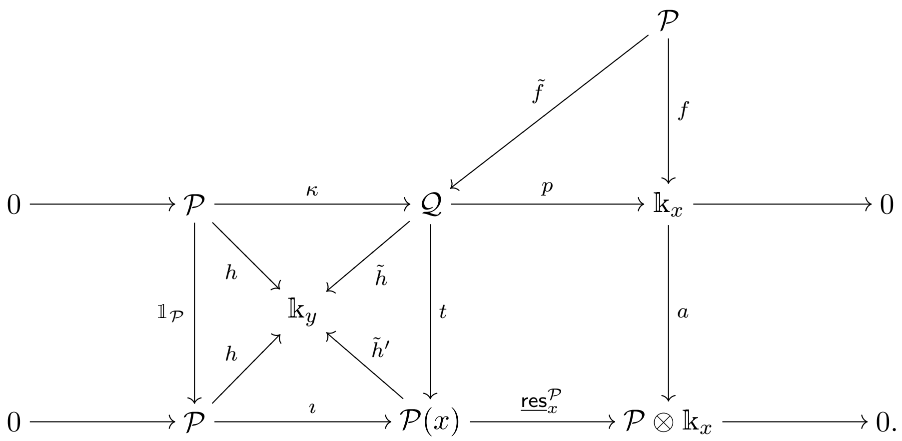
Step 2. The next result is the key part of the proof of Theorem 3.7.

Proposition 3.13. *The following diagram is commutative.*

$$\begin{array}{ccccc} \mathfrak{sl}\big(\mathcal{P}\big|_x\big) & \xrightarrow{T} & K & \xrightarrow{M_H} & \\ \overline{R}\uparrow & & & & \\ \dfrac{\mathsf{Hom}\big(\mathcal{P},\mathcal{P}(x)\big)}{\langle\imath\rangle} & \xrightarrow{\imath^!} & \mathfrak{pgl}\big(\mathsf{Hom}(\mathcal{P},\Bbbk_y)\big) & \xrightarrow{\bar{r}_H} & \mathsf{Lin}\big(H,\mathsf{Hom}(\mathcal{P},\Bbbk_y)/H\big). \end{array}$$

Proof. We show this result by diagram chasing. Recall that the vector space K is the linear span of the simple tensors $f\otimes\omega\in\mathsf{Hom}(\mathcal{P},\Bbbk_x)\otimes\mathsf{Ext}(\Bbbk_x,\mathcal{P})$ such that $\omega\circ f=0$. Let $0\longrightarrow\mathcal{P}\xrightarrow{\kappa}\mathcal{Q}\xrightarrow{p}\Bbbk_x\longrightarrow 0$ be a short exact sequence corresponding to an element $\omega\in\mathsf{Ext}(\Bbbk_x,\mathcal{P})$. Recall that by Proposition 3.3 there exists a unique $a\in\mathsf{Hom}(\Bbbk_x,\mathcal{P}\otimes\Bbbk_x)$ such that $\omega=\delta_x(a)$.

Since $\mathsf{Hom}(\Bbbk_x,\Bbbk_y)=0=\mathsf{Ext}(\Bbbk_x,\Bbbk_y)$, for any $h\in\mathsf{Hom}(\mathcal{P},\Bbbk_y)$ there exist unique elements $\tilde{h}\in\mathsf{Hom}(\mathcal{Q},\Bbbk_y)$ and $\tilde{h}'\in\mathsf{Hom}\big(\mathcal{P}(x),\Bbbk_y\big)$ such that the following diagram is commutative:

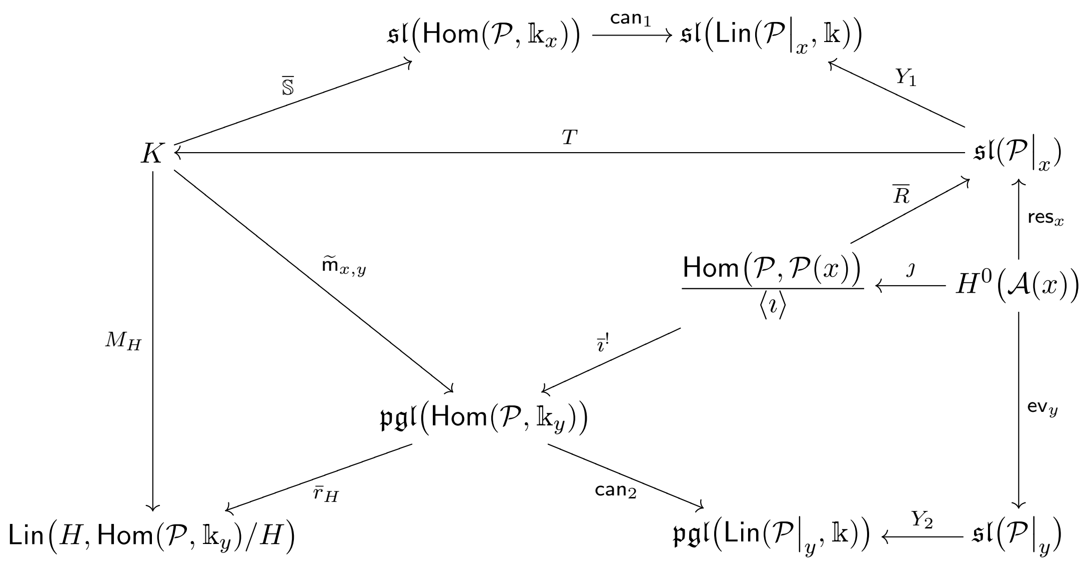


Although a lift $\tilde{f}\in\mathsf{Hom}(\mathcal{P},\mathcal{Q})$ is only defined up to a translation $\tilde{f}\mapsto\tilde{f}+\lambda\kappa$ for some $\lambda\in\Bbbk$, we have a well-defined element $\overline{t\circ\tilde{f}}\in\dfrac{\mathsf{Hom}\big(\mathcal{P},\mathcal{P}(x)\big)}{\langle\imath\rangle}$ such that $\overline{R}\big(\overline{t\circ\tilde{f}}\big)=a\circ f_x$. By definition, $T(a\circ f_x)=f\otimes\omega$. It remains to observe that

$$\big(\bar{r}_H\circ\vec{\imath}^{\,!}([t\tilde{f}])\big)(h)=[\tilde{h}'t\tilde{f}]=[\tilde{h}f]=\big(M_H(f\otimes\omega)\big)(h).$$

Since $\overline{R}$ and T are isomorphisms and the vector space K is generated by simple tensors, this concludes the proof. □

Step 3. Now we are ready to proceed with the proof of Theorem 3.7. Note that the following diagram is commutative.



where $\widetilde{\mathsf{m}}_{x,y} = \mathsf{m}^{\mathcal{P}}_{\Bbbk_x,\Bbbk_y}$ from (14). Indeed, by Lemma 3.5 we have the equality $Y_1 \circ T = \mathsf{can}_1 \circ \mathbb{S}$, which gives commutativity of the top square. Next, the equality $\bar{r}_H \circ \widetilde{\mathsf{m}}_{x,y} = M_H$ just expresses the commutativity of the diagram (15). The equality $\overline{R} \circ \jmath = \mathsf{res}_x$ follows from the definition of the map $\overline{R}$, see (33).

The equality $Y_2 \circ \mathsf{ev}_y = \mathsf{can}_2 \circ \overline{\imath}^! \circ \jmath$ is given by Proposition 3.11, yielding the commutativity of the right lower part. Finally, by Proposition 3.13 we have the equality $\bar{r}_H \circ \overline{\imath}^! = M_H \circ T \circ \overline{R}$. Since this equality is true for any one-dimensional subspace $H \subseteq \mathsf{Hom}(\mathcal{P}, \Bbbk_y)$, Lemma 2.3 implies that $\widetilde{\mathsf{m}}_{x,y} \circ T \circ \overline{R} = \overline{\imath}^!$. This finishes the proof of commutativity of the above diagram. It remains to conclude that the commutativity of the diagram (30) follows as well and Theorem 3.7 is proven. □

Corollary 3.14. *Let E be an elliptic curve over $\Bbbk$, $\mathcal{P}$ a simple vector bundle on E, $\mathcal{A} = \mathsf{Ad}(\mathcal{P})$ and $x, y \in R$ two distinct points. Let $r_{x,y} \in \mathcal{A}\big|_x \otimes \mathcal{A}\big|_y$ be the image of the linear map $\mathsf{ev}_y \circ \mathsf{res}_x^{-1} \in \mathsf{Lin}(\mathcal{A}\big|_x, \mathcal{A}\big|_y)$ under the linear isomorphism $\mathsf{Lin}(\mathcal{A}\big|_x, \mathcal{A}\big|_y) \longrightarrow \mathcal{A}\big|_x \otimes \mathcal{A}\big|_y$ induced by the Killing form $\mathcal{A}\big|_x \times \mathcal{A}\big|_x \longrightarrow \Bbbk, (a, b) \mapsto \mathsf{tr}(a \circ b)$. Then $r_{x,y}$ is a solution of the classical Yang–Baxter equation: for any pairwise distinct points $x_1, x_2, x_3 \in E$ we have:*

$$[r^{12}_{x_1,x_2}, r^{13}_{x_1,x_3}] + [r^{12}_{x_1,x_2}, r^{23}_{x_2,x_3}] + [r^{12}_{x_1,x_2}, r^{13}_{x_1,x_3}] = 0, \tag{34}$$

where both sides of the above identity are viewed as elements of $\mathcal{A}\big|_{x_1} \otimes \mathcal{A}\big|_{x_2} \otimes \mathcal{A}\big|_{x_3}$. Moreover, the tensor r_{x_1,x_2} is unitary:

$$r_{x_2,x_1} = -\tau(r_{x_1,x_2}), \tag{35}$$

where $\tau : \mathcal{A}\big|_{x_1} \otimes \mathcal{A}\big|_{x_2} \longrightarrow \mathcal{A}\big|_{x_2} \otimes \mathcal{A}\big|_{x_1}$ *is the map permuting both factors.*

Proof. By Theorem 3.7, the tensor $r_{x,y}$ is the image of the tensor $m_{x,y}$ from (19) under the isomorphism

$$\mathfrak{pgl}\bigl(\mathsf{Hom}(\mathcal{P}, \Bbbk_x)\bigr) \otimes \mathfrak{pgl}\bigl(\mathsf{Hom}(\mathcal{P}, \Bbbk_y)\bigr) \xrightarrow{\overline{Y}_2 \otimes \overline{Y}_2} \mathcal{A}\big|_x \otimes \mathcal{A}\big|_y.$$

Since $\overline{Y}_2$ is an anti-isomorphism of Lie algebras, the equality (34) is a corollary of (21). In the same way, the equality (35) is a consequence of (22). □

Now we generalize Corollary 3.14 to the case of the singular Weierstraß curves.

4. Genus one fibrations and CYBE

We start with the following geometric data.

- Let $E \xrightarrow{p} T$ be a flat *projective* morphism of relative dimension one between algebraic varieties. We denote by $\breve{E}$ the regular locus of p.
- We assume there exists a section $\imath : T \to \breve{E}$ of p.
- Moreover, we assume that for all points $t \in T$ the fiber E_t is an *irreducible* Calabi–Yau curve.
- The fibration $E \xrightarrow{p} T$ is embeddable into a smooth fibration of projective surfaces over T and $\Omega_{E/T} \cong \mathcal{O}_E$.

Example 4.1. Let $E_T \subset \mathbb{P}^2 \times \mathbb{A}^2 \longrightarrow \mathbb{A}^2 =: T$ be the elliptic fibration given by the equation $wv^2 = 4u^3 + g_2uw^2 + g_3w^3$ and let $\Delta(g_2, g_3) = g_2^3 + 27g_3^2$ be the discriminant of this family. This fibration has a section $(g_2, g_3) \mapsto \bigl((0:1:0), (g_2, g_3)\bigr)$ and satisfies the condition $\Omega_{E/T} \cong \mathcal{O}_E$.

The following result is well-known.

Lemma 4.2. *Consider* $(n, d) \in \mathbb{N} \times \mathbb{Z}$ *such that* $\gcd(n, d) = 1$. *Then there exists* $\mathcal{P} \in \mathsf{VB}(E)$ *such that for any* $t \in T$ *its restriction* $\mathcal{P}\big|_{E_t}$ *is simple of rank* n *and degree* d.

Sketch of the proof. Let $\Sigma := \imath(T) \subset E$ and $\mathcal{I}_\Delta$ be the structure sheaf of the diagonal $\Delta \subset E \times_T E$. Let $\mathsf{FM}^{\mathcal{I}_\Delta}$ be the Fourier–Mukai transform with the kernel $\mathcal{I}_\Delta$. By [12, Theorem 2.12], $\mathsf{FM}^{\mathcal{I}_\Delta}$ is an auto-equivalence of the derived category $\mathsf{FM}^{\mathcal{I}_\Delta}$. By [13, Proposition 4.13(iv)] there exists an auto-equivalence $\mathbb{F}$ of the derived category $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(E)$, which is a certain composition of the functors $\mathsf{FM}^{\mathcal{I}_\Delta}$ and $- \otimes \mathcal{O}(\Sigma)$ such that $\mathbb{F}(\mathcal{O}_\Sigma) \cong \mathcal{P}[0]$, where $\mathcal{P}$ is a vector bundle on E having the required properties. □

Now we fix the following notation. Let $\mathcal{P}$ be as in Lemma 4.2 and $\mathcal{A} = \mathsf{Ad}(\mathcal{P})$. We set $\overline{X} := E \times_T \breve{E} \times_T \breve{E}$ and $\overline{B} := \breve{E} \times_T \breve{E}$. Let $q : \overline{X} \longrightarrow \overline{B}$ be the canonical projection, $\Delta \subset \breve{E} \times_T \breve{E}$ the diagonal, $B := \overline{B} \setminus \Delta$ and $X := q^{-1}(B)$. The elliptic fibration $q : \overline{X} \longrightarrow \overline{B}$ has two canonical sections h_i, $i = 1, 2$, given by $h_i(y_1, y_2) = (y_i, y_1, y_2)$. Let $\Sigma_i := h_i(\overline{B})$ and $\overline{\mathcal{A}}$ be the pull-back of $\mathcal{A}$ on $\overline{X}$. Note that the relative dualizing sheaf $\Omega = \Omega_{\overline{X}/\overline{B}}$ is trivial. Similarly to (24) one has the following canonical short exact sequence

$$0 \longrightarrow \Omega \longrightarrow \Omega(\Sigma_1) \xrightarrow{\mathsf{res}_{\Sigma_1}} \mathcal{O}_{\Sigma_1} \longrightarrow 0, \tag{36}$$

see [14, Subsection 3.1.2] for a precise construction. By the assumptions from the beginning of this section, there exists an isomorphism $\mathcal{O}_{\overline{X}} \longrightarrow \Omega_{\overline{X}/\overline{B}}$ induced by a nowhere vanishing section $w \in H^0(\Omega_{E/T})$. It gives the following short exact sequence

$$0 \longrightarrow \overline{\mathcal{A}} \longrightarrow \overline{\mathcal{A}}(\Sigma_1) \xrightarrow{\mathsf{res}^{\mathcal{A}}_{\Sigma_1}} \overline{\mathcal{A}}\big|_{\Sigma_1} \longrightarrow 0. \tag{37}$$

In a similar way, we have another canonical sequence

$$0 \longrightarrow \overline{\mathcal{A}}(\Sigma_1 - \Sigma_2) \longrightarrow \overline{\mathcal{A}}(\Sigma_1) \longrightarrow \overline{\mathcal{A}}(\Sigma_1)\big|_{\Sigma_2} \longrightarrow 0. \tag{38}$$

Proposition 4.3. *In the above notation, the following results are true.*

- *We have the vanishing* $q_*(\overline{\mathcal{A}}) = 0 = \mathbb{R}^1 q_*(\overline{\mathcal{A}})$.
- *The coherent sheaf* $q_*\big(\overline{\mathcal{A}}(\Sigma_1)\big)$ *is locally free.*
- *Moreover, we have the morphism of locally free sheaves on B given by the composition* $q_*\big(\overline{\mathcal{A}}(\Sigma_1)\big) \longrightarrow q_*\big(\overline{\mathcal{A}}(\Sigma_1)\big|_{\Sigma_2}\big) \longrightarrow q_*\big(\overline{\mathcal{A}}\big|_{\Sigma_2}\big)$, *which is an isomorphism outside of the closed subset* $\Delta_n := \big\{(t,x,y) \mid n \cdot \big([x]-[y]\big) = 0 \in J(E_t)\big\} \subset B$.

Proof. Let $z = (t,x,y) \in \overline{B}$ be an arbitrary point. By the base-change formula we have: $\mathbb{L}\imath_z^*\big(\mathbb{R}q_*(\overline{\mathcal{A}})\big) \cong \mathbb{R}\Gamma(\mathcal{A}\big|_{E_t}) = 0$, where the last vanishing is true by Proposition 2.14. This proves the first part of the theorem.

Thus, applying q_* to the short exact sequence (37), we get an isomorphism

$$\mathsf{res}_1 := q_*\big(\mathsf{res}^{\overline{\mathcal{A}}}_{\Sigma_1}\big) : \quad q_*\big(\overline{\mathcal{A}}(\Sigma_1)\big) \longrightarrow q_*\big(\overline{\mathcal{A}}\big|_{\Sigma_1}\big).$$

For $i = 1, 2$, let $p_i : \overline{B} := \breve{E} \times \breve{E} \longrightarrow E$ be the composition of i-th canonical projection with the canonical inclusion $\breve{E} \subseteq E$. It is easy to see that we have a canonical isomorphism $\gamma : q_*\big(\overline{\mathcal{A}}\big|_{\Sigma_i}\big) \longrightarrow p_i^*(\mathcal{A})$. Hence, the coherent sheaf $q_*\big(\overline{\mathcal{A}}(\Sigma_1)\big)$ is locally free on $\overline{B}$.

To prove the last part, first note that the canonical morphism $q_*\big(\overline{\mathcal{A}}\big|_{\Sigma_2}\big) \longrightarrow q_*\big(\overline{\mathcal{A}}(\Sigma_1)\big|_{\Sigma_2}\big)$ is an isomorphism on B. Moreover, by Proposition 2.14, the subset Δ_n is precisely the support of the complex $\mathbb{R}q_*\big(\mathcal{A}(\Sigma_1 - \Sigma_2)\big)$. In particular, this shows that Δ_n is a proper closed subset of B. Finally, applying q_* to the short exact sequence (38), we get a morphism of locally free sheaves $\mathsf{ev}_2 : q_*\big(\overline{\mathcal{A}}(\Sigma_1)\big) \longrightarrow p_2^*(\mathcal{A})$, which is an isomorphism on the complement of Δ_n. This proves the proposition. □

Theorem 4.4. *In the above notation, let* $r \in \Gamma\big(\overline{B}, p_1^*(\mathcal{A}) \otimes p_2^*(\mathcal{A})\big)$ *be the meromorphic section which is the image of* $\mathsf{ev}_2 \circ \mathsf{res}_1^{-1}$ *under the canonical isomorphism*

$$\mathsf{Hom}\big(p_1^*(\mathcal{A}), p_2^*(\mathcal{A})\big) \longrightarrow H^0\big(p_1^*(\mathcal{A})^\vee \otimes p_2^*(\mathcal{A})\big) \longrightarrow H^0\big(p_1^*(\mathcal{A}) \otimes p_2^*(\mathcal{A})\big).$$

The last isomorphism above is induced by the canonical isomorphism $\mathcal{A} \longrightarrow \mathcal{A}^\vee$ *from Proposition 2.8. Then the following statements are true.*

- *The poles of r lie on the divisor Δ. In particular, r is holomorphic on B.*
- *Moreover, r is non-degenerate on the complement of the set Δ_n.*
- *The section r satisfies a version of the classical Yang–Baxter equation:*

$$\big[r^{12}, r^{13}\big] + \big[r^{12}, r^{23}\big] + \big[r^{13}, r^{23}\big] = 0,$$

where both sides are viewed as meromorphic sections of $p_1^*(\mathcal{A}) \otimes p_2^*(\mathcal{A}) \otimes p_3^*(\mathcal{A})$.

- *Moreover, the section r is unitary. This means that*

$$\sigma^*(r) = -\tilde{r} \in H^0\big(p_2^*(\mathcal{A}) \otimes p_1^*(\mathcal{A})\big), \tag{39}$$

where σ is the canonical involution of $\overline{B} = \breve{E} \times_T \breve{E}$ and $\tilde{r}$ is the section corresponding to the morphism $\mathsf{ev}_1 \circ \mathsf{res}_2^{-1}$.

- *In particular, Corollary 3.14 is also true for singular Weierstraß cubic curves.*

Proof. By Proposition 4.3, we have the following morphisms in $\mathsf{VB}(\overline{B})$:

$$p_1^*(\mathcal{A}) \xleftarrow{\mathsf{res}_1} q_*\big(\overline{\mathcal{A}}(\Sigma_1)\big) \longrightarrow q_*\big(\overline{\mathcal{A}}(\Sigma_1)\big|_{\Sigma_2}\big) \xleftarrow{\imath} q_*\big(\overline{\mathcal{A}}\big|_{\Sigma_2}\big) \xrightarrow{\gamma} p_2^*(\mathcal{A}).$$

Moreover, γ is an isomorphism, whereas res_1 and $\imath$ become isomorphisms after restricting on B. This shows that the section $r \in \Gamma\big(\overline{B}, p_1^*(\mathcal{A}) \otimes p_2^*(\mathcal{A})\big)$ is indeed *meromorphic* with poles lying on the diagonal Δ. Since $\mathsf{ev}_2 \circ \mathsf{res}_1^{-1}$ is an isomorphism on $B \setminus \Delta_n$, the section r is non-degenerate on $B \setminus \Delta_n$.

To prove the last two parts of the theorem, assume first that the generic fiber of E is smooth. Let $t \in T$ be such that E_t is an elliptic curve. Then in the notation of Corollary 3.14, for any $z = (t, x, y) \in B$ have have:

$$\imath_z^*(r) = r_{x,y} \in \big(\mathcal{A}\big|_{E_t}\big)\Big|_x \otimes \big(\mathcal{A}\big|_{E_t}\big)\Big|_y,$$

where we use the canonical isomorphism

$$\imath_z^*\big(p_1^*(\overline{\mathcal{A}}) \otimes p_2^*(\overline{\mathcal{A}})\big) \longrightarrow \big(\mathcal{A}\big|_{E_t}\big)\Big|_x \otimes \big(\mathcal{A}\big|_{E_t}\big)\Big|_y.$$

Let x_1, x_2 and x_3 be three pairwise distinct points of E_t and $\bar{x} = (t, x_1, x_2, x_3) \in \breve{E} \times_T \breve{E} \times_T \breve{E}$. By Corollary 3.14 we have:

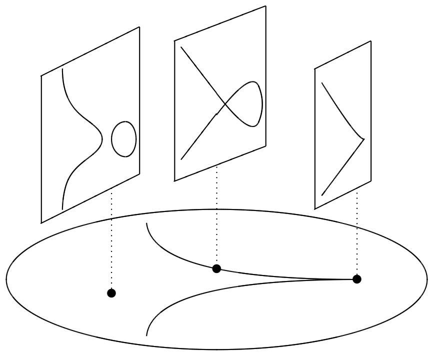
$$\imath_{\bar{x}}^*\Big(\big[r^{12}, r^{13}\big] + \big[r^{12}, r^{23}\big] + \big[r^{13}, r^{23}\big]\Big) = 0. \tag{40}$$

In a similar way, we have the equality:

$$\imath_z^*\big(\sigma^*(r) + \tilde{r}\big) = 0. \tag{41}$$

Since the section r is continuous on B, the equalities (40) and (41) are true for the singular fibers of E as well. In particular, the statement of Corollary 3.14 is also true for singular Weierstraß cubic curves. This implies that Theorem 4.4 is true for an arbitrary genus one fibrations satisfying the conditions from the beginning of this section. □

Summary. Let $E \xrightarrow{p} T$, $\imath : T \longrightarrow E$ and $w \in H^0\big(\Omega_{E/T}\big)$ be as at the beginning of the section, $\mathcal{P}$ be a relatively stable vector bundle on E of rank n and degree d (recall that we automatically have $\gcd(n, d) = 1$) and $\mathcal{A} = \mathsf{Ad}(\mathcal{P})$.



For any closed point of the base $t \in T$ let U be a small neighborhood of the point $\imath(t) \in E_{t_0}$, V be a small neighborhood of $\bigl(t, \imath(t), \imath(t)\bigr) \in E \times_T E$, $O = \Gamma(U, \mathcal{O})$ and $M = \Gamma(V, \mathcal{M})$, where $\mathcal{M}$ is the sheaf of meromorphic functions on $E \times_T E$. Taking an isomorphism of Lie algebras $\xi : \mathcal{A}(U) \longrightarrow \mathfrak{sl}_n(O)$, we get the tensor-valued meromorphic function

$$r^\xi = r^\xi_{(E,(n,d))} \in \mathfrak{sl}_n(M) \otimes_M \mathfrak{sl}_n(M),$$

which is the image of the *canonical* meromorphic section $r \in \Gamma\bigl(E \times_T E, p_1^*(\mathcal{A}) \otimes p_2^*(\mathcal{A})\bigr)$ from Theorem 4.4. Then the following statements are true.

- The poles of r^ξ lie on the diagonal $\Delta \subset E \times_T E$.
- Moreover, for a fixed $t \in T$ this function is a *unitary* solution of the classical Yang–Baxter equation (1) in variables $(y_1, y_2) \in \{t\} \times (U \cap E_t) \times (U \cap E_t) \subset V \subset E \times_T E$. In other words, we get a family of solutions $r^\xi_t(y_1, y_2)$ of the classical Yang–Baxter equation, which is *analytic* as the function of the parameter $t \in T$.
- Let $\xi' : \mathcal{A}(U) \longrightarrow \mathfrak{sl}_n(O)$ be another isomorphism of Lie algebras and $\rho := \xi' \circ \xi^{-1}$. Then we have the following commutative diagram:

$$\begin{array}{ccc} & \mathcal{A}(U) & \\ \xi \swarrow & & \searrow \xi' \\ \mathfrak{sl}_n(O) & \xrightarrow{\quad\rho\quad} & \mathfrak{sl}_n(O). \end{array}$$

  Moreover, for any $(t, y_1, y_2) \in V \setminus \Delta$ we have:

$$r^{\xi'}(y_1, y_2) = \bigl(\rho(y_1) \otimes \rho(y_2)\bigr) \cdot r^\xi(y_1, y_2) \cdot \bigl(\rho^{-1}(y_1) \otimes \rho^{-1}(y_2)\bigr).$$

  In other words, the solutions r^ξ and $r^{\xi'}$ are *gauge equivalent*.

Remark 4.5. One possibility to generalize Theorem 4.4 and for an arbitrary Calabi–Yau curve E can be achieved by showing that any simple vector bundle on E can be obtained from the structure sheaf $\mathcal{O}$ by applying an appropriate auto-equivalence of the triangulated category $\mathsf{Perf}(E)$ (some progress in this direction has been recently achieved by Hernández Ruipérez, López Martín, Sánchez Gómez and Tejero Prieto in [23]). Once it is done, going along the same lines as in Lemma 4.2, one can construct a sheaf of Lie algebras $\mathcal{A}$ on a genus one fibration $E \xrightarrow{p} T$ such that for the smooth fibers $\mathcal{A}\big|_{E_t} \cong \mathcal{A}_{n,d}$ and for the singular ones $\mathcal{A}\big|_{E_t} \cong \mathcal{A}_{n,\mathbb{d}}$ for n, d and $\mathbb{d}$ as in Proposition 2.14.

At this moment one can pose the following natural question: How constructive is the suggested method of finding of solutions of the classical Yang–Baxter equation (1)? Actually, one can work out a completely explicit recipe to compute the tensor $r^{\xi}_{(E,(n,d))}$ for all types of Weierstraß cubic curves, see for example [14], where an analogous approach to the associative Yang–Baxter equation has been elaborated. The following result can be found in [14, Chapter 6] and also in [31].

Example 4.6. Fix the following basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

of the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. For the pair $(n,d) = (2,1)$ we get the following solutions $r_{(E,(2,1))}$ of the classical Yang–Baxter equation (3).

- In the case E is elliptic, we get the elliptic solution of Baxter:

$$r_{\mathrm{ell}}(z) = \frac{\mathrm{cn}(z)}{\mathrm{sn}(z)} h \otimes h + \frac{1+\mathrm{dn}(z)}{\mathrm{sn}(z)}(e \otimes f + f \otimes e) + \frac{1-\mathrm{dn}(z)}{\mathrm{sn}(z)}(e \otimes e + f \otimes f), \tag{42}$$

- In the case E is nodal, we get the trigonometric solution of Cherednik

$$r_{\mathrm{trg}}(z) = \frac{1}{2}\cot(z) h \otimes h + \frac{1}{\sin(z)}(e \otimes f + f \otimes e) + \sin(z) e \otimes e. \tag{43}$$

- In the case E is cuspidal, we get the rational solution

$$r_{\mathrm{rat}}(z) = \frac{1}{z}\left(\frac{1}{2} h \otimes h + e_{12} \otimes e_{21} + e_{21} \otimes e_{12}\right) + z(f \otimes h + h \otimes f) - z^3 f \otimes f. \tag{44}$$

Remark 4.7. It is a non-trivial analytic consequence of Theorem 4.4 that up a certain (unknown) gauge transformation and a change of variables, the rational solution (44) is a degeneration of the elliptic solution (42) and the trigonometric solution (43).

In the second part of this article, we describe solutions of (1) corresponding to the smooth respectively cuspidal Weierstraß curves. All of them turn out to be elliptic respectively rational. We shall recover all elliptic solutions respectively certain distinguished rational solutions. Note that rational solutions of (1) are most complicated and less understood from the point of view of the Belavin–Drinfeld classification [3].

5. Vector bundles on elliptic curves and elliptic solutions of the classical Yang–Baxter equation

Let $\tau \in \mathbb{C}$ be such that $\mathrm{Im}(\tau) > 0$ and $E = \mathbb{C}/\langle 1, \tau \rangle$ the corresponding complex torus. Let $0 < d < n$ be two coprime integers and $\mathcal{A} = \mathcal{A}_{n,d}$ be the sheaf of Lie algebras from Proposition 2.14.

Proposition 5.1. *The sheaf $\mathcal{A}$ has the following complex–analytic description:*

$$\mathcal{A} \cong \mathbb{C} \times \mathfrak{g} / \sim, \quad \text{with} \quad (z, G) \sim (z+1, XGX^{-1}) \sim (z+\tau, YGY^{-1}), \tag{45}$$

where X and Y are the matrices (5).

Proof. We first recall some well-known technique to work with holomorphic vector bundles on complex tori, see for example [5, 27].

• Let $\mathbb{C} \supset \Lambda = \Lambda_\tau := \langle 1, \tau\rangle \cong \mathbb{Z}^2$. An automorphy factor is a pair (A, V), where V is a finite dimensional vector space over $\mathbb{C}$ and $A : \Lambda \times \mathbb{C} \longrightarrow \mathsf{GL}(V)$ is a holomorphic function such that $A(\lambda+\mu, z) = A(\lambda, z+\mu)A(\mu, z)$ for all $\lambda, \mu \in \Lambda$ and $z \in \mathbb{C}$. Such a pair defines the following holomorphic vector bundle on the torus E:

$$\mathcal{E}(A, V) := \mathbb{C} \times V/\sim, \text{ where } \quad (z, v) \sim \bigl(z+\lambda, A(\lambda, z)v\bigr) \quad \forall(\lambda, z, v) \in \Lambda \times \mathbb{C} \times V.$$

Two such vector bundles $\mathcal{E}(A, V)$ and $\mathcal{E}(B, V)$ are isomorphic if and only if there exists a holomorphic function $H : \mathbb{C} \to \mathsf{GL}(V)$ such that

$$B(\lambda, z) = H(z+\lambda)A(\lambda, z)H(z)^{-1} \quad \text{for all} \quad (\lambda, z) \in \Lambda \times \mathbb{C}.$$

Assume that $\mathcal{E} = \mathcal{E}(\mathbb{C}^n, A)$. Then $\mathsf{Ad}(\mathcal{E}) \cong \mathcal{E}\bigl(\mathfrak{g}, \mathsf{ad}(A)\bigr)$, where $\mathsf{ad}(A)(\lambda, z)(G) := A(\lambda, z) \cdot G \cdot A(\lambda, z)^{-1}$ for $G \in \mathfrak{g}$.

• Quite frequently, it is convenient to restrict ourselves on the following setting. Let $\Phi : \mathbb{C} \longrightarrow \mathsf{GL}_n(\mathbb{C})$ be a holomorphic function such that $\Phi(z+1) = \Phi(z)$ for all $z \in \mathbb{C}$. In other words, we assume that Φ factors through the covering map $\mathbb{C} \xrightarrow{\exp(2\pi i(-))} \mathbb{C}^*$. Then one can define the automorphy factor $(A, \mathbb{C}^n)$ in the following way.

– $A(0, z) = I_n$ is the identity matrix.

– For any $a \in \mathbb{Z}_{>0}$ we set:

$$A(a\tau, z) = \Phi\bigl(z+(a-1)\tau\bigr) \cdot \dots \cdot \Phi(z) \text{ and } A(-a\tau, z) = A(a\tau, z-a\tau)^{-1}.$$

– For any $a, b \in \mathbb{Z}$ we set: $A(a\tau+b, z) = A(a\tau, z)$.

Let $\mathcal{E}(\Phi) := \mathcal{E}(A, \mathbb{C}^n)$ be the corresponding vector bundle on E.

• Consider the holomorphic function $\psi(z) = \exp\bigl(-\pi i d\tau - \frac{2\pi i d}{n}z\bigr)$ and the matrix

$$\Psi = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ \psi^n & 0 & \dots & 0 \end{pmatrix}.$$

It follows from Oda's description of simple vector bundles on elliptic curves [28], that the vector bundle $\mathcal{E}(\Psi)$ is simple of rank n and degree d. See also [14, Proposition 4.1.6] for a proof of this result.

• Denote $\varepsilon = \exp(\frac{2\pi i d}{n})$, $\eta = \varepsilon^{-1}$ and $\rho = \exp(-\frac{2\pi i d}{n}\tau)$. Consider the function $H = \mathsf{diag}\bigl(\psi^{n-1}, \dots, \psi, 1\bigr) : \mathbb{C} \longrightarrow \mathsf{GL}_n(\mathbb{C})$ and the matrices $X' = \mathsf{diag}(\eta^{n-1}, \dots, \eta, 1)$, $Z' = \mathsf{diag}(\rho^{n-1}, \dots, \rho, 1)$, and

$$Y' = \begin{pmatrix} 0 & \rho^{n-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \rho \\ 1 & 0 & \dots & 0 \end{pmatrix}.$$

Let $B(\lambda, z) = H(z+\lambda)A(\lambda, z)H(z)^{-1}$, where $A(\lambda, z)$ is the automorphy factor defined by the function Φ. Then we have: $B(1, z) = X'$ and $B(\tau, z) = \psi \cdot Y'$.

• Note that $\mathsf{ad}(B) = \mathsf{ad}(\varphi \cdot B) \in \mathsf{End}(\mathfrak{g})$ for an arbitrary holomorphic function φ. Hence, after the conjugation of X' and Y' with an appropriate constant diagonal matrix and a subsequent rescaling, we get: $\mathcal{A} \cong \mathcal{E}\big(\mathsf{ad}(C), \mathfrak{g}\big)$, where $C(1, z) = X$ and $C(\tau, z) = Y$. This concludes the proof. □

Let $I := \big\{(p, q) \in \mathbb{Z}^2 \big| 0 \le p \le n-1, 0 \le q \le n-1, (p, q) \ne (0, 0)\big\}$. For any $(k, l) \in I$ denote $Z_{k,l} = Y^k X^{-l}$ and $Z^\vee_{k,l} = \frac{1}{n} X^l Y^{-k}$. Recall the following standard result.

Lemma 5.2. *The following is true.*

- *The operators* $\mathsf{ad}(X), \mathsf{ad}(Y) \in \mathsf{End}(\mathfrak{g})$ *commute.*
- *The set* $\big\{Z_{k,l}\big\}_{(k,l)\in I}$ *is a basis of* $\mathfrak{g}$.
- *Moreover, for any* $(k, l) \in I$ *we have:*

$$\mathsf{ad}(X)(Z_{k,l}) = \varepsilon^k Z_{k,l} \quad \textit{and} \quad \mathsf{ad}(Y)(Z_{k,l}) = \varepsilon^l Z_{k,l}.$$

- *Let* $\mathsf{can} : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathsf{End}(\mathfrak{g})$ *be the canonical isomorphism sending a simple tensor* $G' \otimes G''$ *to the linear map* $G \mapsto \mathsf{tr}(G' \cdot G) \cdot G''$. *Then we have:*

$$\mathsf{can}(Z^\vee_{k,l} \otimes Z_{k,l})(Z_{k',l'}) = \begin{cases} Z_{k,l} & \textit{if } (k', l') = (k, l) \\ 0 & \textit{otherwise.} \end{cases}$$

Next, recall the definition of the first and third Jacobian theta-functions [27].

$$(46) \qquad \begin{cases} \bar{\theta}(z) = \theta_1(z|\tau) = 2q^{\frac{1}{4}} \sum\limits_{n=0}^{\infty} (-1)^n q^{n(n+1)} \sin\big((2n+1)\pi z\big), \\ \theta(z) = \theta_3(z|\tau) = 1 + 2\sum\limits_{n=1}^{\infty} q^{n^2} \cos(2\pi n z), \end{cases}$$

where $q = \exp(\pi i \tau)$. They are related by the following identity:

$$(47) \qquad \theta\Big(z + \frac{1+\tau}{2}\Big) = i \exp\Big(-\pi i\big(z + \frac{\tau}{4}\big)\Big)\bar{\theta}(z).$$

For any $x \in \mathbb{C}$ consider the function $\varphi_x(z) = -\exp\big(-2\pi i(z + \tau - x)\big)$. The next result is well-known, see [27] or [14, Section 4.1].

Lemma 5.3. *The following results are true.*

- *The vector space*

$$\left\{ \mathbb{C} \xrightarrow{f} \mathbb{C} \;\middle|\; \begin{array}{l} f \textit{ is holomorphic} \\ f(z+1) = f(z) \\ f(z+\tau) = \varphi_x(z) f(z) \end{array} \right\}$$

is one-dimensional and generated by the theta-function $\theta_x(z) := \theta\big(z + \frac{1+\tau}{2} - x\big)$.

- *We have:* $\mathcal{E}(\varphi_x) \cong \mathcal{O}_E\big([x]\big)$.

Let $U \subset \mathbb{C}$ be a small open neighborhood of 0 and $O = \Gamma(U, \mathcal{O}_\mathbb{C})$ be the ring of holomorphic functions on U. Let z be a coordinate on U, $\mathbb{C} \xrightarrow{\pi} E$ the canonical covering map, $w = dz \in H^0(E, \Omega)$ a no-where vanishing differential form on E, $\Gamma(U, \mathcal{A}) \xrightarrow{\xi} \mathfrak{sl}_n(O)$ the canonical isomorphism induced by the automorphy data (X, Y) and $x, y \in U$ a pair of distinct points. Consider the following vector space

$$\mathsf{Sol}\big((n,d),x\big) = \left\{ \mathbb{C} \stackrel{F}{\longrightarrow} \mathfrak{g} \;\middle|\; \begin{array}{l} F \quad \text{is holomorphic} \\ F(z+1) = XF(z)X^{-1} \\ F(z+\tau) = \varphi_x(z)YF(z)Y^{-1} \end{array} \right\}.$$

Proposition 5.4. *The following diagram*

$$\begin{array}{ccccc} \mathcal{A}\big|_x & \xleftarrow{\mathsf{res}_x^{\mathcal{A}}(w)} & H^0\big(\mathcal{A}(x)\big) & \xrightarrow{\mathsf{ev}_y^{\mathcal{A}}} & \mathcal{A}\big|_y \\ \downarrow{\scriptstyle \jmath_x} & & \downarrow{\scriptstyle \jmath} & & \downarrow{\scriptstyle \jmath_y} \\ \mathfrak{g} & \xleftarrow{\overline{\mathsf{res}}_x} & \mathsf{Sol}\big((n,d),x\big) & \xrightarrow{\overline{\mathsf{ev}}_y} & \mathfrak{g} \end{array}$$

is commutative, where for $F \in \mathsf{Sol}\big((n,d),x\big)$ we have:

$$\overline{\mathsf{res}}_x(F) = \frac{F(x)}{\theta'\big(\frac{1+\tau}{2}\big)} \quad \text{and} \quad \overline{\mathsf{ev}}_y(F) = \frac{F(y)}{\theta\big(y - x + \frac{1+\tau}{2}\big)}.$$

The linear isomorphism $\jmath$ is induced by the pull-back map π^*.

Comment on the proof. This result can be proven along the same lines as in [14, Section 4.2], see in particular [14, Corollary 4.2.1], hence we omit the details here. □

Now we are ready to prove the main result of this section.

Theorem 5.5. *The solution $r_{(E,(n,d))}(x, y)$ of the classical Yang–Baxter equation (1) constructed in Section 4, is given by the following expression:*

$$r_{(E,(n,d))}(x,y) = \sum_{(k,l)\in I} \exp\Big(-\frac{2\pi i d}{n} kv\Big)\sigma\Big(\frac{d}{n}\big(l - k\tau\big), v\Big) Z_{k,l}^{\vee} \otimes Z_{k,l}, \tag{48}$$

where $v = x - y$ and $\sigma(u, z)$ is the Kronecker elliptic function (7).

Proof. We first have to compute an explicit basis of the vector space $\mathsf{Sol}\big((n,d),x\big)$. For this, we write:

$$F(z) = \sum_{(k,l)\in I} f_{k,l}(z) Z_{k,l}.$$

The condition $F \in \mathsf{Sol}\big((n,d),x\big)$ yields the following constraints on the coefficients $f_{k,l}$:

$$\left\{ \begin{array}{lcl} f_{k,l}(z+1) & = & \varepsilon^k f_{k,l}(z) \\ f_{k,l}(z+\tau) & = & \varepsilon^l \varphi_x(z) f_{k,l}(z). \end{array} \right. \tag{49}$$

It follows from Lemma 5.3 that the space of solutions of the system (49) is one–dimensional and generated by the holomorphic function

$$f_{k,l}(z) = \exp\Big(-\frac{2\pi i d}{n} kz\Big)\theta\Big(z + \frac{1+\tau}{2} - x - \frac{d}{n}(k\tau - l)\Big).$$

From Proposition 5.4 and Lemma 5.2 it follows that the solution $r_{(E,(n,d))}(x,y)$ is given by the following formula:

$$r_{(E,(n,d))}(x,y) = \sum_{(k,l)\in I} r_{k,l}(v) Z_{k,l}^{\vee} \otimes Z_{k,l},$$

where $v = y - x$ and

$$r_{k,l}(v) = \exp\Bigl(-\frac{2\pi i d}{n} k v\Bigr) \frac{\theta'\Bigl(\frac{1+\tau}{2}\Bigr)\theta\Bigl(v + \frac{1+\tau}{2} - \frac{d}{n}(k\tau - l)\Bigr)}{\theta\Bigl(-\frac{d}{n}(k\tau - l)\Bigr)\theta(v)}.$$

Relation (47) implies that

$$r_{k,l}(v) = \exp\Bigl(-\frac{2\pi i d}{n} k v\Bigr) \frac{\bar\theta'(0)\bar\theta\Bigl(v - \frac{d}{n}(k\tau - l)\Bigr)}{\bar\theta\Bigl(-\frac{d}{n}(k\tau - l)\Bigr)\bar\theta(v)}$$

Let $\sigma(u, z)$ be the Kronecker elliptic function (7). It remains to observe that formula (48) follows now from the identity

$$\sigma(u, x) = \frac{\bar\theta'(0)\bar\theta_1(u + x)}{\bar\theta(u)\bar\theta(x)}.$$

□

6. Vector bundles on the cuspidal Weierstrass curve and the classical Yang–Baxter equation

The goal of this section is to derive an explicit algorithm to compute the solution $r_{(E,(n,d))}$ of (1), corresponding to a pair of coprime integers $0 < d < n$ and the cuspidal Weierstraß curve E, which has been constructed in Section 4.

6.1. **Some results on vector bundles on singular curves.** We first recall some general technique to describe vector bundles on singular projective curves, see [6, 10, 18] and especially [14, Section 5.1].

Let X be a reduced singular (projective) curve, $\pi : \widetilde{X} \longrightarrow X$ its normalisation, $\mathcal{I} := \mathcal{H}om_{\mathcal{O}}\bigl(\pi_*(\mathcal{O}_{\widetilde{X}}), \mathcal{O}\bigr) = \mathcal{A}nn_{\mathcal{O}}\bigl(\pi_*(\mathcal{O}_{\widetilde{X}})/\mathcal{O}\bigr)$ the conductor ideal sheaf. Denote by $\eta : Z = V(\mathcal{I}) \longrightarrow X$ the closed Artinian subscheme defined by $\mathcal{I}$ (its topological support is precisely the singular locus of X) and by $\tilde\eta : \widetilde{Z} \longrightarrow \widetilde{X}$ its preimage in $\widetilde{X}$, defined by the Cartesian diagram

$$\begin{array}{ccc} \widetilde{Z} & \xrightarrow{\tilde\eta} & \widetilde{X} \\ {\scriptstyle\tilde\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ Z & \xrightarrow{\eta} & X. \end{array} \qquad (50)$$

In what follows we shall denote $\nu = \eta\tilde\pi = \pi\tilde\eta$.

In order to relate vector bundles on X and $\widetilde{X}$ we need the following construction.

Definition 6.1. The category $\mathsf{Tri}(X)$ is defined as follows.

- Its objects are triples $\bigl(\widetilde{\mathcal{F}}, \mathcal{V}, \theta\bigr)$, where $\widetilde{\mathcal{F}} \in \mathsf{VB}(\widetilde{X})$, $\mathcal{V} \in \mathsf{VB}(Z)$ and
$$\theta:\ \tilde{\pi}^*\mathcal{V} \longrightarrow \tilde{\eta}^*\widetilde{\mathcal{F}}$$
is an isomorphism of $\mathcal{O}_{\widetilde{Z}}$–modules.
- The set of morphisms $\mathsf{Hom}_{\mathsf{Tri}(X)}\bigl((\widetilde{\mathcal{F}}_1, \mathcal{V}_1, \theta_1), (\widetilde{\mathcal{F}}_2, \mathcal{V}_2, \theta_2)\bigr)$ consists of all pairs (f, g), where $f : \widetilde{\mathcal{F}}_1 \longrightarrow \widetilde{\mathcal{F}}_2$ and $g : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ are morphisms of vector bundles such that the following commutative
$$\begin{array}{ccc} \tilde{\pi}^*\mathcal{V}_1 & \xrightarrow{\theta_1} & \tilde{\eta}^*\widetilde{\mathcal{F}}_1 \\ {\scriptstyle \tilde{\pi}^*(g)}\downarrow & & \downarrow{\scriptstyle \tilde{\eta}^*(f)} \\ \tilde{\pi}^*\mathcal{V}_2 & \xrightarrow{\theta_2} & \tilde{\eta}^*\widetilde{\mathcal{F}}_2 \end{array}$$
is commutative.

The importance of Definition 6.1 is explained by the following theorem.

Theorem 6.2. *Let X be a reduced curve. Then the following results are true.*

- *Let $\mathbb{F} : \mathsf{VB}(X) \longrightarrow \mathsf{Tri}(X)$ be the functor assigning to a vector bundle $\mathcal{F}$ the triple $(\pi^*\mathcal{F}, \eta^*\mathcal{F}, \theta_{\mathcal{F}})$, where $\theta_{\mathcal{F}} : \tilde{\pi}^*(\eta^*\mathcal{F}) \longrightarrow \tilde{\eta}^*(\pi^*\mathcal{F})$ is the canonical isomorphism. Then $\mathbb{F}$ is an equivalence of categories.*
- *Let $\mathbb{G} : \mathsf{Tri}(X) \longrightarrow \mathsf{Coh}(X)$ be the functor assigning to a triple $(\widetilde{\mathcal{F}}, \mathcal{V}, \theta)$ the coherent sheaf $\mathcal{F} := \ker\bigl(\pi_*\widetilde{\mathcal{F}} \oplus \eta_*\mathcal{V} \xrightarrow{(\mathsf{c},\, -\bar{\theta})} \nu_*\tilde{\eta}^*\widetilde{\mathcal{F}}\bigr)$, where $\mathsf{c} = \mathsf{c}^{\widetilde{\mathcal{F}}}$ is the canonical morphism $\pi_*\widetilde{\mathcal{F}} \longrightarrow \pi_*\tilde{\eta}_*\tilde{\eta}^*\widetilde{\mathcal{F}} = \nu_*\tilde{\eta}^*\widetilde{\mathcal{F}}$ and $\bar{\theta}$ is the composition $\eta_*\mathcal{V} \xrightarrow{\mathsf{can}} \eta_*\tilde{\pi}_*\tilde{\pi}^*\mathcal{V} \xrightarrow{=} \nu_*\tilde{\pi}^*\mathcal{V} \xrightarrow{\nu_*(\theta)} \nu_*\tilde{\eta}^*\widetilde{\mathcal{F}}$. Then the coherent sheaf $\mathcal{F}$ is locally free. Moreover, the functor $\mathbb{G}$ is quasi-inverse to $\mathbb{F}$.*

A proof of this Theorem can be found in [10, Theorem 1.3]. □

Let $\mathcal{T} = \bigl(\widetilde{\mathcal{F}}, \mathcal{V}, \theta\bigr)$ be an object of $\mathsf{Tri}(X)$. Consider the morphism
$$\overline{\mathsf{conj}}(\theta):\quad \mathcal{E}nd_{\widetilde{Z}}(\tilde{\pi}^*\mathcal{V}) \longrightarrow \mathcal{E}nd_{\widetilde{Z}}(\tilde{\eta}^*\widetilde{\mathcal{F}}),$$
sending a local section φ to $\theta\,\varphi\,\theta^{-1}$. Then we have the following result.

Proposition 6.3. *Let $\mathcal{F} := \mathbb{G}(\mathcal{T})$. Then we have:*
$$\mathcal{E}nd_X(\mathcal{F}) \cong \mathbb{G}\bigl(\mathcal{E}nd_{\widetilde{X}}(\widetilde{\mathcal{F}}), \mathcal{E}nd_Z(\mathcal{V}), \mathsf{conj}(\theta)\bigr),$$
where $\mathsf{conj}(\theta)$ is the morphism making the following diagram
$$\begin{array}{ccc} \tilde{\pi}^*\mathcal{E}nd_Z(\mathcal{V}) & \xrightarrow{\mathsf{conj}(\theta)} & \tilde{\eta}^*\mathcal{E}nd_{\widetilde{X}}(\widetilde{\mathcal{F}}) \\ {\scriptstyle \mathsf{can}}\downarrow & & \downarrow{\scriptstyle \mathsf{can}} \\ \mathcal{E}nd_{\widetilde{Z}}(\tilde{\pi}^*\mathcal{V}) & \xrightarrow{\overline{\mathsf{conj}}(\theta)} & \mathcal{E}nd_{\widetilde{Z}}(\tilde{\eta}^*\widetilde{\mathcal{F}}) \end{array}$$
commutative. Similarly, we have: $\mathsf{Ad}(\mathcal{F}) \cong \mathbb{G}\bigl(\mathsf{Ad}(\widetilde{\mathcal{F}}), \mathsf{Ad}(\mathcal{V}), \mathsf{conj}(\theta)\bigr)$.

A proof of Proposition 6.3 can be deduced from Theorem 6.2 using the standard technique of sheaf theory and is therefore omitted.

6.2. **Simple vector bundles on the cuspidal Weierstraß curve.** Now we recall the description of the simple vector bundles on the cuspidal Weierstraß curve following the approach of Bodnarchuk and Drozd [7], see also [14, Section 5.1.3].

1. Throughout this section, $E = V(wv^2 - u^3) \subseteq \mathbb{P}^2$ is the cuspidal Weierstraß curve.

2. Let $\pi : \mathbb{P}^1 \longrightarrow E$ be the normalization of E. We choose homogeneous coordinates $(z_0 : z_1)$ on $\mathbb{P}^1$ in such a way that $\pi\big((0:1)\big)$ is the singular point of E. In what follows, we denote $\infty = (0:1)$ and $0 = (1:0)$. Abusing the notation, for any $x \in \Bbbk$ we also denote by $x \in \breve{E}$ the image of the point $\tilde{x} = (1:x) \in \mathbb{P}^1$, identifying in this way $\breve{E}$ with $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\} =: U_\infty$. Let $t = \frac{z_0}{z_1}$, then we have: $\Bbbk[U_\infty] = \Bbbk[t]$. Let $R = \Bbbk[\varepsilon]/\varepsilon^2$ and $\Bbbk[t] \to R$ be the canonical projection. Then in the notation of the previous subsection we have: $Z \cong \mathsf{Spec}(\Bbbk)$ and $\widetilde{Z} \cong \mathsf{Spec}(R)$.

3. By the theorem of Birkhoff–Grothendieck, for any $\mathcal{F} \in \mathsf{VB}(E)$ we have:

$$\pi^*\mathcal{F} \cong \bigoplus_{c \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^1}(c)^{\oplus n_c}.$$

A choice of homogeneous coordinates on $\mathbb{P}^1$ yields two distinguished sections $z_0, z_1 \in H^0\big(\mathcal{O}_{\mathbb{P}^1}(1)\big)$. Hence, for any $e \in \mathbb{N}$ we get a distinguished basis of the vector space $\mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(e)\big)$ given by the monomials $z_0^e, z_0^{e-1}z_1, \dots, z_1^e$. Next, for any $c \in \mathbb{Z}$ we fix the following isomorphism

$$\zeta^{\mathcal{O}_{\mathbb{P}^1}(c)} : \quad \mathcal{O}_{\mathbb{P}^1}(c)\big|_{\widetilde{Z}} \longrightarrow \mathcal{O}_{\widetilde{Z}}$$

sending a local section p to $\dfrac{p}{z_1^c\big|_{\widetilde{Z}}}$. Thus, for any vector bundle $\widetilde{\mathcal{F}} = \bigoplus_{c \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^1}(c)^{\oplus n_c}$ of rank n on $\mathbb{P}^1$ we have the induced isomorphism $\zeta^{\widetilde{\mathcal{F}}} : \widetilde{\mathcal{F}}\big|_{\widetilde{Z}} \longrightarrow \mathcal{O}_{\widetilde{Z}}^{\oplus n}$.

4. Consider the set $\Sigma := \big\{(a,b) \in \mathbb{N} \times \mathbb{N} \,\big|\, \gcd(a,b) = 1\big\}$ and for any $(a,b) \in \Sigma \setminus \big\{(1,1)\big\}$ denote:

$$\epsilon(a,b) = \begin{cases} (a-b, b), & a > b \\ (a, b-a), & a < b. \end{cases}$$

Now, starting with a pair $(e,d) \in \Sigma$, we construct a finite sequence of elements of Σ ending with $(1,1)$, defined as follows. We put $(a_0, b_0) = (e,d)$ and, as long as $(a_i, b_i) \neq (1,1)$, we set $(a_{i+1}, b_{i+1}) = \epsilon(a_i, b_i)$. Let

$$J_{(1,1)} = \left(\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array}\right) \in \mathsf{Mat}_{2\times 2}(\mathbb{C}). \tag{51}$$

Assume that the matrix

$$J_{(a,b)} = \left(\begin{array}{c|c} A_1 & A_2 \\ \hline 0 & A_3 \end{array}\right)$$

with $A_1 \in \mathrm{Mat}_{a\times a}(\Bbbk)$ and $A_3 \in \mathrm{Mat}_{b\times b}(\Bbbk)$ has already been defined. Then for $(p,q) \in \Sigma$ such that $\epsilon(p,q) = (a,b)$, we set

$$J_{(p,q)} = \begin{cases} \left(\begin{array}{c|cc} 0 & \mathbb{1} & 0 \\ \hline 0 & A_1 & A_2 \\ 0 & 0 & A_3 \end{array}\right), & p = a \\[2ex] \left(\begin{array}{cc|c} A_1 & A_2 & 0 \\ 0 & A_3 & \mathbb{1} \\ \hline 0 & 0 & 0 \end{array}\right), & q = b. \end{cases} \tag{52}$$

Hence, to any tuple $(e,d) \in \Sigma$ we can assign a certain uniquely determined matrix $J = J_{(e,d)}$ of size $(e+d)\times(e+d)$, obtained by the above recursive procedure from the sequence $\bigl\{(e,d), ..., (1,1)\bigr\}$.

Example 6.4. Let $(e,d) = (3,2)$. Then the corresponding sequence of elements of Σ is $\bigl\{(3,2),(1,2),(1,1)\bigr\}$ and the matrix $J = J_{(3,2)}$ is constructed as follows

$$\left(\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array}\right) \to \left(\begin{array}{c|cc} 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

5. Given $0 < d < n$ mutually prime and $\lambda \in \Bbbk$, we take the matrix

$$\Theta_\lambda = \Theta_{n,d,\lambda} = \mathbb{1} + \varepsilon\bigl(\lambda\mathbb{1} + J_{(e,d)}\bigr) \in \mathsf{GL}_n(R),\ e = n - d. \tag{53}$$

The matrix Θ_λ defines a morphism $\bar{\theta}_\lambda : \eta_*\mathcal{O}_Z \longrightarrow \nu_*\mathcal{O}_{\widetilde{Z}}$. Let $\widetilde{\mathcal{P}} = \widetilde{\mathcal{P}}_{n,d} := \mathcal{O}_{\mathbb{P}^1}^{\oplus e} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d}$. Consider the following vector bundle $\mathcal{P}_\lambda = \mathcal{P}_{n,d,\lambda}$ on E:

$$0 \longrightarrow \mathcal{P}_\lambda \xrightarrow{\binom{\imath}{q}} \pi_*\widetilde{\mathcal{P}} \oplus \eta_*\mathcal{O}_Z^{\oplus n} \xrightarrow{(\zeta^{\widetilde{\mathcal{P}}},\, -\bar{\theta}_\lambda)} \nu_*\mathcal{O}_{\widetilde{Z}}^{\oplus n} \longrightarrow 0. \tag{54}$$

Then $\mathcal{P}_\lambda$ is simple with rank n and degree d. Moreover, in an appropriate sense, $\bigl\{\mathcal{P}_\lambda\bigr\}_{\lambda\in\Bbbk^*}$ is a universal family of simple vector bundles of rank n and degree d on the curve E, see [14, Theorem 5.1.40]. The next result follows from Proposition 6.3.

Corollary 6.5. *Let $0 < d < n$ be a pair of coprime integers, $e = n - d$ and $J = J_{(e,d)} \in \mathsf{Mat}_{n\times n}(\Bbbk)$ be the matrix given by the recursion (52). Consider the vector bundle $\mathcal{A}$ given by the following short exact sequence*

$$0 \longrightarrow \mathcal{A} \xrightarrow{\binom{\jmath}{r}} \pi_*\widetilde{\mathcal{A}} \oplus \eta_*\bigl(\mathsf{Ad}(\mathcal{O}_Z^{\oplus n})\bigr) \xrightarrow{\left(\zeta^{\mathsf{Ad}(\widetilde{\mathcal{P}})},\, -\mathsf{conj}(\Theta_0)\right)} \eta_*\bigl(\mathsf{Ad}(\mathcal{O}_{\widetilde{Z}}^{\oplus n})\bigr) \longrightarrow 0, \tag{55}$$

where $\widetilde{\mathcal{A}} = \mathsf{Ad}(\widetilde{\mathcal{P}})$. Then $\mathcal{A} \cong \mathsf{Ad}(\mathcal{P}_0)$. Moreover, for any trivialization $\xi : \widetilde{\mathcal{P}}\big|_{U_\infty} \longrightarrow \mathcal{O}_{U_\infty}^{\oplus n}$ we get the following isomorphisms of sheaves of Lie algebras

$$\mathcal{A}\big|_{\breve{E}} \xrightarrow{\jmath} \pi_*\bigl(\mathsf{Ad}(\widetilde{\mathcal{P}})\bigr)\big|_{\breve{E}} \longrightarrow \pi_*\mathsf{Ad}\bigl(\mathcal{O}_{U_\infty}^{\oplus n}\bigr) \xrightarrow{\mathsf{can}} \mathsf{Ad}\bigl(\mathcal{O}_{\breve{E}}^{\oplus n}\bigr), \tag{56}$$

where the second morphism is induced by ξ.

6. In the above notation, for any $x \in \breve{E} \cong \mathbb{A}^1$ the corresponding line bundle $\mathcal{O}_E\bigl([x]\bigr)$ is given by the triple $\bigl(\mathcal{O}_{\mathbb{P}^1}(1), \mathbb{k}, 1 - x \cdot \varepsilon\bigr)$, see [14, Lemma 5.1.27].

6.3. **From simple vector bundles on the cuspidal Weierstraß curve to solutions of the classical Yang–Baxter equation.** In this subsection we derive the recipe to compute the solution of the classical Yang–Baxter equation corresponding to the triple $(E, (n, d))$, where E is the cuspidal Weierstraß curve and $0 < d < n$ is a pair of coprime integers. Keeping the same notation as in Subsection 6.2, we additionally introduce the following one.

1. We choose the following regular differential one-form $w := dz$ on E, where $z = \frac{z_1}{z_0}$ is a coordinate on the open chart U_0.

2. Let $\mathfrak{g}[z] := \mathfrak{g} \otimes \mathbb{k}[z]$. Then for any $x \in \mathbb{k}$ we have the $\mathbb{k}$–linear evaluation map $\phi_x : \mathfrak{g}[z] \to \mathfrak{g}$, where $\mathfrak{g}[z] \ni az^p \mapsto x^p \cdot a \in \mathfrak{g}$ for $a \in \mathfrak{g}$. For $x \neq y \in \mathbb{k}$ consider the following $\mathbb{k}$–linear maps:

$$\overline{\mathsf{res}}_x := \phi_x \quad \text{and} \quad \overline{\mathsf{ev}}_y := \frac{1}{y - x}\phi_y. \tag{57}$$

3. Let (e, d) be a pair of coprime positive integers, $n = e + d$ and $\mathfrak{a} := \mathsf{Mat}_{n \times n}(\mathbb{k})$. For the block partition of $\mathfrak{a}$ induced by the decomposition $n = e + d$, consider the following subspace of $\mathfrak{g}[z]$:

$$V_{e,d} = \left\{ F = \left(\begin{array}{c|c} W & X \\ \hline Y & Z \end{array}\right) + \left(\begin{array}{c|c} W' & 0 \\ \hline Y' & Z' \end{array}\right) z + \left(\begin{array}{c|c} 0 & 0 \\ \hline Y'' & 0 \end{array}\right) z^2 \right\}. \tag{58}$$

For a given $F \in V_{e,d}$ denote

$$F_0 = \left(\begin{array}{c|c} W' & X \\ \hline Y'' & Z' \end{array}\right) \text{ and } F_\epsilon = \left(\begin{array}{c|c} W & 0 \\ \hline Y' & Z \end{array}\right). \tag{59}$$

4. For $x \in \mathbb{k}$ consider the following subspace of $V_{e,d}$:

$$\mathsf{Sol}\bigl((e, d), x\bigr) := \Bigl\{ F \in V_{e,d} \,\Big|\, [F_0, J] + xF_0 + F_\epsilon = 0 \Bigr\}. \tag{60}$$

The following theorem is the main result of this section.

Theorem 6.6. *Let $\mathcal{A}$ be the sheaf of Lie algebras given by (55) and $x, y \in \breve{E}$ a pair of distinct points. Then there exists an isomorphism of Lie algebras $\jmath^{\mathcal{A}} : \Gamma(\breve{E}, \mathcal{A}) \to \mathfrak{g}[z]$ and a $\mathbb{k}$–linear isomorphism $\jmath : H^0\bigl(\mathcal{A}(x)\bigr) \to \mathsf{Sol}\bigl((e, d), x\bigr)$ such that the following diagram*

$$\begin{array}{ccccc}
\mathcal{A}\big|_x & \xleftarrow{\ \mathsf{res}_x^{\mathcal{A}}(w)\ } & H^0\bigl(\mathcal{A}(x)\bigr) & \xrightarrow{\ \mathsf{ev}_y^{\mathcal{A}}\ } & \mathcal{A}\big|_y \\
\Big\downarrow{\scriptstyle \jmath_x^{\mathcal{A}}} & & \Big\downarrow{\scriptstyle \jmath} & & \Big\downarrow{\scriptstyle \jmath_y^{\mathcal{A}}} \\
\mathfrak{g} & \xleftarrow{\ \overline{\mathsf{res}}_x\ } & \mathsf{Sol}\bigl((e, d), x\bigr) & \xrightarrow{\ \overline{\mathsf{ev}}_y\ } & \mathfrak{g}
\end{array}$$

is commutative.

Proof. We first introduce the following (final) portion of notations.

1. For $x \in \Bbbk$ consider the section $\sigma = z_1 - xz_0 \in H^0\bigl(\mathcal{O}_{\mathbb{P}^1}(1)\bigr)$. Using the identification $\Bbbk \xrightarrow{\cong} U_0$, $\Bbbk \ni x \mapsto \tilde{x} := (1 : x) \in \mathbb{P}^1$, the section σ induces an isomorphism of line bundles $t_\sigma : \mathcal{O}_{\mathbb{P}^1}\bigl([\tilde{x}]\bigr) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1)$.

2. For any $c \in \mathbb{Z}$ fix the trivialization $\xi^{\mathcal{O}_{\mathbb{P}^1}(c)} : \mathcal{O}_{\mathbb{P}^1}(c)\big|_{U_0} \longrightarrow \mathcal{O}_{U_0}$ given on the level of local sections by the rule $p \mapsto \dfrac{p}{z_0^c\big|_{U_0}}$. Thus, for any vector bundle $\widetilde{\mathcal{F}} = \oplus \mathcal{O}_{\mathbb{P}^1}(c)^{\oplus n_c}$ of rank n we get the induced trivialization $\xi^{\widetilde{\mathcal{F}}} : \widetilde{\mathcal{F}}\big|_{U_0} \longrightarrow \mathcal{O}_{U_0}^{\oplus n}$.

3. Let $\widetilde{\mathcal{E}} = \begin{pmatrix} \widetilde{\mathcal{E}}_1 & \widetilde{\mathcal{E}}_2 \\ \widetilde{\mathcal{E}}_3 & \widetilde{\mathcal{E}}_4 \end{pmatrix}$ be the sheaf of algebras on $\mathbb{P}^1$ with $\widetilde{\mathcal{E}}_1 = \mathcal{M}at_{e\times e}(\mathcal{O}_{\mathbb{P}^1})$, $\widetilde{\mathcal{E}}_4 = \mathcal{M}at_{d\times d}(\mathcal{O}_{\mathbb{P}^1})$, $\widetilde{\mathcal{E}}_2 = \mathcal{M}at_{e\times d}(\mathcal{O}_{\mathbb{P}^1}(-1))$ and $\widetilde{\mathcal{E}}_3 = \mathcal{M}at_{d\times e}(\mathcal{O}_{\mathbb{P}^1}(1))$. The ring structure on $\widetilde{\mathcal{E}}$ is induced by the canonical isomorphism $\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{O}_{\mathbb{P}^1}(1) \xrightarrow{\mathsf{can}} \mathcal{O}_{\mathbb{P}^1}$. Let $\widetilde{\mathcal{A}} = \ker\bigl(\widetilde{\mathcal{E}} \xrightarrow{\mathsf{tr}} \mathcal{O}_{\mathbb{P}^1}\bigr)$, where tr only involves the diagonal entries of $\widetilde{\mathcal{E}}$ and is given by the matrix $(1, 1, \dots, 1)$. Of coarse, $\widetilde{\mathcal{E}} \cong \mathcal{E}nd(\widetilde{\mathcal{P}})$ and $\widetilde{\mathcal{A}} \cong \mathsf{Ad}(\widetilde{\mathcal{P}})$ for $\widetilde{\mathcal{P}} = \mathcal{O}_{\mathbb{P}^1}^{\oplus e} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d}$.

4. Consider the sheaf of algebras $\mathcal{E}$ on E given by the short exact sequence

$$0 \longrightarrow \mathcal{E} \xrightarrow{\binom{\jmath}{r}} \pi_*\widetilde{\mathcal{E}} \oplus \eta_*\bigl(\mathcal{M}_n(Z)\bigr) \xrightarrow{\left(\zeta^{\widetilde{\mathcal{E}}},\, -\mathsf{conj}(\Theta_0)\right)} \eta_*\bigl(\mathcal{M}_n(\widetilde{Z})\bigr) \longrightarrow 0,$$

where $\mathcal{M}_n(T) := \mathcal{E}nd_T(\mathcal{O}_T^{\oplus n})$ for a scheme T. Of coarse $\mathcal{E} \cong \mathcal{E}nd_E(\mathcal{P}_0)$, where $\mathcal{P}_0$ is the simple vector bundle of rank n and degree d on E given by (54).

5. In the above notation we have:

$$H^0\bigl(\widetilde{\mathcal{E}}(1)\bigr) = \left\{ F = \left(\begin{array}{c|c} z_0W + z_1W' & X \\ \hline z_0^2Y + z_0z_1Y' + z_1^2Y'' & z_0Z + z_1Z' \end{array} \right) \right\}, \tag{61}$$

where $W, W' \in \mathsf{Mat}_{e\times e}(\Bbbk)$, $Z, Z' \in \mathsf{Mat}_{d\times d}(\Bbbk)$, $Y, Y', Y'' \in \mathsf{Mat}_{d\times e}(\Bbbk)$ and $X \in \mathsf{Mat}_{e\times d}(\Bbbk)$.

6. For any $F \in H^0\bigl(\widetilde{\mathcal{E}}(1)\bigr)$ as in (61) we denote:

$$\overline{\mathsf{res}}_x(F) = F(1, x) \quad \text{and} \quad \overline{\mathsf{ev}}_y(F) = \frac{1}{y - x}F(1, y). \tag{62}$$

7. Finally, let $\jmath^{\mathcal{E}} : \mathcal{E}\big|_{\breve{E}} \longrightarrow \mathcal{M}_n(\breve{E})$ be the trivialization induced by the trivialization $\xi^{\widetilde{\mathcal{E}}} : \widetilde{\mathcal{E}}\big|_{U_0} \longrightarrow \mathcal{M}_n(U_0)$. This trivialization actually induces an isomorphism of Lie algebras $\jmath^{\mathcal{A}} : \Gamma(\breve{E}, \mathcal{A}) \longrightarrow \mathfrak{g}[z]$ we are looking for.

Now observe that the following diagram is commutative:

(63)
$$
\begin{array}{ccccccccc}
 & & \mathcal{A}\big|_x & \xleftarrow{\mathsf{res}_x^{\mathcal{A}}(w)} & H^0\big(\mathcal{A}(x)\big) & \xrightarrow{\mathsf{ev}_y^{\mathcal{A}}} & \mathcal{A}\big|_y & & \\
 & & \downarrow{\scriptstyle \hat{\pi}_x^*} & & \downarrow{\scriptstyle \hat{\pi}^*} & & \downarrow{\scriptstyle \hat{\pi}_y^*} & & \\
 & & \widetilde{\mathcal{A}}\big|_{\tilde{x}} & \xleftarrow{\mathsf{res}_{\tilde{x}}^{\widetilde{\mathcal{A}}}(w)} & H^0\big(\widetilde{\mathcal{A}}(\tilde{x})\big) & \xrightarrow{\mathsf{ev}_{\tilde{y}}^{\widetilde{\mathcal{A}}}} & \widetilde{\mathcal{A}}\big|_{\tilde{y}} & & \\
 {\scriptstyle \imath_x^{\mathcal{A}}} & & \hookdownarrow & & \hookdownarrow & & \hookdownarrow & & {\scriptstyle \imath_y^{\mathcal{A}}} \\
 & & \widetilde{\mathcal{E}}\big|_{\tilde{x}} & \xleftarrow{\mathsf{res}_{\tilde{x}}^{\widetilde{\mathcal{E}}}(w)} & H^0\big(\widetilde{\mathcal{E}}(\tilde{x})\big) & \xrightarrow{\mathsf{ev}_{\tilde{y}}^{\widetilde{\mathcal{E}}}} & \widetilde{\mathcal{E}}\big|_{\tilde{y}} & & \\
 & & \downarrow{\scriptstyle \xi_{\tilde{x}}^{\widetilde{\mathcal{E}}}} & & \downarrow{\scriptstyle (t_\sigma)_*} & & \downarrow{\scriptstyle \xi_{\tilde{y}}^{\widetilde{\mathcal{E}}}} & & \\
 \mathfrak{g} & \hookrightarrow & \mathfrak{a} & \xleftarrow{\overline{\mathsf{res}}_x} & H^0\big(\widetilde{\mathcal{E}}(1)\big) & \xrightarrow{\overline{\mathsf{ev}}_y} & \mathfrak{a} & \hookleftarrow & \mathfrak{g}\,.
\end{array}
$$

Following the notation of (55), the composition

$$\gamma_{\mathcal{A}}: \quad \pi^*\mathcal{A} \xrightarrow{\pi^*(\imath)} \pi^*\pi_*\widetilde{\mathcal{A}} \xrightarrow{\mathsf{can}} \widetilde{\mathcal{A}}$$

is an isomorphism of vector bundles on $\mathbb{P}^1$. The morphisms $\hat{\pi}_x^*$ and and $\hat{\pi}_y^*$ are the maps, obtained by composing π^* and $\gamma_{\mathcal{A}}$ and then taking the induced map in the corresponding fibers. Similarly, $\hat{\pi}^*$ is the induced map of global sections. The commutativity of both top squares of (63) follows from the "locality" of the morphisms $\mathsf{res}_x^{\mathcal{A}}(w)$ and $\underline{\mathsf{ev}}_y^{\mathcal{A}}$, see [14, Proposition 2.2.8 and Proposition 2.2.12] as well as [14, Section 5.2] for a detailed proof.

The commutativity of both middle squares of (63) is obvious. The commutativity of both lower squares follows from [14, Corollary 5.2.1] and [14, Corollary 5.2.2] respectively. In particular, the explicit formulae (62) for the maps $\overline{\mathsf{res}}_x$ and $\overline{\mathsf{ev}}_y$ are given there. Finally, see [14, Subsection 5.2.2] for the proof of commutativity of both side diagrams.

Now we have to describe the image of the linear map $H^0\big(\mathcal{A}(x)\big) \longrightarrow H^0\big(\widetilde{\mathcal{E}}(1)\big)$ obtained by composing of the three middle vertical arrows in (63). It is convenient to describe first the image of the corresponding linear map $H^0\big(\mathcal{E}(x)\big) \longrightarrow H^0\big(\widetilde{\mathcal{E}}(1)\big)$. Recall that

- The sheaf $\mathcal{E}$ is given by the triple $\big(\widetilde{\mathcal{E}}, \mathsf{Mat}_n(\Bbbk), \mathsf{conj}(\Theta_0)\big)$.
- The line bundle $\mathcal{O}_E\big([x]\big)$ is given by the triple $\big(\mathcal{O}_{\mathbb{P}^1}(1), \Bbbk, \mathbb{1} - x\cdot\varepsilon\big)$.
- The tensor product in $\mathsf{VB}(E)$ corresponds to the tensor product in $\mathsf{Tri}(E)$.

These facts lead to the following consequence. Let $F \in H^0\big(\widetilde{\mathcal{E}}(1)\big)$ be written as in (61). Then F belongs to the image of the linear map $H^0\big(\mathcal{E}(x)\big) \longrightarrow H^0\big(\widetilde{\mathcal{E}}(1)\big)$ if any only if there exists some $A \in \mathfrak{a}$ such that the following equality in $\mathfrak{a}[\varepsilon]$ is true:

$$F\big|_{\widetilde{Z}} = (1 - x\cdot\varepsilon)\cdot\Theta_0\cdot A\cdot\Theta_0^{-1}, \tag{64}$$

where $F\big|_{\widetilde{Z}} := F_0 + \varepsilon F_\epsilon$ and F_0, F_ϵ are given by (59). Since $\Theta_0^{-1} = \mathbb{1} - \varepsilon J_{(e,d)}$, the equation (64) is equivalent to the following constraint:

$$\big[F_0, J_{(e,d)}\big] + xF_0 + F_\epsilon = 0.$$

See also [14, Subsection 5.2.5] for a computation in a similar situation.

Finally, consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0\big(\mathcal{A}(x)\big) & \longrightarrow & H^0\big(\mathcal{E}(x)\big) & \longrightarrow & H^0\big(\mathcal{O}_E(x)\big) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \hat{\pi}} & & \Big\downarrow{\scriptstyle \hat{\pi}} & & \Big\downarrow{\scriptstyle \hat{\pi}} & & \\
0 & \longrightarrow & H^0\big(\widetilde{\mathcal{A}}(\tilde{x})\big) & \longrightarrow & H^0\big(\widetilde{\mathcal{E}}(\tilde{x})\big) & \longrightarrow & H^0\big(\mathcal{O}_{\mathbb{P}^1}(\tilde{x})\big) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle (t_\sigma)_*} & & \Big\downarrow{\scriptstyle (t_\sigma)_*} & & \Big\downarrow{\scriptstyle (t_\sigma)_*} & & \\
0 & \longrightarrow & H^0\big(\widetilde{\mathcal{A}}(1)\big) & \longrightarrow & H^0\big(\widetilde{\mathcal{E}}(1)\big) & \xrightarrow{T} & H^0\big(\mathcal{O}_{\mathbb{P}^1}(1)\big) & \longrightarrow & 0,
\end{array}$$

where

$$T\left(\begin{pmatrix} A_0 z_0 + A_1 z_1 & * \\ * & B_0 z_0 + B_1 z_1 \end{pmatrix}\right) = \big(\mathsf{tr}(A_0) + \mathsf{tr}(B_0)\big) z_0 + \big(\mathsf{tr}(A_1) + \mathsf{tr}(B_1)\big) z_1.$$

Let $\mathsf{Sol}\big((e,d),x\big) := \mathsf{Im}\Big(H^0\big(\mathcal{A}(x)\big) \longrightarrow H^0\big(\widetilde{\mathcal{E}}(1)\big)\Big)$. Note that we have

$$\mathsf{Sol}\big((e,d),x\big) = \mathsf{Ker}(T) \cap \mathsf{Im}\Big(H^0\big(\mathcal{E}(x)\big) \longrightarrow H^0\big(\widetilde{\mathcal{E}}(1)\big)\Big)$$

Let $J : H^0\big(\mathcal{A}(x)\big) \longrightarrow \mathfrak{g}[z]$ be the composition of $H^0\big(\mathcal{A}(x)\big) \longrightarrow H^0\big(\widetilde{\mathcal{E}}(1)\big)$ with the embedding $H^0\big(\widetilde{\mathcal{E}}(1)\big) \longrightarrow \mathfrak{a}[z]$ (sending z_0 to 1 and z_1 to z). Identifying $\mathsf{Sol}\big((e,d),x\big)$ with the corresponding subspace of $\mathfrak{g}[z]$, we conclude the proof of Theorem 6.6. □

Algorithm 6.7. Let E be the cuspidal Weierstraß curve, $0 < d < n$ a pair of coprime integers and $e = n - d$. The solution $r_{(E,(n,d))}$ of the classical Yang–Baxter equation (1) can be obtained along the following lines.

- First compute the matrix $J = J_{(e,d)}$ given by the recursion (52).
- For $x \in \mathbb{k}$ determine the $\mathbb{k}$–linear subspace $\mathsf{Sol}\big((e,d),x\big) \subset \mathfrak{g}[z]$ introduced in (60).
- Choose a basis of $\mathfrak{g}$ and compute the images of the basis vectors under the linear map
$$\mathfrak{g} \xrightarrow{\overline{\mathsf{res}}_x^{-1}} \mathsf{Sol}\big((e,d),x\big) \xrightarrow{\overline{\mathsf{ev}}_y} \mathfrak{g}\,.$$
Here, $\overline{\mathsf{res}}_x(F) = F(x)$ and $\overline{\mathsf{ev}}_y(F) = \dfrac{1}{y-x} F(y)$.
- For fixed $x \neq y \in \mathbb{k}^*$, set $r_{(E,(n,d))}(x,y) = \mathsf{can}^{-1}\big(\overline{\mathsf{ev}}_y \circ \overline{\mathsf{res}}_x^{-1}\big) \in \mathfrak{g} \otimes \mathfrak{g}$, where can is the canonical isomorphism of vector spaces
$$\mathfrak{g} \otimes \mathfrak{g} \to \mathsf{End}(\mathfrak{g}),\ X \otimes Y \mapsto \big(Z \mapsto \mathsf{tr}(XZ)Y\big).$$
- Then $r_{(E,(n,d))}$ is the solution of the classical Yang–Baxter equation (1) corresponding to the triple $(E,(n,d))$. □

It will be necessary to have a more concrete expression for the coefficients of the tensor $r_{(E,(n,d))}$. In what follows, we take the standard basis $\{e_{i,j}\}_{1 \le i \ne j \le n} \cup \{h_l\}_{1 \le l \le n-1}$ of the Lie algebra $\mathfrak{g}$. Since the linear map $\overline{\mathsf{res}}_x : \mathsf{Sol}\big((e,d),x\big) \to \mathfrak{g}$ given by $F \mapsto F(x)$ is an isomorphism, we have:

$$\begin{cases} \overline{\mathsf{res}}_x^{-1}(e_{i,j}) & = & e_{i,j} + G^x_{i,j}(z) & 1 \le i \ne j \le n, \\ \overline{\mathsf{res}}_x^{-1}(h_l) & = & h_l + G^x_l(z) & 1 \le l \le n-1, \end{cases}$$

where the elements $G^x_{i,j}(z), G^x_l(z) \in V_{e,d}$ are uniquely determined by the properties

$$e_{i,j} + G^x_{i,j}(z), h_l + G^x_l(z) \in \mathsf{Sol}\big((e,d),x\big), \quad G^x_{i,j}(x) = 0 = G^x_l(x). \tag{65}$$

Lemma 6.8. *In the notations as above, we have*

$$r_{(E,(n,d))}(x,y) = \frac{1}{y-x}\left[c + \left(\sum_{1\le i\ne j\le n} e_{j,i}\otimes G^x_{i,j}(y)\right) + \left(\sum_{1\le l\le n-1} \check{h}_l \otimes G^x_l(y)\right)\right],$$

where $\check{h}_l$ is the dual of h_l with respect to the trace form and c is the Casimir element in $\mathfrak{g}\otimes\mathfrak{g}$. In particular, $r_{(E,(n,d))}$ is a rational solution of (1) in the sense of [33, 35].

Proof. It follows directly from the definitions that

$$\begin{cases} \overline{\mathsf{ev}}_y \circ \overline{\mathsf{res}}_x^{-1}(e_{i,j}) &= \dfrac{1}{y-x}\Big(e_{i,j} + G^x_{i,j}(y)\Big) & 1 \le i \ne j \le n \\ \overline{\mathsf{ev}}_y \circ \overline{\mathsf{res}}_x^{-1}(h_l) &= \dfrac{1}{y-x}\big(h_l + G^x_l(y)\big) & 1 \le l \le n-1. \end{cases}$$

Since $e_{j,i}$ respectively $\check{h}_l$ is the dual of $e_{i,j}$ respectively h_l with respect to the trace form on $\mathfrak{g}$, the linear map can^{-1} acts as follows:

$$\begin{cases} \mathsf{End}(\mathfrak{g}) \ni \left(e_{i,j} \mapsto \dfrac{1}{y-x}\Big(e_{i,j} + G^x_{i,j}(y)\Big)\right) & \mapsto \; e_{j,i}\otimes \dfrac{1}{y-x}\Big(e_{i,j} + G^x_{i,j}(y)\Big) \in \mathfrak{g}\otimes\mathfrak{g} \\ \mathsf{End}(\mathfrak{g}) \ni \left(h_l \mapsto \dfrac{1}{y-x}\big(h_l + G^x_l(y)\big)\right) & \mapsto \; \check{h}_l \otimes \dfrac{1}{y-x}\big(h_l + G^x_l(y)\big) \in \mathfrak{g}\otimes\mathfrak{g} \end{cases}$$

for $1 \le i \ne j \le n$ and $1 \le l \le n-1$. It remains to recall that the Casimir element in $\mathfrak{g}\otimes\mathfrak{g}$ is given by the formula

$$c = \sum_{1\le i\ne j\le n} e_{i,j}\otimes e_{j,i} + \sum_{1\le l\le n-1} \check{h}_l \otimes h_l. \tag{66}$$

Lemma is proven. □

7. Frobenius structure on parabolic subalgebras

Definition 7.1 (see [29])**.** A finite dimensional Lie algebra $\mathfrak{f}$ over $\mathbb{k}$ is *Frobenius* if there exists a functional $\hat{l} \in \mathfrak{f}^*$ such that the skew-symmetric bilinear form

$$\mathfrak{f}\times\mathfrak{f} \longrightarrow \mathbb{k} \quad (a,b) \mapsto \hat{l}([a,b]) \tag{67}$$

is non-degenerate.

Let (e,d) be a pair of coprime positive integers, $n = e+d$ and $\mathfrak{p} = \mathfrak{p}_e$ be the e-th parabolic subalgebra of $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{k})$, i.e.

$$\mathfrak{p} := \left\{ \left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right) \,\middle|\, \begin{array}{c} A \in \mathsf{Mat}_{e\times e}(\mathbb{k}), B \in \mathsf{Mat}_{e\times d}(\mathbb{k}) \\ C \in \mathsf{Mat}_{d\times d}(\mathbb{k}) \end{array} \text{ and } \mathsf{tr}(A) + \mathsf{tr}(C) = 0 \right\}. \tag{68}$$

The goal of this section is to prove the following result.

Theorem 7.2. *Let $J = J_{(e,d)}$ be the matrix from (52). Then the pairing*

$$\omega_J : \mathfrak{p}\times\mathfrak{p} \longrightarrow \mathbb{k}, \quad (a,b) \mapsto \mathsf{tr}\big(J^t \cdot [a,b]\big) \tag{69}$$

is non-degenerate. In other words, $\mathfrak{p}$ is a Frobenius Lie algebra and

$$l_J : \mathfrak{p} \to \mathbb{k}, \quad a \mapsto \mathsf{tr}(J^t \cdot a) \tag{70}$$

is a Frobenius functional on $\mathfrak{p}$.

In this section, we shall use the following notations and conventions. For a finite dimensional vector space $\mathfrak{w}$ be denote by $\mathfrak{w}^*$ the dual vector space. If $\mathfrak{w} = \mathfrak{w}_1 \oplus \mathfrak{w}_2$ then we have a canonical isomorphism $\mathfrak{w}^* \cong \mathfrak{w}_1^* \oplus \mathfrak{w}_2^*$. For a functional $\hat{w}_i \in \mathfrak{w}_i^*, i = 1, 2$ we denote by the same symbol its *extension by zero* on the whole $\mathfrak{w}$.

Assume we have the following set-up.

- $\mathfrak{f}$ is a finite dimensional Lie algebra.
- $\mathfrak{l} \subset \mathfrak{f}$ is a Lie subalgebra and $\mathfrak{n} \subset \mathfrak{f}$ is a commutative Lie ideal such that $\mathfrak{f} = \mathfrak{l} \dotplus \mathfrak{n}$, i.e. $\mathfrak{f} = \mathfrak{l} + \mathfrak{n}$ and $\mathfrak{l} \cap \mathfrak{n} = 0$.
- There exists $\hat{n} \in \mathfrak{n}^*$ such that for any $\hat{n}' \in \mathfrak{n}^*$ there exists $l \in \mathfrak{l}$ such that $\hat{n}' = \hat{n}\big([-, l]\big)$ in $\mathfrak{f}^*$. Note that $\hat{n}\big([l', l]\big) = 0$ for any $l' \in \mathfrak{l}$, hence it is sufficient to check that for any $m \in \mathfrak{n}$ we have: $\hat{n}'(m) = \hat{n}\big([m, l]\big)$. The relation $\hat{n}' = \hat{n}\big([-, l]\big)$ is compatible with the above convention on zero extension of functionals from $\mathfrak{l}$ to $\mathfrak{f}$.

First note the following easy fact.

Lemma 7.3. *Let* $\hat{m} \in \mathfrak{n}^*$ *be any functional and* $\mathfrak{s} = \mathfrak{s}_{\hat{m}} := \Big\{ l \in \mathfrak{l} \,\Big|\, \mathfrak{f}^* \ni \hat{m}\big([\,-\,, l]\big) = 0 \Big\}$. *Then* $\mathfrak{s}$ *is a Lie subalgebra of* $\mathfrak{l}$.

A version of the following result is due to Elashvili [19]. It was explained to us by Stolin.

Proposition 7.4. *Let* $\mathfrak{f} = \mathfrak{l} \dotplus \mathfrak{n}$ *and* $\hat{n} \in \mathfrak{n}^*$ *be as above. Assume there exists* $\hat{s} \in \mathfrak{l}^*$ *such that its restriction on* $\mathfrak{s} = \mathfrak{s}_{\hat{n}}$ *is Frobenius. Then* $\hat{s} + \hat{n}$ *is a Frobenius functional on* $\mathfrak{f}$.

Proof. Assume $\hat{s} + \hat{n}$ is not Frobenius. Then there exist $l_1 \in \mathfrak{l}$ and $n_1 \in \mathfrak{n}$ such that

$$\mathfrak{f}^* \ni (\hat{s} + \hat{n})\big([l_1 + n_1, \,-\,]\big) = 0.$$

It is equivalent to say that for all $l_2 \in \mathfrak{l}$ and $n_2 \in \mathfrak{n}$ we have:

$$\hat{n}\big([l_1, n_2] + [n_1, l_2]\big) + \hat{s}\big([l_1, l_2]\big) = 0. \tag{71}$$

At the first step, take $l_2 = 0$. Then the equality (71) implies that for all $n_2 \in \mathfrak{n}$ we have: $\hat{n}\big([l_1, n_2]\big) = 0$. This means that $\mathfrak{f}^* \ni \hat{n}\big([\,-\,, l_1]\big) = 0$ and hence, by definition of $\mathfrak{s}$, $l_1 \in \mathfrak{s}$. Assume $l_1 \neq 0$. By assumption, $\hat{s}\big|_{\mathfrak{s}}$ is a Frobenius functional. Hence, there exists $s_1 \in \mathfrak{s}$ such that $\hat{s}\big([l_1, s_1]\big) \neq 0$. Since $s_1 \in \mathfrak{s}$, we have: $\hat{n}\big([n_1, s_1]\big) = 0$. Altogether, it implies:

$$(\hat{s} + \hat{n})\big([l_1 + n_1, s_1]\big) = \hat{s}\big([l_1, s_1]\big) \neq 0.$$

Contradiction. Hence, $l_1 = 0$ and the equation (71) reads as follows:

$$\hat{n}\big([n_1, l_2]\big) = 0 \quad \text{for all} \quad l_2 \in \mathfrak{l}.$$

Assume $n_1 \neq 0$. Then there exists a functional $\hat{n}_1 \in \mathfrak{n}^*$ such that $\hat{n}_1(n_1) \neq 0$. However, by our assumptions, $\hat{n}_1 = \hat{n}\big([\,-\,, l]\big)$ for some $l \in \mathfrak{l}$. But this implies that

$$\hat{n}_1(n_1) = \hat{n}\big([n_1, l]\big) \neq 0.$$

We again obtain a contradiction. Thus, $n_1 = 0$ as well, what finishes the proof. □

Consider the following nilpotent subalgebras of $\mathfrak{g}$:

(72) $$\mathfrak{n} = \left\{N = \left(\begin{array}{c|c} 0 & A \\ \hline 0 & 0 \end{array}\right) \Big| A \in \mathsf{Mat}_{e\times d}(\mathbb{k})\right\} \quad \bar{\mathfrak{n}} = \left\{\bar{N} = \left(\begin{array}{c|c} 0 & 0 \\ \hline A & 0 \end{array}\right) \Big| A \in \mathsf{Mat}_{d\times e}(\mathbb{k})\right\}.$$

Note the following easy fact.

Lemma 7.5. *The linear map* $\bar{\mathfrak{n}} \longrightarrow \mathfrak{n}^*$, $\bar{N} \mapsto \mathsf{tr}\big(\bar{N} \cdot -\big)$ *is an isomorphism.*

Next, consider the following Lie algebra

(73) $$\mathfrak{l} = \left\{L = \left(\begin{array}{c|c} L_1 & 0 \\ \hline 0 & L_2 \end{array}\right) \Bigg| \begin{array}{l} L_1 \in \mathsf{Mat}_{e\times e}(\mathbb{k}) \\ L_2 \in \mathsf{Mat}_{e\times e}(\mathbb{k}) \end{array} \ \mathsf{tr}(L_1) + \mathsf{tr}(L_2) = 0\right\}.$$

Obviously, $\mathfrak{p} = \mathfrak{l} \dotplus \mathfrak{n}$, $\mathfrak{p}$ is a Lie subalgebra of $\mathfrak{p}$ and $\mathfrak{n}$ is a commutative Lie ideal in $\mathfrak{p}$.

Lemma 7.6. *Let* $\bar{N} \in \bar{\mathfrak{n}}$ *and* $\hat{n} = \mathsf{tr}(\bar{N} \cdot -) \in \mathfrak{n}^*$ *be the corresponding functional. Then the condition that for any* $\hat{n}' \in \mathfrak{n}^*$ *there exists* $L \in \mathfrak{l}$ *such that* $\hat{n}' = \hat{n}\big([L, -]\big)$ *in* $\mathfrak{f}^*$ *reads as follows: for any* $\bar{N}' \in \bar{\mathfrak{n}}$ *there exists* $L \in \mathfrak{l}$ *such that* $\bar{N}' = [\bar{N}, L]$.

Proof. By Lemma 7.5 there exists $\bar{N}' \in \bar{\mathfrak{n}}$ such that $\hat{u} = \mathsf{tr}(\bar{N}' \cdot -)$. Note that

$$\mathsf{tr}\big(\bar{N} \cdot [L, -]\big) = \mathsf{tr}\big([\bar{N}, L] \cdot -\big).$$

The equality of functionals $\mathsf{tr}(\bar{N}' \cdot -) = \mathsf{tr}\big([\bar{N}, L] \cdot -\big)$ implies that $\bar{N}' = [\bar{N}, L]$. □

Proof of Theorem 7.2. We prove this result by induction on

$$(e, d) \in \Sigma = \big\{(a, b) \in \mathbb{N} \times \mathbb{N} \,\big|\, \gcd(a, b) = 1\big\}.$$

Basis of induction. Let $(e, d) = (1, 1)$. Then we have: $J = J_{(1,1)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Let $a = \begin{pmatrix} \alpha_1 & \alpha_2 \\ 0 & -\alpha_1 \end{pmatrix}$ and $b = \begin{pmatrix} \beta_1 & \beta_2 \\ 0 & -\beta_1 \end{pmatrix}$ be two elements of $\mathfrak{p}$. Then we have:

$$\omega_J(a, b) = 2 \cdot (\alpha_1\beta_2 - \beta_1\alpha_2).$$

This form is obviously non-degenerate.

Induction step. Assume the result is proven for $(e, d) \in \Sigma$. Recall that for

$$J_{(e,d)} = \left(\begin{array}{c|c} A_1 & A_2 \\ \hline 0 & A_3 \end{array}\right)$$

with $A_1 \in \mathrm{Mat}_{e\times e}(\mathbb{k})$ and $A_3 \in \mathrm{Mat}_{d\times d}(\mathbb{k})$ we have:

$$J_{(e,d+e)} = \left(\begin{array}{c||c|c} 0 & \mathbb{1} & 0 \\ \hline\hline 0 & A_1 & A_2 \\ \hline 0 & 0 & A_3 \end{array}\right) \quad \text{and} \quad J_{(d+e,d)} = \left(\begin{array}{c|c||c} A_1 & A_2 & 0 \\ \hline 0 & A_3 & \mathbb{1} \\ \hline\hline 0 & 0 & 0 \end{array}\right).$$

For simplicity, we shall only treat the implication $(e, d) \Longrightarrow (e, d+e)$. Consider the matrix

$$\bar{N} = \left(\begin{array}{c||c|c} 0 & 0 & 0 \\ \hline\hline \mathbb{1} & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \in \bar{\mathfrak{n}}.$$

Then the following facts follows from a direct computation:

- $\bar{N}$ satisfies the condition of Lemma 7.6.
- The Lie subalgebra $\mathfrak{s} = \mathfrak{s}_{\bar{N}}$ has the following description:

$$\mathfrak{s} = \left\{ \left(\begin{array}{c||c|c} A & 0 & 0 \\ \hline\hline 0 & A & B \\ \hline 0 & 0 & C \end{array}\right) \;\middle|\; \begin{array}{c} A \in \mathsf{Mat}_{e\times e}(\Bbbk), B \in \mathsf{Mat}_{e\times d}(\Bbbk), \\ C \in \mathsf{Mat}_{d\times d}(\Bbbk) \end{array} \quad 2\mathsf{tr}(A) + \mathsf{tr}(C) = 0 \right\}. \tag{74}$$

The implication $(e, d) \Longrightarrow (e, d + e)$ follows from Proposition 7.4 and the following result.

Lemma 7.7. *Let* $\hat{J} = \left(\begin{array}{c||c|c} 0 & 0 & 0 \\ \hline\hline 0 & A & B \\ \hline 0 & 0 & C \end{array}\right)$. *Then there exists an isomorphism of Lie algebras* $\nu : \mathfrak{p} \longrightarrow \mathfrak{s}$ *such that for any* $P \in \mathfrak{p}$ *we have:* $\mathsf{tr}(J^t \cdot P) = \mathsf{tr}\big(\hat{J}^t \cdot \nu(P)\big)$.

The proof of this lemma is lengthy but completely elementary, therefore we leave it to an interested reader. Theorem 7.2 is proven. □

Lemma 7.8. *For any* $G \in \mathfrak{g}$ *there exist uniquely determined* $P \in \mathfrak{p}$ *and* $N \in \mathfrak{n}$ *such that*

$$G = \big[J^t, P\big] + N.$$

Proof. Consider the functional $\mathsf{tr}(G \cdot -) \in \mathfrak{p}^*$. Since the functional $l_J \in \mathfrak{p}^*$ from (70) is Frobenius, there exists a uniquely determined $P \in \mathfrak{p}$ such that $\mathsf{tr}(G \cdot -) = \mathsf{tr}\big([J^t, P] \cdot -\big)$ viewed as elements of $\mathfrak{p}^*$. Note that we have a short exact sequence of vector spaces

$$0 \longrightarrow \mathfrak{n} \xrightarrow{\imath} \mathfrak{g}^* \xrightarrow{\rho} \mathfrak{p}^* \longrightarrow 0,$$

where ρ maps a functional on $\mathfrak{g}$ to its restriction on $\mathfrak{p}$ and $\imath(N) = \mathsf{tr}(N \cdot -)$. Thus, for some uniquely determined $N \in \mathfrak{n}$, we get the following equality in $\mathfrak{g}^*$: $\mathsf{tr}(G \cdot -) = \mathsf{tr}\big(([J^t, P] + N) \cdot -\big)$. Since the trace form is non-degenerate on $\mathfrak{g}$, we get the claim. □

8. Review of Stolin's theory of rational solutions of the classical Yang–Baxter equation

In this section, we review Stolin's results on the classification of rational solutions of the classical Yang–Baxter equation for the Lie algebra $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, see [33, 34, 35].

Definition 8.1. A solution $r : (\mathbb{C}^2, 0) \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ of (1) is called rational if it is non-degenerate, unitary and of the form

$$r(x, y) = \frac{c}{y - x} + s(x, y), \tag{75}$$

where $c \in \mathfrak{g} \otimes \mathfrak{g}$ is the Casimir element and $s(x, y) \in \mathfrak{g}[x] \otimes \mathfrak{g}[y]$.

8.1. **Lagrangian orders.** Let $\widehat{\mathfrak{g}} = \mathfrak{g}((z^{-1}))$. Consider the following non-degenerate $\mathbb{C}$–bilinear form on $\widehat{\mathfrak{g}}$:

$$(\,-\,,\,-\,) : \quad \widehat{\mathfrak{g}} \times \widehat{\mathfrak{g}} \longrightarrow \mathbb{C}, \quad (a, b) \mapsto \mathsf{res}_{z=0}\big(\mathsf{tr}(ab)\big). \tag{76}$$

Definition 8.2. A Lie subalgebra $\mathfrak{w} \subset \widehat{\mathfrak{g}}$ is a *Lagrangian order* if the following three conditions are satisfied.

- $\mathfrak{w} \dotplus \mathfrak{g}[z] = \widehat{\mathfrak{g}}$.

- $\mathfrak{w} = \mathfrak{w}^{\perp}$ with respect to the pairing (76).
- There exists $p \geq 0$ such that $z^{-p-2}\,\mathfrak{g}[\![z^{-1}]\!] \subseteq \mathfrak{w}$.

Observe that from this Definition automatically follows that

$$\mathfrak{w} = \mathfrak{w}^{\perp} \subseteq \big(z^{-p-2}\,\mathfrak{g}[\![z^{-1}]\!]\big)^{\perp} = z^{p}\,\mathfrak{g}[\![z^{-1}]\!].$$

Moreover, the restricted pairing

$$(-\,,\,-): \;\mathfrak{w} \times \mathfrak{g}[z] \longrightarrow \mathbb{C} \tag{77}$$

is non-degenerate, too. Let $\big\{\alpha_l\big\}_{l=1}^{n^2-1}$ be a basis of $\mathfrak{g}$ and $\alpha_{l,k} = \alpha_l z^k \in \mathfrak{g}[z]$ for $1 \leq l \leq n^2-1$, $k \geq 0$. Let $\beta_{l,k} := \alpha_{l,k}^{\vee} \in \mathfrak{w}$ be the dual element of $\alpha_{l,k} \in \mathfrak{g}[z]$ with respect to the pairing (77). Consider the following formal power series:

$$r_{\mathfrak{w}}(x,y) = \sum_{k=0}^{\infty} x^k \left(\sum_{l=1}^{n^2-1} \alpha_l \otimes \beta_{l,k}(y)\right). \tag{78}$$

Theorem 8.3 (see [33, 34])**.** *The following results are true.*

- *The formal power series (78) converges to a rational function.*
- *Moreover, $r_{\mathfrak{w}}$ is a rational solution of (1) satisfying Ansatz (75).*
- *A different choice of a basis of $\mathfrak{g}$ leads to a gauge-equivalent solution.*
- *Other way around, for any solution r of (1) satisfying (75), there exists a Lagrangian order $\mathfrak{w} \subset \widehat{\mathfrak{g}}$ such that $r = r_{\mathfrak{w}}$.*
- *Let σ be any $\mathbb{C}[z]$–linear automorphism of $\mathfrak{g}[z]$ and $\mathfrak{u} = \sigma(\mathfrak{w}) \subset \widehat{\mathfrak{g}}$ be the transformed order. Then the solutions $r_{\mathfrak{w}}$ and $r_{\mathfrak{u}}$ are gauge-equivalent:*
$$r_{\mathfrak{u}}(x,y) = \big(\sigma(x) \otimes \sigma(y)\big) r_{\mathfrak{w}}(x,y).$$
- *The described correspondence $\mathfrak{w} \mapsto r_{\mathfrak{w}}$ provides a bijection between the gauge equivalence classes of rational solutions of (1) satisfying (75) and the orbits of Lagrangian orders in $\widehat{\mathfrak{g}}$ with respect to the action of of $\mathsf{Aut}_{\mathbb{C}[z]}\big(\mathfrak{g}[z]\big)$.*

Example 8.4. Let $\mathfrak{w} = z^{-1}\,\mathfrak{g}[\![z^{-1}]\!]$. It is easy to see that $\mathfrak{w}$ is a Lagrangian order in $\widehat{\mathfrak{g}}$. Let $\big\{\alpha_l\big\}_{l=1}^{n^2-1}$ be any basis of $\mathfrak{g}$. Then we have: $\beta_{l,k} := \big(\alpha_l z^k\big)^{\vee} = \alpha_l^{\vee} z^{-k-1}$. This implies:

$$r_{\mathfrak{w}}(x,y) = \sum_{k=0}^{\infty} x^k \sum_{l=1}^{n^2-1} \alpha_l \otimes \alpha_l^{\vee} y^{-k-1} = \frac{c}{y-x}, \tag{79}$$

where $c \in \mathfrak{g} \otimes \mathfrak{g}$ is the Casimir element. The tensor-valued function $r_{\mathfrak{w}}$ is the celebrated Yang's solution of the classical Yang–Baxter equation (1).

Lemma 8.5. *For any $1 \leq l \leq n^2-1$ and $k \geq 0$ there exists a unique $w_{l,k} \in \mathfrak{g}[z]$ such that*

$$\beta_{l,k} = z^{-k-1}\alpha_l^{\vee} + w_{l,k}.$$

Proof. It is an easy consequence of the assumption $\mathfrak{w} \dotplus \mathfrak{g}[z] = \widehat{\mathfrak{g}}$ and the fact that the pairing (77) is non-degenerate. □

8.2. **Stolin triples.** As we have seen in the previous subsection, the classification of rational solutions of (1) reduces to a description of Lagrangian orders. This correspondence is actually valid for arbitrary simple complex Lie algebras [34]. In the special case $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, there is an explicit parametrization of Lagrangian orders in the following Lie–theoretic terms [33, 35].

Definition 8.6. A *Stolin triple* $(\mathfrak{l}, k, \omega)$ consists of

- a Lie subalgebra $\mathfrak{l} \subseteq \mathfrak{g}$,
- an integer k such that $0 \le k \le n$,
- a skew symmetric bilinear form $\omega : \mathfrak{l} \times \mathfrak{l} \to \mathbb{C}$ which is a 2-cocycle, i.e.
$$\omega\bigl([a, b], c\bigr) + \omega\bigl([b, c], a\bigr) + \omega\bigl([c, a], b\bigr) = 0$$
for all $a, b, c \in \mathfrak{l}$,

such that for the k-th parabolic Lie subalgebra $\mathfrak{p}_k$ of $\mathfrak{g}$ (with $\mathfrak{p}_0 = \mathfrak{p}_n = \mathfrak{g}$) the following two conditions are fulfilled:

- $\mathfrak{l} + \mathfrak{p}_k = \mathfrak{g}$,
- ω is non-degenerate on $(\mathfrak{l} \cap \mathfrak{p}_k) \times (\mathfrak{l} \cap \mathfrak{p}_k)$.

According to Stolin [33], up to the action of $\mathsf{Aut}_{\mathbb{C}[z]}\bigl(\mathfrak{g}[z]\bigr)$, any Lagrangian order in $\widehat{\mathfrak{g}}$ is given by some triple $(\mathfrak{l}, k, \omega)$. In this article, we shall only need the case $\mathfrak{l} = \mathfrak{g}$.

Algorithm 8.7. One can pass from a Stolin triple $(\mathfrak{g}, k, \omega)$ to the corresponding Lagrangian order $\mathfrak{w} \subset \mathfrak{g}((z^{-1}))$ in the following way.

- Consider the following linear subspace

$$\mathfrak{v}_\omega = \bigl\{z^{-1}a + b \,\big|\, \mathsf{tr}(a \cdot -) = \omega(b, -) \in \mathfrak{l}^*\bigr\} \subset z^{-1}\,\mathfrak{g} \dotplus \mathfrak{l} \subset z^{-1}\,\mathfrak{g} \dotplus \mathfrak{g} \subset \widehat{\mathfrak{g}}. \tag{80}$$

- The subspace $\mathfrak{v}_\omega$ defines the following linear subspace

$$\mathfrak{w}' = z^{-2}\,\mathfrak{g}[\![z^{-1}]\!] \dotplus \mathfrak{v}_\omega \subset \widehat{\mathfrak{g}}. \tag{81}$$

- Consider the matrix

$$\eta = \left(\begin{array}{c|c} \mathbb{1}_{k\times k} & 0 \\ \hline 0 & z \cdot \mathbb{1}_{(n-k)\times(n-k)} \end{array}\right) \in \mathsf{GL}_n\bigl(\mathbb{C}[z, z^{-1}]\bigr). \tag{82}$$

and put:

$$\mathfrak{w} = \mathfrak{w}_{(\mathfrak{l},k,\omega)} := \eta^{-1}\,\mathfrak{w}'\,\eta \subset \widehat{\mathfrak{g}}. \tag{83}$$

The next theorem is due to Stolin [33, 34], see also [15, Section 3.2] for a more detailed account of the theory of rational solutions of the classical Yang–Baxter equation (1).

Theorem 8.8. *The following results are true.*

- *The linear subspace* $\mathfrak{w} \subset \widehat{\mathfrak{g}}$ *is a Lagrangian order.*
- *For any Lagrangian order* $\mathfrak{w} \subset \widehat{\mathfrak{g}}$ *there exists* $\alpha \in \mathsf{Aut}_{\mathbb{C}[z]}\bigl(\mathfrak{g}[z]\bigr)$ *and a Stolin triple* $(\mathfrak{l}, k, \omega)$ *such that* $\mathfrak{w} = \alpha\bigl(\mathfrak{w}_{(\mathfrak{l},k,\omega)}\bigr)$.
- *Two Stolin triples* $(\mathfrak{l}, k, \omega)$ *and* $(\mathfrak{l}', k, \omega')$ *define equivalent Lagrangian orders in* $\widehat{\mathfrak{g}}$ *with respect to the* $\mathsf{Aut}_{\mathbb{C}[z]}(\mathfrak{g}[z])$*–action if and only if there exists a Lie algebra automorphism* γ *of* $\mathfrak{g}$ *such that* $\gamma(\mathfrak{l}) = \mathfrak{l}'$ *and* $\gamma^*\bigl([\omega]\bigr) = \omega' \in H^2(\mathfrak{l})$.

Remark 8.9. Unfortunately, the described correspondence between Stolin triples and Lagrangian orders has the following defect: the parameter k is not an invariant of $\mathfrak{w}$. This leads to the fact that two completely different Stolin triples $(\mathfrak{l}, k, \omega)$ and $(\mathfrak{l}', k', \omega')$ can define the same Lagrangian order $\mathfrak{w}$.

Remark 8.10. Consider an arbitrary even-dimensional abelian Lie subalgebra $\mathfrak{b} \subset \mathfrak{g}$ equipped with an arbitrary non-degenerate skew-symmetric bilinear form $\omega : \mathfrak{b} \times \mathfrak{b} \longrightarrow \mathbb{C}$. Obviously, ω is a two-cocycle and we get a Stolin triple $(\mathfrak{b}, 0, \omega)$. Two such triples $(\mathfrak{b}, 0, \omega)$ and $(\mathfrak{b}', 0, \omega')$ define equivalent Lagrangian orders if and only if there exists $\alpha \in \mathsf{Aut}(\mathfrak{g})$ such that $\alpha(\mathfrak{b}) = \mathfrak{b}'$. However, the classification of abelian subalgebras in $\mathfrak{g}$ is essentially equivalent to the classification of finite dimensional $\mathbb{C}[u, v]$–modules. By a result of Drozd [17], the last problem is *representation-wild*. Thus, as it was already pointed out by Belavin and Drinfeld in [3, Section 7], one can not hope to achieve a full classification of all rational solutions of the classical Yang–Baxter equation (1).

Remark 8.11. In this article, we only deal with those Stolin triple $(\mathfrak{g}, e, \omega)$ for which $\mathfrak{l} = \mathfrak{g}$. It leads to the following significant simplifications. Consider the linear map

$$\chi : \mathfrak{g} \xrightarrow{l_\omega} \mathfrak{g}^* \xrightarrow{\mathsf{tr}} \mathfrak{g}, \tag{84}$$

where $l_\omega(a) = \omega(a, -)$ and tr is the isomorphism induced by the trace form. Then

$$\mathfrak{v}_\omega = \big\langle \alpha + z^{-1}\chi(\alpha) \big\rangle_{\alpha \in \mathfrak{g}}.$$

Next, by Whitehead's Theorem, we have the vanishing $H^2(\mathfrak{g}) = 0$. This means that for any two-cocycle $\omega : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}$ there exist a matrix $K \in \mathsf{Mat}_{n\times n}(\mathbb{C})$ such that for all $a, b \in \mathfrak{g}$ we have: $\omega(a, b) = \omega_K(a, b) := \mathsf{tr}\big(K^t \cdot \big([a, b]\big)$. Let $1 \le e \le n$ be such that $\gcd(n, e) = 1$. Then the parabolic subalgebra $\mathfrak{p}_e$ is Frobenius. If $(\mathfrak{g}, e, \omega)$ is a Stolin triple then ω_K has to define a Frobenius pairing on $\mathfrak{p}_e$. If $K' \in \mathsf{Mat}_{n\times n}(\mathbb{C})$ is any other matrix such that $\omega_{K'}$ is non-degenerate on $\mathfrak{p}_e \times \mathfrak{p}_e$ then the triples $(\mathfrak{g}, e, \omega_K)$ and $(\mathfrak{g}, e, \omega_{K'})$ define gauge equivalent solutions of the classical Yang–Baxter equation. This means that the gauge equivalence class of the solution $r_{(\mathfrak{g},e,\omega)}$ does not depend on a particular choice of ω! However, in order to get nice closed formulae for $r_{(\mathfrak{g},e,\omega)}$, we actually need the canonical matrix $J_{(e,d)} \in \mathsf{Mat}_{n\times n}(\mathbb{C})$ constructed by recursion (52).

9. From vector bundles to the cuspidal Weierstrass curve to Stolin triples

For reader's convenience, we recall once again our notation.

- E is the cuspidal Weierstraß curve.
- (e, d) is a pair of positive coprime e integers and $n = e + d$.
- $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{a} = \mathfrak{gl}_n(\mathbb{C})$, $\mathfrak{p} = \mathfrak{p}_e \subset \mathfrak{g}$ is the e-th parabolic subalgebra of $\mathfrak{g}$. We have a decomposition $\mathfrak{p} = \mathfrak{l} \dotplus \mathfrak{n}$, where $\mathfrak{n}$ (respectively $\mathfrak{l}$) is defined by (72) (respectively (73)), $\bar{\mathfrak{n}}$ is the transpose of $\mathfrak{n}$.
- $J = J_{(e,d)} \in \mathfrak{a}$ is the matrix constructed by recursion (52) and $\omega : \mathfrak{p} \times \mathfrak{p} \longrightarrow \mathbb{C}$ is the corresponding Frobenius pairing (69).
- For $1 \le i, j \le n$, let $e_{i,j} \in \mathfrak{a}$ be the corresponding matrix unit, $h_l = e_{l,l} - e_{l+1,l+1}$ for $1 \le l \le n-1$ and $\breve{h}_l$ be its dual with respect to the trace form. Let $c \in \mathfrak{g} \otimes \mathfrak{g}$ be the Casimir element with respect to the trace form.

- Finally, the decomposition $n = e + d$ divides the set $\{(i,j) \in \mathbb{Z}^2 \,|\, 1 \le i, j \le n\}$ in four parts, according to the following convention: $\left(\begin{array}{c|c} \mathrm{IV} & \mathrm{I} \\ \hline \mathrm{III} & \mathrm{II} \end{array}\right)$.

The main results of this section are the following:

- We derive an explicit formula for the rational solution $r_{(\mathfrak{g},e,\omega)}$ of the classical Yang–Baxter equation (1) attached to Stolin triple $(\mathfrak{g}, e, \omega)$.
- We prove that the solutions $r_{(E,(n,d))}$ and $r_{(\mathfrak{g},e,\omega)}$ are gauge–equivalent.

9.1. **Description of the rational solution** $r_{(\mathfrak{g},e,\omega)}$.

Lemma 9.1. *The linear map* $\chi : \mathfrak{g} \longrightarrow \mathfrak{g}$ *from (84) is given by the rule* $a \mapsto \big[J^t, a\big]$.

Proof. For $a, b \in \mathfrak{g}$ we have: $\omega(a,b) = \mathsf{tr}\big(J^t \cdot [a,b]\big) = \mathsf{tr}\big([J^t, a] \cdot b\big)$. Hence, the linear map $l_\omega : \mathfrak{g} \longrightarrow \mathfrak{g}^*$ is given by the formula $a \mapsto \mathsf{tr}\big([J^t, a] \cdot -\big)$. This implies the claim. □

Lemma 9.2. *Let* $\mathfrak{w} \subset \widehat{\mathfrak{g}}$ *be the Lagrangian order constructed from Stolin triple* $(\mathfrak{g}, e, \omega)$ *following Algorithm 8.7. Then we have the following inclusions:*

$$\mathfrak{w}_1 := z^{-3}\bar{n}[\![z^{-1}]\!] \oplus z^{-2}\,\mathfrak{l}[\![z^{-1}]\!] \oplus z^{-1}n[\![z^{-1}]\!] \subset \mathfrak{w} \subset z^{-1}\bar{n}[\![z^{-1}]\!] \oplus \mathfrak{l}[\![z^{-1}]\!] \oplus z\,\mathfrak{n}[\![z^{-1}]\!] := \mathfrak{w}_2\,.$$

Proof. This result is an immediate consequence of the inclusions $z^{-2}\,\mathfrak{g}[\![z^{-1}]\!] \subset \mathfrak{w}' \subset \mathfrak{g}[\![z^{-1}]\!]$, and the fact that $\mathfrak{w} = \eta^{-1}\,\mathfrak{w}'\,\eta$. □

Lemma 9.3. *For any* $1 \le i \ne j \le n$, $1 \le l \le n-1$ *and* $k \ge 0$, *consider the elements* $u_{(i,j;k)}, u_{(l;k)} \in \mathfrak{g}[z]$ *such that*

$$(85) \qquad (z^k e_{i,j})^\vee = z^{-k-1}e_{j,i} + u_{(i,j;k)} \in \mathfrak{w} \quad \text{and} \quad (z^k \check{h}_l)^\vee = z^{-k-1}h_l + u_{(l;k)} \in \mathfrak{w}\,.$$

Then the following statements are true.

- *For all* $1 \le i \ne j \le n$ *and* $k \ge 2$ *we have:* $u_{(i,j;k)} = 0$.
- *For all* $(i,j) \in \mathrm{II} \cup \mathrm{IV}$, $i \ne j$, *we have:* $u_{(i,j;1)} = 0$.
- *Similarly, for all* $1 \le l \le n-1$ *and* $k \ge 1$ *we have:* $u_{(l;k)} = 0$.
- *For all* $(i,j) \in \mathrm{III}$ *and* $k = 0, 1$ *we have:* $u_{(i,j;k)} = 0$.
- *Finally, all non-zero elements* $u_{(i,j;k)}$ *and* $u_{(l;k)}$ *belong to* $\mathfrak{p} \dotplus z\,\mathfrak{n}$.

Proof. According to Lemma 8.5, the elements $u_{(i,j;k)}$ (respectively $u_{(l;k)}$) are uniquely determined by the property that $z^{-k-1}e_{j,i} + u_{(i,j;k)} \in \mathfrak{w}$ (respectively, $z^{-k-1}h_l + u_{(l;k)} \in \mathfrak{w}$). Hence, the first four statements are immediate corollaries of the inclusion $\mathfrak{w}_1 \subset \mathfrak{w}$. On the other hand, the last result follows from the inclusion $\mathfrak{w} \subset \mathfrak{w}_2$. □

In order to get a more concrete description of non-zero elements $u_{(i,j;k)}$ and $u_{(l;k)}$, note the following result.

Lemma 9.4. *Let* $K \in \mathsf{Mat}_{n\times n}(\mathbb{C})$ *be any matrix defining a non-degenerate pairing* ω_K *on* $\mathfrak{p} \times \mathfrak{p}$. *The following statements are true.*

- *For any* $(i,j) \in \mathrm{II} \cup \mathrm{IV}$, $i \neq j$, *there exist uniquely determined* $\left(\begin{array}{c|c} A^{(0)}_{(i,j)} & B^{(0)}_{(i,j)} \\ \hline 0 & D^{(0)}_{(i,j)} \end{array}\right) \in \mathfrak{p}$ *and* $\left(\begin{array}{c|c} 0 & \widetilde{B}^{(0)}_{(i,j)} \\ \hline 0 & 0 \end{array}\right) \in \mathfrak{n}$ *such that*

$$e_{j,i} - \left[K^t, \left(\begin{array}{c|c} A^{(0)}_{(i,j)} & \widetilde{B}^{(0)}_{(i,j)} \\ \hline 0 & D^{(0)}_{(i,j)} \end{array}\right)\right] + \left(\begin{array}{c|c} 0 & B^{(0)}_{(i,j)} \\ \hline 0 & 0 \end{array}\right) = 0. \tag{86}$$

- *Similarly, for any* $1 \leq l \leq n-1$, *there exist uniquely determined* $\left(\begin{array}{c|c} A_{(l)} & B_{(l)} \\ \hline 0 & D_{(l)} \end{array}\right) \in \mathfrak{p}$ *and* $\left(\begin{array}{c|c} 0 & \widetilde{B}_{(l)} \\ \hline 0 & 0 \end{array}\right) \in \mathfrak{n}$ *such that*

$$h_l - \left[K^t, \left(\begin{array}{c|c} A_{(l)} & \widetilde{B}_{(l)} \\ \hline 0 & D_{(l)} \end{array}\right)\right] + \left(\begin{array}{c|c} 0 & B_{(l)} \\ \hline 0 & 0 \end{array}\right) = 0. \tag{87}$$

- *Finally, for any* $(i,j) \in \mathrm{I}$ *and* $k = 0, 1$, *there exist uniquely determined matrices* $\left(\begin{array}{c|c} A^{(k)}_{(i,j)} & B^{(k)}_{(i,j)} \\ \hline 0 & D^{(k)}_{(i,j)} \end{array}\right) \in \mathfrak{p}$ *and* $\left(\begin{array}{c|c} 0 & \widetilde{B}^{(k)}_{(i,j)} \\ \hline 0 & 0 \end{array}\right) \in \mathfrak{n}$ *such that*

$$\left[K^t, e_{j,i} + \left(\begin{array}{c|c} A^{(0)}_{(i,j)} & \widetilde{B}^{(0)}_{(i,j)} \\ \hline 0 & D^{(0)}_{(i,j)} \end{array}\right)\right] = \left(\begin{array}{c|c} 0 & B^{(0)}_{(i,j)} \\ \hline 0 & 0 \end{array}\right) \tag{88}$$

and

$$e_{j,i} - \left[K^t, \left(\begin{array}{c|c} A^{(1)}_{(i,j)} & \widetilde{B}^{(1)}_{(i,j)} \\ \hline 0 & D^{(1)}_{(i,j)} \end{array}\right)\right] + \left(\begin{array}{c|c} 0 & B^{(1)}_{(i,j)} \\ \hline 0 & 0 \end{array}\right) = 0. \tag{89}$$

Proof. All these results follow directly from Lemma 7.8. □

Definition 9.5. Consider the following elements in the Lie algebra $\mathfrak{g}[z]$:

- For $(i,j) \in \mathrm{III}$, we put : $w_{(i,j;0)} = 0 = w_{(i,j;1)}$.
- For $(i,j) \in \mathrm{II} \cup \mathrm{IV}$ such that $i \neq j$, we set:

$$w_{(i,j;0)} = \left(\begin{array}{c|c} A^{(0)}_{(i,j)} & B^{(0)}_{(i,j)} \\ \hline 0 & D^{(0)}_{(i,j)} \end{array}\right) + z \left(\begin{array}{c|c} 0 & \widetilde{B}^{(0)}_{(i,j)} \\ \hline 0 & 0 \end{array}\right), \tag{90}$$

where $\left(\begin{array}{c|c} A^{(0)}_{(i,j)} & B^{(0)}_{(i,j)} \\ \hline 0 & D^{(0)}_{(i,j)} \end{array}\right)$ and $\left(\begin{array}{c|c} 0 & \widetilde{B}^{(0)}_{(i,j)} \\ \hline 0 & 0 \end{array}\right)$ are given by (86). Moreover, we set $w_{(i,j;1)} = 0$.
- Similarly, for $1 \leq l \leq n-1$, following (87), we put

$$w_{(l;0)} = \left(\begin{array}{c|c} A_{(l)} & B_{(l)} \\ \hline 0 & D_{(l)} \end{array}\right) + z \left(\begin{array}{c|c} 0 & \widetilde{B}_{(l)} \\ \hline 0 & 0 \end{array}\right), \tag{91}$$

whereas $w_{(l;1)} = 0$.

- Finally, for $(i,j) \in \mathrm{I}$ and $k = 0,1$, following (88) and (89), we write

$$w_{(i,j;k)} = \left(\begin{array}{c|c} A^{(k)}_{(i,j)} & B^{(k)}_{(i,j)} \\ \hline 0 & D^{(k)}_{(i,j)} \end{array}\right) + z \left(\begin{array}{c|c} 0 & \widetilde{B}^{(k)}_{(i,j)} \\ \hline 0 & 0 \end{array}\right). \tag{92}$$

Now we are ready to prove the main result of this subsection.

Theorem 9.6. *Stolin triple $(\mathfrak{g}, e, \omega_K)$ defines the following solution of (1):*

$$r_{(\mathfrak{g},e,\omega_K)}(x,y) = \frac{c}{y-x} + \sum_{1\le i\ne j\le n} e_{i,j} \otimes w_{(i,j;0)}(y) + \sum_{1\le l\le n-1} \check{h}_l \otimes w_{(l;0)}(y) + x \sum_{1\le i\ne j\le n} e_{i,j} \otimes w_{(i,j;1)}(y).$$

Proof. It is sufficient to show that for any $1 \le i \ne j \le n$, $1 \le l \le n-1$ and $k = 0,1$, we have the following equalities:

$$u_{(i,j;k)} = w_{(i,j;k)} \quad \text{and} \quad u_{(l;k)} = w_{(l;k)}. \tag{93}$$

Recall that $\mathfrak{w} = \mathfrak{w}_1 \dotplus \eta^{-1} \left\langle \alpha + z^{-1}\chi(\alpha) \right\rangle_{\alpha\in\mathfrak{g}} \eta \subset \widehat{\mathfrak{g}}$. It implies that

- For any $(i,j) \in \mathrm{II} \cup \mathrm{IV}$, $i \ne j$, there exists $\mu_{i,j} \in \mathfrak{g}$ such that

$$z^{-1} e_{j,i} + u_{(i,j;0)} = \eta^{-1}\bigl(\mu_{i,j} + z^{-1}\bigl[K^t, \mu_{i,j}\bigr]\bigr)\eta. \tag{94}$$

- Similarly, for any $1 \le l \le n-1$, there exists $\nu_l \in \mathfrak{g}$ such that

$$z^{-1} h_l + u_{(l;0)} = \eta^{-1}\bigl(\nu_l + z^{-1}\bigl[K^t, \nu_l\bigr]\bigr)\eta. \tag{95}$$

- Finally, for any $(i,j) \in \mathrm{I}$ and $k = 0,1$, there exists $\kappa^{(k)}_{i,j} \in \mathfrak{g}$ such that

$$z^{-k-1} e_{j,i} + u_{(i,j;k)} = \eta^{-1}\bigl(\kappa^{(k)}_{i,j} + z^{-1}\bigl[K^t, \kappa^{(k)}_{i,j}\bigr]\bigr)\eta. \tag{96}$$

A straightforward case-by-case analysis shows that equation (94) (respectively, (95) and (96)) is equivalent to equation (86) (respectively, (87) and (88), (89)). Thus, equalities (93) are true and theorem is proven. □

Example 9.7. Let $e = n-1$. We take the matrix

$$K = J_{(n-1,1)} = \left(\begin{array}{cccc|c} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ \hline 0 & 0 & \dots & 0 & 0 \end{array}\right).$$

Solving the equations (86)–(89) yields the following closed formula:

$$\begin{aligned} r_{(\mathfrak{g},e,\omega_K)} &= \frac{c}{y-x} + \\ &+ x\left[e_{1,2} \otimes \check{h}_1 - \sum_{j=3}^{n} e_{1,j} \otimes \left(\sum_{k=1}^{n-j+1} e_{j+k-1,k+1}\right)\right] - y\left[\check{h}_1 \otimes e_{1,2} - \sum_{j=3}^{n}\left(\sum_{k=1}^{n-j+1} e_{j+k-1,k+1}\right) \otimes e_{1,j}\right] \\ &+ \sum_{j=2}^{n-1} e_{1,j} \otimes \left(\sum_{k=1}^{n-j} e_{j+k,k+1}\right) + \sum_{i=2}^{n-1} e_{i,i+1} \otimes \check{h}_i - \sum_{j=2}^{n-1}\left(\sum_{k=1}^{n-j} e_{j+k,k+1}\right) \otimes e_{1,j} - \sum_{i=2}^{n-1} \check{h}_i \otimes e_{i,i+1} \\ &+ \sum_{i=2}^{n-2}\left(\sum_{k=2}^{n-i}\left(\sum_{l=1}^{n-i-k+1} e_{i+k+l-1,l+i}\right) \otimes e_{i,i+k}\right) - \sum_{i=2}^{n-2}\left(\sum_{k=2}^{n-i} e_{i,i+k} \otimes \left(\sum_{l=1}^{n-i-k+1} e_{i+k+l-1,l+i}\right)\right). \end{aligned}$$

In particular, for $n = 2$, we get the following rational solution

$$r(x,y) = \frac{1}{y-x}\left(\frac{1}{2}h\otimes h + e_{12}\otimes e_{21} + e_{21}\otimes e_{12}\right) + \frac{x}{2}e_{12}\otimes h - \frac{y}{2}h\otimes e_{21}.$$

This solution was first time discovered by Stolin in [33]. It is gauge equivalent to the solution (44).

9.2. **Comparison Theorem.** Now we prove the third main result of this article.

Theorem 9.8. *Consider the involutive Lie algebra automorphism* $\tilde{\varphi} : \mathfrak{g} \to \mathfrak{g}, A \mapsto -A^t$. *Then we have:* $(\tilde{\varphi}\otimes\tilde{\varphi})r_{(E,(n,d))} = r_{(\mathfrak{g},e,\omega_K)}$, *where* $K = -J_{(e,d)}$.

Proof. For $x \in \mathbb{C}$, $1 \le i \ne j \le n$ and $1 \le l \le n-1$ consider the following elements of $\mathfrak{g}[z]$:

$$U^{(x)}_{(i,j)} = (z-x)\big(w_{(i,j;0)} + x w_{(i,j;1)}\big) \quad \text{and} \quad U^{(x)}_{(l)} = (z-x)w_{(l;0)},$$

where $w_{(i,j;k)}$ and $w_{(l;0)}$ are element introduced in Definition 9.5. Then we have:

$$r_{(\mathfrak{g},e,\omega_K)} = \frac{1}{y-x}\Big[c + \sum_{1\le i\ne j\le n} e_{i,j}\otimes U^{(x)}_{(i,j)}(y) + \sum_{1\le l\le n-1} \check{h}_l \otimes U^{(x)}_{(l)}(y)\Big].$$

Note that for $(i,j) \in \mathrm{III}$ we have: $U^{(x)}_{(i,j)} = 0$.
In what follows, instead of $\tilde{\varphi}$ we shall use the anti-isomorphism of Lie algebras $\varphi = -\tilde{\varphi}$. We have: $\varphi(e_{i,j}) = e_{j,i}, \varphi(h_l) = h_l, \varphi(\check{h}_l) = \check{h}_l$ and $\varphi\otimes\varphi = \tilde{\varphi}\otimes\tilde{\varphi} \in \mathsf{End}(\mathfrak{g}\otimes\mathfrak{g})$. Hence, we need to show that for all $1 \le i \ne j \le n$ and $1 \le l \le n-1$ we have:

$$\varphi\big(G^x_{(i,j)}\big) = U^{(x)}_{(i,j)} \quad \text{and} \quad \varphi\big(G^{(x)}_{(l)}\big) = U^{(x)}_l.$$

From the definition of elements $G^{(x)}_{(i,j)}$ and $G^{(x)}_{(l)}$ it follows that these equalities are equivalent to the following statements.

- $U^{(x)}_{(i,j)}(x) = 0 = U^{(x)}_{(l)}$ and
- $e_{j,i} + U^{(x)}_{(i,j)}, h_l + U^{(x)}_{(l)} \in \overline{\mathsf{Sol}\big((e,d),x\big)} := \varphi\Big(\mathsf{Sol}\big((e,d),x\big)\Big)$.

The first equality is obviously fulfilled. To show the second, observe that

$$\overline{\mathsf{Sol}\big((e,d),x\big)} := \Big\{P \in \overline{V}_{e,d} \,\Big|\, \big[J^t, P_0\big] + xP_0 + P_\epsilon = 0\Big\} \subset \overline{V}_{e,d},$$

where

$$\overline{V}_{e,d} = \left\{P = \left(\begin{array}{c|c} W & Y \\ \hline X & Z\end{array}\right) + \left(\begin{array}{c|c} W' & Y' \\ \hline 0 & Z'\end{array}\right) z + \left(\begin{array}{c|c} 0 & Y'' \\ \hline 0 & 0\end{array}\right) z^2\right\} \subset \mathfrak{g}[z]$$

and for a given $P \in \overline{V}_{e,d}$ we denote:

$$P_0 = \left(\begin{array}{c|c} W' & Y'' \\ \hline X & Z'\end{array}\right) \text{ and } P_\epsilon = \left(\begin{array}{c|c} W & Y' \\ \hline 0 & Z\end{array}\right). \tag{97}$$

Observe that in the above notations, there are no constraints on the matrix Y.

For any $1 \le i \ne j \le n$ denote: $A_{(i,j)} = A^{(0)}_{(i,j)} + xA^{(1)}_{(i,j)}$. Similarly, we set $B_{(i,j)} = B^{(0)}_{(i,j)} + xB^{(1)}_{(i,j)}$, $\widetilde{B}_{(i,j)} = \widetilde{B}^{(0)}_{(i,j)} + x\widetilde{B}^{(1)}_{(i,j)}$ and $D_{(i,j)} = D^{(0)}_{(i,j)} + xD^{(1)}_{(i,j)}$. Then we have:

$$U^{(x)}_{(i,j)} = -x\left(\begin{array}{c|c} A_{(i,j)} & B_{(i,j)} \\ \hline 0 & D_{(i,j)}\end{array}\right) + z\left(\begin{array}{c|c} A_{(i,j)} & B_{(i,j)} - x\widetilde{B}_{(i,j)} \\ \hline 0 & D_{(i,j)}\end{array}\right) + z^2\left(\begin{array}{c|c} 0 & \widetilde{B}_{(i,j)} \\ \hline 0 & 0\end{array}\right).$$

Similarly,

$$U^{(x)}_{(i,j)} = -x\left(\begin{array}{c|c} A_{(l)} & B_{(l)} \\ \hline 0 & D_{(l)}\end{array}\right) + z\left(\begin{array}{c|c} A_{(l)} & B_{(l)} - x\widetilde{B}_{(l)} \\ \hline 0 & D_{(l)}\end{array}\right) + z^2\left(\begin{array}{c|c} 0 & \widetilde{B}_{(l)} \\ \hline 0 & 0\end{array}\right).$$

First observe that for $(i,j) \in \mathrm{III}$ we have: $U^{(x)}_{(i,j)} = 0$. Since $e_{j,i} \in \overline{\mathsf{Sol}\big((e,d),x\big)}$, we are done with this case. Now we assume that $(i,j) \in \mathrm{II} \cup \mathrm{III} \cup \mathrm{IV}$ and $i \ne j$. Then in the notations of (97), for $e_{j,i} + U^{(x)}_{(i,j)} \in \overline{V}_{e,d}$ we have:

$$P^{(i,j)}_0 = \left(\begin{array}{c|c} A_{(i,j)} & \widetilde{B}_{(i,j)} \\ \hline 0 & D_{(i,j)}\end{array}\right) + \delta_{\mathrm{I}}(i,j)e_{j,i}$$

and

$$P^{(i,j)}_\epsilon = \left(\begin{array}{c|c} -xA_{(i,j)} & B_{(i,j)} - x\widetilde{B}_{(i,j)} \\ \hline 0 & -xD_{(i,j)}\end{array}\right) + \big(\delta_{\mathrm{II}} + \delta_{\mathrm{IV}}\big)(i,j)e_{j,i}.$$

Here, $\delta_{\mathrm{I}}(i,j) = 1$ if $(i,j) \in \mathrm{I}$ and zero otherwise, whereas δ_{II} and δ_{IV} have a similar meaning. The condition $e_{j,i} + U^{(x)}_{(i,j)} \in \overline{\mathsf{Sol}\big((e,d),x\big)}$ is equivalent to the equality

$$\left[J^t, \left(\begin{array}{c|c} A_{(i,j)} & \widetilde{B}_{(i,j)} \\ \hline 0 & D_{(i,j)}\end{array}\right) + \delta_{\mathrm{I}}(i,j)e_{j,i}\right] + x\delta_{\mathrm{I}}(i,j)e_{j,i} + \big(\delta_{\mathrm{II}} + \delta_{\mathrm{IV}}\big)(i,j)e_{j,i} + \left(\begin{array}{c|c} 0 & \widetilde{B}_{(i,j)} \\ \hline 0 & 0\end{array}\right) = 0.$$

Considering separately the case $(i,j) \in \mathrm{I}$ and $(i,j) \in \mathrm{II} \cup \mathrm{IV}$, one can verify that this equation follows from the equations (86), (88) and (89). A similar argument shows that the condition $h_l + U^{(x)}_{(l)} \in \overline{\mathsf{Sol}\big((e,d),x\big)}$ is equivalent to (87). Theorem is proven. □

Remark 9.9. Since the solutions $r_{(\mathfrak{g},e,\omega_K)}$ and $r_{(\mathfrak{g},e,\omega_J)}$ are gauge equivalent, we obtain a gauge equivalence of $r_{(\mathfrak{g},e,\omega_J)}$ and $r_{(E,(n,d))}$.

Corollary 9.10. It follows now from Theorem 3.7 that up to a (not explicitly known) gauge transformation and a change of variables, the rational solution from Example 9.7 is a degeneration of the Belavin's elliptic r–matrix (48) for $\varepsilon = \exp\big(\frac{2\pi i}{n}\big)$. It seems to be quite difficult to prove this result using just direct analytic methods.

We conclude this paper by the following result, which has been pointed out to us by Alexander Stolin.

Proposition 9.11. *The solutions $r_{(E,(n,d))}$ and $r_{(E,(n,e))}$ are gauge equivalent.*

Proof. Consider the Lie algebra automorphism $\mathfrak{a} \stackrel{\psi}{\mathfrak{a}}, e_{i,j} \mapsto e_{n+1-i,n+1-j}$. Obviously, ψ is an automorphism of $\mathfrak{g}$, too. Moreover, it is not difficult to see that $\psi(J_{(e,d)}) = J_{(d,e)}$. The automorphism ψ extends to an automorphism of $\mathfrak{g}[z]$. Moreover, the following diagram is commutative:

$$\begin{array}{ccccc} \mathfrak{g} & \xleftarrow{\overline{\mathsf{res}}_x} & \mathsf{Sol}\big((e,d),x\big) & \xrightarrow{\overline{\mathsf{ev}}_y} & \mathfrak{g} \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} \\ \mathfrak{g} & \xleftarrow{\overline{\mathsf{res}}_x} & \mathsf{Sol}\big((d,e),x\big) & \xrightarrow{\overline{\mathsf{ev}}_y} & \mathfrak{g} \end{array}$$

Hence, $(\psi \otimes \psi) r_{(E,(n,d))} = r_{(E,(n,e))}$. Proposition is proven. □

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