ON THE HALL ALGEBRA OF AN ELLIPTIC CURVE, I

IGOR BURBAN* AND OLIVIER SCHIFFMANN†

Forests may fall,
But not the dusk they shield.
H.P. Lovecraft

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Introduction

Among the oldest and still most fundamental objects in representation theory and combinatorics are the rings of symmetric polynomials

$$\mathbf{\Lambda}^+ = \mathbb{C}[x_1, x_2, \ldots]^{\mathfrak{S}_\infty} := \varprojlim \mathbb{C}[x_1, \ldots, x_r]^{\mathfrak{S}_r},$$

and symmetric Laurent polynomials

$$\mathbf{\Lambda} = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots]^{\mathfrak{S}_\infty}.$$

These rings admit numerous algebraic and geometric realizations, but one of the historically first constructions, dating to the work of Steinitz in 1900 completed later by Hall, was given in terms of what is now called the classical Hall algebra $\mathbf{H}$ (see [Ma], Chapter II). This algebra has a basis consisting of isomorphism classes of abelian q-groups, where q is a fixed prime power, and the structure constants are defined by counting extensions between such abelian groups. In fact, these structure constants are polynomials in q, and we can therefore consider $\mathbf{H}$ as a $\mathbb{C}[q^{\pm 1}]$-algebra. A theorem of Steinitz and Hall provides an isomorphism $\mathbf{H} \simeq \mathbf{\Lambda}_q^+ = \mathbb{C}[q^{\pm 1}][x_1, x_2, \ldots]^{\mathfrak{S}_\infty}$. Under this isomorphism, the natural basis of $\mathbf{H}$ (resp. the natural scalar product) is mapped to the basis of Hall-Littlewood polynomials (resp. the Hall-Littlewood scalar product). In addition, Zelevinsky [Z] endowed $\mathbf{\Lambda}_q^+$ with a structure of a cocommutative Hopf algebra and the whole algebra $\mathbf{\Lambda}_q = \mathbf{\Lambda} \otimes \mathbb{C}[q^{\pm 1}]$

* Mathematisches Institut, Friedrich-Wilhelms-Universität Bonn, Endenicher Allee 60, D-53115, Bonn, Germany, e-mail: burban@math.uni-bonn.de
† Institut Mathématique de Jussieu, Paris 6, 175 rue du Chevaleret, 75013 Paris, France,
e-mail: olive@math.jussieu.fr.

can be recovered from $\mathbf{\Lambda}_q^+$ by the Drinfeld double construction. This Hopf algebra structure is also intrinsically defined by means of the Hall algebra.

One aim of the present work is to initiate a similar approach for the rings of *diagonal* symmetric polynomials

$$\mathbf{\Lambda}^{++} = \mathbb{C}[x_1, x_2, \ldots, y_1, y_2, \ldots]^{\mathfrak{S}_\infty}, \qquad \mathbf{\Lambda}^{+} = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, y_1, y_2, \ldots]^{\mathfrak{S}_\infty}$$

and

$$\mathbf{\Lambda} = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, y_1^{\pm 1}, y_2^{\pm 1}, \ldots]^{\mathfrak{S}_\infty},$$

with $\mathfrak{S}_\infty$ acting simultaneously on the variables x_i and y_i, based on the category of coherent sheaves on an elliptic curve. These rings have recently attracted a lot of attention due to its close relations to Macdonald's polynomials and double affine Hecke algebras.

To any abelian category $\mathcal{A}$ defined over a finite field $\boldsymbol{k} = \mathbb{F}_q$ and satisfying certain finiteness conditions one can attach an associative algebra $\mathbf{H}_{\mathcal{A}}$ defined over the field $\mathbb{Q}(v)$, $v = \sqrt{q}^{-1}$ called the *Hall algebra* of the category $\mathcal{A}$. As a $\mathbb{Q}(v)$–vector space $\mathbf{H}_{\mathcal{A}}$ has a basis parameterized by isomorphism classes of objects of $\mathcal{A}$ and its structure constants are expressed via the number of extensions between the objects of $\mathcal{A}$. The interest in this construction grew considerably after Ringel studied in [R1] the Hall algebra of the category of representations of an arbitrary quiver $\vec{Q}$ and showed that it contains the positive part $\mathbf{U}_v^+(\mathfrak{g})$ of the quantized enveloping algebra of the Kac-Moody algebra $\mathfrak{g}$ associated to $\vec{Q}$.

In a similar direction, Kapranov considered in [K1] a natural subalgebra $\mathbf{H}_X^{sph}$ of the Hall algebra $\mathbf{H}_X$ of the category of coherent sheaves $Coh(X)$ on a smooth projective curve X defined over a finite field $\boldsymbol{k}$. This *spherical Hall algebra* $\mathbf{H}_X^{sph}$ plays an important role in the Langlands program for the function field of X because it can be interpreted as the algebra of (everywhere unramified, principal) Eisenstein series for $GL(n)$ for all n, with the product coming from the parabolic induction functor. In the case $X = \mathbb{P}^1$ the algebra $\mathbf{H}_X^{sph}$ is isomorphic to the positive part of the quantum loop algebra $U_v(\mathfrak{Lsl}_2)$ (see [K1] and also [BK]). In higher genus, Kapranov defined a surjective map from another algebra $\mathbb{U}_X^+$ (defined by generators and relations) to $\mathbf{H}_X^{sph}$. Unfortunately, this map has a nontrivial kernel, and it is not known how to describe it explicitly.

In this paper, we study in details the Hall algebra $\mathbf{H}_X$ of an elliptic curve X defined over $\boldsymbol{k}$ and a certain subalgebra $\mathbf{U}_X^+$ of $\mathbf{H}_X$ which turns out to coincide with the spherical Hall algebra $\mathbf{H}_X^{sph}$ of Kapranov. We show that $\mathbf{U}_X^+$ is naturally a deformation of the ring of *diagonal* symmetric polynomials

$$\mathbf{\Lambda}^{+} := \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, y_1, y_2, \ldots]^{\mathfrak{S}_\infty}.$$

In Theorem 5.4 we provide an explicit description of the bialgebra $\mathbf{U}_X^+$ by generators and relations. It is neither commutative, nor cocommutative. In order to obtain a more symmetric and canonical object, we consider the Drinfeld double $\mathbf{U}_X$ of $\mathbf{U}_X^+$, which is now a deformation of the ring $\mathbf{\Lambda} = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, y_1^{\pm 1}, y_2^{\pm 1}, \ldots]^{\mathfrak{S}_\infty}$. We prove (Theorem 3.8) that the group of exact auto-equivalences of the derived category $D^b\bigl(Coh(X)\bigr)$ naturally acts on $\mathbf{U}_X$ by *algebra* automorphisms, yielding an action of $SL(2,\mathbb{Z})$ on $\mathbf{U}_X$. In Section 5 we construct a natural "monomial" basis of $\mathbf{U}_X^+$ (resp. of $\mathbf{U}_X$) indexed by the set of finite *convex* paths in the region $(\mathbb{Z}^2)^+ = \bigl\{(p,q) \in \mathbb{Z}^2 \mid p \geq 1 \text{ or } p = 0, q \geq 0\bigr\}$ (resp. in $\mathbb{Z}^2$). This basis is equivariant with respect to the $SL(2,\mathbb{Z})$-action.

We show that the structure constants of $\mathbf{U}_X$ are Laurent polynomials in $\sigma^{1/2}$ and $\bar{\sigma}^{1/2}$, where $\sigma, \bar{\sigma}$ are the Frobenius eigenvalues on the l-adic cohomology group

$H^1(X_{\overline{\boldsymbol{k}}}, \overline{\mathbb{Q}_l})$ (observe that $v = (\sigma\bar{\sigma})^{-1/2}$). This allows us to consider $\mathbf{U}_X$ as a $\mathbb{C}\big[\sigma^{\pm 1/2}, \bar{\sigma}^{\pm 1/2}\big]$-algebra. More precisely, we introduce a generic version $\boldsymbol{\mathcal{E}}_{\mathbf{R}}$ of the Hall algebras $\mathbf{U}_X^+$, which is defined over the ring $\mathbf{R} = \mathbb{C}\big[\sigma^{\pm 1/2}, \bar{\sigma}^{\pm 1/2}\big]$, where $\sigma, \bar{\sigma}$ are now formal parameters and which specializes to all the algebras $\mathbf{U}_X$. Moreover, for the values $\sigma = \bar{\sigma} = 1$ one gets the ring

$$(\boldsymbol{\mathcal{E}}_{\mathbf{R}})_{|\sigma=\bar{\sigma}=1} \simeq \boldsymbol{\Lambda} = \mathbb{C}\big[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, y_1^{\pm 1}, y_2^{\pm 1}, \ldots\big]^{\mathfrak{S}_\infty}$$

of diagonal symmetric polynomials and $\boldsymbol{\mathcal{E}}_{\mathbf{R}}$ is a flat deformation of $\boldsymbol{\Lambda}$. We show that as in the case of $\mathbf{U}_X$, the algebra $\boldsymbol{\mathcal{E}}_{\mathbf{R}}$ has a monomial basis, a triangular decomposition, and carries an action of $SL(2,\mathbb{Z})$ by automorphisms.

A very interesting two-parameter deformation of the ring

$$\boldsymbol{\Lambda}_n = \mathbb{C}\big[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1^{\pm 1}, \ldots, y_n^{\pm 1}\big]^{\mathfrak{S}_n}$$

is provided by the *spherical* double affine Hecke algebra (DAHA) $\mathbf{S}\ddot{\mathbf{H}}_n$ of type $\mathfrak{gl}(n)$ (see [Ch]). In a joint work [SV1] of the second-named author with E. Vasserot it is shown that there are surjective homomorphisms $\boldsymbol{\mathcal{E}}_{\mathbf{R}} \twoheadrightarrow \mathbf{S}\ddot{\mathbf{H}}_n$ for any positive integer n, so that $\boldsymbol{\mathcal{E}}_{\mathbf{R}}$ may be thought of as the "stable limit" $\mathbf{S}\ddot{\mathbf{H}}_\infty$ of the type A spherical DAHA. In the companion paper [S3], we shall use a geometric version of the Hall algebra to construct certain "canonical bases" of $\boldsymbol{\mathcal{E}}_{\mathbf{R}}$, which may be thought of as some "double" analogues of Kazhdan-Lusztig polynomials of type A.

The elliptic Hall algebra $\boldsymbol{\mathcal{E}}_{\mathbf{R}}$ has recently found applications in the geometric construction of Macdonald polynomials via Eisenstein series (see [SV1]), and in the computation of convolution algebras in the equivariant K-theory of Hilbert schemes of $\mathbb{A}^2$ and of the commuting variety (see [SV2]).

Let us now briefly describe the content of this paper. After recalling Atiyah's classification of coherent sheaves on an elliptic curve X and the structure of the group of exact auto-equivalences of the derived category $D^b\big(Coh(X)\big)$ in Section 1, we introduce, following Ringel and Green, the Hall bialgebra $\mathbf{H}_X$ of the category $Coh(X)$ in Section 2. In Section 3 we deal with the Drinfeld double $\mathbf{DH}_X$ of $\mathbf{H}_X$ and constructs an embedding of the group of exact auto-equivalences of $D^b\big(Coh(X)\big)$ into $\mathsf{Aut}(\mathbf{DH}_X)$. The subalgebra $\mathbf{U}_X$ of $\mathbf{DH}_X$ we are interested in is defined in Section 4. The main theorem of this article, describing $\mathbf{U}_X$ by generators and relations is proven in Section 5. Section 6 contains various important properties of $\mathbf{U}_X$ (integral form, central extension, etc). In the last Section 7 sum up main properties of the algebra $\mathbf{U}_X$ proven in this article. Appendix A is devoted a discussion of Fourier-Mukai transforms for elliptic curves defined over finite fields, whereas in Appendix B we prove some basic properties of the Drinfeld double of a topological bialgebra.

1. Coherent sheaves on elliptic curves

1.1. Let $\boldsymbol{k}$ be any field. Throughout the paper X denotes a smooth elliptic curve defined over $\boldsymbol{k}$, that is, X is a smooth projective curve of genus one having a rational point. Note, that by Weil's inequality in the case of a finite field $\boldsymbol{k} = \mathbb{F}_q$ we have $\big||X(\boldsymbol{k})| - (q+1)\big| \le 2\sqrt{q}$, hence any smooth projective curve of genus one has such a point. We denote by $Coh(X)$ its category of coherent sheaves. Let us first outline, following Atiyah, the classification of coherent sheaves on elliptic curves (in [A] it is assumed that $\boldsymbol{k}$ is algebraically closed, but the proof can be applied for an arbitrary field $\boldsymbol{k}$). Recall that the slope of a sheaf $\mathcal{F} \in Coh(X)$ is $\mu(\mathcal{F}) = \mathsf{deg}(\mathcal{F})/\,\mathsf{rank}(\mathcal{F})$, and that a sheaf $\mathcal{F}$ is *semi-stable* (resp. *stable*) if for any subsheaf $\mathcal{G} \subset \mathcal{F}$ we have $\mu(\mathcal{G}) \le \mu(\mathcal{F})$ (resp. $\mu(\mathcal{G}) < \mu(\mathcal{F})$). The full subcategory C_μ of $Coh(X)$ consisting of all semi-stable sheaves of a fixed slope $\mu \in \mathbb{Q} \cup \{\infty\}$ is abelian, artinian and closed under extensions. Moreover, if $\mathcal{F}, \mathcal{G}$ are semi-stable with $\mu(\mathcal{F}) < \mu(\mathcal{G})$ then $\mathrm{Hom}(\mathcal{G}, \mathcal{F}) = \mathrm{Ext}(\mathcal{F}, \mathcal{G}) = 0$. Any sheaf $\mathcal{F}$ possesses a unique filtration (the Harder-Narasimhan filtration, or HN filtration)

$$0 = \mathcal{F}^{r+1} \subset \mathcal{F}^r \subset \cdots \subset \mathcal{F}^1 = \mathcal{F}$$

for which $\mathcal{F}^i/\mathcal{F}^{i+1}$ is semi-stable of slope, say μ_i, and $\mu_1 < \cdots < \mu_r$. Observe that C_∞ is just the category of torsion sheaves, and hence is equivalent to the product category $\prod_x \mathcal{T}or_x$, where x runs through the set of closed points of X and $\mathcal{T}or_x$ denotes the category of torsion sheaves supported at x. Since $\mathcal{T}or_x$ is equivalent to the category of finite length modules over the local ring R_x of the point x, there is a unique simple sheaf $\mathcal{O}_x$ in $\mathcal{T}or_x$.

Theorem 1.1 ([A]). *The following holds :*

i) *the HN filtration of any coherent sheaf splits (non-canonically). In particular, any indecomposable coherent sheaf is semi-stable,*
ii) *the set of stable sheaves of slope μ is the set of simple objects of C_μ,*
iii) *there are canonical exact equivalences of abelian categories $\epsilon_{\nu,\mu} : \mathsf{C}_\mu \xrightarrow{\sim} \mathsf{C}_\nu$ for any $\mu, \nu \in \mathbb{Q} \cup \{\infty\}$.*

The Grothendieck group $K_0\big(Coh(X)\big)$ of $Coh(X)$ is equipped with the Euler bilinear form $\langle\ ,\ \rangle : K_0\big(Coh(X)\big) \otimes K_0\big(Coh(X)\big) \longrightarrow \mathbb{Z}$ defined by the formula

$$\overline{\mathcal{F}} \otimes \overline{\mathcal{G}} \mapsto \dim \mathrm{Hom}(\mathcal{F}, \mathcal{G}) - \dim \mathrm{Ext}(\mathcal{F}, \mathcal{G}).$$

There is a natural map $K_0\big(Coh(X)\big) \longrightarrow K_0'\big(Coh(X)\big) := \mathbb{Z}^2$, given by

$$\overline{\mathcal{F}} \mapsto \big(\mathsf{rank}(\mathcal{F}), \mathsf{deg}(\mathcal{F})\big)$$

whose kernel coincides with the radical of the form $\langle\ ,\ \rangle$. As we shall be mainly interested in the class of a sheaf in the numerical K–group $K_0'\big(Coh(X)\big)$, we also denote by $\overline{\mathcal{F}}$ the pair $\big(\mathsf{rank}(\mathcal{F}), \mathsf{deg}(\mathcal{F})\big)$. By the Riemann-Roch formula one has

$$\big\langle (r_1, d_1), (r_2, d_2) \big\rangle = r_1 d_2 - r_2 d_1.$$

In particular, the Euler form is skew-symmetric in our case.

1.2. Let $D^b\big(Coh(X)\big)$ stand for the bounded derived category of coherent sheaves on X. As $Coh(X)$ has global dimension one, the structure of $D^b\big(Coh(X)\big)$ is very simple to describe: any object of this category is isomorphic to its cohomology, i.e. $\mathcal{F}^\bullet \simeq \bigoplus_n H^n(\mathcal{F}^\bullet)[-n]$.

We also consider the so-called root category $\mathcal{R}_X = D^b\big(Coh(X)\big)/[2]$, where $[1]$ is the shift functor. This category can be described as follows

(1) $\mathrm{Ob}(\mathcal{R}_X) = \big\{\mathcal{F}^\pm | \mathcal{F} \in \mathrm{Ob}(Coh_X)\big\}$
(2) $\mathrm{Hom}_{\mathcal{R}_X}(\mathcal{F}^\pm, \mathcal{G}^\pm) = \mathrm{Hom}_X(\mathcal{F}, \mathcal{G})$ and $\mathrm{Hom}_{\mathcal{R}_X}(\mathcal{F}^\pm, \mathcal{G}^\mp) = \mathrm{Ext}^1_X(\mathcal{F}, \mathcal{G})$.

The category $\mathcal{R}_X$ is triangulated and there is a canonical exact functor

$$\Psi : D^b\big(Coh(X)\big) \longrightarrow \mathcal{R}_X$$

inducing a group isomorphism $K_0(X) \longrightarrow K_0(\mathcal{R}_X)$. Since the shift $[2]$ preserves the Euler form $\langle\ ,\ \rangle$, we can define a morphism $K_0(\mathcal{R}_X) \longrightarrow K_0'\big(Coh(X)\big)$, mapping $\overline{\mathcal{F}^\pm}$ to the class $\pm\overline{\mathcal{F}}$. Moreover, one can view the root category $\mathcal{R}_X$ as the category of two-periodic complexes with the functor Ψ being a Galois covering functor in the sense of Gabriel, see [PX] for further details.

Next, let us consider auto-equivalences of triangulated categories $D^b\big(Coh(X)\big)$ and $\mathcal{R}_X$. Let $\mathcal{E}$ be a *spherical object* in the derived category $D^b\big(Coh(X)\big)$, i.e. an object satisfying $\mathrm{Hom}(\mathcal{E},\mathcal{E}) = \mathrm{Hom}\big(\mathcal{E},\mathcal{E}[1]\big) = \boldsymbol{k}$. For example the structure sheaf the curve $\mathcal{O}$ or the structure sheaf of a $\boldsymbol{k}$-rational point $\mathcal{O}_{x_0}$. Seidel and Thomas considered in [ST] the functor

$$T_{\mathcal{E}} : D^b(Coh_X) \longrightarrow D^b(Coh_X)$$

defined by $T_{\mathcal{E}}(\mathcal{F}) = \mathrm{cone}\big(\mathrm{RHom}(\mathcal{E},\mathcal{F}) \overset{\boldsymbol{k}}{\otimes} \mathcal{E} \xrightarrow{ev} \mathcal{F}\big)$. The functor $T_{\mathcal{E}}$ is exact and if coherent sheaves $\mathcal{E}$ and $\mathcal{F}$ satisfy the condition $\mathrm{Ext}^1(\mathcal{E},\mathcal{F}) = 0$, then $T_{\mathcal{E}}(\mathcal{F})$ is quasi-isomorphic to the complex

$$\big(\mathrm{Hom}(\mathcal{E},\mathcal{F}) \overset{\boldsymbol{k}}{\otimes} \mathcal{E} \xrightarrow{ev} \mathcal{F}\big) = \big(\mathcal{E}^n \xrightarrow{ev} \mathcal{F}\big),$$

where $n = \dim \mathrm{Hom}(\mathcal{E},\mathcal{F})$. On the level of $K_0\big(Coh(X)\big)$ the functor $T_{\mathcal{E}}$ induces the group homomorphism $t_{\mathcal{E}} : K_0\big(Coh(X)\big) \longrightarrow K_0\big(Coh(X)\big)$, given by

$$\gamma \mapsto \gamma - \langle \mathcal{E}, \gamma \rangle \overline{\mathcal{E}},$$

where $\langle\ ,\ \rangle$ denotes the Euler form on $K_0\big(Coh(X)\big)$.

Let x_0 be a rational point of X. In the basis $\big\{\overline{\mathcal{O}}, \overline{\mathcal{O}}_{x_0}\big\}$ of the numerical K-group $K_0'\big(Coh(X)\big)$, the twist functors $T_{\mathcal{O}}$, $T_{\mathcal{O}_{x_0}}$ and the shift [1] induce linear transformations given by the matrices

$$t_{\mathcal{O}} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \qquad t_{\mathcal{O}_{x_0}} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad t_{[1]} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Observe that for any $\boldsymbol{k}$-rational point x_0 the equivalence $T_{\mathcal{O}_{x_0}}$ preserves $Coh(X)$ and is simply given by $\mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{O}(x_0)$, see [ST, formula (3.11)].

Due to [ST, Proposition 2.10] the functor $T_{\mathcal{E}}$ is an equivalence of categories for any spherical object $\mathcal{E}$ and by [ST, Lemma 3.2] it is isomorphic to a Fourier-Mukai transform with the kernel $\mathrm{cone}(\mathcal{E}^\vee \boxtimes \mathcal{E} \longrightarrow \mathcal{O}_\Delta) \in D^b\big(Coh(X \times X)\big)$. Moreover, by [ST, Proposition 2.13] we have the following braid group relation:

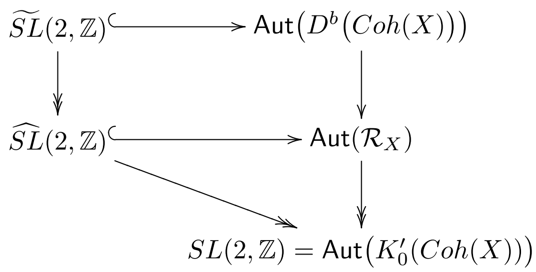
$$T_{\mathcal{O}_{x_0}} T_{\mathcal{O}} T_{\mathcal{O}_{x_0}} \cong T_{\mathcal{O}} T_{\mathcal{O}_{x_0}} T_{\mathcal{O}}.$$

Proposition 1.2 (see [Mu, ST])**.** *Let $\Phi := T_{\mathcal{O}_{x_0}} T_{\mathcal{O}} T_{\mathcal{O}_{x_0}}$, then $\Phi^2 \cong i^*[1]$, where i is an involution of X preserving x_0. Moreover, for the duality functor $D = R\mathcal{H}om(-,\mathcal{O})$ we have an isomorphism*

$$D \circ \Phi \cong i^* \circ [1] \circ \Phi \circ D.$$

Proof. The braid group relation between $T_{\mathcal{O}}$ and $T_{\mathcal{O}_{x_0}}$ was proven in [ST] without any restrictions on the base field. However, in the proof of two other isomorphisms, given in [Mu] the assumption for $\boldsymbol{k}$ to be algebraically closed was used. We refer to Appendix A for a proof in the case of an arbitrary field.

From the above relations one deduces that the group generated by $T_{\mathcal{O}}, T_{\mathcal{O}_{x_0}}$ and [1] is the universal covering $\widetilde{SL}(2,\mathbb{Z})$ of $SL(2,\mathbb{Z})$ given by a central extension of $SL(2,\mathbb{Z})$ by $\mathbb{Z}$. Since in $\mathsf{Aut}(\mathcal{R}_X)$ we have $[1]^2 \simeq \mathsf{id}$, the action of the group $\big\langle T_{\mathcal{O}}, T_{\mathcal{O}_{x_0}}, [1]\big\rangle$ on the root category $\mathcal{R}_X$ breaks up to the action of $\widehat{SL}(2,\mathbb{Z})$, where $\widehat{SL}(2,\mathbb{Z})$ is a two-fold covering of $SL(2,\mathbb{Z})$. That all may be summed up in the following commutative diagram:



For any $\nu \in \mathbb{Q} \cup \{\pm\infty\}$ denote by $Coh_{\leq\nu}$ (resp. $Coh_{>\nu}$) the full subcategory of $Coh(X)$ consisting of sheaves all of whose indecomposable (= semi-stable) constituents have slope at most ν (resp. strictly greater then ν). Next, let $Coh^{\nu}(X)$ be the full subcategory of $D^b\big(Coh(X)\big)$ whose objects consist of direct sums $\mathcal{F} \oplus \mathcal{G}[1]$ where $\mathcal{F} \in Coh_{>\nu}, \mathcal{G} \in Coh_{\leq\nu}$. This has the structure of an abelian category as the heart of the t-structure on $D^b\big(Coh(X)\big)$ associated to the torsion pair $(Coh_{>\nu}, Coh_{\leq\nu})$. One can view the category $Coh^{\nu}(X)$ as a full subcategory of the root category $\mathcal{R}_X$.

For a spherical sheaf $\mathcal{E}$ of class $(r,d) \in K_0'\big(Coh(X)\big)$ and slope $\mu = \frac{d}{r}$ the auto-equivalence $T_{\mathcal{E}}$ establishes an equivalence between $Coh(X)$ and $Coh^{\nu}(X)$, where $\nu = -\infty$ if $\mu = \infty$ and $\nu = \mu - \frac{1}{r^2}$ if $\mu \neq \infty$. More generally, if $\hat{\gamma} \in \widehat{SL}(2,\mathbb{Z})$ is a lift of $\gamma \in SL(2,\mathbb{Z})$ then $\hat{\gamma}$ sends $Coh(X)$ to $Coh^{\nu}(X)$ where $\nu = \frac{p'}{q'}$, and $(q',p') = \gamma(0,-1)$. Finally, each equivalence $\epsilon_{\nu,\mu}$ in Atiyah's Theorem 1.1 can be obtained as the restriction to C_μ of one of the above auto-equivalences of $D^b\big(Coh(X)\big)$ and $\mathcal{R}_X$. We can visualize the structure of the category $\mathcal{R}_X$ by the following picture, where $Coh(X)^+ = Coh(X)$ and $Coh(X)^- = Coh(X)[1]$.

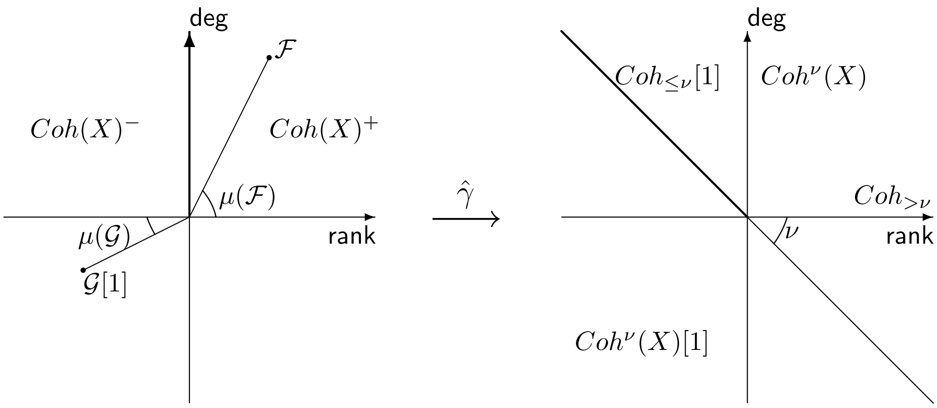


Figure 1. The root category $\mathcal{R}_X$ and its auto-equivalences

2. Hall algebra of an elliptic curve

2.1. From now on we assume that $\boldsymbol{k} = \mathbb{F}_q$ is a finite field, fix a square root v of q^{-1} and work over the quadratic field extension $K = \mathbb{Q}(\sqrt{q}) = \mathbb{Q}(v)$. Note that $Coh(X)$ is a hereditary abelian category. Consider the free K-module $\mathbf{H}_X$ with linear basis $\big\{[\mathcal{F}]\big\}$ where $\mathcal{F}$ runs through the set of isomorphism classes of objects in $Coh(X)$. There is a natural $\mathbb{Z}^2$-grading on $\mathbf{H}_X$ given by $\mathbf{H}_X[\alpha] = \bigoplus_{\overline{\mathcal{F}}=\alpha} K[\mathcal{F}]$. To a triple $(\mathcal{F},\mathcal{G},\mathcal{H})$ of coherent sheaves we associate the finite set $\mathcal{P}^{\mathcal{H}}_{\mathcal{F},\mathcal{G}}$ of exact sequences

$0 \to \mathcal{G} \to \mathcal{H} \to \mathcal{F} \to 0$. Next, we set $P^{\mathcal{H}}_{\mathcal{F},\mathcal{G}} = \#\mathcal{P}^{\mathcal{H}}_{\mathcal{F},\mathcal{G}}$ and $F^{\mathcal{H}}_{\mathcal{F},\mathcal{G}} = \dfrac{P^{\mathcal{H}}_{\mathcal{F},\mathcal{G}}}{a_{\mathcal{F}}a_{\mathcal{G}}}$, where $a_{\mathcal{K}} = \#\mathrm{Aut}(\mathcal{K})$ for a coherent sheaf $\mathcal{K}$. As in [R1] we now define an associative product on $\mathbf{H}_X$ by the formula

$$[\mathcal{F}] \cdot [\mathcal{G}] = v^{-\langle \mathcal{F},\mathcal{G}\rangle} \sum_{\mathcal{H}} F^{\mathcal{H}}_{\mathcal{F},\mathcal{G}}[\mathcal{H}], \tag{2.1}$$

and, following [G], a coassociative coproduct

$$\Delta\big([\mathcal{H}]\big) = \sum_{\mathcal{F},\mathcal{G}} v^{-\langle \mathcal{F},\mathcal{G}\rangle} \frac{P^{\mathcal{H}}_{\mathcal{F},\mathcal{G}}}{a_{\mathcal{H}}} [\mathcal{F}] \otimes [\mathcal{G}]. \tag{2.2}$$

(note that we are using the opposite of the algebra and coalgebra structures considered in [K1]). The counit $\varepsilon : \mathbf{H}_X \longrightarrow K$ is defined as follows

$$\varepsilon\big([\mathcal{F}]\big) = \begin{cases} 1 & \text{if} \quad \mathcal{F} \cong 0 \\ 0 & \text{if} \quad \mathcal{F} \not\cong 0. \end{cases}$$

Finally, the bilinear form given by

$$\big([\mathcal{F}], [\mathcal{G}]\big) = \delta_{\mathcal{F},\mathcal{G}} \frac{1}{a_{\mathcal{F}}}$$

is a non-degenerate Hopf pairing on $\mathbf{H}_X$, i.e. we have $(ab, c) = \big(a \otimes b, \Delta(c)\big)$ for any $a, b, c \in \mathbf{H}_X$ (see [G]).

2.2. The comultiplication Δ only takes value in a certain completion of $\mathbf{H}_X \otimes \mathbf{H}_X$ (the sum on the right-hand side of (2.2) is infinite unless $\mathcal{H}$ is a torsion sheaf). Note also that the space $\mathbf{H}_X[\alpha]$ is infinite dimensional for $\alpha = (r, d) \in \mathbb{Z}^2, r > 0$.

We denote $(\mathbb{Z}^2)^+ = \big\{(q, p) \in \mathbb{Z}^2 \mid q \geq 1 \text{ or } q = 0, p \geq 0\big\}$ and for a given class $\alpha \in (\mathbb{Z}^2)^+$ define $\mathbf{H}_X^{\not\geq m}[\alpha] = \mathsf{span}\big\{[\mathcal{F}] | \overline{\mathcal{F}} = \alpha \text{ and } \mathcal{F} \notin Coh_{\geq m}\big\}$ and $\mathbf{H}_X^{\geq m}[\alpha] = \mathsf{span}\big\{[\mathcal{F}] | \overline{\mathcal{F}} = \alpha \text{ and } \mathcal{F} \in Coh_{\geq m}\big\}$.

Lemma 2.1. *For any class $\alpha \in (\mathbb{Z}^2)^+$ and any integer m the vector space $\mathbf{H}_X^{\geq m}[\alpha]$ is finite-dimensional.*

Proof. Note that for any $m \in \mathbb{Z}$ there are only finitely many elements $\alpha_1, \alpha_2, \ldots, \alpha_t$ of $(\mathbb{Z}^2)^+$ such that $m \leq \mu(\alpha_1) < \cdots < \mu(\alpha_t)$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_t = \alpha$. Moreover, it follows from the Atiyah's classification that for any class $\beta \in (\mathbb{Z}^2)^+$ there are only finitely many semi-stable coherent sheaves of class β. Since any coherent sheaf on an elliptic curve splits into a direct sum of semi-stable ones, the claim easily follows. ✓

For any integer m we have a surjective linear map of vector spaces jet_m : $\mathbf{H}_X[\alpha] \to \mathbf{H}_X^{\geq m}[\alpha]$ inducing an isomorphism $\pi_m : \mathbf{H}_X[\alpha]/\mathbf{H}_X^{\not\geq m}[\alpha] \to \mathbf{H}_X^{\geq m}[\alpha]$. For any $m \leq n$ the canonical embedding $\mathbf{H}_X^{\not\geq m}[\alpha] \to \mathbf{H}_X^{\not\geq n}[\alpha]$ induces a commutative diagram

$$\begin{array}{ccc} \mathbf{H}_X[\alpha]/\mathbf{H}_X^{\not\geq n}[\alpha] & \xrightarrow{\quad \pi_n \quad} & \mathbf{H}_X^{\geq n}[\alpha] \\ \uparrow & & \uparrow{\scriptstyle \varphi_{m,n}} \\ \mathbf{H}_X[\alpha]/\mathbf{H}_X^{\not\geq m}[\alpha] & \xrightarrow{\quad \pi_m \quad} & \mathbf{H}_X^{\geq m}[\alpha] \end{array}$$

Obviously, $\big(\mathbf{H}_X^{\geq n}[\alpha], \varphi_{m,n}\big)$ forms a projective system, and we can define

$$\widehat{\mathbf{H}}_X[\alpha] := \lim_{\substack{\longleftarrow \\ n}} \big(\mathbf{H}_X^{\geq n}[\alpha]\big). \tag{2.3}$$

One can view $\widehat{\mathbf{H}}_X[\alpha]$ as the set of infinite sums $\big\{\sum a_{\mathcal{F}}[\mathcal{F}] | a_{\mathcal{F}} \in K, \overline{\mathcal{F}} = \alpha\big\}$. For the sake of convenience we also denote by jet_n the canonical morphism $\widehat{\mathbf{H}}_X[\alpha] \to \mathbf{H}_X^{\geq n}[\alpha]$. By the universal property of the projective limit there is an (injective) linear map $\mathbf{H}_X[\alpha] \to \widehat{\mathbf{H}}_X[\alpha]$, and since the surjection $\mathbf{H}_X[\alpha] \to \mathbf{H}_X^{\ngeq n}[\alpha]$ splits, we may consider $\mathbf{H}_X^{\ngeq n}[\alpha]$ as a subspace of $\widehat{\mathbf{H}}_X[\alpha]$ via the inclusion $\mathbf{H}_X^{\ngeq n}[\alpha] \to \mathbf{H}_X[\alpha] \longrightarrow \widehat{\mathbf{H}}_X[\alpha]$. So, the projection $\mathsf{jet}_n : \widehat{\mathbf{H}}_X[\alpha] \to \mathbf{H}_X^{\geq n}[\alpha]$ is an idempotent morphism and if we denote $r_n = 1 - \mathsf{jet}_n$, then any element $h \in \widehat{\mathbf{H}}_X[\alpha]$ can be written as $\mathsf{jet}_n(h) + r_n(h)$, where $\mathsf{jet}_n(h) \in \mathbf{H}_X^{\geq n}[\alpha]$ and $\mathsf{jet}_n\big(r_n(h)\big) = 0$. Using this formalism, the space $\mathbf{H}_X[\alpha]$ viewed as a subset of $\widehat{\mathbf{H}}_X[\alpha]$ can be identified with the set of those sequences $h = (h_n)$ for which $r_n(h_n) = 0$ for $n \gg 0$.

So, we define $\widehat{\mathbf{H}}_X := \bigoplus_{\alpha \in (\mathbb{Z}^2)^+} \widehat{\mathbf{H}}_X[\alpha]$. In a similar way, for $\alpha, \beta \in (\mathbb{Z}^2)^+$ the sequence of vector spaces $\big(\mathbf{H}_X^{\geq n}[\alpha] \otimes \mathbf{H}_X^{\geq m}[\beta]\big) = \big(\mathbf{H}_X[\alpha]/\mathbf{H}_X^{\ngeq n}[\alpha] \otimes \mathbf{H}_X[\beta]/\mathbf{H}_X^{\ngeq n}[\beta]\big)$ forms a projective system and we put

$$\mathbf{H}_X[\alpha] \widehat{\otimes} \mathbf{H}_X[\beta] := \varprojlim_{n,m} (\mathbf{H}_X^{\geq n}[\alpha] \otimes \mathbf{H}_X^{\geq m}[\beta]). \tag{2.4}$$

In this case as well $\mathbf{H}_X[\alpha] \widehat{\otimes} \mathbf{H}_X[\beta]$ can be identified with the set of infinite sums $\big\{\sum_{\mathcal{F},\mathcal{G}} b_{\mathcal{F},\mathcal{G}}[\mathcal{F}] \otimes [\mathcal{G}] | \overline{\mathcal{F}} = \alpha, \overline{\mathcal{G}} = \beta,\ b_{\mathcal{F},\mathcal{G}} \in K\big\}$. For $\gamma \in (\mathbb{Z}^2)^+$ we set

$$(\mathbf{H}_X \widehat{\otimes} \mathbf{H}_X)[\gamma] := \prod_{\substack{\alpha+\beta=\gamma \\ \alpha,\beta \in (\mathbb{Z}^2)^+}} \mathbf{H}_X[\alpha] \widehat{\otimes} \mathbf{H}_X[\beta] \tag{2.5}$$

and finally

$$\mathbf{H}_X \widehat{\otimes} \mathbf{H}_X := \bigoplus_{\gamma \in (\mathbb{Z}^2)^+} \mathbf{H}_X \widehat{\otimes} \mathbf{H}_X[\gamma]. \tag{2.6}$$

Proposition 2.2. *In the notation as above the following properties hold*

1. $\widehat{\mathbf{H}}_X$ *and* $\mathbf{H}_X \widehat{\otimes} \mathbf{H}_X$ *are associative algebras;*
2. *the comultiplication* $\Delta : \mathbf{H}_X \longrightarrow \mathbf{H}_X \widehat{\otimes} \mathbf{H}_X$ *is a ring homomorphism and extends to a map* $\Delta : \widehat{\mathbf{H}}_X \longrightarrow \mathbf{H}_X \widehat{\otimes} \mathbf{H}_X$*;*
3. *let* $\Delta_{\alpha,\beta} : \mathbf{H}_X[\alpha+\beta] \to \mathbf{H}_X[\alpha] \widehat{\otimes} \mathbf{H}_X[\beta]$ *stand for the* (α,β)*-component of* Δ*, then* $\Delta_{\alpha,\beta}\big(\mathbf{H}_X[\alpha+\beta]\big) \subset \mathbf{H}_X[\alpha] \otimes \mathbf{H}_X[\beta]$.

Proof. Let us show that the composition map $\widehat{\mathbf{H}}_X[\alpha] \otimes \widehat{\mathbf{H}}_X[\beta] \xrightarrow{m} \widehat{\mathbf{H}}_X[\alpha+\beta]$ given by the rule $\big(\sum a_{\mathcal{H}}[\mathcal{H}]\big) \otimes \big(\sum b_{\mathcal{G}}[\mathcal{G}]\big) \mapsto \big(\sum a_{\mathcal{H}} b_{\mathcal{G}}[\mathcal{H}][\mathcal{G}]\big)$ is well-defined. Indeed, for a fixed coherent sheaf $\mathcal{F}$ of class $\overline{\mathcal{F}} = \alpha + \beta$ there are finitely many exact sequences

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow \mathcal{H} \longrightarrow 0$$

such that $\overline{\mathcal{H}} = \alpha$ and $\overline{\mathcal{G}} = \beta$. To see this, let $\mathcal{F} = \bigoplus_{i=1}^{n} \mathcal{F}_i$ and $\mathcal{H} = \bigoplus_{j=1}^{m} \mathcal{H}_j$ be the splittings of $\mathcal{F}$ and $\mathcal{H}$ into a direct sum of semi-stable objects, then the existence of an epimorphism $\mathcal{F} \twoheadrightarrow \mathcal{H}$ implies the conditions $\mathsf{rank}(\mathcal{H}) \leq \mathsf{rank}(\mathcal{F})$ and $\mu(\mathcal{H}_j) \geq \min\big\{\mu(\mathcal{F}_i) | 1 \leq i \leq n\big\}$ for all $1 \leq j \leq m$. Hence it follows that the degrees of all sheaves $\mathcal{H}_j$ are bounded below and as $\sum_{j=1}^{m} \mathsf{deg}(\mathcal{H}_j) = \mathsf{deg}(\alpha)$, they are also bounded above. By Atiyah's classification, there are finitely many semi-stable sheaves of a given class and hence there are finitely many sheaves $\mathcal{H}$ of class α which are quotients of $\mathcal{F}$. In the same way, there are only finitely many subsheaves of $\mathcal{F}$ of class β. This means that only finitely many sheaves from $\mathbf{H}_X[\alpha]$ and $\mathbf{H}_X[\beta]$

contribute to the element $[\mathcal{F}]$ from $\widehat{\mathbf{H}}_X[\alpha+b]$, which shows that the map m is well-defined.

In a similar fashion, one deals with $\mathbf{H}_X\widehat{\otimes}\mathbf{H}_X$. In this case the map

$$\prod_{\alpha_1+\beta_1=\gamma_1}\mathbf{H}_X[\alpha_1]\widehat{\otimes}\mathbf{H}_X[\beta_1] \quad\otimes \prod_{\alpha_2+\beta_2=\gamma_1}\mathbf{H}_X[\alpha_2]\widehat{\otimes}\mathbf{H}_X[\beta_2] \xrightarrow{m} \prod_{\substack{\alpha_2+\beta_2=\gamma_2\\ \alpha_1+\beta_1=\gamma_1}}\mathbf{H}_X[\alpha_1+\alpha_2]\widehat{\otimes}\mathbf{H}_X[\beta_1+\beta_2] \tag{2.7}$$

is convergent since for a given $[\mathcal{F}]\otimes[\mathcal{G}]\in\mathbf{H}_X[\gamma_1]\otimes\mathbf{H}_X[\gamma_2]$ there are finitely many surjective morphisms $\mathcal{F}\twoheadrightarrow\mathcal{M}$ and $\mathcal{G}\twoheadrightarrow\mathcal{N}$, where $\mathcal{M}$ and $\mathcal{N}$ are coherent sheaves satisfying $\overline{\mathcal{M}}+\overline{\mathcal{N}}=\gamma_1$. The proof that $\Delta_{\alpha,\beta}(\mathbf{H}_X)[\alpha+\beta]\subseteq\mathbf{H}_X[\alpha]\otimes\mathbf{H}_X[\beta]$ is completely analogous.

To see that Δ is a ring homomorphism, fix a pair of tuples $(\alpha,\beta,\alpha',\beta')$ and $(\gamma,\gamma',\delta,\delta')$ of elements of $K_0'(Coh(X))$ satisfying

$$\gamma+\gamma'=\alpha,\quad \delta+\delta'=\beta,\quad \gamma+\delta=\alpha',\quad \gamma'+\delta'=\beta'$$

and put

$$c_{\underline{\gamma},\underline{\delta}}=(m\otimes m)\circ P_{23}\circ(\Delta_{\gamma,\gamma'}\otimes\Delta_{\delta,\delta'}):\mathbf{H}_X[\alpha]\otimes\mathbf{H}_X[\beta]\to\mathbf{H}_X[\alpha']\otimes\mathbf{H}_X[\beta'],$$

where P_{23} is the operator of permutation of the second and third components. For any tuples of sheaves $(\mathcal{F},\mathcal{G},\mathcal{H},\mathcal{K})$ such that $\overline{\mathcal{F}}=\alpha,\overline{\mathcal{G}}=\beta,\overline{\mathcal{H}}=\alpha',\overline{\mathcal{K}}=\beta'$ let $c_{\underline{\gamma},\underline{\delta}}(\mathcal{F},\mathcal{G},\mathcal{H},\mathcal{K})$ be the coefficient of $[\mathcal{H}]\otimes[\mathcal{K}]$ in $c_{\underline{\gamma},\underline{\delta}}([\mathcal{F}]\otimes[\mathcal{G}])$. It is easy to see that $c_{\underline{\gamma},\underline{\delta}}(\mathcal{F},\mathcal{G},\mathcal{H},\mathcal{K})=0$ for all but finitely many tuples $(\gamma,\gamma',\delta,\delta')$.

Lemma 2.3. [G] *The map*

$$(m\circ\Delta)_{\alpha,\beta}^{\alpha',\beta'}=\bigoplus_{\underline{\gamma},\underline{\delta}}c_{\underline{\gamma},\underline{\delta}}:\mathbf{H}_X[\alpha]\otimes\mathbf{H}_X[\beta]\to\mathbf{H}_X[\alpha']\otimes\mathbf{H}_X[\beta'].$$

satisfies the equality $(m\circ\Delta)_{\alpha,\beta}^{\alpha',\beta'}=\Delta_{\alpha',\beta'}\circ m.$

Note to the proof of Lemma. As in the case of quivers, this result is equivalent to the following formula. Let $\mathcal{F},\mathcal{G},\mathcal{M},\mathcal{N}$ be arbitrary coherent sheaves on X. Then the following equality of Hall numbers is true:

$$\sum_{\mathcal{H}}\frac{P^{\mathcal{H}}_{\mathcal{M},\mathcal{N}}\cdot P^{\mathcal{H}}_{\mathcal{F},\mathcal{G}}}{a_{\mathcal{H}}}=\sum_{\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}}q^{-\langle\mathcal{A},\mathcal{D}\rangle}\frac{P^{\mathcal{M}}_{\mathcal{A},\mathcal{B}}\cdot P^{\mathcal{N}}_{\mathcal{C},\mathcal{D}}\cdot P^{\mathcal{F}}_{\mathcal{A},\mathcal{C}}\cdot P^{\mathcal{G}}_{\mathcal{B},\mathcal{D}}}{a_{\mathcal{A}}\cdot a_{\mathcal{B}}\cdot a_{\mathcal{C}}\cdot a_{\mathcal{D}}}. \tag{2.8}$$

This formula can be proved by essentially the same computation as in [G] (a more detailed proof in [R2], in which all arguments involving the dimension vector are replaced by the corresponding ones involving $K_0'\big(Coh(X)\big)$, can be applied in our case literally). See also [S4]. ✓

Observe that since the Euler form $\langle\ ,\ \rangle$ is antisymmetric it is not necessary to twist the multiplication in $\mathbf{H}_X\otimes\mathbf{H}_X$ as it was done in [G]. Lemma 2.3 implies that the linear map Δ is a ring homomorphism. The Proposition 2.2 is proven. ✓

A graded algebra with a graded coproduct satisfying the properties of Proposition 2.2 will be called a *topological* bialgebra. The next important lemma says that the linear maps

$$m:\widehat{\mathbf{H}}_X[\alpha]\otimes\widehat{\mathbf{H}}_X[\beta]\longrightarrow\widehat{\mathbf{H}}_X[\alpha+\beta],\qquad \Delta_{\alpha,\beta}:\widehat{\mathbf{H}}_X[\alpha+\beta]\longrightarrow\mathbf{H}_X[\alpha]\widehat{\otimes}\mathbf{H}_X[\beta]$$

and

$$m:(\mathbf{H}_X[\alpha_1]\widehat{\otimes}\mathbf{H}_X[\beta_1])\otimes(\mathbf{H}_X[\alpha_2]\widehat{\otimes}\mathbf{H}_X[\beta_2])\longrightarrow\mathbf{H}_X[\alpha_1+\beta_1]\widehat{\otimes}\mathbf{H}_X[a_2+\beta_2]$$

are *continuous*. Recall that for an element $a \in \widehat{\mathbf{H}}_X[\alpha]$ and $n \in \mathbb{Z}$ we can write $a = \mathsf{jet}_n(a) + r_n(a)$, where $\mathsf{jet}_n(a) \in \widehat{\mathbf{H}}_X^{\geq n}$ and $\mathsf{jet}_n\big((r_n(a)\big) = 0$.

Lemma 2.4. *For any two classes $\alpha, \beta \in (\mathbb{Z}^2)^+$ and any $m \in \mathbb{Z}$ there exists another integer n such that for any $a \in \widehat{\mathbf{H}}_X[\alpha]$ and $b \in \widehat{\mathbf{H}}[\beta]$ we have $\mathsf{jet}_m(ab) = \mathsf{jet}_m\big(\mathsf{jet}_n(a)\,\mathsf{jet}_n(b)\big)$. Similarly, for any pair of integers m, n there exists another pair k, l such that for all elements $f \in \mathbf{H}_X[\alpha_1]\widehat{\otimes}\mathbf{H}_X[\beta_1]$, $g \in \mathbf{H}_X[\alpha_2]\widehat{\otimes}\mathbf{H}_X[\beta_2]$ we have*

$$\mathsf{jet}_{m,n}(fg) = \mathsf{jet}_{m,n}\big(\mathsf{jet}_{k,l}(f)\,\mathsf{jet}_{k,l}(g)\big).$$

Finally, for any pair of integers m, n there exists k such that for and any $a \in \widehat{\mathbf{H}}_X[\alpha + \beta]$ we have $\mathsf{jet}_{m,n}\big(\Delta_{\alpha,\beta}(a)\big) = \Delta_{\alpha,\beta}\big(\mathsf{jet}_k(a)\big)$.

Proof. For any coherent sheaf $\mathcal{H}$ of class $\alpha + \beta$ there are only finitely many sheaves $\mathcal{F}$ of class α such that there is a surjection $\mathcal{H} \twoheadrightarrow \mathcal{F}$. Hence, we have a finite number of exact sequences $0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow \mathcal{F} \longrightarrow 0$ with $\overline{\mathcal{F}} = \alpha$ and $\overline{\mathcal{G}} = \beta$. Since the vector space $\mathbf{H}_X^{\geq m}[\alpha + \beta]$ is finite-dimensional, we see that there exists n such that $r_n(a)$ and $r_n(b)$ do not contribute to $\mathsf{jet}_m(ab)$. The proof of two other statements is completely analogous. ✓

Later, we shall need the following property of the Hopf pairing in $\mathbf{H}_X$.

Lemma 2.5. *Let $\sum x_i' \otimes x_i'' \in \mathbf{H}_X\widehat{\otimes}\mathbf{H}_X[\gamma]$ and $y \in \mathbf{H}_X[\gamma]$ and suppose that $\sum(x_i'x_i'', y) < \infty$. Then*

$$\sum_i (x_i'x_i'', y) = \sum_{i,j} (x_i', y_j^{(1)})(x_i'', y_j^{(2)})$$

where $\sum\limits_j y_j^{(1)} \otimes y_j^{(2)} = \Delta(y)$.

2.4. There exists a natural "PBW-type" decomposition for $\mathbf{H}_X$. For any $\mu \in \mathbb{Q}\cup\{\infty\}$ we consider the subspace $\mathbf{H}_X^{(\mu)} \subset \mathbf{H}_X$ linearly spanned by classes $\big\{[\mathcal{F}] \mid \mathcal{F} \in \mathsf{C}_\mu\big\}$. Since the category C_μ is stable under extensions, $\mathbf{H}_X^{(\mu)}$ is a subalgebra of $\mathbf{H}_X$ (but not a subbialgebra !). The exact equivalence ϵ_{μ_1,μ_2} defined in Theorem 1.1 gives rise to an algebra isomorphism $\epsilon_{\mu_1,\mu_2}:\ \mathbf{H}_X^{(\mu_2)} \xrightarrow{\sim} \mathbf{H}_X^{(\mu_1)}$. Let $\vec{\bigotimes\limits_\mu}\mathbf{H}_X^{(\mu)}$ stand for the (restricted) tensor product of spaces $\mathbf{H}_X^{(\mu)}$ with $\mu \in \mathbb{Q} \cup \{\infty\}$, ordered from left to right in increasing order, i.e. for the vector space spanned by elements of the form $a_{\mu_1} \otimes \cdots \otimes a_{\mu_r}$ with $a_{\mu_i} \in \mathbf{H}_X^{(\mu_i)}$ and $\mu_1 < \cdots < \mu_r$.

Lemma 2.6. *The multiplication map $m :\ \vec{\bigotimes\limits_\mu}\mathbf{H}_X^{(\mu)} \to \mathbf{H}_X$ is an isomorphism.*

Proof. As the spaces $\mathrm{Ext}(\mathcal{F}, \mathcal{G})$ vanish for $\mathcal{F} \in \mathsf{C}_\mu$, $\mathcal{G}_\nu \in \mathsf{C}_\nu$ and $\mu < \nu$, we have, up to a power of v, $[\mathcal{F}_1] \cdot [\mathcal{F}_2] \cdots [\mathcal{F}_r] = [\mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_r]$ if $\mathcal{F}_i \in \mathsf{C}_{\mu_i}$ and $\mu_1 < \ldots < \mu_r$. Any sheaf can be decomposed into a direct sum of semi-stable summands, and these are determined up to isomorphism. The statement easily follows. ✓

Let $\mathsf{C}[\mu_1, \mu_2]$ be the full subcategory of sheaves whose HN decomposition only contains slopes $\mu \in [\mu_1, \mu_2]$. This category is exact and in particular stable under extensions. Moreover, we have the following remark.

Remark 2.7. For any $\mu_1 \leq \mu_2$ the Hall algebra of the exact category $\mathsf{C}[\mu_1, \mu_2]$ is a subalgebra of $\mathbf{H}_X$, isomorphic to $\vec{\bigotimes\limits_{\mu_1\leq\mu\leq\mu_2}}\mathbf{H}_X^{(\mu)}$.

We conclude this section by the following proposition.

Proposition 2.8. *Consider the algebra*

$$T = K\langle X, Y^{\pm}\rangle/(Y^{\pm}Y^{\mp} = 1,\ XY^{\pm} = v^{\pm 2}Y^{\pm}X).$$

Then there exists a K–linear algebra homomorphism $\chi : \mathbf{H}_X \to T$, called Reineke's character, given by the formula

$$\chi\big([\mathcal{F}]\big) = q^{-\alpha\beta}\frac{X^{\alpha}Y^{\beta}}{a_{\mathcal{F}}},$$

where $(\alpha, \beta) = \big(\mathsf{rank}(\mathcal{F}), \mathsf{deg}(\mathcal{F})\big) \in \mathbb{Z}^2$.

Proof. Let $\Gamma \subset K_0'\big(Coh(X)\big)$ be the semi-group generated by the images of classes of coherent sheaves on X. Following Reineke [Re], consider the associative algebra

$$K\big(\Gamma, \langle\ ,\ \rangle\big) = \Big\{\sum_{\gamma\in\Gamma} a_{\gamma}t^{\gamma} | a_{\gamma} \in K\Big\}$$

where the multiplication is given by the rule $t^{\alpha}t^{\beta} = v^{\langle\alpha,\beta\rangle}t^{\alpha+\beta}$. Let $X = t^{\overline{\mathcal{O}}}$ and $Y^{\pm} = t^{\pm\overline{\mathcal{O}}_{x_0}}$. Then we have: $Y^{\pm}Y^{\mp} = 1$ and $t^{\overline{\mathcal{O}}+\overline{\mathcal{O}}_{x_0}} = vYX = v^{-1}XY$, hence $K\big(\Gamma, \langle\ ,\ \rangle\big) \cong T$. Finally, by [Re, Lemma 6.1] the linear map $\chi : \mathbf{H}_X \to T$ mapping $[\mathcal{F}]$ to $\frac{t^{\overline{\mathcal{F}}}}{a_{\mathcal{F}}} = q^{-\alpha\beta}\frac{X^{\alpha}Y^{\beta}}{a_{\mathcal{F}}}$ is an algebra homomorphism. ✓

3. Drinfeld double of $\mathbf{H}_X$

3.1. As in the case of quivers, it is natural to consider the Drinfeld double of the bialgebra $\mathbf{H}_X$. This is what we do in this Section.

Lemma 3.1. *$\mathbf{H}_X$ is isomorphic to the K-algebra generated by the collection of elements $\{x_{\mathcal{F}}\ |\mathcal{F}$ is semi-stable $\}$ subject to the set of relations*

$$x_{\mathcal{F}} \cdot x_{\mathcal{G}} = v^{-\langle\mathcal{F},\mathcal{G}\rangle}\sum_{\mathcal{H}} F^{\mathcal{H}}_{\mathcal{F},\mathcal{G}}\underline{x}_{\mathcal{H}}, \qquad \forall\ \mathcal{F}, \mathcal{G} \text{ semi-stable} \tag{3.1}$$

where by definition $\underline{x}_{\mathcal{H}} = v^{\sum_{i<j}\langle\mathcal{H}_i,\mathcal{H}_j\rangle}x_{\mathcal{H}_1}\cdots x_{\mathcal{H}_r}$ if $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_r$ with all $\mathcal{H}_i$ being semi-stable and $\mu(\mathcal{H}_1) < \cdots < \mu(\mathcal{H}_r)$.

Proof. Let $\mathbf{G}$ be the algebra defined above. By construction, there is a morphism $\phi : \mathbf{G} \to \mathbf{H}_X$, which is surjective by virtue of Lemma 2.6 (we have $\phi(\underline{x}_{\mathcal{H}}) = [\mathcal{H}]$). Let $\mathbf{G}' \subset \mathbf{G}$ denote the linear span of elements $\underline{x}_{\mathcal{H}}$ for $\mathcal{H} \in Coh(X)$. It is clear that ϕ restricts to an isomorphism of vector spaces between $\mathbf{G}'$ and $\mathbf{H}_X$, hence it is enough to show that $\mathbf{G} = \mathbf{G}'$.

If $\mathcal{F} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_r$ is a decomposition of a sheaf $\mathcal{F}$ into a direct sum of semi-stable objects with $\mu(\mathcal{H}_1) < \cdots < \mu(\mathcal{H}_r)$, we denote $HN(\mathcal{F}) = (\overline{\mathcal{H}_1}, \ldots, \overline{\mathcal{H}_r})$ and call this vector the *HN-type* of $\mathcal{F}$. One can introduce an order on the set of HN types as follows : $\big((r_1, d_1), \ldots, (r_s, d_s)\big) \preceq \big((r_1', d_1'), \ldots, (r_t', d_t')\big)$ if there exists l such that $(r_{s-i}, d_{s-i}) = (r_{t-i}', d_{t-i}')$ for $i < l$ while $\frac{d_{s-l}}{r_{s-l}} > \frac{d_{t-l}'}{r_{t-l}'}$ or $\frac{d_{s-l}}{r_{s-l}} = \frac{d_{t-l}'}{r_{t-l}'}$ and $d_{s-l} > d_{t-l}'$.

Fix $\alpha \in K_0'\big(Coh(X)\big)$. We shall prove that any monomial $x_{\mathcal{F}_1}\cdots x_{\mathcal{F}_r}$ of weight α belongs to $\mathbf{G}'$. For this, we argue successively by induction on the HN type $HN(\underline{\mathcal{F}})$ of the sheaf $\mathcal{F} = \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_r$ and then on the number $n_{\underline{\mathcal{F}}}$ of inversions in the sequence $\big(\mu(\mathcal{F}_1), \ldots, \mu(\mathcal{F}_r)\big)$. Note that if $HN(\underline{\mathcal{F}})$ is maximal, i.e. if $\mu(\mathcal{F}_1) = \cdots = \mu(\mathcal{F}_r) = \nu$ then $x_{\mathcal{F}_1}\cdots x_{\mathcal{F}_r} \in \bigoplus_{\mathcal{H}\in\mathsf{C}_\nu} Kx_{\mathcal{H}} \subset \mathbf{G}'$; on the other hand, if $n_{\underline{\mathcal{F}}} = 0$ then $\mu(\mathcal{F}_1) \leq \cdots \leq \mu(\mathcal{F}_r)$ and $x_{\mathcal{F}_1}\cdots x_{\mathcal{F}_r} \in \mathbf{G}'$ by definition. So let $x_{\mathcal{F}_1}\cdots x_{\mathcal{F}_r}$ be a monomial of weight α and assume that $x_{\mathcal{G}_1}\cdots x_{\mathcal{G}_s}$ belongs to $\mathbf{G}'$ whenever

$HN(\underline{\mathcal{G}}) \succ HN(\underline{\mathcal{F}})$ or $HN(\underline{\mathcal{G}}) = HN(\underline{\mathcal{F}})$ and $n_{\underline{\mathcal{G}}} < n_{\underline{\mathcal{F}}}$. If $n_{\underline{\mathcal{F}}} = 0$ then we are done, so we may assume that $\mu(\mathcal{F}_i) > \mu(\mathcal{F}_{i+1})$ for some i. By Remark 2.7, we have

$$x_{\mathcal{F}_i} \cdot x_{\mathcal{F}_{i+1}} \in K x_{\mathcal{F}_{i+1}} \cdot x_{\mathcal{F}_i} \oplus \bigoplus_{\substack{\mathcal{H} \in \mathsf{C}[\mu(\mathcal{F}_{i+1}), \mu(\mathcal{F}_i)] \\ \mathcal{H} \neq \mathcal{F}_i \oplus \mathcal{F}_{i+1}}} K \underline{x}_{\mathcal{H}}.$$

Now observe that the number of inversions of $x_{\mathcal{F}_1} \cdots x_{\mathcal{F}_{i+1}} \cdot x_{\mathcal{F}_i} \cdots x_{\mathcal{F}_r}$ is one less than $n_{\underline{\mathcal{F}}}$, while the HN-type the sheaf $\mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_{i-1} \oplus \mathcal{H} \oplus \mathcal{F}_{i+2} \cdots \oplus \mathcal{F}_r$ is strictly greater than $HN(\underline{\mathcal{F}})$ as soon as $\mathcal{H} \in \mathsf{C}\big[\mu(\mathcal{F}_{i+1}), \mu(\mathcal{F}_i)\big]$ is of class $\overline{\mathcal{F}_i \oplus \mathcal{F}_{i+1}}$ and $\mathcal{H} \neq \mathcal{F}_i \oplus \mathcal{F}_{i+1}$. We deduce using the induction hypothesis that $x_{\mathcal{F}_1} \cdots x_{\mathcal{F}_r}$ belongs to $\mathbf{G}'$, as desired. ✓

Let $\mathbf{DH}_X$ be the Drinfeld double of the topological bialgebra $\mathbf{H}_X$ with respect to the Hopf pairing (,). Recall (see e.g. [X1]) that this is an algebra generated by two copies of $\mathbf{H}_X$, which we denote by $\mathbf{H}_X^+$ and $\mathbf{H}_X^-$ to avoid confusion, subject to the following set of relations for any pair $g \in \mathbf{H}_X^+$ and $h \in \mathbf{H}_X^-$:

$$(R(g,h)) \qquad \sum_{i,j} h_i^{(1)\,-} g_j^{(2)\,+} \big(h_i^{(2)}, g_j^{(1)}\big) = \sum_{i,j} g_j^{(1)\,+} h_i^{(2)\,-} \big(h_i^{(1)}, g_j^{(2)}\big)$$

(we use here the usual Sweedler notation $\Delta(x^{\pm}) = \sum_i x_i^{(1)\,\pm} \otimes x_i^{(2)\,\pm}$). Observe that although the coproduct takes value in a completion of $\mathbf{H}_X \otimes \mathbf{H}_X$, the relation $\big(R(g,h)\big)$ contains only finitely many terms. Indeed it is enough to consider the case $g = [\mathcal{G}], h = [\mathcal{H}]$, and then $h_i^{(2)}$ involves only sheaves which are subsheaves of $\mathcal{H}$, while $g_j^{(1)}$ involves only sheaves which are quotients of $\mathcal{G}$. As $\mathrm{Hom}(\mathcal{G}, \mathcal{H})$ is a finite set, there are only finitely many sheaves which are both quotients of $\mathcal{G}$ and subsheaves of $\mathcal{H}$, hence the scalar product $(h_i^{(2)}, g_j^{(1)})$ vanishes for almost all values of (i,j). The same holds for the right-hand side of $\big(R(g,h)\big)$. If $h \in \mathbf{H}_X$ then we write h^+, h^- for the corresponding elements in $\mathbf{H}_X^+$ and $\mathbf{H}_X^-$ respectively.

Proposition 3.2. *The algebra $\mathbf{DH}_X$ is isomorphic to the K-algebra generated by two copies $\mathbf{H}_X^+, \mathbf{H}_X^-$ of the Hall algebra $\mathbf{H}_X$ subject to the set of relations*

$$(3.2) \qquad R\big([\mathcal{G}]^+, [\mathcal{H}]^-\big) \qquad \text{for any semi-stable } \mathcal{G}, \mathcal{H} \in Coh(X).$$

Proof. We have to show that the set of relations (3.2) for semistable $a = [\mathcal{G}]^+, b = [\mathcal{H}]^-$ implies the set of relations $R(a,b)$ for arbitrary a, b. By bilinearity of the relations $R(a,b)$ it is enough to prove this in the case $a = [\mathcal{F}]^+, b = [\mathcal{K}]^-$ for some (arbitrary) sheaves $\mathcal{F}, \mathcal{K}$.

Lemma 3.3. *Let $a, b \in \mathbf{H}_X^+, c, d \in \mathbf{H}_X^-$. The relation $R(ab, c)$ is implied by the collection of all relations $R(a, c_k^{(1)})$ and $R(b, c_k^{(2)})$ for all $k \geq 1$. Similarly, $R(a, cd)$ follows from the collection of relations $R(a_k^{(1)}, c)$ and $R(a_k^{(2)}, d)$.*

We refer to Appendix B for a proof of this lemma.

Now, let us consider the algebra $\mathbf{A}$ generated by $\mathbf{H}_X^+$ and $\mathbf{H}_X^-$ modulo relations (3.2). For any coherent sheaf $\mathcal{F}$ there exist semi-stable sheaves $\mathcal{G}_1, \ldots, \mathcal{G}_r$ such that $[\mathcal{F}] = v^{\sum_{i<j} \langle \mathcal{G}_i, \mathcal{G}_j \rangle} [\mathcal{G}_1] \cdots [\mathcal{G}_r]$. Thus, in view of the above Lemma 3.3, it is enough to prove that $R\big([\mathcal{G}], [\mathcal{K}]\big)$ holds for semi-stable $\mathcal{G}$ and arbitrary $\mathcal{K}$. We shall prove this by induction on the rank r of $\mathcal{K}$. As any torsion sheaf is semi-stable, the statement is clear for $r = 0$. So let us assume that $R\big([\mathcal{G}], [\mathcal{K}']\big)$ holds for all semi-stable $\mathcal{G}$ and arbitrary $\mathcal{K}'$ of rank less than r, and let $\mathcal{K}$ be a sheaf of rank r. If $\mathcal{K}$ is semi-stable then there is nothing to prove, so we may assume that $\mathcal{K}$ splits into a non-trivial direct sum of semi-stable objects $\mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_l$. Assume first

that $\mathsf{rank}(\mathcal{K}_i) < r$ for all i. Then $[\mathcal{K}] = v^{\sum_{i<j}\langle \mathcal{K}_i,\mathcal{K}_j\rangle}[\mathcal{K}_1]\cdots[\mathcal{K}_l]$, and by Lemma 3.3 $R\big([\mathcal{G}],[\mathcal{K}]\big)$ is a consequence of the set of relations $R\big([\mathcal{G}]_j^{(i)},[\mathcal{K}_i]\big)$. These hold in $\mathbf{A}$ by the induction hypothesis since $\mathsf{rank}(\mathcal{K}_i) < r$. The last case to consider is that of a sum $\mathcal{K} = \mathcal{I} \oplus \mathcal{T}$ where $\mathcal{I}$ is a semi-stable vector bundle and $\mathcal{T}$ is a torsion sheaf. As above, $R\big([\mathcal{G}],[\mathcal{K}]\big)$ is implied by the relations $R\big([\mathcal{G}]_i^{(1)},[\mathcal{I}]\big)$ and $R\big([\mathcal{G}]_j^{(2)},[\mathcal{T}]\big)$. The second set of relations is satisfied by the induction hypothesis. For the first set, let us again decompose $[\mathcal{G}]_i^{(1)} = v^{d_i}[\mathcal{V}_1]\cdots[\mathcal{V}_t]$ for some semi-stable sheaves $\mathcal{V}_j$. As before, it is enough to see that $R\big([\mathcal{V}_j],[\mathcal{I}]_k^{(j)}\big)$ holds for all j,k. But as $\mathcal{I}$ is a vector bundle, any sheaf appearing in $[\mathcal{I}]_k^{(j)}$ is either semi-stable, or splits as a direct sum of smaller rank sheaves. In both cases the induction hypothesis applies. The Proposition is proved. ✓

Proposition 3.4. *The multiplication map $\mathbf{H}_X^+ \otimes \mathbf{H}_X^- \xrightarrow{m} \mathbf{DH}_X$ is a vector space isomorphism.*

Proof. This statement is classical for Hopf algebras (see [J], 3.2.4). However, in our situation of topological bialgebras, extra care needs to be taken because the coproduct Δ takes values in the completion $\mathbf{H}_X\widehat{\otimes}\mathbf{H}_X$. A proof of Proposition 3.4 is given in Appendix B. ✓

3.2. It is useful to view $\mathbf{DH}_X$ as the (yet inexistent) Hall algebra of the root category $\mathcal{R}_X$, where $\mathbf{H}_X^+$ corresponds to the Hall algebra of $Coh(X)$ and $\mathbf{H}_X^-$ corresponds to the Hall algebra of $Coh(X)[1]$ (see, however [T] or [XX] for a recent approach to Hall algebras for derived categories). For $\mathcal{F} \in Coh(X)$ we put

$$\big[\mathcal{F}[\epsilon]\big] = \begin{cases} [\mathcal{F}]^+ & \text{if } \epsilon = 0 \\ [\mathcal{F}]^- & \text{if } \epsilon = 1. \end{cases}$$

We define the set of semi-stable objects of the root category $\mathcal{R}_X$ as $\big\{\mathcal{F}[\epsilon]\big\}$, where $\mathcal{F}$ is semi-stable and $\epsilon \in \mathbb{Z}/2\mathbb{Z}$. Observe that this set is invariant under auto-equivalences of $\mathcal{R}_X$.

Corollary 3.5. *The algebra $\mathbf{DH}_X$ is generated by the set of elements $[\mathcal{F}]$, where $\mathcal{F}$ runs among all semi-stable objects $\mathcal{F} \in \mathcal{R}_X$.*

Proof. This is a consequence of Lemma 3.1 and Proposition 3.2. ✓

Similarly to the case of the usual Hall numbers, for any triple of objects $\mathcal{F},\mathcal{G},\mathcal{H}$ of the derived category $D^b\big(Coh(X)\big)$ we denote by $P_{\mathcal{F},\mathcal{G}}^{\mathcal{H}}$ the number of the distinguished triangles $\big\{\mathcal{G} \to \mathcal{H} \to \mathcal{F} \to \mathcal{G}[1]\big\}$ and $F_{\mathcal{F},\mathcal{G}}^{\mathcal{H}} = \dfrac{P_{\mathcal{F},\mathcal{G}}^{\mathcal{H}}}{a_{\mathcal{F}}\cdot a_{\mathcal{G}}}$. Next, for any four objects $\mathcal{M},\mathcal{N},\mathcal{A}$ and $\mathcal{B}$ of $Coh(X)$ we denote by $C_{\mathcal{A},\mathcal{B}}^{\mathcal{M},\mathcal{N}}$ the number of the long exact sequences of the form

$$\big\{0 \to \mathcal{N} \to \mathcal{B} \to \mathcal{A} \to \mathcal{M} \to 0\big\}.$$

The following result of Kapranov [K2, Lemma 2.4.3] plays a key role in our study of the Drinfeld double $\mathbf{DH}_X$.

Lemma 3.6. *For for any four objects $\mathcal{M},\mathcal{N},\mathcal{A}$ and $\mathcal{B}$ of $Coh(X)$ we have:*

$$C_{\mathcal{A},\mathcal{B}}^{\mathcal{M},\mathcal{N}} = \frac{P_{\mathcal{B}[1],\mathcal{A}}^{\mathcal{N}[1]\oplus\mathcal{M}}}{|\mathrm{Ext}(\mathcal{M},\mathcal{N})|}.$$

Our next goal is to obtain an explicit form of the relations $R\big([\mathcal{F}]^-,[\mathcal{G}]^+\big)$, where $\mathcal{F}$ and $\mathcal{G}$ are semi-stable sheaves on X.

Proposition 3.7. *Let $\mathcal{F}$ and $\mathcal{G}$ be a pair of semi-stable sheaves on an elliptic curve X with slopes $\mu = \mu(\mathcal{F})$ and $\nu = \mu(\mathcal{G})$.*
(1) *If $\mu < \nu$ then $R\big([\mathcal{F}]^-, [\mathcal{G}]^+\big)$ can be rewritten as*

$$[\mathcal{F}]^- \cdot [\mathcal{G}]^+ = v^{\langle \mathcal{F},\mathcal{G}\rangle} \sum_{\mathcal{B},\mathcal{C}} v^{-\langle \mathcal{C},\mathcal{B}\rangle} F^{\mathcal{C}[1]\oplus\mathcal{B}}_{\mathcal{F}[1],\mathcal{G}} [\mathcal{C}]^+ \cdot [\mathcal{B}]^-.$$

(2) *If $\mu > \nu$ then $R\big([\mathcal{F}]^-, [\mathcal{G}]^+\big)$ reads as*

$$[\mathcal{G}]^+ \cdot [\mathcal{F}]^- = v^{\langle \mathcal{G},\mathcal{F}\rangle} \sum_{\mathcal{A},\mathcal{D}} v^{-\langle \mathcal{D},\mathcal{A}\rangle} F^{\mathcal{D}\oplus\mathcal{A}[-1]}_{\mathcal{G},\mathcal{F}[-1]} [\mathcal{D}]^- \cdot [\mathcal{A}]^+.$$

(3) *Finally, if $\mu = \nu$ then we have*

$$\sum_{\mathcal{A},\mathcal{D}\in \mathsf{C}_\mu} C^{\mathcal{A},\mathcal{D}}_{\mathcal{F},\mathcal{G}} [\mathcal{A}]^+ \cdot [\mathcal{D}]^- = \sum_{\mathcal{B},\mathcal{C}\in \mathsf{C}_\mu} C^{\mathcal{C},\mathcal{B}}_{\mathcal{G},\mathcal{F}} [\mathcal{C}]^- \cdot [\mathcal{B}]^-.$$

Proof. (1) Consider the first case when $\mu < \nu$. Let

$$\Delta\big([\mathcal{F}]\big) = \sum_{\mathcal{A},\mathcal{B}} v^{-\langle \mathcal{A},\mathcal{B}\rangle} \frac{P^{\mathcal{F}}_{\mathcal{A},\mathcal{B}}}{a_{\mathcal{F}}} [\mathcal{A}] \otimes [\mathcal{B}] \quad \text{and} \quad \Delta\big([\mathcal{G}]\big) = \sum_{\mathcal{C},\mathcal{D}} v^{-\langle \mathcal{C},\mathcal{D}\rangle} \frac{P^{\mathcal{G}}_{\mathcal{C},\mathcal{D}}}{a_{\mathcal{G}}} [\mathcal{C}] \otimes [\mathcal{D}].$$

Since $\mathcal{F}$ and $\mathcal{G}$ are semi-stable and $\mu(\mathcal{F}) < \mu(\mathcal{G})$, we have $\big([\mathcal{B}],[\mathcal{C}]\big) = 0$ for any proper subobject $\mathcal{B}$ of $\mathcal{F}$ and any proper quotient object $\mathcal{C}$ of $\mathcal{G}$. Hence, the relation $R\big([\mathcal{F}]^-, [\mathcal{G}]^+\big)$ has the following shape:

$$[\mathcal{F}]^- \cdot [\mathcal{G}]^+ = \sum_{\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}} v^{-\langle \mathcal{A},\mathcal{B}\rangle - \langle \mathcal{C},\mathcal{D}\rangle} \frac{P^{\mathcal{F}}_{\mathcal{A},\mathcal{B}} P^{\mathcal{G}}_{\mathcal{C},\mathcal{D}}}{a_{\mathcal{F}} a_{\mathcal{G}}} \big([\mathcal{A}],[\mathcal{D}]\big) [\mathcal{C}]^+ \cdot [\mathcal{B}]^-$$

$$= \sum_{\mathcal{B},\mathcal{C}} v^{-\langle \mathcal{F}-\mathcal{B},\mathcal{B}\rangle - \langle \mathcal{C},\mathcal{G}-\mathcal{C}\rangle} \frac{1}{a_{\mathcal{F}} a_{\mathcal{G}}} \sum_{\mathcal{A}} \frac{P^{\mathcal{F}}_{\mathcal{A},\mathcal{B}} P^{\mathcal{G}}_{\mathcal{C},\mathcal{A}}}{a_{\mathcal{F}}} [\mathcal{C}]^+ \cdot [\mathcal{B}]^-.$$

Recall that the Euler form on $Coh(X)$ is skew-symmetric, hence for any object $\mathcal{I}$ of $Coh(X)$ we have: $\langle \mathcal{I},\mathcal{I}\rangle = 0$. Next, note the following equality of Hall coefficients:

$$\sum_{\mathcal{A}} \frac{P^{\mathcal{F}}_{\mathcal{A},\mathcal{B}} P^{\mathcal{G}}_{\mathcal{C},\mathcal{A}}}{a_{\mathcal{F}}} = C^{\mathcal{C},\mathcal{B}}_{\mathcal{G},\mathcal{F}}.$$

Hence, the whole expression can be rewritten as

$$[\mathcal{F}]^- \cdot [\mathcal{G}]^+ = \sum_{\mathcal{B},\mathcal{C}} v^{-\langle \mathcal{F},\mathcal{B}\rangle - \langle \mathcal{C},\mathcal{G}\rangle} \frac{C^{\mathcal{C},\mathcal{B}}_{\mathcal{G},\mathcal{F}}}{a_{\mathcal{F}} a_{\mathcal{G}}} [\mathcal{C}]^+ \cdot [\mathcal{B}]^-.$$

Since $\mathrm{Hom}(\mathcal{C},\mathcal{B}) = 0$ for any subobject $\mathcal{B}$ of $\mathcal{F}$ and any quotient object $\mathcal{C}$ of $\mathcal{G}$, Lemma 3.6 implies that $C^{\mathcal{C},\mathcal{B}}_{\mathcal{G},\mathcal{F}} = v^{-2\langle \mathcal{C},\mathcal{B}\rangle} P^{\mathcal{C}[1]\oplus\mathcal{B}}_{\mathcal{F}[1],\mathcal{G}}$. Hence, we obtain:

$$[\mathcal{F}]^- \cdot [\mathcal{G}]^+ = \sum_{\mathcal{B},\mathcal{C}} v^{-\langle \mathcal{F},\mathcal{B}\rangle - \langle \mathcal{C},\mathcal{G}\rangle - 2\langle \mathcal{C},\mathcal{B}\rangle} F^{\mathcal{C}[1]\oplus\mathcal{B}}_{\mathcal{F}[1],\mathcal{G}} [\mathcal{C}]^+ \cdot [\mathcal{B}]^-.$$

To get the claim, it remains to note that

$$\langle \mathcal{F},\mathcal{G}\rangle + \langle \mathcal{F},\mathcal{B}\rangle + \langle \mathcal{C},\mathcal{G}\rangle + \langle \mathcal{C},\mathcal{B}\rangle = \langle \mathcal{F}+\mathcal{C}, \mathcal{G}+\mathcal{B}\rangle = 0.$$

(2) In the case $\mu > \nu$, the derivation of the formula for $R\big([\mathcal{F}]^-, [\mathcal{G}]^+\big)$ is similar to the case (1) and is therefore left to the reader.
(3) Finally, consider the case $\mu(\mathcal{F}) = \mu = \mu(\mathcal{G})$. Let sequences

$$0 \longrightarrow \mathcal{B} \longrightarrow \mathcal{F} \longrightarrow \mathcal{A} \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \mathcal{D} \longrightarrow \mathcal{G} \longrightarrow \mathcal{C} \longrightarrow 0$$

be exact. Assume that $\big([\mathcal{B}],[\mathcal{C}]\big) \neq 0$, i.e. $\mathcal{B} \cong \mathcal{C}$. Then $\mathcal{B}$ and $\mathcal{C}$ are necessarily semi-stable of slope μ. Hence, $\mathcal{A}$ and $\mathcal{D}$ are semi-stable of slope μ as well. In other words, all four objects $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ belong to the same abelian category C_μ. The relation $R\big([\mathcal{F}]^-, [\mathcal{G}]^+\big)$ can be rewritten as follows:

$$\sum_{\substack{\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}\in\mathsf{C}_\mu \\ \mathcal{B}\cong\mathcal{C}}} v^{-\langle\mathcal{A},\mathcal{B}\rangle-\langle\mathcal{C},\mathcal{D}\rangle} \frac{P^{\mathcal{F}}_{\mathcal{A},\mathcal{B}} P^{\mathcal{G}}_{\mathcal{C},\mathcal{D}}}{a_{\mathcal{F}} a_{\mathcal{G}} a_{\mathcal{B}}} [\mathcal{A}]^+ \cdot [\mathcal{D}]^- =$$
$$= \sum_{\substack{\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}\in\mathsf{C}_\mu \\ \mathcal{A}\cong\mathcal{D}}} v^{-\langle\mathcal{A},\mathcal{B}\rangle-\langle\mathcal{C},\mathcal{D}\rangle} \frac{P^{\mathcal{F}}_{\mathcal{A},\mathcal{B}} P^{\mathcal{G}}_{\mathcal{C},\mathcal{D}}}{a_{\mathcal{F}} a_{\mathcal{G}} a_{\mathcal{A}}} [\mathcal{C}]^- \cdot [\mathcal{B}]^+.$$

By Theorem 1.1, C_μ is equivalent to the category of coherent torsion sheaves on X. Hence, the Euler form $\langle\ ,\ \rangle$ vanishes on C_μ and we obtain the relation

$$\sum_{\mathcal{A},\mathcal{D}\in\mathsf{C}_\mu} \Big(\sum_{\mathcal{B}\in\mathsf{C}_\mu} \frac{P^{\mathcal{F}}_{\mathcal{A},\mathcal{B}} P^{\mathcal{G}}_{\mathcal{B},\mathcal{D}}}{a_{\mathcal{B}}}\Big) [\mathcal{A}]^+ \cdot [\mathcal{D}]^- = \sum_{\mathcal{B},\mathcal{C}\in\mathsf{C}_\mu} \Big(\sum_{\mathcal{A}\in\mathsf{C}_\mu} \frac{P^{\mathcal{F}}_{\mathcal{A},\mathcal{B}} P^{\mathcal{G}}_{\mathcal{C},\mathcal{A}}}{a_{\mathcal{A}}}\Big) [\mathcal{C}]^- \cdot [\mathcal{B}]^+,$$

which is obviously equivalent to the relation (3) of Proposition 3.7. ✓

Theorem 3.8. *Let Φ be an auto-equivalence of $D^b\big(Coh(X)\big)$. Then the assignment $[\mathcal{F}] \mapsto [\Phi(\mathcal{F})]$, where $\mathcal{F}$ is a semi-stable object of the root category $\mathcal{R}_X$, extends to a uniquely determined algebra automorphism of $\mathbf{DH}_X$.*

Proof. Recall that $\mathbf{DH}_X$ is a K-algebra generated by the symbols $[\mathcal{F}]^\pm$, where $\mathcal{F}$ is a semi-stable coherent sheaf on X subject to the relations $P\big([\mathcal{F}]^\pm, [\mathcal{G}]^\pm\big)$

$$[\mathcal{F}]^\pm \cdot [\mathcal{G}]^\pm = v^{-\langle\mathcal{F},\mathcal{G}\rangle} \sum_{\mathcal{H}\cong\mathcal{H}_1\oplus\mathcal{H}_2\oplus\dots\oplus\mathcal{H}_t} v^{\sum_{i<j}\langle\mathcal{H}_i,\mathcal{H}_j\rangle} F^{\mathcal{H}}_{\mathcal{F},\mathcal{G}} [\mathcal{H}_1]^\pm \dots [\mathcal{H}_t]^\pm,$$

where $\mathcal{F}$ and $\mathcal{G}$ are semi-stable, $\mu(\mathcal{F}) < \mu(\mathcal{G})$ and $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots \oplus \mathcal{H}_t$ is a splitting into a direct sum of semi-stable objects such that $\mu(\mathcal{H}_1) < \mu(\mathcal{H}_2) < \dots < \mu(\mathcal{H}_t)$; together with the relations $R\big([\mathcal{F}]^\pm, [\mathcal{G}]^\pm\big)$ of Proposition 3.7. In order to show that the group $\mathsf{Aut}\big(D^b(Coh(X))\big)$ acts on $\mathbf{DH}_X$ by algebra automorphisms, it is sufficient to check that all relations $P\big([\mathcal{F}]^\pm, [\mathcal{G}]^\pm\big)$ and $R\big([\mathcal{F}]^-, [\mathcal{G}]^+\big)$ are preserved for $\mathcal{F}$ and $\mathcal{G}$ semi-stable.

Consider first the case of the relations $P\big([\mathcal{F}], [\mathcal{G}]\big)$. Let $\mu = \mu(\mathcal{F}), \nu = \mu(\mathcal{G})$ and Φ be an auto-equivalence of $D^b\big(Coh(X)\big)$.

<u>Case 1</u>. First assume that $\Phi(\mathcal{F}) \cong \widehat{\mathcal{F}}[i]$ and $\Phi(\mathcal{G}) \cong \widehat{\mathcal{G}}[i]$ for some $i \in \mathbb{Z}$. Let $\hat{\mu} = \mu(\widehat{\mathcal{F}})$ and $\hat{\nu} = \mu(\widehat{\mathcal{G}})$. If we assume that $\mu > \nu$ then it automatically follows that $\hat{\mu} > \hat{\nu}$. Moreover, Φ induces an equivalence of exact categories $\mathsf{C}[\nu,\mu] \to \mathsf{C}[\hat{\nu},\hat{\mu}]$. Hence, we have an isomorphism of Hall algebras $H\big(\mathsf{C}[\nu,\mu]\big) \to H\big(\mathsf{C}[\hat{\nu},\hat{\mu}]\big)$ preserving all Hall constants. In other words, the relation $P\big([\mathcal{F}]^\pm, [\mathcal{G}]^\pm\big)$ is mapped to the relation $P\big([\widehat{\mathcal{F}}]^\pm, [\widehat{\mathcal{G}}]^\pm\big)$.

<u>Case 2</u>. Assume $\Phi(\mathcal{F}) \cong \widehat{\mathcal{F}}[2i+1]$ and $\Phi(\mathcal{G}) \cong \widehat{\mathcal{G}}[2i]$ for some $i \in \mathbb{Z}$.

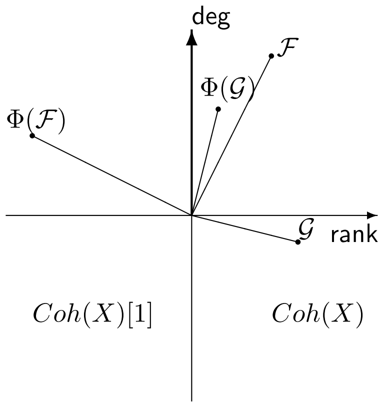


Figure 2. Relation $P([\mathcal{F}],[\mathcal{G}])$, where $\mu(\mathcal{F}) > \mu(\mathcal{G})$

First note that there exists a slope κ, where $\nu \le \kappa < \mu$ such that $\Phi(\mathsf{C}_\varphi) \in Coh(X)[2i+1]$ for $\kappa < \varphi \le \mu$ and $\Phi(\mathsf{C}_\varphi) \in Coh(X)[2i]$ for $\nu \le \varphi \le \kappa$. Next, for any short exact sequence $0 \to \mathcal{G} \to \mathcal{H} \to \mathcal{F} \to 0$ we can write

$$\mathcal{H} \cong \bigl(\mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_t\bigr) \oplus \bigl(\mathcal{H}_{t+1} \oplus \dots \oplus \mathcal{H}_n\bigr) \cong \mathcal{H}' \oplus \mathcal{H}'',$$

where all objects $\mathcal{H}_i$ are semi-stable, $\mu(\mathcal{H}_1) < \mu(\mathcal{H}_2) < \dots < \mu(\mathcal{H}_t) = \kappa < \mathcal{H}_{t+1} < \dots < \mu(\mathcal{H}_n)$, $\mathcal{H}' = \mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_t$ and $\mathcal{H}'' = \mathcal{H}_{t+1} \oplus \dots \oplus \mathcal{H}_n$. In these notations we have:

$$[\mathcal{F}] * [\mathcal{G}] = v^{-\langle \mathcal{F},\mathcal{G}\rangle} \sum_{\mathcal{H}',\mathcal{H}''} F^{\mathcal{H}'\oplus\mathcal{H}''}_{\mathcal{F},\mathcal{G}} v^{\langle \mathcal{H}',\mathcal{H}''\rangle} [\mathcal{H}'] * [\mathcal{H}'']. \tag{3.3}$$

Since $\Phi(\mathcal{F}) \cong \widehat{\mathcal{F}}[2i+1], \Phi(\mathcal{H}'') \cong \widehat{\mathcal{H}}''[2i+1]$, whereas $\Phi(\mathcal{G}) \cong \widehat{\mathcal{G}}[2i]$ and $\Phi(\mathcal{H}') \cong \widehat{\mathcal{H}}'[2i]$, we have: $\langle \mathcal{F},\mathcal{G}\rangle = -\langle \widehat{\mathcal{F}},\widehat{\mathcal{G}}\rangle$ and $\langle \mathcal{H}',\mathcal{H}''\rangle = -\langle \widehat{\mathcal{H}}',\widehat{\mathcal{H}}''\rangle$. Hence, the image of the relation (3.3) is the following equality in the Drinfeld double:

$$[\widehat{\mathcal{F}}]^- * [\widehat{\mathcal{G}}]^+ = v^{\langle \widehat{\mathcal{F}},\widehat{\mathcal{G}}\rangle} \sum_{\widehat{\mathcal{H}}',\widehat{\mathcal{H}}''} v^{-\langle \widehat{\mathcal{H}}',\widehat{\mathcal{H}}''\rangle} F^{\widehat{\mathcal{H}}'\oplus\widehat{\mathcal{H}}''[1]}_{\widehat{\mathcal{F}}[1],\widehat{\mathcal{G}}} [\mathcal{H}']^+ * [\mathcal{H}'']^-. \tag{3.4}$$

It remains to note that $\mu(\widehat{\mathcal{F}}) < \mu(\widehat{\mathcal{G}})$ and the equality (3.4) is nothing but the relation of the Drinfeld double $R\bigl([\widehat{\mathcal{F}}]^-, [\widehat{\mathcal{G}}]^+\bigr)$.

Case 3. In a similar way, if $\Phi(\mathcal{F}) \cong \widehat{\mathcal{F}}[2i]$ and $\Phi(\mathcal{G}) \cong \widehat{\mathcal{G}}[2i-1]$ for some $i \in \mathbb{Z}$ then the relation $P\bigl([\mathcal{F}],[\mathcal{G}]\bigr)$ is mapped to the relation $R\bigl([\widehat{\mathcal{F}}]^+, [\widehat{\mathcal{G}}]^-\bigr)$.

Now we check the preservation of the relations of the Drinfeld double $R\bigl([\widehat{\mathcal{F}}]^\pm, [\widehat{\mathcal{G}}]^\mp\bigr)$ for all semi-stable objects $\mathcal{F}$ and $\mathcal{G}$.

Case 1. First assume $\mu(\mathcal{F}) = \mu(\mathcal{G}) = \mu$. Recall that any auto-equivalence $\Phi \in \mathsf{Aut}\bigl(D^b(Coh(X))\bigr)$ induces an equivalence of abelian categories $\mathsf{C}_\mu \cong \mathsf{C}_\nu$ for an appropriate slope ν. In particular, we obtain:

$$\Phi\bigl(R\bigl([\mathcal{F}]^\pm, [\mathcal{G}]^\mp\bigr)\bigr) = R\bigl([\widehat{\mathcal{F}}]^\pm, [\widehat{\mathcal{G}}]^\mp\bigr)$$

where $\Phi(\mathcal{F}) \cong \widehat{\mathcal{F}}[i]$ and $\Phi(\mathcal{G}) \cong \widehat{\mathcal{G}}[i]$ for an appropriate $i \in \mathbb{Z}$.

Case 2. Assume assume $\mu(\mathcal{F}) < \mu(\mathcal{G})$. Then there exists an auto-equivalence Ψ such that both complexes $\widehat{\mathcal{F}} := \Psi\bigl(\mathcal{F}[1]\bigr)$ and $\widehat{\mathcal{G}} := \Psi(\mathcal{G})$ belong to the heart of the standard t-structure $Coh(X)$. Since we have already shown that $R\bigl([\mathcal{F}]^-, [\mathcal{G}]^+\bigr) = \Psi\bigl(P([\widehat{\mathcal{F}}]^+, [\widehat{\mathcal{G}}]^+)\bigr)$, we have:

$$\Phi\bigl(R\bigl([\mathcal{F}]^-, [\mathcal{G}]^+\bigr)\bigr) = \Phi \circ \Psi^{-1}\bigl(P([\widehat{\mathcal{F}}]^+, [\widehat{\mathcal{G}}]^+)\bigr).$$

Hence, this is again a relation either of the type $P\big([\widehat{\mathcal{F}}]^{\pm},[\widehat{\mathcal{G}}]^{\pm}\big)$ or of the type $R\big([\widehat{\mathcal{F}}]^{\mp},[\widehat{\mathcal{G}}]^{\pm}\big)$.

Case 3. The remaining case $\mu(\mathcal{F})>\mu(\mathcal{G})$ is similar to the former one and is left to the reader. Theorem 3.8 is proven. ✓

Corollary 3.9. *The group $\widehat{SL}(2,\mathbb{Z})$ acts by algebra automorphisms on $\mathbf{DH}_X$.*

Remark 3.10. Theorem 3.8 is close in spirit to [K2] (see also [X2] and [PT]). Recently, it has been generalized by Cramer, who proved that any derived auto-equivalence between hereditary abelian categories gives rise to an isomorphism at the level of Hall algebras, see [Cr].

It turns out that the Drinfeld double $\mathbf{DH}_X$ carries one more symmetry :

Proposition 3.11. *The duality functor $D=\mathbf{R}\mathcal{H}om(-,\mathcal{O})$ induces an involutive anti-isomorphism $[\mathcal{F}]\mapsto\big[D(\mathcal{F})\big]$ of the algebra $\mathbf{DH}_X$ satisfying the relation*

$$D\circ\Phi=i^*\circ[1]\circ\Phi\circ D,$$

where $\Phi=T_{\mathcal{O}}T_{\mathcal{O}_{x_o}}T_{\mathcal{O}}$ and i is an involution of the curve X preserving x_0.

Proof. The proof of the fact that D is an anti-homomorphism of the algebra $\mathbf{DH}_X$ is completely analogous to the proof of the Theorem 3.8 and is therefore skipped. The equality relating the dualizing functor and the Fourier-Mukai transform is a corollary of the Proposition A.2. (see Appendix A). ✓

Remark 3.12. Since the map D sends vector bundles to vector bundles, it restricts to an antiinvolution of the subalgebra $\mathbf{H}^{+,\mathsf{vec}}:=\bigotimes^{\rightarrow}_{-\infty<\mu<\infty}\mathbf{H}^{+,(\mu)}$.

4. The algebra $\mathbf{U}_X$

Our main object of study is a subalgebra $\mathbf{U}_X$ of $\mathbf{DH}_X$, generated by certain "averages" of semi-stable sheaves. Before defining $\mathbf{U}_X$ and giving some of its first properties, we state some useful results on the classical Hall algebra, associated to the category of torsion sheaves supported at a point (or equivalently to the category of nilpotent representations of the Jordan quiver).

4.1. We shall need the usual notions of ν-integers : if $\nu\neq\pm1$ we set

$$[s]_\nu=\frac{\nu^s-\nu^{-s}}{\nu-\nu^{-1}}.$$

We shall usually only use $[s]:=[s]_v$ where $v^2=\#\boldsymbol{k}^{-1}$ is as in Section 2.1. For a finite field $\boldsymbol{l}$ fix $u\in\mathbb{C}$ such that $u^2=(\#\boldsymbol{l})^{-1}$. Denote by $\mathcal{N}_{\boldsymbol{l}}$ the category of nilpotent representations over $\boldsymbol{l}$ of the quiver consisting of a single vertex and a single loop. Then there is exactly one indecomposable object $I_{(r)}$ of length r for any $r\in\mathbb{N}$, and for a partition $\lambda=(\lambda_1,\ldots,\lambda_s)$ we write $I_\lambda=I_{(\lambda_1)}\oplus\cdots\oplus I_{(\lambda_s)}$. The set $\{I_\lambda\}$, where λ runs among all partitions is a complete collection of non-isomorphic objects in $\mathcal{N}_{\boldsymbol{l}}$. The structure of the Hall algebra $\mathbf{H}_{\mathcal{N}_{\boldsymbol{l}}}$ of the category $\mathcal{N}_{\boldsymbol{l}}$ is completely described in [Ma], Chap. II, (see also [Ma] Chap. III. 3.4). The following proposition summarizes those properties of $\mathbf{H}_{\mathcal{N}_{\boldsymbol{l}}}$ that will be needed later on. Let us denote by Λ_t Macdonald's ring of symmetric functions, defined over the ring $\mathbb{Q}[t^{\pm1}]$, and by e_λ (resp. p_λ) the elementary (resp. power-sum) symmetric functions.

Proposition 4.1 ([Ma])**.** *The assignment* $[I_{(1)^r}] \mapsto u^{r(r-1)}e_r$ *extends to a bialgebra isomorphism* $\Psi_{\boldsymbol{l}} : \mathbf{H}_{\mathcal{N}_{\boldsymbol{l}}} \xrightarrow{\sim} (\Lambda_t)_{|t=u^2}$*. Set* $F_r = \Psi_{\boldsymbol{l}}^{-1}(p_r)$*. Then*

i) $F_r = \sum_{|\lambda|=r} n_u\big(l(\lambda)-1\big)[I_\lambda]$, *where* $n_u(l) = \prod_{i=1}^{l}(1-u^{-2i})$,

ii) $\Delta(F_r) = F_r \otimes 1 + 1 \otimes F_r$,

iii) $(F_r, F_s) = \delta_{r,s}\dfrac{ru^r}{u^{-r}-u^r}$.

Proof. Statements i), ii) and iii) may be found in [Ma], III. 7. Ex.2, I.5 Ex. 25 and III.4 (4.11) respectively ✓

In particular, the scalar product (,) on $\mathbf{H}_{\mathcal{N}_{\boldsymbol{l}}}$ coincides, up to a renormalization, with the Hall-Littlewood scalar product.

4.2. Let x be a closed point of X. Since the residue field $\boldsymbol{k}_x$ at the point x is of the same characteristic as $\boldsymbol{k}$, there is an equivalence of categories $\mathcal{N}_{\boldsymbol{k}_x} \xrightarrow{\sim} \mathcal{T}or_x$ which provides us with an isomorphism $\Psi_{\boldsymbol{k}_x} : \mathbf{H}_{\mathcal{T}or_x} \xrightarrow{\sim} (\Lambda_t)_{|t=v^{2\deg(x)}}$, where $v^2 = \#\boldsymbol{k}^{-1}$. For $r \in \mathbb{N}$ we define an element $T^{(\infty)}_{r,x} \in \mathbf{H}_X$ by the equation

$$\frac{T^{(\infty)}_{r,x}}{[r]} = \begin{cases} 0 & \text{if } r \not\equiv 0 \ (\text{mod } \mathsf{deg}(x)) \\ \frac{\mathsf{deg}(x)}{r}\Psi^{-1}_{\boldsymbol{k}_x}\big(p_{\frac{r}{\mathsf{deg}(x)}}\big) & \text{if } r \equiv 0 \ (\text{mod } \mathsf{deg}(x)) \end{cases}$$

and we put $T^{(\infty)}_r = \sum_x T^{(\infty)}_{r,x}$. Note that this sum is finite since there are only finitely many closed points on X of a given degree.

Recall the subalgebras $\mathbf{H}^{(\mu)}_X$ of $\mathbf{H}_X$ defined in Section 2.4. In particular, $\mathbf{H}^{(\infty)}_X$ is the Hall algebra of the category of torsion sheaves on X. As the Hall algebra of $\mathcal{N}_{\boldsymbol{k}_x}$ is commutative for any x, it follows that $\mathbf{H}^{(\infty)}_X$ is commutative and hence for any slope μ the algebra $\mathbf{H}^{(\mu)}_X$ is commutative as well.

By definition, $T^{(\infty)}_r \in \mathbf{H}^{(\infty)}_X$. For an arbitrary $\mu \in \mathbb{Q}$ we put $T^{(\mu)}_r = \epsilon_{\mu,\infty}\big(T^{(\infty)}_r\big)$. As $\epsilon_{\mu_1,\mu_2} \circ \epsilon_{\mu_2,\mu_3} \simeq \epsilon_{\mu_1,\mu_3}$, we have $\epsilon_{\mu_1,\mu_2}\big(T^{(\mu_2)}_r\big) = T^{(\mu_1)}_r$ for any μ_1, μ_2.

Definition 4.2. Let $\mathbf{U}^+_X \subset \mathbf{H}^+_X$ be the K-subalgebra generated by all elements $T^{(\mu)}_r$ for $r \geq 1$ and $\mu \in \mathbb{Q} \cup \{\infty\}$, and let $\mathbf{U}^-_X \subset \mathbf{H}^-_X$ be the (isomorphic) subalgebra defined in a similar way. We denote by $\mathbf{U}_X$ the subalgebra of $\mathbf{DH}_X$ generated by $\mathbf{U}^+_X$ and $\mathbf{U}^-_X$.

It will be convenient for us to introduce one more type of notation : if $\mu = \frac{l}{n}$ with $n \geq 1$ and l, n relatively prime, we put $T_{(\pm rn, \pm rl)} = (T^{(\mu)}_r)^{\pm} \in \mathbf{U}^{\pm}_X$. Similarly, we put $T_{(0,\pm r)} = (T^{(\infty)}_r)^{\pm}$ and $T_{(0,0)} = 1$. We also set $\mathbf{Z} = \mathbb{Z}^2$, so that

$$\mathbf{Z}^{\pm} = \big\{(q,p) \in \mathbb{Z}^2 \mid \pm q > 0 \text{ or } q = 0, \pm p \geq 0\big\}, \qquad \mathbf{Z}^* = \mathbb{Z}^2 \backslash \{(0,0\}$$

and $\mathbf{Z} = \mathbf{Z}^+ \cup \mathbf{Z}^-$. Thus, by definition, $\mathbf{U}^{\pm}_X$ is the subalgebra of $\mathbf{DH}_X$ generated by $T_{(q,p)}$ for $(q,p) \in (\mathbf{Z})^{\pm}$. Note also that by construction the $\widehat{SL}(2,\mathbb{Z})$-action on $\mathbf{DH}_X$ preserves $\mathbf{U}_X$. In fact, since $i^*\big(T_{(q,p)}\big) = T_{(q,p)}$ for any involution i of X, this action factors through $SL(2,\mathbb{Z})$.

Finally, it will be necessary to consider a new system of generators for $\mathbf{U}^+_X$. Namely, for $\alpha \in \mathbf{Z}^+$ we put

$$\mathbf{1}^{\mathsf{ss}}_\alpha = \sum_{\overline{\mathcal{H}}=\alpha; \mathcal{H} \in \mathsf{C}_{\mu(\alpha)}} [\mathcal{H}] \in \mathbf{H}^+_X[\alpha].$$

This sum is finite. If $\alpha = (q,p)$ with p,q relatively prime then (see e.g. [S1], Section 6.3.) we have

$$1 + \sum_{r \geq 1} \mathbf{1}^{\mathsf{ss}}_{r\alpha} s^r = \exp\bigg(\sum_{r\geq 1} \frac{T_{r\alpha}}{[r]} s^r\bigg). \tag{4.1}$$

In particular, $\mathbf{1}^{\mathsf{ss}}_{\alpha} \in \mathbf{U}^+_X$ and the set $\{\mathbf{1}^{\mathsf{ss}}_{\alpha} \mid \alpha \in \mathbf{Z}^+\}$ indeed generates $\mathbf{U}^+_X$.

4.3. Let us now introduce completions of $\mathbf{U}^+_X$ and $\mathbf{U}^+_X \otimes \mathbf{U}^+_X$. Put $\mathbf{U}^{\not\geq n}_X[\alpha] := \mathbf{U}^+_X[\alpha] \cap \mathbf{H}^{\not\geq n}_X[\alpha]$ and $\mathbf{U}^{\geq n}_X[\alpha] := \mathbf{U}^+_X / \mathbf{U}^{\not\geq n}_X[\alpha]$. We can define

$$\widehat{\mathbf{U}}^+_X[\alpha] := \lim_{\substack{\longleftarrow \\ n}} \mathbf{U}^{\geq n}_X[\alpha], \tag{4.2}$$

then clearly $\widehat{\mathbf{U}}^+_X[\alpha] \subseteq \widehat{\mathbf{H}}^+_X[\alpha]$. In the same way, we denote

$$\mathbf{U}^+_X[\alpha] \widehat{\otimes} \mathbf{U}^+_X[\beta] := \lim_{\substack{\longleftarrow \\ n,m}} \mathbf{U}^{\geq n}_X[\alpha] \otimes \mathbf{U}^{\geq m}_X[\beta] \subseteq \mathbf{H}^+_X[\alpha] \widehat{\otimes} \mathbf{H}^+_X[\beta]. \tag{4.3}$$

By definition, an element $a \in \widehat{\mathbf{H}}^+_X[\alpha]$ belongs to $\widehat{\mathbf{U}}^+_X[\alpha]$ if and only if $\mathsf{jet}_n(a) \in \mathbf{U}^{\geq n}_X[\alpha]$ for all n. Similarly, $a \in \mathbf{H}^+_X[\alpha] \widehat{\otimes} \mathbf{H}^+_X[\beta]$ belongs to $\mathbf{U}^+_X[\alpha] \widehat{\otimes} \mathbf{U}^+_X[\beta]$ if and only if $\mathsf{jet}_{m,n}(a) \in \mathbf{U}^{\geq m}_X[\alpha] \otimes \mathbf{U}^{\geq n}_X[\beta]$ for all m,n.

Next, we set

$$\widehat{\mathbf{U}}^+_X := \bigoplus_{\alpha \in \mathbf{Z}^+} \widehat{\mathbf{U}}^+_X[\alpha] \quad \text{and} \quad \mathbf{U}^+_X \widehat{\otimes} \mathbf{U}^+_X = \bigoplus_{\alpha \in \mathbf{Z}^+} \big(\prod_{\beta+\gamma=\alpha} \mathbf{U}^+_X[\beta] \widehat{\otimes} \mathbf{U}^+_X[\gamma]\big).$$

The aim of this section is to prove the following result :

Proposition 4.3. *$\widehat{\mathbf{U}}^+_X$ is a topological sub-bialgebra of $\widehat{\mathbf{H}}^+_X$. That is, $\widehat{\mathbf{U}}^+_X$ is stable under the product, and we have $\Delta_{\alpha,\beta}\big(\widehat{\mathbf{U}}^+_X[\alpha+\beta]\big) \subset \mathbf{U}^+_X[\alpha] \widehat{\otimes} \mathbf{U}^+_X[\beta]$.*

Proof. We first show that $\widehat{\mathbf{U}}^+_X$ is stable under multiplication. Let $a \in \widehat{\mathbf{U}}^+_X[\alpha]$ and $b \in \widehat{\mathbf{U}}^+_X[\beta]$, and fix $u_n \in \mathbf{U}^+_X[\alpha]$, $v_n \in \mathbf{U}^+_X[\beta]$ so that $\mathsf{jet}_n(a) = \mathsf{jet}_n(u_n)$, and $\mathsf{jet}_n(b) = \mathsf{jet}_n(v_n)$ for all n. Then by the continuity of the product (Lemma 2.4) for all m we can find n such that $\mathsf{jet}_m(ab) = \mathsf{jet}_m(u_n v_n)$, hence $ab \in \widehat{\mathbf{U}}^+_X[\alpha+\beta]$. The same proof shows that $\mathbf{U}^+_X \widehat{\otimes} \mathbf{U}^+_X$ is a subalgebra of $\mathbf{H}^+_X \widehat{\otimes} \mathbf{H}^+_X$.

To prove the stability of $\widehat{\mathbf{U}}^+_X$ under the coproduct, it is enough to show that $\Delta(\mathbf{1}^{\mathsf{ss}}_{\alpha}) \in \mathbf{U}^+_X \widehat{\otimes} \mathbf{U}^+_X$ for any α. For this, we introduce another set of generators, this time for $\widehat{\mathbf{U}}^+_X$. Namely, we denote

$$\mathbf{1}_{\alpha} := \sum_{\mathcal{F};\, \overline{\mathcal{F}}=\alpha} [\mathcal{F}] \in \widehat{\mathbf{H}}^+_X[\alpha]$$

From the existence and splitting of the Harder-Narasimhan filtrations we deduce the following equality in $\widehat{\mathbf{H}}_X[\alpha]$

$$\mathbf{1}_{\alpha} = \mathbf{1}^{\mathsf{ss}}_{\alpha} + \sum_{t>1} \sum_{\substack{\alpha_1+\cdots+\alpha_t=\alpha \\ \mu(\alpha_1)<\cdots<\mu(\alpha_t)}} v^{\sum_{i<j}\langle \alpha_i,\alpha_j\rangle} \mathbf{1}^{\mathsf{ss}}_{\alpha_1} \cdots \mathbf{1}^{\mathsf{ss}}_{\alpha_t}, \tag{4.4}$$

from which we conclude that $\mathsf{jet}_n(\mathbf{1}_{\alpha}) \in \mathbf{U}^{\geq n}_X[\alpha]$ for any n, and thus $\mathbf{1}_{\alpha}$ belongs to $\widehat{\mathbf{U}}^+_X$. Next, we use the following well-known property of Hall algebras (see e.g. [S4, Lemma 1.7]) : for any $\alpha,\beta \in \mathbf{Z}^+$,

$$\Delta_{\alpha,\beta}(\mathbf{1}_{\alpha+\beta}) = v^{\langle \alpha,\beta\rangle} \mathbf{1}_{\alpha} \otimes \mathbf{1}_{\beta}. \tag{4.5}$$

It follows that for any polynomial $u = u(\mathbf{1}_{\alpha_1}, \ldots, \mathbf{1}_{\alpha_r})$ we have $\Delta(u) \subset \mathbf{U}_X^+ \widehat{\otimes} \mathbf{U}_X^+$. The inclusion $\Delta(\mathbf{1}_\alpha^{\mathsf{ss}}) \in \mathbf{U}_X^+ \widehat{\otimes} \mathbf{U}_X^+$ is thus a consequence of the continuity of the map Δ together with the next Lemma :

Lemma 4.4. *For any $\alpha \in \mathbf{Z}^+$ and any $n \in \mathbb{Z}$ there exists an integer $m(n)$, a polynomial $u_n \in K\big[t_1, t_2, \ldots, t_{m(n)}\big]$ and classes $\alpha_1, \alpha_2, \ldots, \alpha_{m(n)} \in \mathbf{Z}^+$ satisfying $\sum\limits_{i=1}^{m(n)} \alpha_i = \alpha$ such that $\mathsf{jet}_n(\mathbf{1}_\alpha^{\mathsf{ss}}) = \mathsf{jet}_n\big(u_n(\mathbf{1}_{\alpha_1}, \mathbf{1}_{\alpha_2}, \ldots, \mathbf{1}_{\alpha_{m(n)}})\big)$.*

Proof. We prove this lemma by induction on $\mathsf{rank}(\alpha)$. The case $\mathsf{rank}(\alpha) = 0$ is clear since $\mathbf{1}_\alpha^{\mathsf{ss}} = \mathbf{1}_\alpha$. Assume that $\alpha = (r, d)$ and that the assertion is proven for all classes β such that $\mathsf{rank}(\beta) < r$. From the formula (4.4) we get the following expression in $\widehat{\mathbf{H}}_X[\alpha]$:

$$\mathbf{1}_\alpha^{\mathsf{ss}} = \mathbf{1}_\alpha - \sum_{\substack{\mathsf{rank}(\alpha)=\mathsf{rank}(\gamma) \\ n \le \mu(\gamma) < \mu(\alpha)}} v^{\langle \gamma, \alpha-\gamma\rangle} \mathbf{1}_\gamma^{\mathsf{ss}} \mathbf{1}_{\alpha-\gamma}^{\mathsf{ss}} - \sum_{t>1} \sum_{\substack{\beta_1+\cdots+\beta_t=\alpha \\ \mathsf{rank}(\beta_i)<\mathsf{rank}(\alpha) \\ n \le \mu(\beta_1) < \cdots < \mu(\beta_t)}} v^{\sum_{i<j}\langle \beta_i, \beta_j\rangle} \mathbf{1}_{\beta_1}^{\mathsf{ss}} \cdots \mathbf{1}_{\beta_t}^{\mathsf{ss}} + r_n,$$

where $\mathsf{jet}_n(r_n) = 0$. Note that both sums in the right hand side of the equality are finite. In particular, there are finitely many classes $\gamma = (r, d')$ such that $d > d'$ and $\mu(\gamma) \ge n$. Applying the above formula an appropriate number of times to the element $\mathbf{1}_\gamma^{\mathsf{ss}}$ we obtain

$$\mathbf{1}_\alpha^{\mathsf{ss}} = \mathbf{1}_\alpha + \sum_{\substack{i=1 \\ \mathsf{rank}(\gamma_i)=\mathsf{rank}(\alpha)}}^{k} \mathbf{1}_{\gamma_i} p_i + \sum_{\substack{j=1 \\ \beta_1+\cdots+\beta_{n(j)}=\alpha \\ \mathsf{rank}(\beta_i)<\mathsf{rank}(\alpha)}}^{l} q_j\, \mathbf{1}_{\beta_1}^{\mathsf{ss}} \mathbf{1}_{\beta_2}^{\mathsf{ss}} \ldots \mathbf{1}_{\beta_{n(j)}}^{\mathsf{ss}} + r'_n, \tag{4.6}$$

where $\mathsf{jet}_n(r'_n) = 0$, p_i are polynomials in elements of type $\mathbf{1}_{(0,l)}$ and q_j are scalars. Now by the continuity of the product (Lemma 2.4) there exists an integer N such that for all classes $\beta_1, \beta_2, \ldots, \beta_{n(j)}$ occurring in the decomposition (4.6) of $\mathbf{1}_\alpha^{\mathsf{ss}}$ and for any $x_1 \in \widehat{\mathbf{H}}_X[\beta_1], x_2 \in \widehat{\mathbf{H}}_X[\beta_2], \ldots, x_{n(j)} \in \widehat{\mathbf{H}}_X[\beta_{n_j}]$ we have

$$\mathsf{jet}_n(x_1 x_2 \ldots x_{n(j)}) = \mathsf{jet}_n\big(\mathsf{jet}_N(x_1)\,\mathsf{jet}_N(x_2) \ldots \mathsf{jet}_N(x_{n(j)})\big).$$

Approximating the elements $\mathbf{1}_{\beta_1}^{\mathsf{ss}}, \mathbf{1}_{\beta_2}^{\mathsf{ss}}, \ldots, \mathbf{1}_{\beta_{n(j)}}^{\mathsf{ss}}$ up to the order N by polynomials $u_{\beta_1}, u_{\beta_2}, \ldots, u_{\beta_{n(j)}}$ in classes $\mathbf{1}_\gamma$, we obtain the desired polynomial u_n, approximating $\mathbf{1}_\alpha^{\mathsf{ss}}$ up to the order n. This concludes the proof of the Lemma and of Proposition 4.3. ✓

4.4. We now come to the main result of this Section. In Section 2.4. we described a PBW-type basis of $\mathbf{H}_X^+$, which by Proposition 3.4 extends to a PBW basis of $\mathbf{H}_X$.

We give a similar construction for $\mathbf{U}_X^+$. For $\mu \in \mathbb{Q} \cup \{\infty\}$ let us denote by $\mathbf{U}_X^{\pm,(\mu)} \subset \mathbf{U}_X^\pm$ the subalgebra generated by $\big\{(T_r^{(\mu)})^\pm \mid r \ge 1\big\}$. We also let $\vec{\bigotimes}_\mu$ stand for the restricted ordered tensor product (see Section 2.4.).

Theorem 4.5. *The multiplication map induces isomorphisms of K-vector spaces*

$$\vec{\bigotimes_\mu} \mathbf{U}_X^{\pm,(\mu)} \xrightarrow{\sim} \mathbf{U}_X^\pm, \qquad \vec{\bigotimes_\mu} \mathbf{U}_X^{+,(\mu)} \otimes \vec{\bigotimes_\mu} \mathbf{U}_X^{-,(\mu)} \xrightarrow{\sim} \mathbf{U}_X. \tag{4.7}$$

Moreover, $\mathbf{U}_X^\pm$ is a topological bialgebra: $\Delta_{\alpha,\beta}(\mathbf{U}_X^\pm[\alpha+\beta]) \subset \mathbf{U}_X^\pm[\alpha] \otimes \mathbf{U}_X^\pm[\beta]$.

Proof. In order to prove the above result, it will be convenient to consider $\overline{\mathbf{U}}_X^+[\alpha] := \widehat{\mathbf{U}}_X^+[\alpha] \cap \mathbf{H}_X^+[\alpha]$, where the intersection is taken in $\widehat{\mathbf{H}}_X^+[\alpha]$. Of course, $\mathbf{U}_X^+[\alpha] \subset \overline{\mathbf{U}}_X^+[\alpha]$ and as it will turn out in the end $\overline{\mathbf{U}}_X^+[\alpha] = \mathbf{U}_X^+[\alpha]$, but *a priori* $\overline{\mathbf{U}}_X^+[\alpha]$

might be bigger. Observe however that $\overline{\mathbf{U}}^+_X[\alpha]=\mathbf{U}^+_X[\alpha]$ for any class $\alpha=(0,d)$, $d\in\mathbb{Z}_{>0}$. It is easy to see that $\overline{\mathbf{U}}^+_X=\bigoplus_{\alpha\in\mathbf{Z}^+}\overline{\mathbf{U}}^+_X[\alpha]$ is a subalgebra of $\mathbf{H}^+_X$. In addition, it is also a sub-bialgebra :

Lemma 4.6. *For any α,β we have $\Delta_{\alpha,\beta}\big(\overline{\mathbf{U}}^+_X[\alpha+\beta]\big)\subset\overline{\mathbf{U}}^+_X[\alpha]\otimes\overline{\mathbf{U}}^+_X[\beta]$.*

Proof. By Proposition 4.3 It is enough to show that

$$\big(\mathbf{U}^+_X[\alpha]\widehat{\otimes}\mathbf{U}^+_X[\beta]\big)\cap\big(\mathbf{H}^+_X[\alpha]\otimes\mathbf{H}^+_X[\beta]\big)=\overline{\mathbf{U}}^+_X[\alpha]\otimes\overline{\mathbf{U}}^+_X[\beta],\tag{4.8}$$

where the intersection is taken in $\mathbf{H}^+_X[\alpha]\widehat{\otimes}\mathbf{H}^+_X[\beta]$. Let $V_{\alpha,\beta}$ stand for the left hand side of (4.8). The inclusion $\overline{\mathbf{U}}^+_X[\alpha]\otimes\overline{\mathbf{U}}^+_X[\beta]\subset V_{\alpha,\beta}$ is obvious since $\overline{\mathbf{U}}^+_X[\alpha]\otimes\overline{\mathbf{U}}^+_X[\beta]\subset\mathbf{U}^+_X[\alpha]\widehat{\otimes}\mathbf{U}^+_X[\beta]$ and $\overline{\mathbf{U}}^+_X[\alpha]\otimes\overline{\mathbf{U}}^+_X[\beta]\subset\mathbf{H}^+_X[\alpha]\otimes\mathbf{H}^+_X[\beta]$. For any sheaf $\mathcal{F}$ of class γ let $\mathsf{pr}_{\mathcal{F}}:\widehat{\mathbf{H}}^+_X[\gamma]\to\mathbb{C}$ be the linear form picking the coefficient of $[\mathcal{F}]$. To prove the reverse inclusion, it is enough to show that for any $\mathcal{F}$ of class α and any $\mathcal{G}$ of class β we have

$$(\mathsf{pr}_{\mathcal{F}}\otimes 1)(V_{\alpha,\beta})\subset\overline{\mathbf{U}}^+_X[\beta]\qquad\text{and}\qquad(1\otimes\mathsf{pr}_{\mathcal{G}})(V_{\alpha,\beta})\subset\overline{\mathbf{U}}^+_X[\alpha].\tag{4.9}$$

Indeed, if (4.9) holds then

$$V_{\alpha,\beta}\subset\big(\overline{\mathbf{U}}^+_X[\alpha]\otimes\mathbf{H}^+_X[\beta]\big)\cap\big(\mathbf{H}^+_X[\alpha]\otimes\overline{\mathbf{U}}^+_X[\beta]\big)=\overline{\mathbf{U}}^+_X[\alpha]\otimes\overline{\mathbf{U}}^+_X[\beta].$$

Finally, we prove (4.9). Let $v\in V_{\alpha,\beta}$, and let $\mathcal{F},\mathcal{G}$ be sheaves of class α and β respectively. Choose $m\in\mathbb{Z}$ such that $\mathcal{F},\mathcal{G}\in Coh_{\geq m}$. As $v\in\mathbf{U}^+_X[\alpha]\widehat{\otimes}\mathbf{U}^+_X[\beta]$ for any $m'<m$ we have $v\in\mathbf{U}^+_X[\alpha]\otimes\mathbf{U}^+_X[\beta]+\big(\mathbf{H}^{\not\geq m'}_X[\alpha]\widehat{\otimes}\mathbf{H}^+_X[\beta]+\mathbf{H}^+_X[\alpha]\widehat{\otimes}\mathbf{H}^{\not\geq m'}_X[\beta]\big)$ from which we deduce that $(\mathsf{pr}_{\mathcal{F}}\otimes 1)(v)\in\mathbf{U}^+_X[\beta]+\widehat{\mathbf{H}}^{\not\geq m'}_X[\beta]$ and $(1\otimes\mathsf{pr}_{\mathcal{G}})(v)\in\mathbf{U}^+_X[\alpha]+\widehat{\mathbf{H}}^{\not\geq m'}_X[\alpha]$. Equation (4.9) follows and the Lemma is proved. ✓

Let $\overline{\mathbf{U}}_X\subset\mathbf{DH}_X$ be the subalgebra generated by two copies $\overline{\mathbf{U}}^\pm_X\subset\mathbf{H}^\pm_X$ of $\overline{\mathbf{U}}^+_X$.

Corollary 4.7. *The algebra $\overline{\mathbf{U}}_X$ is isomorphic to the Drinfeld double of $\overline{\mathbf{U}}^+_X$, and the multiplication map induces an isomorphism $\overline{\mathbf{U}}^+_X\otimes\overline{\mathbf{U}}^-_X\simeq\overline{\mathbf{U}}_X$.*

Proof. Since $\overline{\mathbf{U}}^+_X$ is a topological bialgebra, by the same proof as for Proposition 3.4 we see that $\mathbf{D}\overline{\mathbf{U}}^+_X\cong\overline{\mathbf{U}}^+_X\otimes\overline{\mathbf{U}}^-_X$ and $\mathbf{D}\overline{\mathbf{U}}^+_X$ is isomorphic to the subalgebra of $\mathbf{DH}_X$ generated by $\overline{\mathbf{U}}^+_X$ and $\overline{\mathbf{U}}^-_X$. ✓

Recall that we defined for any $\nu\in\mathbb{Q}\cup\{\infty\}$ a subalgebra $\mathbf{H}^{+,(\nu)}_X$ and that we set $\mathbf{U}^{+,(\nu)}_X=\mathbf{U}^+_X\cap\mathbf{H}^{+,(\nu)}_X$. By Lemma 2.6, we have a tensor product decomposition $m:\vec{\bigotimes_\nu}\mathbf{H}^{+,(\nu)}_X\xrightarrow{\sim}\mathbf{H}^+_X$. Let $\mathbf{H}^{+,\mathsf{vec}}_X=m\big(\vec{\bigotimes}_{\nu<\infty}\mathbf{H}^{+,(\nu)}_X\big)$ be the subspace spanned by the classes of the vector bundles, so that the multiplication map $m:\mathbf{H}^{+,\mathsf{vec}}_X\otimes\mathbf{H}^{+,(\infty)}_X\to\mathbf{H}^+_X$ is an isomorphism. The following property of $\overline{\mathbf{U}}^+_X$ will be crucial for our purposes.

Lemma 4.8. *We have $\overline{\mathbf{U}}^+_X\subset\mathbf{H}^{+,\mathsf{vec}}_X\otimes\mathbf{U}^{+,(\infty)}_X$.*

Proof. Consider an element $u\in\overline{\mathbf{U}}^+_X[\alpha]$. Viewing it as an element of $m\big(\mathbf{H}^{+,\mathsf{vec}}_X\otimes\mathbf{H}^{+,(\infty)}_X\big)$ we can expand it as a finite sum $u=\sum_l u_l$ with $u_l=\sum_i u'_{l,i}\cdot u''_{l,i}$, where $u'_{l,i}\in\mathbf{H}^{+,\mathsf{vec}}_X$ and $u''_{l,i}\in\mathbf{H}^{+,(\infty)}_X\big[(0,l)\big]$ for all i,l. Let $\pi:\mathbf{H}^+_X\to\mathbf{H}^{+,\mathsf{vec}}_X$ denote the projection of the Hall algebra on its subspace. Observe that as any coherent sheaf $\mathcal{F}$ has a unique maximal torsion subsheaf, we get

$$(\pi\otimes 1)\Delta_{\alpha-(0,l),(0,l)}(u)=v^{\langle\alpha-(0,l),(0,l)\rangle}\sum_i u'_{l,i}\otimes u''_{l,i}.\tag{4.10}$$

On the other hand, by Lemma 4.6 we have

$$\Delta_{\alpha-(0,l),(0,l)}(u) \in \overline{\mathbf{U}}_X^+\big[\alpha-(0,l)\big]\otimes \overline{\mathbf{U}}_X^+\big[(0,l)\big] = \overline{\mathbf{U}}_X^+\big[\alpha-(0,l)\big]\otimes \mathbf{U}_X^+\big[(0,l)\big].$$

But then from (4.10) we obtain $\sum_i u'_{l,i}\otimes u''_{l,i} \in \mathbf{H}_X^{+,\mathsf{vec}}\otimes \mathbf{U}_X^+\big[(0,l)\big]$for all l, and hence $u \in \mathbf{H}_X^{+,\mathsf{vec}}\otimes \mathbf{U}_X^{+,(\infty)}$ as wanted. ✓

After these preliminaries we are now ready to prove that the multiplication map induces an isomorphism $\vec{\bigotimes}_\nu \mathbf{U}_X^{+,(\nu)} \simeq \mathbf{U}_X^+$. For any $\nu \in \mathbb{Q}\cup\{\infty\}$ there exists $\gamma \in SL(2,\mathbb{Z})$ such that $\gamma(\nu)=\infty$. Recall that the group $\widehat{SL}(2,\mathbb{Z})$ acts on $\mathbf{DH}_X$ and preserves $\mathbf{U}_X$. Moreover, this action is compatible with the decomposition $\mathbf{DH}_X \simeq \vec{\bigotimes}_\mu \mathbf{H}_X^{+,(\mu)}\otimes \vec{\bigotimes}_\mu \mathbf{H}_X^{-,(\mu)}$ (i.e. it permutes the subalgebras $\mathbf{H}_X^{\pm,(\mu)}$). Hence, using Corollary 4.7 and Lemma 4.8 we obtain the chain of inclusions

$$\gamma(\mathbf{U}_X^+) \subset \overline{\mathbf{U}}_X^+\otimes\overline{\mathbf{U}}_X^- \subset \Big(\vec{\bigotimes_{\mu<\infty}} \mathbf{H}_X^{+,(\mu)}\otimes \mathbf{U}_X^{+,(\infty)}\Big)\otimes\Big(\vec{\bigotimes_{\mu<\infty}} \mathbf{H}_X^{-,(\mu)}\otimes \mathbf{U}_X^{-,(\infty)}\Big). \tag{4.11}$$

But then, applying γ^{-1} to (4.11) and using the equality $\gamma^{-1}\big(\mathbf{U}_X^{+,(\infty)}\big) = \mathbf{U}_X^{+,(\nu)}$ we see that $\mathbf{U}_X^+ \subset \vec{\bigotimes}_{\mu<\nu} \mathbf{H}_X^{+,(\mu)}\otimes \mathbf{U}_X^{+,(\nu)}\otimes \vec{\bigotimes}_{\mu>\nu}\mathbf{H}_X^{+,(\mu)}$. As this is true for all ν, we get $\mathbf{U}_X^+ \subset \vec{\bigotimes}_\mu \mathbf{U}_X^{+,(\mu)}$ and finally $\mathbf{U}_X^+ = \vec{\bigotimes}_\mu \mathbf{U}_X^{+,(\mu)}$. Of course, this also proves the equality $\mathbf{U}_X^- = \vec{\bigotimes}_\mu \mathbf{U}_X^{-,(\mu)}$. The second statement in Theorem 4.5 is now a consequence of Corollary 4.7 and the next result :

Lemma 4.9. *The two algebras* $\mathbf{U}_X^+$ *and* $\overline{\mathbf{U}}_X^+$ *coincide.*

Proof. Recall that the condition $u \in \overline{\mathbf{U}}_X^+[\alpha]$ means that $u\in \mathbf{H}_X^+[\alpha]$ and for all n there exists $u_n \in \mathbf{U}_X^+[\alpha]$ such that $\mathsf{jet}_n(u)=\mathsf{jet}_n(u_n)$. But note that for $n \gg 0$ we have $u=\mathsf{jet}_n(u)$ and as $\mathbf{U}_X^+\simeq \vec{\bigotimes}_\nu \mathbf{U}_X^{+,(\nu)}$, we get $\mathsf{jet}_n(u_n)\in \mathbf{U}_X^+[\alpha]$. Therefore, $\mathbf{U}_X^+ = \overline{\mathbf{U}}_X^+$ and as a corollary,

$$\overline{\mathbf{U}}_X = \mathbf{D}\overline{\mathbf{U}}_X^+ = \overline{\mathbf{U}}_X^+\otimes\overline{\mathbf{U}}_X^- = \mathbf{U}_X^+\otimes\mathbf{U}_X^- = \mathbf{D}\mathbf{U}_X^+ = \mathbf{U}_X.$$

This concludes the proof of Theorem 4.5. ✓

4.5. We finish this section with several important computations regarding $\mathbf{U}_X^+$. They will be used in a crucial way in the next section. Let us set, for $i \geq 1$

$$c_i(X) = \#X(\mathbb{F}_{q^i})\, v^i[i]/i.$$

Lemma 4.10. *For any* $\mathbf{x}=(q,p)\in\mathbf{Z}^+$ *we have*

$$(T_\mathbf{x},T_\mathbf{x}) = \frac{c_r(X)}{(v^{-1}-v)},$$

where $r=\mathsf{gcd}(q,p)\in\mathbb{N}$.

Proof. Using the $SL(2,\mathbb{Z})$ action, we may restrict ourselves to the case of $\mathbf{x}=(0,r)$ with $r>0$. We have

$$\big(T_{(0,r)},T_{(0,r)}\big) = \sum_{d|r}\sum_{x:\ \mathsf{deg}(x)=d}\big(T_{r,x}^{(\infty)},T_{r,x}^{(\infty)}\big).$$

If x is of degree d, it follows from Proposition 4.1, iii) that

$$\big(T_{r,x}^{(\infty)},T_{r,x}^{(\infty)}\big) = \frac{d[r]^2}{r(q^r-1)} = \frac{v^r[r]d}{r(v^{-1}-v)}.$$

The statement of the Lemma follows from the equation

$$\sum_{d|r} \sum_{x:\ \mathsf{deg}(x)=d} d = \#X(\mathbb{F}_{q^r}).$$

✓

We now turn to the coproduct. Define elements $\Theta_{\mathbf{x}} \in \mathbf{U}_X$ by equating the coefficients of the following generating series:

$$\sum_i \Theta_{i\mathbf{x}_0} s^i = \exp\Big((v^{-1}-v)\sum_{r\geq 1} T_{r\mathbf{x}_0} s^r\Big), \tag{4.12}$$

for any $\mathbf{x}_0 \in \mathbf{Z}^*$ such that $\mathsf{deg}(\mathbf{x}_0) = 1$.

Lemma 4.11. *For any $p \in \mathbb{Z}$ we have:*

$$\Delta(T_{(1,p)}) = T_{(1,p)} \otimes 1 + \sum_{l \geq 0} \Theta_{(0,l)} \otimes T_{(1,p-l)}.$$

Proof. Up to a twist by a line bundle, it is enough to consider $T_{(1,0)} = \mathbf{1}^{\mathsf{ss}}_{(1,0)}$. By the proof of Proposition 4.3, we have

$$\begin{aligned}
\Delta_{(1,-n),(0,n)}(\mathbf{1}_{(1,0)}) &= v^n \mathbf{1}_{(1,-n)} \otimes \mathbf{1}_{(0,n)},\\
\Delta_{(0,n),(1,-n)}(\mathbf{1}_{(1,0)}) &= v^{-n} \mathbf{1}_{(0,n)} \otimes \mathbf{1}_{(1,-n)},\\
\Delta_{(0,n),(0,m)}(\mathbf{1}_{(0,n+m)}) &= \mathbf{1}_{(0,n)} \otimes \mathbf{1}_{(0,m)}
\end{aligned}$$

and moreover

$$\mathbf{1}^{\mathsf{ss}}_{(1,0)} = \sum_{n \geq 0} v^n \mathbf{1}_{(1,-n)} \chi_n,$$

where $\chi_n = \sum_{r>0} (-1)^r \sum_{l_1+\cdots+l_r=n} \mathbf{1}_{(0,l_1)} \cdots \mathbf{1}_{(0,l_r)}$. Denote by

$$\mathbf{1}(s) = \sum_{l \geq 0} \mathbf{1}_{(0,l)} s^l, \qquad \chi(s) = \sum_{l \geq 0} \chi_l s^l$$

the generating functions of $\mathbf{1}_{(0,n)}$ and $\{\chi_n\}$. It is easy to see that the elements $\{\chi_n\}$ are completely determined by the relations $\sum_{i+j=l} \mathbf{1}_{(0,i)} \chi_j = \delta_{l,0}$, which can be rewritten in the form $\mathbf{1}(s)\chi(s) = 1$. In particular, from the formula for the coproduct we have $\Delta(\mathbf{1}(s)) = \mathbf{1}(s) \otimes \mathbf{1}(s)$ from which we deduce that $\Delta\big(\chi(s)\big) = \chi(s) \otimes \chi(s)$, i.e $\Delta_{(0,n),(0,m)}(\chi_{n+m}) = \chi_n \otimes \chi_m$. This implies that

$$\Delta_{(1,-l),(0,l)}\big(\mathbf{1}^{ss}_{(1,0)}\big) = \sum_{k \geq 0} v^{l+k} \mathbf{1}_{(1,-l-k)} \chi_k \otimes \big(\mathbf{1}_{(0,l)} + \mathbf{1}_{(0,l-1)}\chi_1 + \cdots + \chi_l\big).$$

Using $\sum_{i+j=l} \mathbf{1}_{(0,i)} \chi_j = \delta_{l,0}$, we get $\Delta_{(1,-l),(0,l)}\big(\mathbf{1}^{\mathsf{ss}}_{(1,0)}\big) = \delta_{l,0} \mathbf{1}^{\mathsf{ss}}_{(1,0)} \otimes 1$. A similar computation shows that

$$\Delta_{(0,l),(1,-l)}(\mathbf{1}^{\mathsf{ss}}_{(1,0)}) = \sum_{k \geq 0} \big(v^{k-l}\mathbf{1}_{(0,l)} + v^{2+k-l}\mathbf{1}_{(0,l-1)}\chi_1 + \cdots + v^{k+l}\chi_l\big) \otimes \mathbf{1}_{(1,-l-k)}\chi_k.$$

Hence, setting $\theta_l = \sum_{k=0}^{l} v^{2k-l} \mathbf{1}_{(0,l-k)} \chi_k$ we obtain

$$\Delta(\mathbf{1}^{\mathsf{ss}}_{(1,0)}) = \mathbf{1}^{\mathsf{ss}}_{(1,0)} \otimes 1 + \sum_{l \geq 0} \theta_l \otimes \mathbf{1}^{\mathsf{ss}}_{(1,-l)}.$$

Finally, we claim that the elements θ_l can be characterized through the relation $\sum_{l \geq 0} \theta_l s^l = \exp\big((v^{-1}-v)\sum_{r \geq 1} T_{(0,r)} s^r\big)$. To see this, note that by definition

$\sum_{l\geq 0} \mathbf{1}_{(0,l)} s^l = \exp\left(\sum_{r\geq 1} \frac{T_{(0,r)}}{[r]} s^r\right)$, hence $\sum_{l\geq 0} \chi_l s^l = \exp\left(-\sum_{r\geq 1} \frac{T_{(0,r)}}{[r]} s^r\right)$. But then

$$\begin{aligned}\sum_{l\geq 0} \theta_l s^l =& \mathbf{1}(v^{-1}s)\chi(vs) = \exp\left(\sum_{r\geq 1} v^{-r}\frac{T_{(0,r)}}{[r]} s^r - \sum_{r\geq 1} v^{r}\frac{T_{(0,r)}}{[r]} s^r\right)\\ =& \exp\left((v^{-1}-v)\sum_{r\geq 1} T_{(0,r)} s^r\right)\end{aligned}$$

as desired. ✓

The final computation which we shall need is the following. Set

$$\mathbf{1}^{\mathsf{vec}}_\alpha = \sum_{\substack{\mathcal{F}\ \textit{vec. bdle}\\ \overline{\mathcal{F}}=\alpha}} [\mathcal{F}] \in \widehat{\mathbf{U}}^+_X.$$

Lemma 4.12. *For any $n \geq 0$ and for any $\alpha = (r,d) \in \mathbf{Z}^+$ we have*

$$\big[T_{(0,n)}, \mathbf{1}_\alpha\big] = c_n(X)\frac{v^{rn}-v^{-rn}}{v^n - v^{-n}} \mathbf{1}_{\alpha+(0,n)}, \tag{4.13}$$

$$\big[T_{(0,n)}, \mathbf{1}^{\mathsf{vec}}_\alpha\big] = c_n(X)\frac{v^{rn}-v^{-rn}}{v^n - v^{-n}} \mathbf{1}^{\mathsf{vec}}_{\alpha+(0,n)}. \tag{4.14}$$

Proof. Since $\mathbf{1}_\alpha = \sum_{d\geq 0} v^{\langle \alpha,(0,d)\rangle}\mathbf{1}^{\mathsf{vec}}_{\alpha-(0,d)}\mathbf{1}_{(0,d)}$ and $\big[T_{(0,n)}, \mathbf{1}_{(0,d)}\big] = 0$ for all n and d, the equation (4.13) is a consequence of (4.14). We shall thus only deal with (4.14).

Assume first that $\mathsf{rank}(\alpha) = 1$. Up to twisting by a line bundle, we may assume that $\alpha = (1,0)$. Note that $\mathbf{1}^{\mathsf{vec}}_{(1,0)} = T_{(1,0)}$. There exist elements $S_0, \ldots, S_n$ with S_i belonging to the algebra generated by $T_{(0,1)}, \ldots, T_{(0,i)}$ such that $T_{(0,n)}T_{(1,0)}$ is equal to a linear combination

$$T_{(0,n)}T_{(1,0)} = \sum_{i=0}^{n} T_{(1,n-i)}S_i, \tag{4.15}$$

We first compute S_n. Let us write $T_{(0,n)} = \sum_{\mathcal{T}} w_{\mathcal{T}}[\mathcal{T}]$ and $S_n = \sum_{\mathcal{T}} u_{\mathcal{T}}[\mathcal{T}]$, for some scalars $w_{\mathcal{T}}, u_{\mathcal{T}} \in K$. Observe that a term of the form $[\mathcal{L}\oplus\mathcal{T}]$, for a line bundle $\mathcal{L}$ of degree zero and a torsion sheaf $\mathcal{T}$ of degree n, only appears on the right hand side of (4.15) in $T_{(1,0)}S_n$, and with a coefficient equal to $u_{\mathcal{T}}v^{-n}$. On the other hand, the coefficient of $[\mathcal{L}\oplus\mathcal{T}]$ in the left hand side is equal to $v^n w_{\mathcal{T}} F^{\mathcal{L}\oplus\mathcal{T}}_{\mathcal{T},\mathcal{L}} = v^{-n}w_{\mathcal{T}}$. Hence $u_{\mathcal{T}} = w_{\mathcal{T}}$ for all $\mathcal{T}$ and $S_n = T_{(0,n)}$.

Now we show that $S_i = 0$ for $i \neq 0, n$. By Proposition 2.2, $\Delta\big([T_{(0,n)}, T_{(1,0)}]\big) = \big[\Delta(T_{(0,n)}), \Delta(T_{(1,0)})\big]$. By Proposition 4.1, ii), $\Delta\big(T_{(0,n)}\big) = T_{(0,n)}\otimes 1 + 1\otimes T_{(0,n)}$. Let $C = \Delta\big([T_{(0,n)}, T_{(1,0)}]\big)$. From Lemma 4.11 we deduce the formula

$$C = \big[T_{(0,n)}, T_{(1,0)}\big]\otimes 1 + 1 \otimes \big[T_{(0,n)}, T_{(1,0)}\big] + \sum_{l\geq 1}\Theta_{(0,l)}\otimes\big[T_{(0,n)}, T_{(1,-l)}\big]. \tag{4.16}$$

Let i_0 be the maximal value of i distinct from n for which $S_i \neq 0$. Note that $\Delta_{(1,n-i_0),(0,i_0)}\big(T_{(1,n-i)}S_i\big) = 0$ if $i < i_0$, while we have $\Delta_{(1,n-i_0),(0,i_0)}\big(T_{(1,n-i_0)}S_{i_0}\big) = v^n T_{(1,n-i_0)}\otimes S_{i_0}$. But on the other hand, for any $j > 0$, (4.16) implies that $\Delta_{(1,n-j),(0,j)}\big([T_{(0,n)}, T_{(1,0)}]\big) = 0$. Hence $i_0 = 0$, $\big[T_{(0,n)}, T_{(1,0)}\big] = z_0 T_{(1,n)}$ for some $z_0 \in K$. In order to determine the value of z_0 we compute the scalar product $\big(T_{(0,n)}T_{(1,0)}, \mathbf{1}_{(1,n)}\big)$ in two different ways. By Proposition 4.1 ii), $T_{(0,n)}$ is orthogonal to the subalgebra generated by $T_{(0,i)}$ for $i < n$. Hence, using (4.1) and (4.4), we obtain $\mathbf{1}_{(1,n)} = T_{(1,n)} + \frac{v^n}{[n]} T_{(1,0)}T_{(0,n)} + u$ where $u \in \big(KT_{(1,n)} \oplus KT_{(1,0)}T_{(0,n)}\big)^{\perp}$, and using Lemma 4.10 we get

$$
\begin{aligned}
\big(T_{(0,n)}T_{(1,0)}, \mathbf{1}_{(1,n)}\big) &= \frac{v^n}{[n]}\big(T_{(1,0)}T_{(0,n)}, T_{(1,0)}T_{(0,n)}\big) + z_0\big(T_{(1,n)}, T_{(1,n)}\big) \\
&= \frac{v^n}{[n]}\frac{c_n(X)c_1(X)}{(v^{-1}-v)^2} + z_0\frac{c_1(X)}{v^{-1}-v}.
\end{aligned}
\tag{4.17}
$$

On the other hand, we have $\big(T_{(0,n)}T_{(1,0)}, \mathbf{1}_{(1,n)}\big) = \big(T_{(0,n)} \otimes T_{(1,0)}, \Delta(\mathbf{1}_{(1,n)})\big)$ and by (4.5) we have $\Delta(\mathbf{1}_{(1,n)}) = \frac{v^{-n}}{[n]} T_{(0,n)} \otimes T_{(1,0)} + u'$ where $u' \in \big(KT_{(0,n)} \otimes T_{(1,0)}\big)^{\perp}$. It follows that

$$
\big(T_{(0,n)}T_{(1,0)}, \mathbf{1}_{(1,n)}\big) = \frac{v^{-n}}{[n]}\big(T_{(0,n)}, T_{(0,n)}\big)\big(T_{(1,0)}, T_{(1,0)}\big) = \frac{v^{-n}}{[n]}\frac{c_1(X)c_n(X)}{(v^{-1}-v)^2}.
\tag{4.18}
$$

Combining (4.17) and (4.18) we finally obtain $z_0 = c_n(X)$ as wanted.

Now let $r = \mathsf{rank}(\alpha)$ be arbitrary. Repeating the argument above, we have $\big[T_{(0,n)}, \mathbf{1}^{\mathsf{vec}}_{(r,d)}\big] \in \widehat{\mathbf{U}}^{\mathsf{vec}}_X$. Let $\mathcal{V}$ be a vector bundle of class $\alpha + (0,n)$ and let $\mathcal{T}$ be a torsion sheaf of degree n. The coefficient of $[\mathcal{V}]$ in $\big[[\mathcal{T}], \mathbf{1}^{\mathsf{vec}}_{\alpha}\big]$ is easily seen to be equal to $v^{rn}\big|\mathrm{Hom}^{\mathsf{surj}}(\mathcal{V}, \mathcal{T})\big|/a_{\mathcal{T}}$, where $\mathrm{Hom}^{\mathsf{surj}}(\mathcal{V}, \mathcal{T})$ stands for the set of surjective maps $\mathcal{V} \twoheadrightarrow \mathcal{T}$. By the inclusion-exclusion principle, we have

$$
\begin{aligned}
\big|\mathrm{Hom}^{\mathsf{surj}}(\mathcal{V}, \mathcal{T})\big| &= \big|\mathrm{Hom}(\mathcal{V}, \mathcal{T})\big| - \sum_{\mathcal{T}' \subsetneq \mathcal{T}} \big|\mathrm{Hom}(\mathcal{V}, \mathcal{T}')\big| + \sum_{\mathcal{T}'' \subsetneq \mathcal{T}' \subsetneq \mathcal{T}} \big|\mathrm{Hom}(\mathcal{V}, \mathcal{T}'')\big| - \cdots \\
&= v^{-2rn} - \sum_{\mathcal{T}' \subsetneq \mathcal{T}} v^{-2r\mathsf{deg}(\mathcal{T}')} + \sum_{\mathcal{T}'' \subsetneq \mathcal{T}' \subsetneq \mathcal{T}} v^{-2r\mathsf{deg}(\mathcal{T}'')} - \cdots
\end{aligned}
$$

(the sum is finite since there are only finitely many subsheaves of $\mathcal{T}$). The above expression only depends on $\mathcal{T}$ and the rank r. Hence $\big[T_{(0,n)}, \mathbf{1}^{\mathsf{vec}}_{\alpha}\big] = u_r \mathbf{1}^{\mathsf{vec}}_{\alpha+(0,n)}$ for some $u_r \in K$, which remains to be determined. For this, we use the iterated coproduct map $\Delta_{1,\ldots,1}$. We have, by (4.5)

$$
\Delta_{1,\ldots,1}\big(\mathbf{1}_{(r,l)}\big) = \sum_{l_1+\cdots+l_n=l} v^{\sum_{i<j}(l_j-l_i)} \mathbf{1}_{(1,l_1)} \otimes \cdots \otimes \mathbf{1}_{(1,l_r)},
\tag{4.19}
$$

while by Proposition 4.1, ii),

$$
\begin{aligned}
&\Delta_{1,\ldots,1}\big([T_{(0,n)}, \mathbf{1}_{(r,d)}]\big) \\
&\quad= \Big[\sum_{j=1}^{r} 1 \otimes \cdots \otimes T_{(0,n)} \otimes \cdots \otimes 1, \Delta_{1,\ldots,1}\big(\mathbf{1}_{(r,d)}\big)\Big] \\
&\quad= u_1 \sum_{j=1}^{r} \sum_{d_1+\cdots+d_n=d} v^{\sum_{i<j}(d_j-d_i)} \mathbf{1}_{(1,d_1)} \otimes \cdots \otimes \mathbf{1}_{(1,d_j+n)} \otimes \cdots \otimes \mathbf{1}_{(1,d_r)}.
\end{aligned}
\tag{4.20}
$$

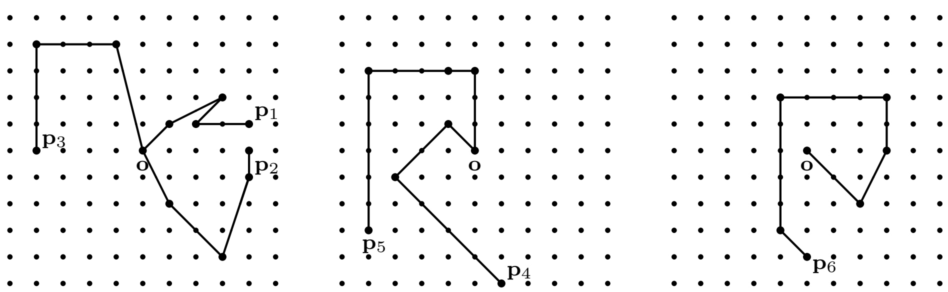
Comparing (4.19) with (4.20) and using the case $r = 1$ treated above we get

$$
u_r = u_1 v^{(r+1)n} \sum_{j=1}^{r} v^{-2jn} = c_n(X)\frac{v^{rn} - v^{-rn}}{v^n - v^{-n}}
$$

as wanted. We are done. ✓

5. The algebra $\boldsymbol{\mathcal{E}}_{\sigma,\bar{\sigma}}$

5.1. The aim of this section is to give a presentation for $\mathbf{U}_X$ by generators and relations. Since it is convenient to depict elements of $\mathbf{U}_X$ graphically, a few notational preparations are in order. Let $\mathbf{o}$ stand for the origin in $\mathbf{Z}$. By a path in $\mathbf{Z}$ we shall mean a finite sequence $\mathbf{p} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r)$ of non-zero elements of $\mathbf{Z}$, which we represent as the piecewise-linear curve in $\mathbf{Z}$ joining $\mathbf{o}, \mathbf{x}_1, \mathbf{x}_1 + \mathbf{x}_2, \ldots, \mathbf{x}_1 + \cdots + \mathbf{x}_r$. Let $\widehat{\mathbf{xy}} \in [0, 2\pi[$ denote the angle between the segments $\mathbf{ox}$ and $\mathbf{oy}$. A path $\mathbf{p} = (\mathbf{x}_1, \ldots, \mathbf{x}_r)$ will be called *convex* if $0 \le \widehat{\mathbf{x}_1\mathbf{x}_2} \le \widehat{\mathbf{x}_1\mathbf{x}_3} \le \cdots \le \widehat{\mathbf{x}_1\mathbf{x}_r} < 2\pi$. Put $L_0 = \mathbb{N}(0,-1)$ and let $\mathbf{Conv}'$ be the collection of all convex paths $\mathbf{p} = (\mathbf{x}_1, \ldots, \mathbf{x}_r)$ satisfying $\widehat{\mathbf{x}_1 L_0} \ge \widehat{\mathbf{x}_2 L_0} \ge \cdots \ge \widehat{\mathbf{x}_r L_0} \ge 0$. Two convex paths $\mathbf{p} = (\mathbf{x}_1, \ldots, \mathbf{x}_r)$ and $\mathbf{q} = (\mathbf{y}_1, \ldots, \mathbf{y}_s)$ in $\mathbf{Conv}'$ will be called equivalent if $\{\mathbf{x}_1, \ldots, \mathbf{x}_r\} = \{\mathbf{y}_1, \ldots, \mathbf{y}_s\}$, i.e. if $\mathbf{p}$ is the result of permuting together several segments of $\mathbf{q}$ of the same slope. We denote by $\mathbf{Conv}$ the set of equivalence classes of convex paths in $\mathbf{Conv}'$. We shall only consider convex paths up to equivalence, and we shall simply refer to elements of $\mathbf{Conv}$ as "paths". We also introduce $\mathbf{Conv}^+$ (resp. $\mathbf{Conv}^-$) as the set of convex paths $(\mathbf{x}_1, \ldots, \mathbf{x}_s)$ satisfying $\widehat{\mathbf{x}_1 L_0} \ge \cdots \ge \widehat{\mathbf{x}_s L_0} \ge \pi$ (resp. $\pi > \widehat{\mathbf{x}_1 L_0} \ge \cdots \ge \widehat{\mathbf{x}_s L_0} \ge 0$). Concatenation of paths then yields an identification $\mathbf{Conv} \simeq \mathbf{Conv}^+ \times \mathbf{Conv}^-$.



$\mathbf{p}_1$ is not convex
$\mathbf{p}_2 \in \mathbf{Conv}^+$, $\mathbf{p}_3 \in \mathbf{Conv}^-$

$\mathbf{p}_4$ is convex but $\mathbf{p}_4 \notin \mathbf{Conv}$
$\mathbf{p}_5 \in \mathbf{Conv}$, but $\mathbf{p}_5 \notin \mathbf{Conv}^{\pm}$

$\mathbf{p}_6$ is not convex

Figure 3. Examples of paths

Observe that distinct paths could give rise to the same polygonal line in $\mathbf{Z}$: for instance $\mathbf{p} = \big((0,1),(0,1)\big)$ and $\mathbf{p}' = \big((0,2)\big)$. To a path $\mathbf{p} = (\mathbf{x}_1, \ldots, \mathbf{x}_r)$ we associate the element $T_{\mathbf{p}} := T_{\mathbf{x}_1} \cdots T_{\mathbf{x}_r} \in \mathbf{U}_X$. This expression is well-defined since $\mathbf{U}_X^{\pm,(\mu)}$ is commutative for all slopes μ. Moreover, it follows from Theorem 4.5 that the set of elements $\big\{T_{\mathbf{p}} \,|\mathbf{p} \in \mathbf{Conv}^{\pm}\big\}$ is a K-basis of $\mathbf{U}_X^{\pm}$.

Remark 5.1. The group $SL(2,\mathbb{Z})$ naturally acts on the set of paths. For any ray L in $\mathbf{Z}$ starting at the origin we can define the set $\mathbf{Conv}^L$ by replacing L_0 by L, and any $\sigma \in SL(2,\mathbb{Z})$ maps bijectively $\mathbf{Conv}^L$ to $\mathbf{Conv}^{\sigma(L)}$. In particular, $\{T_{\mathbf{p}} \,|\mathbf{p} \in \mathbf{Conv}^L\}$ is a K-basis of $\mathbf{U}_X$ for any L. Such a choice of L corresponds to a choice of a t-structure in the derived category $D^b\big(Coh(X)\big)$.

For $\mathbf{x}, \mathbf{y} \in \mathbf{Z}^*$ we let $\boldsymbol{\Delta}_{\mathbf{x},\mathbf{y}}$ stand for the triangle with corners $\mathbf{o}, \mathbf{x}, \mathbf{x}+\mathbf{y}$. If $\widehat{\mathbf{xy}} < \pi$ then $T_{\mathbf{y}} T_{\mathbf{x}}$ (corresponding to the path $(\mathbf{y}, \mathbf{x})$) can be written as a linear combination of elements $T_{\mathbf{p}}$ where $\mathbf{p}$ runs through the set of *convex* paths lying in $\boldsymbol{\Delta}_{\mathbf{x},\mathbf{y}}$. Indeed, this is a reformulation of Remark 2.7 when $\mathbf{x}, \mathbf{y} \in \mathbf{Z}^+$, and follows for an arbitrary pair $(\mathbf{x}, \mathbf{y})$ by $SL(2,\mathbb{Z})$-invariance of $\mathbf{U}_X$.

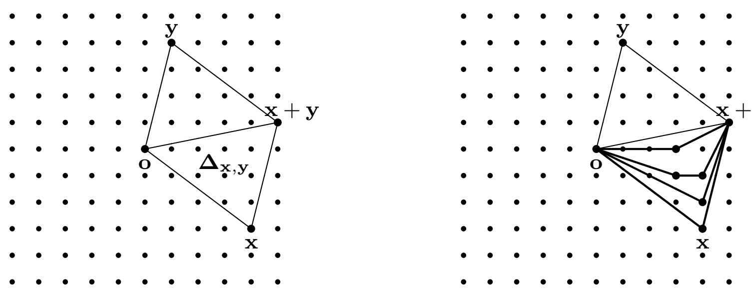


Figure 4. The triangle $\mathbf{\Delta_{x,y}}$ and some convex paths in it

Several arguments in this Section are based on Pick's formula, which we recall : for any pair of non-colinear points $\mathbf{x}, \mathbf{y} \in \mathbf{Z}$:

$$|\det(\mathbf{x},\mathbf{y})| = \mathsf{deg}(\mathbf{x}) + \mathsf{deg}(\mathbf{y}) + \mathsf{deg}(\mathbf{x}+\mathbf{y}) - 2 + 2\#(\Delta_{\mathbf{x},\mathbf{y}} \cap \mathbf{Z}). \tag{5.1}$$

5.2. We shall describe $\mathbf{U}_X$ as an abstract algebra of paths modulo a minimal set of "straightening" relations given below. If $\mathbf{x} = (q,p) \in \mathbf{Z}^*$ we write $\mathsf{deg}(\mathbf{x}) = \mathsf{gcd}(q,p) \in \mathbb{N}$. For non-collinear $\mathbf{x}, \mathbf{y} \in \mathbf{Z}^*$ we set $\epsilon_{\mathbf{x},\mathbf{y}} = \mathsf{sign}\big(\det(\mathbf{x},\mathbf{y})\big) \in \{\pm 1\}$.

Definition 5.2. Fix $\sigma, \bar\sigma \in \mathbb{C}^*$ with $\sigma, \bar\sigma \notin \{\pm 1\}$ and set $\nu = (\sigma\bar\sigma)^{-1/2}$ and

$$c_i(\sigma,\bar\sigma) = (\sigma^{i/2} - \sigma^{-i/2})(\bar\sigma^{i/2} - \bar\sigma^{-i/2})[i]_\nu / i.$$

Let $\boldsymbol{\mathcal{E}}_{\sigma,\bar\sigma}$ be the $\mathbb{C}$-algebra generated by $\{t_{\mathbf{x}} \mid \mathbf{x} \in \mathbf{Z}^*\}$ modulo the following set of relations

i) If $\mathbf{x}, \mathbf{x}'$ belong to the same line in $\mathbf{Z}$ then

$$[t_{\mathbf{x}}, t_{\mathbf{x}'}] = 0,$$

ii) Assume that $\mathbf{x}, \mathbf{y} \in \mathbf{Z}^*$ are such that $\mathsf{deg}(\mathbf{x}) = 1$ and that $\mathbf{\Delta_{x,y}}$ has no interior lattice point. Then

$$[t_{\mathbf{y}}, t_{\mathbf{x}}] = \epsilon_{\mathbf{x},\mathbf{y}} c_{\mathsf{deg}(\mathbf{y})}(\sigma,\bar\sigma) \frac{\theta_{\mathbf{x}+\mathbf{y}}}{\nu^{-1} - \nu}$$

where the elements $\theta_{\mathbf{z}}$, $\mathbf{z} \in \mathbf{Z}^*$ are defined by the following generating series

$$\sum_i \theta_{i\mathbf{x}_0} s^i = \exp\left((\nu^{-1} - \nu) \sum_{r \geq 1} t_{r\mathbf{x}_0} s^r \right), \tag{5.2}$$

for any $\mathbf{x}_0 \in \mathbf{Z}^*$ such that $\mathsf{deg}(\mathbf{x}_0) = 1$.

Observe that $\theta_{\mathbf{z}} = (\nu^{-1} - \nu) t_{\mathbf{z}}$ whenever $\mathsf{deg}(\mathbf{z}) = 1$. We also denote by $\boldsymbol{\mathcal{E}}^{\pm}_{\sigma,\bar\sigma}$ the subalgebra of $\boldsymbol{\mathcal{E}}_{\sigma,\bar\sigma}$ generated by $t_{\mathbf{x}}$ for $\mathbf{x} \in \mathbf{Z}^{\pm}$.

Lemma 5.3. *For any $\gamma \in SL(2,\mathbb{Z})$ we have an algebra automorphism $\Phi_\gamma : \boldsymbol{\mathcal{E}}_{\sigma,\bar\sigma} \to \boldsymbol{\mathcal{E}}_{\sigma,\bar\sigma}$ given by the formula $\Phi_\gamma(t_{\mathbf{x}}) = t_{\gamma(\mathbf{x})}$ for any $\mathbf{x} \in \mathbf{Z}^*$.*

Proof. Obvious. ✓

5.3. Now let $\#X(\mathbb{F}_{q^r})$ stand for the number of rational points of X over $\mathbb{F}_{q^r}$ and recall that $v = q^{-1/2}$. By a theorem of Hasse (see e.g. [Ha], App. C) there exist conjugate algebraic numbers $\sigma, \overline{\sigma}$, satisfying $\sigma\overline{\sigma} = q$, such that

$$\#X(\mathbb{F}_{q^r}) = q^r + 1 - (\sigma^r + \overline{\sigma}^r)$$

for any $r \geq 1$. These numbers $\sigma, \overline{\sigma}$ are the eigenvalues of the Frobenius automorphism acting on $H^1\big(X_{\overline{\mathbb{F}_q}}, \overline{\mathbb{Q}_l}\big)$. Note that $c_i(\sigma,\bar{\sigma}) = v^i[i]\#X(\mathbb{F}_{q^i})/i = c_i(X)$.

Theorem 5.4. *The assignment* $\Omega : t_{\mathbf{x}} \mapsto T_{\mathbf{x}}$ *for* $\mathbf{x} \in \mathbf{Z}^*$ *extends to an isomorphism* $\Omega : \boldsymbol{\mathcal{E}}_{\sigma,\bar{\sigma}} \simeq \mathbf{U}_X \otimes_K \mathbb{C}$.

Example 5.5. Before giving the proof of this theorem, let us illustrate the use of the straightening relation ii). We shall compute $t_{(1,2)}t_{(1,-1)}$, which corresponds to the path $\big((1,2),(1,-1)\big)$ not belonging to **Conv**. By ii) we have $\big[t_{(0,1)}, t_{(1,1)}\big] = c_1 t_{(1,2)}$, hence

$$\begin{aligned}\big[t_{(1,2)}, t_{(1,-1)}\big] &= \frac{1}{c_1}\Big\{ \big[[t_{(0,1)}, t_{(1,-1)}], t_{(1,1)}\big] + \big[t_{(0,1)}, [t_{(1,1)}, t_{(1,-1)}]\big] \Big\} \\ &= [t_{(1,0)}, t_{(1,1)}] + \big[t_{(0,1)}, t_{(2,0)} + \frac{1}{2}(v^{-1}-v)t_{(1,0)}^2\big]\end{aligned}$$

where we have used ii) in each term, and the relation

$$\frac{\theta_{(2,0)}}{v^{-1}-v} = t_{(2,0)} + \frac{1}{2}(v^{-1}-v)t_{(1,0)}^2.$$

Now, by ii) again we have $\big[t_{(1,0)}, t_{(1,1)}\big] = -c_1 t_{(2,1)}$, $\big[t_{(0,1)}, t_{(2,0)}\big] = c_2 t_{(2,1)}$ and $\big[t_{(0,1)}, t_{(1,0)}\big] = t_{(1,1)}$. Hence, we obtain

$$\begin{aligned}\big[t_{(1,2)}, t_{(1,-1)}\big] &= (c_2 - c_1)t_{(2,1)} + \frac{1}{2}(v^{-1}-v)c_1(t_{(1,1)}t_{(1,0)} + t_{(1,0)}t_{(1,1)}) \\ &= (c_2 - c_1)t_{(2,1)} + \frac{1}{2}(v^{-1}-v)c_1\big(c_1 t_{(2,1)} + 2t_{(1,0)}t_{(1,1)}\big)\end{aligned}$$

Gathering terms, we get

$$t_{(1,2)}t_{(1,-1)} = t_{(1,-1)}t_{(1,2)} + \frac{1}{2}(v^{-1}-v)c_1 t_{(1,0)}t_{(1,1)} + c_1\big([3] - vc_1\big)t_{2,1}.$$

Observe that all three paths $\big((1,-1),(1,2)\big)$, $\big((1,0),(1,1)\big)$, $\big((2,1)\big)$ belong to **Conv**.

We begin the proof of Theorem 5.4. Let us first show that the map Ω is well-defined, i.e. that relations i) and ii) hold in $\mathbf{U}_X$. By the $SL(2,\mathbb{Z})$-invariance of $\boldsymbol{\mathcal{E}}_{\sigma,\bar{\sigma}}$ and $\mathbf{U}_X$ it is enough to prove relation i) for $\mathbf{x} = (0,r), \mathbf{x}' = (0,r')$. The subalgebra $\mathbf{H}_X^{(\infty)}$ of $\mathbf{H}_X$ is stable under the coproduct (as any subsheaf or quotient of a torsion sheaf is again a torsion sheaf) and can be described as the product over all points $x \in X$ of the Hall bialgebras of the categories $\mathcal{N}_{\boldsymbol{k}_x}$. By Proposition 4.1, ii), $\Delta\big(T_{r,x}^{(\infty)}\big) = T_{r,x}^{(\infty)} \otimes 1 + 1 \otimes T_{r,x}^{(\infty)}$. Hence, from the definition of the Drinfeld double we get $\big[T_{(0,r)}, T_{(0,r')}\big] = 0$ as desired.

Let us prove the relation ii). Assume that $\mathbf{x}, \mathbf{y}$ are as in ii). Since $\mathsf{deg}(\mathbf{x}) = 1$ we cannot have $\mathsf{deg}(\mathbf{y}) = \mathsf{deg}(\mathbf{x}+\mathbf{y}) = 2$. On the other hand, it is easy to see that if $\mathsf{deg}(\mathbf{y}) \geq 2$ and $\mathsf{deg}(\mathbf{x}+\mathbf{y}) \geq 3$, or if $\mathsf{deg}(\mathbf{x}+\mathbf{y}) \geq 2$ and $\mathsf{deg}(\mathbf{y}) \geq 3$ then $\boldsymbol{\Delta}_{\mathbf{x},\mathbf{y}}$ contains interior lattice points. In conclusion, we either have $\mathsf{deg}(\mathbf{y}) = 1$ or $\mathsf{deg}(\mathbf{x}+\mathbf{y}) = 1$. We split our argument according to this dichotomy.

Case a.1. We have $\mathsf{deg}(\mathbf{x}+\mathbf{y}) = 1$ and $\epsilon_{\mathbf{x},\mathbf{y}} > 0$. Up to the $SL(2,\mathbb{Z})$-action, we may fix $\mathbf{x} = (1,0)$ and if $\det(\mathbf{x},\mathbf{y}) = r$ then we may furthermore assume that $\mathbf{y} = (s,r)$ for some $0 \leq s < r$. Using Pick's formula (5.1), we deduce that there are no points inside $\boldsymbol{\Delta}_{\mathbf{x},\mathbf{y}}$ if and only if $\mathsf{deg}(\mathbf{y}) = r$, which implies $\mathbf{y} = (0,r)$. Then relation ii) follows from Lemma 4.12.

Case a.2 We have $\mathsf{deg}(\mathbf{x}+\mathbf{y}) = 1$ and $\epsilon_{\mathbf{x},\mathbf{y}} < 0$. Without loss of generality, we may assume that $\mathbf{x} = (r_1, d_1), \mathbf{y} = (r_2, d_2)$ with $r_1 > 0$ and $r_2 > 0$. Now let us use the antiautomorphism D of Proposition 3.11. Note that $D(T_{(r,d)}) = T_{(r,-d)}$ and $\epsilon_{D(\mathbf{x}),D(\mathbf{y})} > 0$ hence the desired relation follows from case a.1 above.

Case b. We have $\mathsf{deg}(\mathbf{y}) = 1$. In that situation, simple application of Pick's formula (5.1) shows that $\mathsf{deg}(\mathbf{x}+\mathbf{y}) = \big|\det(\mathbf{x},\mathbf{y})\big|$, and, after exchanging the role of $\mathbf{x}$ and $\mathbf{y}$ if necessary and using the $SL(2,\mathbb{Z})$ invariance we may assume that $\mathbf{x} = (1,n), \mathbf{y} = (-1,l)$. The expression for the commutator $\big[T_{\mathbf{y}}, T_{\mathbf{x}}\big]$ can be now derived from the definition of the Drinfeld double together with Lemma 4.11 : if $n+l > 0$ then $\epsilon_{(1,n),(-1,l)} = 1$ and the relation $R\big(T_{(1,n)}, T_{(-1,l)}\big)$ is

$$T_{(-1,l)}T_{(1,n)} = T_{(1,n)}T_{(-1,l)} + \Theta_{(0,n+l)}(T_{(-1,l)}, T_{(-1,l)}) = T_{(1,n)}T_{(-1,l)} + c_1\frac{\Theta_{(0,n+l)}}{v^{-1}-v},$$

and if $n+l < 0$ then $\epsilon_{(1,n),(-1,l)} = -1$ and the relation $R\big(T_{(1,n)}, T_{(-1,l)}\big)$ is

$$T_{(1,n)}T_{(-1,l)} = T_{(-1,l)}T_{(1,n)} + \Theta_{(0,n+l)}\big(T_{(-1,l)}, T_{(-1,l)}\big) = T_{(-1,l)}T_{(1,n)} + c_1\frac{\Theta_{(0,n+l)}}{v^{-1}-v}.$$

This concludes the proof of relation ii).

By the above, Ω is well-defined and extends to a surjective algebra morphism $\Omega : \boldsymbol{\mathcal{E}}_{\sigma,\bar{\sigma}} \twoheadrightarrow \mathbf{U}_X \otimes \mathbb{C}$. Moreover, this morphism is $SL(2,\mathbb{Z})$-equivariant. In the rest of the proof, we construct an inverse of Ω. We first concentrate on the "positive" subalgebra $\boldsymbol{\mathcal{E}}^+_{\sigma,\bar{\sigma}}$ of $\boldsymbol{\mathcal{E}}_{\sigma,\bar{\sigma}}$. For any path $\mathbf{p} = (\mathbf{x}_1, \ldots, \mathbf{x}_r)$ we set $t_{\mathbf{p}} = t_{\mathbf{x}_1} \cdots t_{\mathbf{x}_r}$. Note that from the surjectivity of Ω and Proposition 4.5 it follows that the elements $\big\{t_{\mathbf{p}} \mid \mathbf{p} \in \mathbf{Conv}^+\big\}$ are linearly independent.

Lemma 5.6. *The subalgebra $\boldsymbol{\mathcal{E}}^+_{\sigma,\bar{\sigma}}$ is equal to $\bigoplus_{\mathbf{p}\in\mathbf{Conv}^+} \mathbb{C}t_{\mathbf{p}}$.*

Proof. The inclusion is obvious in one direction. For the other inclusion, we have to show that any path $\mathbf{p}$ in $\mathbf{Z}^+$ can be "straightened" using the relations ii). By an argument, which is at all steps similar to the proof of Lemma 3.1, it is sufficient to show that for any $\mathbf{x}, \mathbf{y} \in \mathbf{Z}^+$ with $\mu(\mathbf{y}) > \mu(\mathbf{x})$, we have

$$t_{\mathbf{y}}t_{\mathbf{x}} \in \bigoplus_{\mathbf{p}\in I_{\mathbf{x},\mathbf{y}}} \mathbb{C}t_{\mathbf{p}}, \tag{5.3}$$

where by definition $I_{\mathbf{x},\mathbf{y}}$ is the set of convex paths in $\boldsymbol{\Delta}_{\mathbf{x},\mathbf{y}}$ joining $\mathbf{o}$ to $\mathbf{x}+\mathbf{y}$. We shall achieve this by induction on $\det(\mathbf{y},\mathbf{x})$.

If $\det(\mathbf{y},\mathbf{x}) = 1$ then (e.g. by Pick's formula) $\mathsf{deg}(\mathbf{x}) = \mathsf{deg}(\mathbf{y}) = \mathsf{deg}(\mathbf{x}+\mathbf{y}) = 1$ thus $t_{\mathbf{y}}t_{\mathbf{x}} = t_{\mathbf{x}}t_{\mathbf{y}} + c_1 t_{\mathbf{x}+\mathbf{y}}$ by relation ii). So let us fix $d > 1$ and let us assume that (5.3) holds for any $\mathbf{x}', \mathbf{y}'$ satisfying $\det(\mathbf{y}', \mathbf{x}') < d$.

If $\mathbf{p} = (\mathbf{x}_1, \ldots, \mathbf{x}_r)$ is any path in $\mathbf{Z}^+$ we put $\mathbf{p}^\# = (\mathbf{x}_{\sigma(1)}, \ldots, \mathbf{x}_{\sigma(r)})$ where σ is the least length permutation satisfying $\mu(\mathbf{x}_{\sigma(1)}) \leq \mu(\mathbf{x}_{\sigma(2)}) \leq \cdots \leq \mu(\mathbf{x}_{\sigma(r)})$, and we denote by $a(\mathbf{p})$ the area of the polygon bounded by $\mathbf{p}$ and $\mathbf{p}^\#$. Observe that if $\mathbf{p}'$ is a subpath of $\mathbf{p}$ then $a(\mathbf{p}') \leq a(\mathbf{p})$. Also, if $\mathbf{z}, \mathbf{w} \in \mathbf{Z}^+$ are such that $\mu(\mathbf{z}) > \mu(\mathbf{w})$ then $a\big((\mathbf{z},\mathbf{w})\big) = \det(\mathbf{z},\mathbf{w})$.

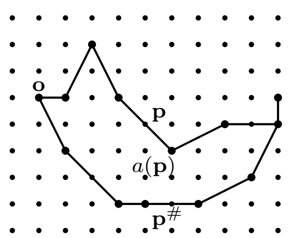


Figure 5. The area $a(\mathbf{p})$ of a path in $\mathbf{Z}^+$.

Claim. For any path $\mathbf{p}$ in $\mathbf{Z}^+$ satisfying $a(\mathbf{p}) < d$ we have $t_{\mathbf{p}} \in \bigoplus_{\mathbf{p}\in\mathbf{Conv}^+} \mathbb{C}t_{\mathbf{p}}$.
Proof of Claim. The assertion is true by definition if $a(\mathbf{p}) = 0$. If $a(\mathbf{p}) > 0$ then $\mathbf{p} = (\mathbf{x}_1, \ldots, \mathbf{x}_r)$ with $\mu(\mathbf{x}_1) \leq \cdots \leq \mu(\mathbf{x}_s) > \mu(\mathbf{x}_{s+1})$ for some s. We have $\det(\mathbf{x}_s, \mathbf{x}_{s+1}) \leq a(\mathbf{p}) < d$ hence $t_{\mathbf{x}_s} t_{\mathbf{x}_{s+1}} = \sum_i u_i t_{\mathbf{q}_i}$ for some $\mathbf{q}_i \in I_{\mathbf{x}_{s+1},\mathbf{x}_s}$ and, setting $\mathbf{p}_i = (\mathbf{x}_1, \ldots, \mathbf{x}_{s-1}, \mathbf{q}_i, \mathbf{x}_{s+2}, \ldots, \mathbf{x}_r)$ we get $t_{\mathbf{p}} = \sum_i u_i t_{\mathbf{p}_i}$. It is clear that for all i both $\mathbf{p}_i$ and $\mathbf{p}_i^{\#}$ strictly lie inside the polygon bounded by $\mathbf{p}$ and $\mathbf{p}^{\#}$, so that $a(\mathbf{p}_i) < a(\mathbf{p})$.

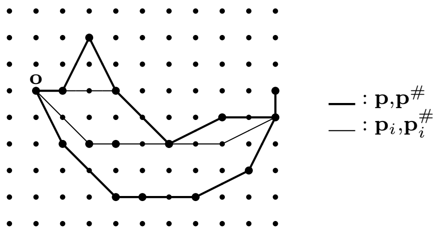


Figure 6. The area of a path $\mathbf{p}$ before and after one straightening.

We may iterate this process until we are only left with paths $\mathbf{q}$ satisfying $a(\mathbf{q}) = 0$. Hence $t_{\mathbf{p}} \in \bigoplus_{\mathbf{p}\in\mathbf{Conv}^+} \mathbb{C}t_{\mathbf{p}}$ and the claim is proven. ✓

Now let us fix $\mathbf{x}, \mathbf{y}$ such that $\mu(\mathbf{y}) > \mu(\mathbf{x})$ and $\det(\mathbf{y}, \mathbf{x}) = d$. If $\boldsymbol{\Delta}_{\mathbf{x},\mathbf{y}}$ has no interior lattice point then (see the proof of relation ii) above) either $\mathsf{deg}(\mathbf{x}) = \mathsf{deg}(\mathbf{y}) = \mathsf{deg}(\mathbf{x}+\mathbf{y}) = 2$, or $\mathsf{deg}(\mathbf{x}) = 1$ or $\mathsf{deg}(\mathbf{y}) = 1$. In the first case, we can assume up to the $SL(2,\mathbb{Z})$-action that $\mathbf{y} = (2,0)$ and $\mathbf{x} = (0,2)$. We leave to the reader to check that repeated applications of ii) as in Example 5.5 lead to the equality

$$t_{(0,2)}t_{(2,0)} = t_{(2,0)}t_{(0,2)} + ct^2_{(1,1)} + c_2\big(\frac{c_2}{c_1} - 2\big)t_{(2,2)},$$

where $c = \frac{(v^{-1}-v)}{2}\left(\frac{c_2}{c_1}(c_2 + \frac{c_2}{c_1} - 1) + \frac{(v^{-1}-v)}{2}c_2(1-c_1)\right)$. In the last two cases, relation ii) directly yields (5.3). So we may assume that $\boldsymbol{\Delta}_{\mathbf{x},\mathbf{y}}$ contains interior lattice points.

Let us choose $\mathbf{z} \in \boldsymbol{\Delta}_{\mathbf{x},\mathbf{y}}$ so that the triangle $\mathbf{ozx}$ has no interior points and $\mathsf{deg}(\mathbf{z}) = \mathsf{deg}(\mathbf{x}-\mathbf{z}) = 1$. Note that (5.3) is stable under the action of $SL(2,\mathbb{Z})$, hence without loss of generality we can assume that $\mathbf{x}-\mathbf{z} \in \mathbf{Z}^+$. By construction, $\mathbf{z}$ and $\mathbf{x}-\mathbf{z}$ satisfy both conditions of the relation ii), hence $[t_{\mathbf{z}}, t_{\mathbf{x}-\mathbf{z}}] = c_1\frac{\theta_{\mathbf{x}}}{v^{-1}-v} = c_1 t_{\mathbf{x}} + u$ for some u belonging to the subalgebra $\langle t_{\mathbf{x}_0}, \ldots, t_{(\mathsf{deg}(\mathbf{x})-1)\mathbf{x}_0}\rangle$ generated by $t_{\mathbf{x}_0}, \ldots, t_{(\mathsf{deg}(\mathbf{x})-1)\mathbf{x}_0}$, where $\mathbf{x}_0 = \frac{\mathbf{x}}{\mathsf{deg}(\mathbf{x})}$. Therefore,

$$c_1[t_{\mathbf{y}}, t_{\mathbf{x}}] = \big[[t_{\mathbf{y}}, t_{\mathbf{z}}], t_{\mathbf{x}-\mathbf{z}}\big] + \big[t_{\mathbf{z}}, [t_{\mathbf{y}}, t_{\mathbf{x}-\mathbf{z}}]\big] - [t_{\mathbf{y}}, u]. \tag{5.4}$$

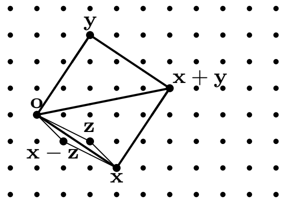


Figure 7. The decomposition $\mathbf{x} = \mathbf{z} + (\mathbf{x} - \mathbf{z})$.

Note that $c_1 \neq 0$ since $|\sigma| = |\bar{\sigma}| = \sqrt{q}$. As $\mathbf{z}$ is an interior point of $\boldsymbol{\Delta}_{\mathbf{x},\mathbf{y}}$ we have $\mathbf{z} = \alpha\mathbf{x} + \beta\mathbf{y}$ for some $\alpha, \beta \in]0,1[$ satisfying $\alpha > \beta$. It follows that $\det(\mathbf{y}, \mathbf{z}) < d$

and $\det(\mathbf{y},\mathbf{x}-\mathbf{z})<d$, and by the induction hypothesis

$$[t_{\mathbf{y}},t_{\mathbf{z}}]\in\bigoplus_{\mathbf{p}\in I_{\mathbf{z},\mathbf{y}}}\mathbb{C}t_{\mathbf{p}},\qquad [t_{\mathbf{y}},t_{\mathbf{x}-\mathbf{z}}]\in\bigoplus_{\mathbf{q}\in I_{\mathbf{x}-\mathbf{z},\mathbf{y}}}\mathbb{C}t_{\mathbf{q}}.$$

Next, as $\mu(\mathbf{x}-\mathbf{z})<\mu(\mathbf{z})<\mu(\mathbf{y})$ we have, for any $\mathbf{p}\in I_{\mathbf{z},\mathbf{y}}$, $\big((\mathbf{x}-\mathbf{z}),\mathbf{p}\big)\in\mathbf{Conv}^+$ and $(\mathbf{p},(\mathbf{x}-\mathbf{z}))^{\#}=\big((\mathbf{x}-\mathbf{z}),\mathbf{p}\big)$. Thus $a\big((\mathbf{x}-\mathbf{z},\mathbf{p})\big)=0$ and $a\big((\mathbf{p},\mathbf{x}-\mathbf{z})\big)=\det(\mathbf{y}+\mathbf{z},\mathbf{x}-\mathbf{z})=(1+\beta-\alpha)\det(\mathbf{y},\mathbf{x})<d$. It follows by the Claim that $\big[[t_{\mathbf{y}},t_{\mathbf{z}}],t_{\mathbf{x}-\mathbf{z}}\big]\in\bigoplus_{\mathbf{Conv}^+}\mathbb{C}t_{\mathbf{p}}$. In a similar manner, for any $\mathbf{q}\in I_{\mathbf{x}-\mathbf{z},\mathbf{y}}$ we have $a\big((\mathbf{z},\mathbf{q})\big)<a\big((\mathbf{z},\mathbf{y},\mathbf{x}-\mathbf{z})\big)=\det(\mathbf{y}+\mathbf{z},\mathbf{x}-\mathbf{z})=\det(\mathbf{y},\mathbf{x})-\det(\mathbf{x}+\mathbf{y},\mathbf{z})<d$ since $\det(\mathbf{x}+\mathbf{y},\mathbf{z})>0$; and $a\big((\mathbf{q},\mathbf{z})\big)<a\big((\mathbf{y},\mathbf{x}-\mathbf{z},\mathbf{z})\big)=\det(\mathbf{y},\mathbf{x})=d$. Thus $\big[t_{\mathbf{z}},[t_{\mathbf{y}},t_{\mathbf{x}-\mathbf{z}}]\big]\in\bigoplus_{\mathbf{Conv}^+}\mathbb{C}t_{\mathbf{p}}$. Finally, let us write $u=\sum_{j=1}^{\mathsf{deg}(\mathbf{x})-1}a_jt_{j\mathbf{x}_0}$ with $a_j\in\big\langle t_{\mathbf{x}_0},\ldots,t_{(\mathsf{deg}(\mathbf{x})-1)\mathbf{x}_0}\big\rangle$ of weight $(\mathsf{deg}(\mathbf{x})-j)\mathbf{x}_0$. By the induction hypothesis, $t_{\mathbf{y}}a_j\in\bigoplus_{\mathbf{p}}\mathbb{C}t_{\mathbf{p}}$ where $\mathbf{p}$ ranges in $I_{(\mathsf{deg}(\mathbf{x})-j)\mathbf{x}_0,\mathbf{y}}$. But as for any such j and $\mathbf{p}$ we have $a\big((\mathbf{p},j\mathbf{x}_0)\big)=\frac{\mathsf{deg}(\mathbf{x})-j}{\mathsf{deg}(\mathbf{x})}d<d$ the Claim implies that $t_{\mathbf{y}}u\in\bigoplus_{\mathbf{Conv}^+}\mathbb{C}t_{\mathbf{p}}$. Hence all together, by (5.4), $t_{\mathbf{y}}t_{\mathbf{x}}\in\bigoplus_{\mathbf{Conv}^+}\mathbb{C}t_{\mathbf{p}}$. Finally, let us write $t_{\mathbf{y}}t_{\mathbf{x}}=\sum_{\mathbf{p}\in\mathbf{Conv}^+}c_{\mathbf{p}}t_{\mathbf{p}}$. Applying Ω, we get $T_{\mathbf{y}}T_{\mathbf{x}}=\sum_{\mathbf{p}}c_{\mathbf{p}}T_{\mathbf{p}}$. By Remark 2.7, we have $T_{\mathbf{y}}T_{\mathbf{x}}\in\bigoplus_{\mathbf{p}\in\mathbf{I}_{\mathbf{x},\mathbf{y}}}\mathbb{C}T_{\mathbf{p}}$ so that $c_{\mathbf{p}}=0$ for $\mathbf{p}\notin\mathbf{I}_{\mathbf{x},\mathbf{y}}$. Therefore $t_{\mathbf{y}}t_{\mathbf{x}}\in\bigoplus_{\mathbf{p}\in\mathbf{I}_{\mathbf{x},\mathbf{y}}}\mathbb{C}t_{\mathbf{p}}$ as desired. This closes the induction step and proves Lemma 5.6. ✓

Now we are ready to finish the proof of Theorem 5.4. Define $\boldsymbol{\mathcal{E}}^-_{\sigma,\bar\sigma}$ in the same way as $\boldsymbol{\mathcal{E}}^+_{\sigma,\bar\sigma}$ by replacing $\mathbf{Z}^+$ by $\mathbf{Z}^-$. By Lemma 5.6, $\boldsymbol{\mathcal{E}}^-_{\sigma,\bar\sigma}$ is equal to $\bigoplus_{\mathbf{p}\in\mathbf{Conv}^-}\mathbb{C}t_{\mathbf{x}}$. The map Ω restricts to isomorphisms $\boldsymbol{\mathcal{E}}^{\pm}_{\sigma,\bar\sigma}\simeq\mathbf{U}^{\pm}_X\otimes\mathbb{C}$. By Theorem 4.5 and Corollary 4.7, $\mathbf{U}_X$ is generated by $\mathbf{U}^{\pm}_X$ modulo the collection of relations $R(g,h)$ for sums of classes of semi-stable sheaves $g\in\mathbf{U}^+_X$ and $h\in\mathbf{U}^-_X$. Now, if g and h are as above and $\mu(g)=\mu(h)$ then $R(g,h)$ expresses the fact that $\mathbf{U}^{+,(\mu)}_X$ and $\mathbf{U}^{-,(\mu)}_X$ commute. By relation ii), $R\big(\Omega^{-1}(g),\Omega^{-1}(h)\big)$ holds in $\boldsymbol{\mathcal{E}}_{\sigma,\bar\sigma}$. If on the other hand $\mu(g)\neq\mu(h)$ then there exists $\gamma\in SL(2,\mathbb{Z})$ such that $\gamma(g),\gamma(h)\in\mathbf{U}^+_X$. In that situation, applying γ to $R(g,h)$ yields a relation $R^{\gamma}\big(\gamma(g),\gamma(h)\big)$ in $\mathbf{U}^+_X$. We deduce that $R^{\gamma}(\Omega^{-1}\circ\gamma(g)),\Omega^{-1}\circ\gamma(h)))$ holds in $\boldsymbol{\mathcal{E}}^+_{\sigma,\bar\sigma}$. As $\boldsymbol{\mathcal{E}}_{\sigma,\bar\sigma}$ carries an action of $SL(2,\mathbb{Z})$ compatible with Ω, it follows that $R\big(\Omega^{-1}(g),\Omega^{-1}(h)\big)$ holds in $\boldsymbol{\mathcal{E}}_{\sigma,\bar\sigma}$. Therefore, Ω^{-1} extends to a morphism $\mathbf{U}_X\otimes\mathbb{C}\to\boldsymbol{\mathcal{E}}_{\sigma,\bar\sigma}$, which is the desired inverse to Ω. The theorem is proved. ✓

5.4. We still assume that $(\sigma,\bar\sigma)$ is associated to an elliptic curve X. The proof of Theorem 5.4 in fact gives the following. Let $'\boldsymbol{\mathcal{E}}^+_{\sigma,\bar\sigma}$ be the $\mathbb{C}$-algebra generated by elements $t_{\mathbf{x}}$ for $\mathbf{x}\in\mathbf{Z}^+$ subject to relations i) and ii) of Section 5.2.

Corollary 5.7. *The assignment* $\Omega\ :t_{\mathbf{x}}\mapsto T_{\mathbf{x}}$ *for* $\mathbf{x}\in\mathbf{Z}^+$ *extends to an algebra isomorphism* $'\boldsymbol{\mathcal{E}}^+_{\sigma,\bar\sigma}\xrightarrow{\sim}\mathbf{U}^+_X$. *In other words, the natural morphism* $'\boldsymbol{\mathcal{E}}^+_{\sigma,\bar\sigma}\to\boldsymbol{\mathcal{E}}^+_{\sigma,\bar\sigma}$ *is an isomorphism.*

6. Further results : integral form and central extension

In this section, we gather some useful properties of the algebras $\boldsymbol{\mathcal{E}}_{\sigma,\bar\sigma}$ and $\mathbf{U}_X$.

6.1. For any smooth projective curve X Kapranov [K1] considered[1] a natural subalgebra $\mathbf{H}^{sph}_X$ of $\mathbf{H}_X$ which we call the *spherical Hall algebra* of X. By definition, $\mathbf{H}^{sph}_X$ is generated by the elements $\big\{\mathbf{1}_{(0,d)}\mid d\in\mathbb{N}\big\}\cup\big\{\mathbf{1}^{\mathsf{ss}}_{(1,l)}\mid l\in\mathbb{Z}\big\}$. In the

[1] at least implicitly

language of automorphic forms used in [K1], these generators are the simplest and most natural cuspidal elements of $\mathbf{H}_X$. In the case of an elliptic curve X it turns out that our algebra $\mathbf{U}_X^+$ coincides with $\mathbf{H}_X^{sph}$. This is an easy consequence of the following corollary of Theorem 5.4 :

Corollary 6.1. *The algebra* $\mathbf{U}_X^+$ *is generated by* $\big\{T_\alpha \,|\, \mathsf{rank}(\alpha) \leq 1\big\}$. *Similarly, the algebra* $\mathbf{U}_X$ *is generated by either of the following two sets :*

$$\big\{T_{(\pm 1,0)}, T_{(0,\pm 1)}\big\}, \qquad \big\{T_{(1,0)}, T_{(0,1)}, T_{(-1,-1)}\big\}.$$

Proof. We prove the first statement by induction. Denote by $\mathfrak{W}$ the subalgebra generated by $\{T_\alpha \,|\, \mathsf{rank}(\alpha) \leq 1\}$ and assume that $T_{r,s} \in \mathfrak{W}$ for any $(r,s) \in \mathbf{Z}^+$ with $r < n \geq 2$. Fix $\mathbf{z} = (n,p) \in \mathbf{Z}^+$, and let $\mathbf{x}$ be the point of $\big\{(r,s) \mid r < n\big\}$ closest to the segment $\mathbf{oz}$. By construction there are no interior lattice points in $\Delta_{\mathbf{x},\mathbf{z}-\mathbf{x}}$ and thus $\big[T_{\mathbf{x}}, T_{\mathbf{z}-\mathbf{x}}\big] = u\theta_{\mathbf{z}}$ for some $u \neq 0$. By the induction hypothesis, we have $T_{\mathbf{x}}, T_{\mathbf{z}-\mathbf{x}} \in \mathfrak{W}$, and $\theta_{\mathbf{z}} \in (v^{-1}-v)T_{\mathbf{z}} \oplus \mathfrak{W}$. We deduce that $T_{\mathbf{z}} \in \mathfrak{W}$ as wanted.

Let us deal with the second assertion. As before, denote by $\mathfrak{W}$ the subalgebra generated by $\big\{T_{(\pm 1,0)}, T_{(0,\pm 1)}\big\}$. We have, for any $l \in \mathbb{Z}$, $\big[T_{(0,\pm 1)}, T_{(1,l)}\big] = \pm c_1 T_{(1,l\pm 1)}$ and it follows that $T_{(1,l)} \in \mathfrak{W}$ for any $l \in \mathbb{Z}$. Similarly, $T_{(-1,l)} \in \mathfrak{W}$ for any $l \in \mathbb{Z}$. But then, considering commutators $\big[T_{(-1,l)}, T_{(1,l')}\big]$, we have $\Theta_{(0,n)} \in \mathfrak{W}$ for any n as well. The subalgebra generated by $\{\Theta_{(0,n)}\}$ and the one generated by $\{T_{(0,n)}\}$ being equal, we see that $\mathfrak{W}$ contains all $T_{(r,n)}$ with $|r| \leq 1$. Applying the first statement of the corollary, we get $\mathbf{U}_X^{\pm} \subset \mathfrak{W}$, from which we deduce that $\mathfrak{W} = U_X$.

The last statement follows the second statement together with the relations $\big[T_{(1,0)}, T_{(-1,-1)}\big] = c_1 T_{(0,-1)}$ and $\big[T_{(-1,-1)}, T_{(0,1)}\big] = c_1 T_{(-1,0)}$. ✓

Kapranov exhibited certain relations satisfied by the generators $\big\{\mathbf{1}_{(0,d)}, \mathbf{1}^{\mathsf{ss}}_{(1,l)} \mid d > 0, l \in \mathbb{Z}\big\}$, for any curve X. These are the so-called *functional equations* for Eisenstein series. When X is an elliptic curve, they take the following form. Put

$$E^+(t) = \sum_{p \in \mathbb{Z}} \mathbf{1}^{\mathsf{ss}}_{(1,p)} t^p, \qquad \psi^+(s) = \sum_{d \geq 0} \mathbf{1}_{(0,d)} s^d.$$

Then (see [K1], Thm. 3.3.)

$$E^+(t_1)E^+(t_2) = \frac{\zeta_X(t_1/t_2)}{\zeta_X(t_2/t_1)} E^+(t_2)E^+(t_1) \tag{6.1}$$

$$\psi^+(t_1)E^+(t_2) = \zeta(\sigma^{-1/2}\bar{\sigma}^{-1/2}t_1/t_2)E^+(t_2)E^+(t_1), \tag{6.2}$$

$$\psi^+(t_1)\psi^+(t_2) = \psi^+(t_2)\psi^+(t_1), \tag{6.3}$$

where $\zeta_X(t) = \dfrac{(1-\sigma t)(1-\bar{\sigma} t)}{(1-t)(1-qt)}$ is the *zeta function* of X. It is known however that relations (6.1 − 6.3) do not exhaust the complete list of relations of $\mathbf{U}_X^+$. In other words, if $\mathbb{U}_X^+$ denotes the algebra generated by some elements $T_{(1,l)}, T_{(0,d)}$ subject to relations (6.1 − 6.3) above then there is a nontrivial surjective algebra homomorphism $\mathbb{U}_X^+ \twoheadrightarrow \mathbf{U}_X^+$. One may hope to use the description of $\mathbf{U}_X^+$ given in this paper to explicitly describe the kernel of this map $\mathbb{U}_X^+ \twoheadrightarrow \mathbf{U}_X^+$. This appears to us to be a very interesting problem: using Kapranov's interpretation of the Hall algebra in terms of automorphic forms for $GL(n)$ over a function field, elements of this kernel correspond to some new, higher rank relations satisfied by residues of Eisenstein series.

The reader will find yet another presentation of $\mathbf{U}_X^+$ in [SV2], Section 9, this time in terms of shuffle (or *Feigin-Odesskii*) algebras.

6.2. We have only defined so far an algebra $\boldsymbol{\mathcal{E}}_{\sigma,\bar\sigma}$ for complex values of σ and $\bar\sigma$. However, it is also natural to consider a version of $\boldsymbol{\mathcal{E}}_{\sigma,\bar\sigma}$, where σ and $\bar\sigma$ are formal parameters. Put $\mathbf{R}=\mathbb{C}\big[\sigma^{\pm 1/2},\bar\sigma^{\pm 1/2}\big]$, $\mathbf{K}=\mathrm{Frac}(\mathbf{R})=\mathbb{C}\big(\sigma^{1/2},\bar\sigma^{1/2}\big)$ where $\sigma,\bar\sigma$ are now formal variables, and consider the $\mathbf{K}$-algebra $\boldsymbol{\mathcal{E}}_{\mathbf{K}}$ generated by elements $\big\{t_{\mathbf{x}}\mid \mathbf{x}\in\mathbf{Z}^*\big\}$ modulo the relations i) and ii) of Section 5.2.We also set $\nu=(\sigma\bar\sigma)^{-1/2}$, $\tilde t_{\mathbf{x}}=t_{\mathbf{x}}/[\mathsf{deg}(\mathbf{x})]_\nu$, and for an arbitrary path $\mathbf{p}=(\mathbf{x}_1,\ldots,\mathbf{x}_r)$ we put $\tilde t_{\mathbf{p}}=t_{\mathbf{p}}/[\mathbf{p}]_\nu$ with $[\mathbf{p}]_\nu=[\mathsf{deg}(\mathbf{x}_1)]_\nu\cdots[\mathsf{deg}(\mathbf{x}_r)]_\nu$. Finally, we let $\boldsymbol{\mathcal{E}}_{\mathbf{R}}$ stand for the $\mathbf{R}$-subalgebra of $\boldsymbol{\mathcal{E}}_{\mathbf{K}}$ generated by $\big\{\tilde t_{\mathbf{x}}\mid \mathbf{x}\in\mathbf{Z}^*\big\}$. Subalgebras $\boldsymbol{\mathcal{E}}^{\pm}_{\mathbf{R}}$ are defined in a similar fashion. There is an obvious action of $SL(2,\mathbb{Z})$ on $\boldsymbol{\mathcal{E}}_{\mathbf{K}}$ and $\boldsymbol{\mathcal{E}}_{\mathbf{R}}$.

Proposition 6.2. *The following hold :*

i) $\boldsymbol{\mathcal{E}}^{\pm}_{\mathbf{K}}=\bigoplus_{\mathbf{p}\in\mathbf{Conv}^{\pm}}\mathbf{K}\tilde t_{\mathbf{p}}$,

ii) *there is a triangular decomposition* $\boldsymbol{\mathcal{E}}_{\mathbf{K}}=\boldsymbol{\mathcal{E}}^{+}_{\mathbf{K}}\otimes\boldsymbol{\mathcal{E}}^{-}_{\mathbf{K}}$*; in particular, we have* $\boldsymbol{\mathcal{E}}_{\mathbf{K}}=\bigoplus_{\mathbf{p}\in\mathbf{Conv}}\mathbf{K}\tilde t_{\mathbf{p}}$.

Proof. We begin with i). Let us first show that the elements $\big\{\tilde t_{\mathbf{p}}\mid \mathbf{p}\in\mathbf{Conv}^+\big\}$ are linearly independent. For this we shall use a specialization argument. Let $'\boldsymbol{\mathcal{E}}^{+}_{\mathbf{R}}$ be the $\mathbf{R}$-algebra generated by some elements $\big\{'\tilde t_{\mathbf{x}}\mid \mathbf{x}\in\mathbf{Z}^+\big\}$ modulo the relations i) and ii) in Section 5.2 (these relations have coefficients in $\mathbf{R}$ when written in terms of the generators $\tilde t_{\mathbf{x}}$). By construction there is a canonical map $'\boldsymbol{\mathcal{E}}^{+}_{\mathbf{R}}\to {}'\boldsymbol{\mathcal{E}}^{+}_{\mathbf{R}}\otimes_{\mathbf{R}}\mathbf{K}=\boldsymbol{\mathcal{E}}_{\mathbf{K}}, u\mapsto u\otimes 1$ whose image is $\boldsymbol{\mathcal{E}}^{+}_{\mathbf{R}}$. Moreover, for any elliptic curve X with Frobenius eigenvalues $\{\alpha,\overline\alpha\}$ there is a specialization morphism

$$\mathsf{ev}_X\ :{}'\boldsymbol{\mathcal{E}}^{+}_{\mathbf{R}}\to ({}'\boldsymbol{\mathcal{E}}^{+}_{\mathbf{R}})_{|\substack{\sigma=\alpha\\ \bar\sigma=\overline\alpha}}=\boldsymbol{\mathcal{E}}^{+}_{\alpha,\overline\alpha}\simeq\mathbf{U}_X^+.$$

Now assume that $\sum_{\mathbf{p}\in\mathbf{Conv}^+}z_{\mathbf{p}}\tilde t_{\mathbf{p}}$ is a nontrivial (finite) linear relation in $\boldsymbol{\mathcal{E}}_{\mathbf{R}}$, with $z_{\mathbf{p}}\in\mathbf{R}$. Then $c:=\sum_{\mathbf{p}\in\mathbf{Conv}^+}z_{\mathbf{p}}{}'\tilde t_{\mathbf{p}}$ is a torsion element of $'\boldsymbol{\mathcal{E}}^{+}_{\mathbf{R}}$. Let Z denote its support, which a strict subvariety of $\mathsf{Spec}(\mathbf{R})\simeq\mathbb{C}^*\times\mathbb{C}^*$. We have

$$\mathsf{ev}_X(c)=\mathsf{ev}_X\left(\sum_{\mathbf{p}}z_{\mathbf{p}}{}'\tilde t_{\mathbf{p}}\right)=\sum_{\mathbf{p}}z_{\mathbf{p}}\widetilde T_{\mathbf{p}}\in\mathbf{U}_X^+ \tag{6.4}$$

where $\widetilde T_{\mathbf{p}}=T_{\mathbf{p}}/[\mathbf{p}]$. If $(\alpha,\overline\alpha)\notin Z$ then $\mathsf{ev}_X(c)=0$ and (6.4) yields a nontrivial linear dependence relation between the elements $\big\{\widetilde T_{\mathbf{p}}\mid\mathbf{p}\in\mathbf{Conv}^+\big\}$, in contradiction with Theorem 4.5. It remains to find an elliptic curve with $(\alpha,\overline\alpha)\notin Z$. For all prime powers q let $N(q)$ be the number of possible Frobenius eigenvalues $\{\alpha,\overline\alpha\}$ for an elliptic curve over $\mathbb{F}_q$ (i.e. the number of isogeny classes of elliptic curves over $\mathbb{F}_q$). Then $\lim_{q\to\infty}N(q)=\infty$ (this is, for instance, a consequence of the main theorem in [Ho]). But by Bezout's theorem the number of intersection points between Z and $Y_q=\big\{(y,y')\mid yy'=q\big\}$ is bounded as $q\to\infty$. This provides the existence of the required elliptic curve, and concludes the proof of the linear independence of the elements $\big\{\tilde t_{\mathbf{p}}\mid\mathbf{p}\in\mathbf{Conv}^+\big\}$ in $\boldsymbol{\mathcal{E}}^{+}_{\mathbf{R}}$ and hence in $\boldsymbol{\mathcal{E}}^{+}_{\mathbf{K}}$. The same arguments as in Lemma 5.6 now show that $\boldsymbol{\mathcal{E}}^{+}_{\mathbf{K}}=\bigoplus_{\mathbf{p}\in\mathbf{Conv}^+}\mathbf{K}\tilde t_{\mathbf{p}}$.

We turn our attention to ii). We shall first show that the multiplication map $\boldsymbol{\mathcal{E}}^{+}_{\mathbf{K}}\otimes\boldsymbol{\mathcal{E}}^{-}_{\mathbf{K}}\to\boldsymbol{\mathcal{E}}_{\mathbf{K}}$ is surjective. For this, using i), it is enough to see that

$$\tilde t_{\mathbf{y}}\tilde t_{\mathbf{p}}\in\boldsymbol{\mathcal{E}}^{+}_{\mathbf{K}}\boldsymbol{\mathcal{E}}^{-}_{\mathbf{K}} \tag{6.5}$$

for any $\mathbf{y}\in\mathbf{Z}^-$ and $\mathbf{p}\in\mathbf{Conv}^+$. We say that a path $\mathbf{p}=(\mathbf{x}_1,\ldots,\mathbf{x}_r)$ is concave if $(\mathbf{x}_r,\ldots,\mathbf{x}_1)$ is convex. Let $\mathbf{Conc}^{\pm}$ denote the set of concave paths in $\mathbf{Z}^{\pm}$, and put $\mathbf{Conc}\simeq\mathbf{Conc}^+\times\mathbf{Conc}^-$. A symmetric version of Lemma 5.6 and i) above shows that $\boldsymbol{\mathcal{E}}^{\pm}_{\mathbf{K}}=\bigoplus_{\mathbf{p}\in\mathbf{Conc}^{\pm}}\mathbf{K}\tilde{t}_{\mathbf{p}}$. In particular, for any $\mathbf{x},\mathbf{y}$ in $\mathbf{Z}$ with $\widehat{\mathbf{xy}}>\pi$ we can write $\tilde{t}_{\mathbf{y}}\tilde{t}_{\mathbf{x}}$ as a linear combination of elements $\tilde{t}_{\mathbf{p}}$ for *concave* paths $\mathbf{p}$ lying in the triangle $\Delta_{\mathbf{x},\mathbf{y}}$ (compare with Section 5.1.). Now choose $\mathbf{y}\in\mathbf{Z}^-$ and $\mathbf{x}\in\mathbf{Z}^+$. If $\widehat{\mathbf{xy}}>\pi$ then by the above remark $\tilde{t}_{\mathbf{y}}\tilde{t}_{\mathbf{x}}\in\bigoplus_{\mathbf{p}\in\mathbf{Conc}}\mathbf{K}\tilde{t}_{\mathbf{p}}$; if $\widehat{\mathbf{xy}}=\pi$ then $[\tilde{t}_{\mathbf{y}},\tilde{t}_{\mathbf{x}}]=0$; and if $\widehat{\mathbf{xy}}<\pi$ then $\tilde{t}_{\mathbf{y}}\tilde{t}_{\mathbf{x}}\in\bigoplus_{\mathbf{p}\in\mathbf{Conv}}\mathbf{K}\tilde{t}_{\mathbf{p}}$. In all cases,

$$\tilde{t}_{\mathbf{y}}\tilde{t}_{\mathbf{x}}\in\boldsymbol{\mathcal{E}}^{+}_{\mathbf{K}}\boldsymbol{\mathcal{E}}^{-}_{\mathbf{K}} \tag{6.6}$$

We shall prove (6.5) by induction on the rank of $-\mathbf{y}$. If $\mathsf{rank}(-\mathbf{y})=0$ (i.e. if $\mathbf{y}=(l,0)$ for some $l\in\mathbb{N}^-$) then $\big[\tilde{t}_{\mathbf{y}},\tilde{t}_{\mathbf{x}}\big]\in\boldsymbol{\mathcal{E}}^{+}_{\mathbf{K}}$ for any $\mathbf{x}\in\mathbf{Z}^+$. Thus $\big[\tilde{t}_{\mathbf{y}},\tilde{t}_{(\mathbf{x}_1,\ldots,\mathbf{x}_r)}\big]=\sum_{i=1}^{r}\tilde{t}_{(\mathbf{x}_1,\ldots,\mathbf{x}_{i-1})}\big[\tilde{t}_{\mathbf{y}},\tilde{t}_{\mathbf{x}_i}\big]\tilde{t}_{(\mathbf{x}_{i+1},\ldots,\mathbf{x}_r)}\in\boldsymbol{\mathcal{E}}^{+}_{\mathbf{K}}$. Now fix $\mathbf{y}\in\mathbf{Z}^-$ such that $-\mathbf{y}$ is of positive rank and assume that (6.5) holds for all $\mathbf{y}'$ of smaller rank. Observe that if $\mathsf{rank}(\mathbf{x})>0$ then from (6.6) we have $[\tilde{t}_{\mathbf{y}},\tilde{t}_{\mathbf{x}}]=\sum_i u_i\tilde{t}_{\mathbf{p}_i^+}\tilde{t}_{\mathbf{p}_i^-}$ with $\mathbf{p}_i^{\pm}\in\mathbf{Conv}^{\pm}$ and $\mathbf{p}_i^-=(\mathbf{z}_1^{(i)},\ldots,\mathbf{z}_{l_i}^{(i)})$ satisfying $\mathsf{rank}(-\mathbf{z}_j^{(i)})<\mathsf{rank}(-\mathbf{y})$. As a consequence, by the induction hypothesis we have $\tilde{t}_{\mathbf{p}_i^-}\boldsymbol{\mathcal{E}}^{+}_{\mathbf{K}}\subset\boldsymbol{\mathcal{E}}^{+}_{\mathbf{K}}\otimes\boldsymbol{\mathcal{E}}^{-}_{\mathbf{K}}$. Next, if $\mathsf{rank}(\mathbf{x})=0$ then $[\tilde{t}_{\mathbf{y}},\tilde{t}_{\mathbf{x}}]\in\boldsymbol{\mathcal{E}}^{-}_{\mathbf{K}}$. From these two facts we deduce that if $(\mathbf{x}_1,\ldots,\mathbf{x}_r)\in\mathbf{Conv}^+$ then $[\tilde{t}_{\mathbf{y}},\tilde{t}_{\mathbf{p}}]=\sum_{i=1}^{r}\tilde{t}_{(\mathbf{x}_1,\ldots,\mathbf{x}_{i-1})}[\tilde{t}_{\mathbf{y}},\tilde{t}_{\mathbf{x}_i}]\tilde{t}_{(\mathbf{x}_{i+1},\ldots,\mathbf{x}_r)}\in\boldsymbol{\mathcal{E}}^{+}_{\mathbf{K}}\boldsymbol{\mathcal{E}}^{-}_{\mathbf{K}}$, as wanted. This closes the induction and proves the surjectivity of the map $\boldsymbol{\mathcal{E}}^{+}_{\mathbf{K}}\otimes\boldsymbol{\mathcal{E}}^{-}_{\mathbf{K}}\to\boldsymbol{\mathcal{E}}_{\mathbf{K}}$. It only remains to see that the elements $\tilde{t}_{\mathbf{p}^+}\tilde{t}_{\mathbf{p}^-}$ for $\mathbf{p}^{\pm}\in\mathbf{Conv}^{\pm}$ and $\alpha\in\mathbf{Z}$ are linearly independent over $\mathbf{K}$. For this, we may argue in the same fashion as in i) above using a specialization argument. ✓

We view $\boldsymbol{\mathcal{E}}_{\mathbf{R}}$ and $\boldsymbol{\mathcal{E}}_{\mathbf{K}}$ as generic versions of the Hall algebra $\mathbf{U}_X$. Moreover, one can lift various notions from $\mathbf{U}_X$ to these generic forms. For instance we set

$$\widehat{\boldsymbol{\mathcal{E}}}^{+}_{\mathbf{K}}=\bigoplus_{\alpha\in\mathbf{Z}}\widehat{\boldsymbol{\mathcal{E}}}^{+}_{\mathbf{K}}[\alpha],\qquad \widehat{\boldsymbol{\mathcal{E}}}^{+}_{\mathbf{K}}[\alpha]=\prod_{\substack{\mathbf{p}\in\mathbf{Conv}^+\\ wt(\mathbf{p})=\alpha}}\mathbf{K}\tilde{t}_{\mathbf{p}}$$

and define elements $\mathbf{1}^{\mathsf{ss}}_{\alpha}\in\boldsymbol{\mathcal{E}}^{+}_{\mathbf{K}},\mathbf{1}_{\alpha}\in\widehat{\boldsymbol{\mathcal{E}}}^{+}_{\mathbf{K}}$ for any $\alpha\in\mathbf{Z}^+$ by the formulas

$$1+\sum_{l\geq 1}\mathbf{1}^{\mathsf{ss}}_{r\alpha_0}s^l=\exp\left(\sum_{l\geq1}\tilde{t}_{l\alpha_0}s^l\right)$$

for any α_0 such that $\mathsf{deg}(\alpha_0)=1$ and

$$\mathbf{1}_{\alpha}=\mathbf{1}^{\mathsf{ss}}_{\alpha}+\sum_{t>1}\sum_{\substack{\alpha_1+\cdots+\alpha_t=\alpha\\ \mu(\alpha_1)<\cdots<\mu(\alpha_t)}}\nu^{\sum_{i<j}\langle\alpha_i,\alpha_j\rangle}\mathbf{1}^{\mathsf{ss}}_{\alpha_1}\cdots\mathbf{1}^{\mathsf{ss}}_{\alpha_t}.$$

The elements $\big\{\mathbf{1}^{\mathsf{ss}}_{\alpha}\mid\alpha\in\mathbf{Z}^+\big\}$ belong to and actually generate over $\mathbf{R}$ the subalgebra $\boldsymbol{\mathcal{E}}^{+}_{\mathbf{R}}$, while the elements $\big\{\mathbf{1}_{\alpha}\mid\alpha\in\mathbf{Z}^+\big\}$ belong to and topologically generate over $\mathbf{R}$ the subalgebra $\widehat{\boldsymbol{\mathcal{E}}}^{+}_{\mathbf{R}}$. It is clear that the elements $\mathbf{1}^{\mathsf{ss}}_{\alpha}$ and $\mathbf{1}_{\alpha}$ specialize, for each given elliptic curve X, to the corresponding elements of the Hall algebras $\mathbf{U}^{+}_{X}$ and $\widehat{\mathbf{U}}^{+}_{X}$. Using the generators $\big\{\mathbf{1}_{\alpha}\mid\alpha\in\mathbf{Z}^+\big\}$ we may define a comutiplication Δ on $\boldsymbol{\mathcal{E}}^{+}_{\mathbf{K}}$ by means of the formula (4.5). This comultiplication preserves $\boldsymbol{\mathcal{E}}^{+}_{\mathbf{R}}$.

Let us now give a more precise description of the integral form $\boldsymbol{\mathcal{E}}_{\mathbf{R}}$:

Proposition 6.3. *The following proposition hold :*

i) $\boldsymbol{\mathcal{E}}^{\pm}_{\mathbf{R}}=\bigoplus_{\mathbf{p}\in\mathbf{Conv}^{\pm}}\mathbf{R}\tilde{t}_{\mathbf{p}}$,

ii) *there is a triangular decomposition* $\boldsymbol{\mathcal{E}}_{\mathbf{R}}=\boldsymbol{\mathcal{E}}^{+}_{\mathbf{R}}\otimes\boldsymbol{\mathcal{E}}^{-}_{\mathbf{R}}$*; in particular, we have* $\boldsymbol{\mathcal{E}}_{\mathbf{R}}=\bigoplus_{\mathbf{p}\in\mathbf{Conv}}\mathbf{R}\tilde{t}_{\mathbf{p}}$,

iii) *for any* $\alpha,\overline{\alpha}\in\mathbb{C}\backslash\{\pm1\}$ *we have* $(\boldsymbol{\mathcal{E}}_{\mathbf{R}})_{|\substack{\sigma=\alpha\\ \bar{\sigma}=\overline{\alpha}}}=\boldsymbol{\mathcal{E}}_{\alpha,\overline{\alpha}}$,

iv) *we have:* $(\boldsymbol{\mathcal{E}}_{\mathbf{R}})_{\big|\substack{\sigma=1\\ \bar{\sigma}=1}} = \mathbb{C}[\tilde{t}_{\mathbf{x}}]_{\mathbf{x}\in\mathbf{Z}^*}$ *is a (commutative) polynomial algebra.*

Proof. To prove statement i), we have to check that $\boldsymbol{\mathcal{E}}^+_{\mathbf{R}}$ is linearly spanned over $\mathbf{R}$ by $\{\tilde{t}_{\mathbf{p}} \mid \mathbf{p} \in \mathbf{Conv}^+\}$. Let us temporarily denote by $\mathbf{V}$ the space $\bigoplus_{\mathbf{p}\in\mathrm{Conv}^+} \mathbf{R}\tilde{t}_{\mathbf{p}}$. It is enough to show that $\mathbf{V}$ is a subalgebra of $\boldsymbol{\mathcal{E}}^+_{\mathbf{K}}$, i.e. that $\tilde{t}_{\mathbf{p}} \in \mathbf{V}$ for an *arbitrary* path $\mathbf{p}$ in $\mathbf{Z}^+$. For this we proceed along the lines of Lemma 5.6, whose notations we shall freely use. It is sufficient to show that

$$\tilde{t}_{\mathbf{x}}\tilde{t}_{\mathbf{y}} \in \mathbf{V} \tag{6.7}$$

for any $\mathbf{x}, \mathbf{y} \in \mathbf{Z}^+$. We argue by induction on $\big|\det(\mathbf{x}, \mathbf{y})\big| \in \mathbb{N}$. The claim (6.7) is clear if $\det(\mathbf{x}, \mathbf{y}) = 0$. Let us fix an integer $l > 0$ and assume that (6.7) holds for any pair $\mathbf{x}', \mathbf{y}'$ with $\big|\det(\mathbf{x}', \mathbf{y}')\big| < l$. Then, as in Lemma 5.6 we have $\tilde{t}_{\mathbf{q}} \in \mathbf{V}$ for any path $\mathbf{q}$ with $a(\mathbf{q}) < l$. Let us fix a pair $\mathbf{x}, \mathbf{y}$ such that $\det(\mathbf{x}, \mathbf{y}) = l$. Up to $SL(2,\mathbb{Z})$-action, we may assume that $\mathbf{x} = (0, n)$ and $\mathbf{y} = (r, d)$. Because the change of basis matrix between $\{\tilde{t}_{\mathbf{z}}\}$ and $\{\mathbf{1}^{\mathsf{ss}}_{\mathbf{z}}\}$ is invertible over $\mathbf{R}$, it is equivalent to prove that $[\tilde{t}_{\mathbf{x}}, \mathbf{1}^{\mathsf{ss}}_{\mathbf{y}}] \in \mathbf{V}$. By Proposition 6.2 i) we have $[\tilde{t}_{\mathbf{x}}, \mathbf{1}^{\mathsf{ss}}_{\mathbf{y}}] \in \bigoplus_{\mathbf{p}\in I_{\mathbf{x},\mathbf{y}}} \mathbf{K}\tilde{t}_{\mathbf{p}}$. We have to show that all the coefficients belong to $\mathbf{R}$. For this we write

$$\begin{aligned}\mathbf{1}_{\mathbf{y}} = \mathbf{1}_{(r,d)} = &\mathbf{1}^{\mathsf{ss}}_{(r,d)} + \sum_{k\geq 1} \nu^{rk}\mathbf{1}^{\mathsf{ss}}_{(r,d-k)}\mathbf{1}^{\mathsf{ss}}_{(0,k)} + \\ &+ \sum_{\substack{(r_1,d_1)+\cdots+(r_l,d_l)=(r,d)\\ \frac{d_1}{r_1}<\cdots<\frac{d_l}{r_l};\; r_1<r}} \nu^{\sum_{i<j}(r_i d_j - r_j d_i)}\mathbf{1}^{\mathsf{ss}}_{(r_1,d_1)}\cdots\mathbf{1}^{\mathsf{ss}}_{(r_l,d_l)}.\end{aligned} \tag{6.8}$$

Thus

$$\begin{aligned}\tilde{t}_{(0,n)}\mathbf{1}^{\mathsf{ss}}_{\mathbf{y}} = &[\tilde{t}_{(0,n)}, \mathbf{1}_{\mathbf{y}}] - \mathbf{1}_{\mathbf{y}}\tilde{t}_{(0,n)} - \sum_{k\geq 1}\nu^{rk}\tilde{t}_{(0,n)}\mathbf{1}^{\mathsf{ss}}_{(r,d-k)}\mathbf{1}^{\mathsf{ss}}_{(0,k)} - \\ &- \sum_{\substack{(r_1,d_1)+\cdots+(r_l,d_l)=(r,d)\\ \frac{d_1}{r_1}<\cdots<\frac{d_l}{r_l};\; r_1<r}} \nu^{\sum_{i<j}(r_i d_j - r_j d_i)}\tilde{t}_{(0,n)}\mathbf{1}^{\mathsf{ss}}_{(r_1,d_1)}\cdots\mathbf{1}^{\mathsf{ss}}_{(r_l,d_l)}.\end{aligned} \tag{6.9}$$

Observe that the infinite sums in (6.9) become finite after projection to $\bigoplus_{\mathbf{p}\in I_{\mathbf{x},\mathbf{y}}} \tilde{t}_{\mathbf{p}}$. In the second sum in (6.9) we have $r_1 < r$ therefore $\det\big((0,n),(r_1,d_1)\big) < rn = l$ and by our induction hypothesis we may straighten $\tilde{t}_{(0,n)}\mathbf{1}^{\mathsf{ss}}_{(r_1,d_1)} = \sum_{\mathbf{q}\in I_{\mathbf{x},(r_1,d_1)}} u_{\mathbf{q}}\tilde{t}_{\mathbf{q}}$. For any convex path $\mathbf{q} \in I_{\mathbf{x},(r_1,d_1)}$ we have $\mathbf{a}\big(\mathbf{q} \cup ((r_2,d_2),\ldots,(r_l,d_l)\big) < l$ and hence $\tilde{t}_{\mathbf{q}}\mathbf{1}^{\mathsf{ss}}_{(r_2,d_2)}\cdots\mathbf{1}^{\mathsf{ss}}_{(r_l,d_l)} \in \mathbf{V}$. By Lemma 4.12 $\big[\tilde{t}_{(0,n)}, \mathbf{1}_{(r,d)}\big] \in \mathbf{V}$. Finally, after projection to $\bigoplus_{\mathbf{p}\in I_{\mathbf{x},\mathbf{y}}} \mathbf{K}\tilde{t}_{\mathbf{p}}$ we have $\mathbf{1}_{\mathbf{y}}\tilde{t}_{(0,n)} \in \mathbf{V}$. All together, working modulo $\mathbf{V}$ and projecting to $\bigoplus_{\mathbf{p}\in I_{\mathbf{x},\mathbf{y}}} \mathbf{K}\tilde{t}_{\mathbf{p}}$ we get:

$$\tilde{t}_{(0,n)}\mathbf{1}^{\mathsf{ss}}_{(r,d)} \equiv -\sum_{k\geq 1}\nu^{rn}\tilde{t}_{(0,n)}\mathbf{1}^{\mathsf{ss}}_{(r,d-k)}\mathbf{1}^{\mathsf{ss}}_{(0,k)}. \tag{6.10}$$

Substituting (6.10) into itself (i.e. developing each $\tilde{t}_{(0,n)}\mathbf{1}^{\mathsf{ss}}_{(r,d-k)}$ according to (6.10)) sufficiently many times yields an expression

$$\tilde{t}_{(0,n)}\mathbf{1}^{\mathsf{ss}}_{(r,d)} \equiv \sum_{k\geq N}\tilde{t}_{(0,n)}\mathbf{1}^{\mathsf{ss}}_{(r,d-k)}w_k, \tag{6.11}$$

where $w_k \in \mathbf{R}\big[\tilde{t}_{(0,1)}, \tilde{t}_{(0,2)}, \ldots\big]$. For $N \gg 0$ the right-hand side of (6.11) vanishes after projection to $\bigoplus_{\mathbf{p}\in I_{\mathbf{x},\mathbf{y}}} \mathbf{K}\tilde{t}_{\mathbf{p}}$. It follows that $\tilde{t}_{(0,n)}\mathbf{1}^{\mathsf{ss}}_{(r,d)} \in \mathbf{V}$ as desired. Statement i) is proven.

The proof of ii) is completely parallel to that of Proposition 6.2 ii). For iii), notice that by Theorem 5.4 there exists an algebra map

$$\mathsf{ev}_X^* \;: \mathbf{U}_X \simeq \boldsymbol{\mathcal{E}}_{\alpha,\overline{\alpha}} \to (\boldsymbol{\mathcal{E}}_{\mathbf{R}})_{|\substack{\sigma=\alpha \\ \bar\sigma=\overline{\alpha}}}, \qquad \tilde{t}_{\mathbf{x}} \mapsto \tilde{t}_{\mathbf{x}}.$$

This map clearly sends the $\mathbb{C}$-basis $\{\tilde{t}_{\mathbf{p}} \mid \mathbf{p} \in \mathbf{Conv}^+\}$ of $\mathbf{U}_X^+ \otimes_K \mathbb{C}$ to the $\mathbb{C}$-basis $\{\tilde{t}_{\mathbf{p}} \mid \mathbf{p} \in \mathbf{Conv}^+\}$ of $(\boldsymbol{\mathcal{E}}_{\mathbf{R}})_{|\substack{\sigma=\alpha \\ \bar\sigma=\overline{\alpha}}}$. It remains to prove iv). For this we shall show that $[\boldsymbol{\mathcal{E}}_{\mathbf{R}}, \boldsymbol{\mathcal{E}}_{\mathbf{R}}] \in c_1(\sigma,\bar\sigma)\boldsymbol{\mathcal{E}}_{\mathbf{R}}$. Obviously, it is enough to prove that $[\tilde{t}_{\mathbf{x}}, \tilde{t}_{\mathbf{y}}] \in c_1(\sigma,\bar\sigma)\boldsymbol{\mathcal{E}}_{\mathbf{R}}$ for any $\mathbf{x},\mathbf{y} \in \mathbf{Z}^*$. Using the $SL(2,\mathbb{Z})$-action we may assume that $\mathbf{x}=(0,n)$ for $n \geq 0$ and that $\mathbf{y} \in \mathbf{Z}^+$. By Lemma 4.12 we have

$$\left[\tilde{t}_{(0,l)}, \mathbf{1}_\alpha\right] = \frac{c_n(\sigma,\bar\sigma)[\mathsf{rank}(\alpha)]_{\nu^n}}{[n]_\nu}\mathbf{1}_{\alpha+(0,n)}$$

and it is easy to check that $c_n(\sigma,\bar\sigma)/[n]_\nu \in c_1(\sigma,\bar\sigma)\mathbf{R}$. Since the elements $\mathbf{1}_\alpha$ topologically generate $\widehat{\boldsymbol{\mathcal{E}}}^+_{\mathbf{R}}$, we may approximate $\tilde{t}_{\mathbf{y}}$ up to any degree of precision by a polynomial with $\mathbf{R}$-coefficients in the $\mathbf{1}_\alpha$. We conclude using the continuity of the multiplication (see Lemma 2.4). ✓

As a consequence of iv) above and Weyl's theorem (see [W]) there is a natural isomorphism

$$(\boldsymbol{\mathcal{E}}_{\mathbf{R}})_{|\substack{\sigma=1 \\ \bar\sigma=1}} \xrightarrow{\sim} \mathbb{C}[x_1^{\pm 1},\ldots,y_1^{\pm 1},\ldots]^{\mathfrak{S}_\infty} = \boldsymbol{\Lambda}, \quad \tilde{t}_{(r,d)} \mapsto \sum_i x_i^r y_i^d.$$

Hence $\boldsymbol{\mathcal{E}}_{\mathbf{R}}$ may be thought of as a flat deformation of the ring of invariants $\boldsymbol{\Lambda}$.

6.3. There is an obvious $\mathfrak{S}_2$-symmetry in $\boldsymbol{\mathcal{E}}_{\mathbf{K}}$: numbers $\sigma,\bar\sigma$ corresponding to the two Frobenius eigenvalues in $H^1(X_{\overline{\mathbf{k}}}, \overline{\mathbb{Q}_l})$, are interchangeable. Less obvious is the fact that this $\mathfrak{S}_2$-symmetry may be upgraded to an $\mathfrak{S}_3$-symmetry. To see this, we simply renormalize the generators. Set

$$u_{\mathbf{x}} = \frac{t_{\mathbf{x}}}{c_{\mathsf{deg}(\mathbf{x})}(\sigma,\bar\sigma)}, \qquad (\mathbf{x} \in \mathbf{Z}^*)$$

and for any $i \geq 1$ put

$$\alpha_i = \alpha_i(\sigma,\bar\sigma) = (1-\sigma^i)(1-\bar\sigma^i)(1-(\sigma\bar\sigma)^{-i})/i. \tag{6.12}$$

The defining relations in Section 5.2 may now be rewritten as

i) For a pair of collinear $\mathbf{x},\mathbf{x}'$ we have
$$[u_{\mathbf{x}}, u_{\mathbf{x}'}] = 0.$$

ii) Assume that $\mathbf{x},\mathbf{y} \in \mathbf{Z}^*$ are such that $\mathsf{deg}(\mathbf{x})=1$ and that $\boldsymbol{\Delta}_{\mathbf{x},\mathbf{y}}$ has no interior lattice point. Then
$$[u_{\mathbf{y}}, u_{\mathbf{x}}] = \epsilon_{\mathbf{x},\mathbf{y}}\frac{\theta_{\mathbf{x}+\mathbf{y}}}{\alpha_1}$$
where the elements $\theta_{\mathbf{z}}$, $\mathbf{z} \in \mathbf{Z}^*$ are obtained by equating the Fourier coefficients of the collection of relations
$$\sum_i \theta_{i\mathbf{x}_0} s^i = \exp\left(\sum_{r\geq 1} \alpha_r u_{r\mathbf{x}_0} s^r\right), \tag{6.13}$$
for any $\mathbf{x}_0 \in \mathbf{Z}^*$ such that $\mathsf{deg}(\mathbf{x}_0)=1$.

In this presentation it is obvious that $\boldsymbol{\mathcal{E}}_{\mathbf{K}}$ is equipped with an $\mathfrak{S}_3$ family of $\mathbb{C}$-automorphisms Θ_γ for $\gamma \in \mathrm{Perm}\{\sigma,\bar\sigma,(\sigma\bar\sigma)^{-1}\}$ simply defined by $\Theta_\gamma(u_{\mathbf{x}}) = u_{\mathbf{x}}$, $\Theta_\gamma(\bullet) = \gamma(\bullet)$ for $\bullet \in \{\sigma,\bar\sigma,(\sigma\bar\sigma)^{-1}\}$. This symmetry may seem puzzling at first glance : for any fixed elliptic curve X over a finite field $\mathbb{F}_q$ we have $|\sigma| = |\bar\sigma| = q^{1/2}$ while $|(\sigma\bar\sigma)^{-1}| = q^{-1}$.

6.4. In order to define the coproduct Δ of a Hall algebra $\mathbf{H}$ or to construct the Drinfeld double of $\mathbf{H}$, it is usually necessary to add an extra commutative 'Cartan' subalgebra $\boldsymbol{\mathcal{K}}$ to $\mathbf{H}$ (see e.g. [S4]). In the present case of the category of coherent sheaves over an elliptic curve we could avoid doing so because the symmetrized Euler form vanishes. However adding the corresponding "Cartan" subalgebra $\boldsymbol{\mathcal{K}}$ provides a natural central extension $\widetilde{\mathbf{H}}_X$ of $\mathbf{H}$ (and similarly for $\mathbf{U}_X$ and $\boldsymbol{\mathcal{E}}_{\mathbf{K}}$). This central extension is also important in applications (see e.g. [SV2])

For the sake of brevity, we only write down the relations in $\widetilde{\boldsymbol{\mathcal{E}}}_{\mathbf{K}}$, using the rescaled presentation of Section 6.3.

Definition 6.4. Let $\widetilde{\boldsymbol{\mathcal{E}}}_{\mathbf{K}}$ be the $\mathbf{K}$-algebra defined by generators $\{\boldsymbol{\kappa}_\alpha \mid \alpha \in \mathbf{Z}\}$ and $\{u_{\mathbf{x}} \mid \mathbf{x} \in \mathbf{Z}^*\}$ modulo the following set of relations :

i) the subalgebra $\boldsymbol{\mathcal{K}}$ generated by $\{\boldsymbol{\kappa}_\alpha \mid \alpha \in \mathbf{Z}\}$ is central and we have
$$\boldsymbol{\kappa}_0 = 1, \qquad \boldsymbol{\kappa}_\alpha \boldsymbol{\kappa}_\beta = \boldsymbol{\kappa}_{\alpha+\beta},$$

ii) if $\mathbf{x}, \mathbf{y}$ belong to the same line in $\mathbf{Z}$ then
$$[u_{\mathbf{y}}, u_{\mathbf{x}}] = \delta_{\mathbf{x},-\mathbf{y}} \frac{\boldsymbol{\kappa}_{\mathbf{x}} - \boldsymbol{\kappa}_{\mathbf{x}}^{-1}}{\alpha_{\mathsf{deg}(\mathbf{x})}}$$

iii) if $\mathbf{x}, \mathbf{y} \in \mathbf{Z}^*$ are such that $\mathsf{deg}(\mathbf{x}) = 1$ and that $\boldsymbol{\Delta}_{\mathbf{x},\mathbf{y}}$ has no interior lattice point then
$$[u_{\mathbf{y}}, u_{\mathbf{x}}] = \epsilon_{\mathbf{x},\mathbf{y}} \boldsymbol{\kappa}_{\alpha(\mathbf{x},\mathbf{y})} \frac{\theta_{\mathbf{x}+\mathbf{y}}}{\alpha_1},$$
where
$$\alpha(\mathbf{x},\mathbf{y}) = \begin{cases} \epsilon_{\mathbf{x}}(\epsilon_{\mathbf{x}}\mathbf{x} + \epsilon_{\mathbf{y}}\mathbf{y} - \epsilon_{\mathbf{x}+\mathbf{y}}(\mathbf{x}+\mathbf{y}))/2 & \text{if } \epsilon_{\mathbf{x},\mathbf{y}} = 1, \\ \epsilon_{\mathbf{y}}(\epsilon_{\mathbf{x}}\mathbf{x} + \epsilon_{\mathbf{y}}\mathbf{y} - \epsilon_{\mathbf{x}+\mathbf{y}}(\mathbf{x}+\mathbf{y}))/2 & \text{if } \epsilon_{\mathbf{x},\mathbf{y}} = -1, \end{cases}$$
and where the elements $\theta_{\mathbf{z}}$, $\mathbf{z} \in \mathbf{Z}^*$, are given by
$$\sum_i \theta_{i\mathbf{x}_0} s^i = \exp\Big(\sum_{r \geq 1} \alpha_r u_{r\mathbf{x}_0} s^r\Big),$$
for any $\mathbf{x}_0 \in \mathbf{Z}^*$ such that $\mathsf{deg}(\mathbf{x}_0) = 1$, where the coefficients α_r are given by (6.12).

Note that by relation ii), the algebra $\widetilde{\boldsymbol{\mathcal{E}}}_{\mathbf{K}}$ contains many copies of the Heisenberg algebra (one for each line in $\mathbf{Z}$). Hence $\widetilde{\boldsymbol{\mathcal{E}}}_{\mathbf{K}}$ can be thought of as a flat deformation of a Heisenberg algebra over $\mathbf{Z}$.

The triangular decomposition of $\widetilde{\boldsymbol{\mathcal{E}}}_{\mathbf{K}}$ now takes the form
$$\widetilde{\boldsymbol{\mathcal{E}}}_{\mathbf{K}} \simeq \boldsymbol{\mathcal{E}}_{\mathbf{K}}^+ \otimes \boldsymbol{\mathcal{K}} \otimes \boldsymbol{\mathcal{E}}_{\mathbf{K}}^-. \tag{6.14}$$
One consequence of the central extension is that the group $SL(2,\mathbb{Z})$ no longer acts on $\widetilde{\boldsymbol{\mathcal{E}}}_{\mathbf{K}}$: only its universal cover $\widetilde{SL}(2,\mathbb{Z})$ does. There is a short exact sequence
$$1 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{SL}(2,\mathbb{Z}) \longrightarrow SL(2,\mathbb{Z}) \longrightarrow 1.$$
For any slope $\frac{q}{p} \in \mathbb{Q} \cup \{\infty\}$ and any $\gamma \in \widetilde{SL}(2,\mathbb{Z})$ we define a winding number $n(\gamma, \frac{q}{p})$ as follows. There is a natural action of $SL(2,\mathbb{Z})$ on the circle $S^1 = (\mathbb{R}^2 \setminus \{0\})/\mathbb{R}^{+\star}$. Using the identification $S^1 = \mathbb{R}/2\mathbb{Z}$, we can uniquely lift this action to an $\widetilde{SL}(2,\mathbb{Z})$-action on $\mathbb{R}$. Any $(q,p) \in \mathbf{Z}^*$ gives rise to an element $(q:p) \in S^1$ and if $\overline{(q:p)} \in \mathbb{R}$ is any lift of $(q:p)$ then
$$n\Big(\tilde{\gamma}, \frac{q}{p}\Big) = \begin{cases} \#\big(\mathbb{Z} \cap [\overline{(q,p)}, \tilde{\gamma}(\overline{(q:p)})]\big) & \text{if } \tilde{\gamma}(\overline{(q:p)}) \geq \overline{(q:p)} \\ -\#\big(\mathbb{Z} \cap [\tilde{\gamma}(\overline{(q,p)}), \overline{(q:p)}]\big) & \text{otherwise.} \end{cases} \tag{6.15}$$

One checks that the following rule gives rise to an $\widetilde{SL}(2,\mathbb{Z})$-action on $\widetilde{\boldsymbol{\mathcal{E}}}_{\mathbf{K}}$ by automorphisms :

$$\Phi(\boldsymbol{\kappa}_{\mathbf{x}}) = \boldsymbol{\kappa}_{\Phi(\mathbf{x})}, \qquad \Phi(u_{\mathbf{x}}) = u_{\Phi(\mathbf{x})}\boldsymbol{\kappa}_{\Phi(\mathbf{x})}^{n(\Phi,\mu(\mathbf{x}))}. \tag{6.16}$$

We leave it to the reader to define the integral forms, specializations, etc of $\widetilde{\boldsymbol{\mathcal{E}}}_{\mathbf{K}}$. All properties (such as (6.14) and (6.16), finite generation etc.) extend to these settings.

7. Summary

Let us sum up the main results obtained in this article. To any elliptic curve X defined over a finite field $\boldsymbol{k} = \mathbb{F}_q$ we have attached an associative algebra $\mathbf{U}_X$ over the field $K = \mathbb{Q}(v)$, where $v^{-2} = q$. Let $\sigma \in \overline{\mathbb{Q}}$ be such that $\bar{\sigma}\sigma = q$ and $\left|X(\mathbb{F}_{q^i})\right| = q^i + 1 - (\sigma^i + \bar{\sigma}^i)$. Then we have:

1. The algebra $\mathbf{U}_X$ is $\mathbb{Z}^2$-graded and $K = \mathbf{U}_X\bigl[(0,0)\bigr]$ is the center of $\mathbf{U}_X$.

2. The algebra $\mathbf{U}_X$ can be described by the following generators and relations:
(1) For $(r,d) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ we have a generator $T_{(r,d)} \in \mathbf{U}_X\bigl[(r,d)\bigr]$.
(2) Let $\mathsf{gcd}(r,d) = 1$, then we defined elements $\Theta_{i(r,d)} \in \mathbf{U}_X\bigl[i(r,d)\bigr]$, $i \geq 1$ using the following equality

$$1 + \sum_{i=1}^{\infty} \Theta_{i(r,d)} s^i = \exp\bigl((v^{-1} - v)\sum_{j=1}^{\infty} T_{j(r,d)} s^j\bigr),$$

where s is a formal parameter.
(3) If the vectors (r,d) and (r',d') are collinear then we have:

$$\bigl[T_{(r,d)}, T_{(r',d')}\bigr] = 0.$$

(4) Assume that $(r,d), (r',d') \in \mathbb{Z}^2 \setminus \{(0,0)\}$ are such that $\mathsf{gcd}(r,d) = 1$ and the triangle with the corners $(0,0), (r,d), (r',d')$ contains no interior points. Then

$$\bigl[T_{(r,d)}, T_{(r',d')}\bigr] = \mathrm{sign}\bigl(rd' - r'd\bigr) c_h \frac{\Theta_{(r+r',d+d')}}{v - v^{-1}},$$

where $h = \mathsf{gcd}(r',d')$ and $c_h = \dfrac{v^h[h]_v}{h}\bigl|X(\mathbb{F}_{q^h})\bigr|$. Relations (3) and (4) form a complete list of relations of $\mathbf{U}_X$, see Theorem 5.4. The structure constants of $\mathbf{U}_X$ are Laurent polynomials in $\sigma^{\pm\frac{1}{2}}$ and $\bar{\sigma}^{\pm\frac{1}{2}}$, so we may also introduce a generic version $\boldsymbol{\mathcal{E}}_{\mathbf{R}}$ of the Hall algebra $\mathbf{U}_X$, defined over the ring $\mathbf{R} = \mathbb{C}[\sigma^{\pm\frac{1}{2}}, \bar{\sigma}^{\pm\frac{1}{2}}]$, see Section 6.2.

3. The algebra $\mathbf{U}_X$ is finitely generated and the elements $T_{(\pm 1,0)}, T_{(0,\pm 1)}$ generate $\mathbf{U}_X$, see Corollary 6.1.

4. The algebra $\mathbf{U}_X$ carries a natural $SL(2,\mathbb{Z})$–action: for any $\gamma \in SL(2,\mathbb{Z})$ the map $T_{(r,d)} \mapsto T_{\gamma(r,d)}$ induces an algebra automorphism of $\mathbf{U}_X$.

5. Let $\mathbf{U}_X^{\pm} = \bigl\langle T_{(r,d)} | (r,d) \in (\mathbb{Z}^2)^{\pm}\bigr\rangle$, then $\mathbf{U}_X^{\pm}$ are graded topological bialgebras, see Lemma 4.6. This means that there is a graded coassociative ring homomorphism

$$\Delta : \mathbf{U}_X^{\pm} \longrightarrow \mathbf{U}_X^{\pm}\widehat{\otimes}\mathbf{U}_X^{\pm},$$

taking value in a certain completion of $\mathbf{U}_X^{\pm} \otimes \mathbf{U}_X^{\pm}$ and given by the collection of linear maps for each $\alpha, \beta \in (\mathbb{Z}^2)^{\pm}$

$$\Delta_{\alpha,\beta} : \mathbf{U}_X^{\pm}\bigl[\alpha + \beta\bigr] \longrightarrow \mathbf{U}_X^{\pm}[\alpha] \otimes \mathbf{U}_X^{\pm}[\beta].$$

6. The algebra $\mathbf{U}_X$ is isomorphic to the Drinfeld double of the topological bialgebra $\mathbf{U}_X^+$ and one has the decomposition $\mathbf{U}_X = \mathbf{U}_X^+ \otimes \mathbf{U}_X^-$, where $\mathbf{U}_X^\pm = \langle T_{(r,d)} | (r,d) \in (\mathbb{Z}^2)^+ \rangle$, see Theorem 4.5.

7. The algebra $\mathbf{U}_X$ has a monomial basis $\{T_{(r_1,d_1)}T_{(r_2,d_2)}\dots T_{(r_n,d_n)}\}$ parameterized by the set of convex paths $\big((r_1,d_1),(r_2,d_2),\dots,(r_n,d_n)\big)$ in $\mathbb{Z}^2$, see Theorem 4.5.

8. The algebra $\mathbf{U}_X$ is a flat deformation of the ring

$$K\big[x_1^\pm, x_2^\pm, \dots, y_1^\pm, y_2^\pm, \dots\big]^{\mathfrak{S}_\infty}$$

of symmetric Laurent series.

9. One can also write down some explicit formulas for the coproduct of certain generators of $\mathbf{U}_X^+$ (Proposition 4.1 and Lemma 4.11):

$$\Delta\big(T_{(0,d)}\big) = T_{(0,d)} \otimes 1 + 1 \otimes T_{(0,d)} \text{ and } \Delta\big(T_{(1,d)}\big) = T_{(1,d)} \otimes 1 + \sum_{l \geq 0} \Theta_{(0,l)} \otimes T_{(1,d-l)}.$$

Appendix A

In this appendix, we provide the details regarding the properties of the Fourier-Mukai transforms on elliptic curves defined over a *finite* field $\boldsymbol{k}$.

For a projective curve $\mathcal{Y}$ defined over the field $\boldsymbol{k}$ consider the functor $\mathrm{Pic}^0_{\mathcal{Y}/\boldsymbol{k}}$: $\mathrm{Sch}_{\boldsymbol{k}} \longrightarrow \mathrm{Sets}$ given by

$$\mathrm{Pic}^0_{\mathcal{Y}/\boldsymbol{k}}(\mathcal{S}) = \Big\{\mathcal{F} \in \mathrm{Coh}_{\mathcal{Y}\times\mathcal{S}} |\ \mathcal{F} \text{ is } \mathcal{S}-\text{flat and for any closed point}$$
$$s : \mathrm{Spec}(\boldsymbol{l}) \longrightarrow S \text{ holds } s_{\boldsymbol{l}}^*(\mathcal{F}) \in \mathrm{Pic}^0(\mathcal{Y}_{\boldsymbol{l}})\Big\}\Big/\sim$$

where $\mathcal{Y}_{\boldsymbol{l}} = \mathcal{Y} \times_{\mathrm{Spec}(\boldsymbol{k})} \mathrm{Spec}(\boldsymbol{l})$ and the map $s_{\boldsymbol{l}} : \mathcal{Y}_{\boldsymbol{l}} \longrightarrow \mathcal{Y} \times \mathcal{S}$ is induced by the base change and the equivalence relation is $\mathcal{F} \sim \mathcal{F} \otimes \pi_{\mathcal{S}}^*(\mathcal{L})$ for any locally free rank one sheaf $\mathcal{L}$ on $\mathcal{S}$.

In the case of an elliptic curve X over $\boldsymbol{k}$ with a rational point p_0 the functor $\mathrm{Pic}^0_{X/\boldsymbol{k}}$ is representable by the pair $(X,\mathcal{P})$, where $\mathcal{P} = \mathcal{O}_{X\times X}(-\Delta + p_0 \times X + X \times p_0)$ and $\Delta \subset X \times X$ is the diagonal, see for example [AK, example 8.9.iii].

The sheaf $\mathcal{P}^\vee$ is locally free on $X \times X$ and hence flat over X. Moreover, for any closed point $p : \mathrm{Spec}(\boldsymbol{l}) \longrightarrow X$ one has an isomorphism $\mathcal{P}^\vee\big|_{X_{\boldsymbol{l}}} \cong \big(\mathcal{P}\big|_{X_{\boldsymbol{l}}}\big)^\vee$. By the universal property of $(X,\mathcal{P})$ there exists a unique map $i : X \longrightarrow X$ and a line bundle $\mathcal{L}$ on X such that $\mathcal{P}^\vee \otimes \pi_2^*\mathcal{L} \cong (1 \times i)^*\mathcal{P}$. Denote by $\sigma = p_0 \times 1 : X \longrightarrow X \times X$. From equalities $\sigma^*\mathcal{P}^\vee \cong \mathcal{O}$ and $\sigma^*(1 \times i)^*\mathcal{P} \cong (1 \times i)^*\sigma^*\mathcal{P} \cong \mathcal{O}$ we conclude that

$$\mathcal{P}^\vee \cong (1 \times i)^*\mathcal{P}.$$

Moreover, the isomorphism $\mathcal{P} \cong \mathcal{P}^{\vee\vee}$ and the universality of $(X,\mathcal{P})$ imply $i^2 = 1$.

Proposition A.1. *Let $\mathcal{Y}$ be a projective variety over $\boldsymbol{k}$ and $\bar{\boldsymbol{k}}$ the algebraic closure of $\boldsymbol{k}$. For any field extension $\boldsymbol{k} \subset \boldsymbol{l}$ denote by $\mathcal{Y}_{\boldsymbol{l}} = \mathcal{Y} \times_{\mathrm{Spec}(\boldsymbol{k})} \mathrm{Spec}(\boldsymbol{l})$ and by $\varphi_{\boldsymbol{l}} : \mathcal{Y}_{\boldsymbol{l}} \longrightarrow \mathcal{Y}$ the base-change map. Let $\mathcal{F}$, $\mathcal{G}$ be two coherent sheaves, denote $\mathcal{F}_{\boldsymbol{l}} = \varphi_{\boldsymbol{l}}^*(\mathcal{F})$ and $\mathcal{G}_{\boldsymbol{l}} = \varphi_{\boldsymbol{l}}^*(\mathcal{G})$. Assume that $\mathcal{F}_{\bar{\boldsymbol{k}}} \cong \mathcal{G}_{\bar{\boldsymbol{k}}}$ then $\mathcal{F} \cong \mathcal{G}$.*

Proof. Let $f : \mathcal{F}_{\bar{\boldsymbol{k}}} \longrightarrow \mathcal{G}_{\bar{\boldsymbol{k}}}$ and $g : \mathcal{G}_{\bar{\boldsymbol{k}}} \longrightarrow \mathcal{F}_{\bar{\boldsymbol{k}}}$ be two maps such that $gf = 1_{\mathcal{F}_{\bar{\boldsymbol{k}}}}$ and $fg = 1_{\mathcal{G}_{\bar{\boldsymbol{k}}}}$. From the isomorphism $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F},\mathcal{G}) \otimes_{\boldsymbol{k}} \bar{\boldsymbol{k}} \cong \mathrm{Hom}_{\mathcal{O}_{\bar{\boldsymbol{k}}}}(\mathcal{F}_{\bar{\boldsymbol{k}}},\mathcal{G}_{\bar{\boldsymbol{k}}})$ follows that $f = \sum_{i=1}^n \bar{a}_i\varphi_i$ and $g = \sum_{j=1}^m \bar{b}_j\psi_j$, where $\bar{a}_1,\dots,\bar{a}_n,\bar{b}_1,\dots,\bar{b}_m \in \bar{\boldsymbol{k}}$, $\varphi_1,\dots,\varphi_n$ is a basis of $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F},\mathcal{G})$ over $\boldsymbol{k}$ and $\psi_1,\dots,\psi_m$ a basis of $\mathrm{Hom}_{\mathcal{O}}(\mathcal{G},\mathcal{F})$ over $\boldsymbol{k}$. Let $\boldsymbol{l}$ be the finite extension of $\boldsymbol{k}$ generated by the elements $\bar{a}_1,\dots,\bar{a}_n;\bar{b}_1,\dots,\bar{b}_m$, then for the base-change map $\varphi_{\boldsymbol{l}} : \mathcal{Y}_{\boldsymbol{l}} \longrightarrow X$ we have $\varphi_{\boldsymbol{l}}^*(\mathcal{F}) \cong \varphi_{\boldsymbol{l}}^*(\mathcal{G})$. By the projection formula we get $\varphi_{\boldsymbol{l}*}(\varphi_{\boldsymbol{l}}^*\mathcal{F}) = \mathcal{F} \otimes \varphi_{\boldsymbol{l}*}(\mathcal{O}_{\boldsymbol{l}})$. Let $d = \deg(\boldsymbol{l}/\boldsymbol{k})$. Since $\mathrm{Spec}(\boldsymbol{l}) = \underbrace{\mathrm{Spec}(\boldsymbol{k}) \sqcup \mathrm{Spec}(\boldsymbol{k}) \sqcup \cdots \sqcup \mathrm{Spec}(\boldsymbol{k})}_{d \text{ times}}$ as a scheme over $\boldsymbol{k}$, we have $\mathcal{Y}_{\boldsymbol{l}} = \underbrace{\mathcal{Y} \sqcup \mathcal{Y} \sqcup \cdots \sqcup \mathcal{Y}}_{d \text{ times}}$ and $\varphi_{\boldsymbol{l}*}(\mathcal{O}_{\boldsymbol{l}}) = \mathcal{O}^d$. Hence $\mathcal{F}^d \cong \mathcal{G}^d$ and the Krull-Schmidt theorem implies $\mathcal{F} \cong \mathcal{G}$. ✓

Proof of Proposition 1.2. In the case of an algebraically closed field $\bar{\boldsymbol{k}}$ this result was shown by Mukai [Mu]. This isomorphism is equivalent to the fact that

$$\mathbf{R}\pi_{13}(\pi_{12}^*\mathcal{P} \otimes \pi_{23}^*\mathcal{P}) \cong \mathcal{O}_{i(\Delta)}[-1],$$

where $\mathcal{O}_{i(\Delta)}$ is the structure sheaf of the subscheme $i(\Delta) \subseteq X \times X$. The case of a finite field $\boldsymbol{k}$ can be derived from the corresponding result about $\bar{\boldsymbol{k}}$ by going into the algebraic closure: $\varphi_{\bar{\boldsymbol{k}}} : X_{\bar{\boldsymbol{k}}} \longrightarrow X$ and using the isomorphism

$$(\varphi_{\bar{\boldsymbol{k}}} \times \varphi_{\bar{\boldsymbol{k}}})^*\mathcal{O}_{X\times X}(-\Delta + p_0 \times X + X \times p_0) \cong \mathcal{O}_{X_{\bar{\boldsymbol{k}}}\times X_{\bar{\boldsymbol{k}}}}(-\Delta_{\bar{\boldsymbol{k}}} + \bar{p}_0 \times X_{\bar{\boldsymbol{k}}} + X_{\bar{\boldsymbol{k}}} \times \bar{p}_0),$$

the flat base-change and the Proposition A.1 above. ✓

Proposition A.2. [see [Mu, Proposition 3.8]] *Let $D = \mathbf{R}\mathcal{H}om(-, \mathcal{O})$ be the dualizing functor. Then there is an isomorphism of functors*

$$D \circ \Phi \cong i^* \circ [1] \circ \Phi \circ D.$$

Proof. This result is a corollary of the isomorphism $\mathcal{P}^\vee \cong (1 \times i)^*\mathcal{P}$ and can be proven along the same lines as in [Mu]. ✓

Appendix B

In the second appendix, we provide proofs for some technical statements regarding the Drinfeld double construction for topological bialgebras and some properties of Hopf algebras, which are crucial for the proof of Theorem 4.5.

Proof of Lemma 3.3. For simplicity, we drop the exponents ± in the notation. Since both statements in the Lemma are similar, we give a proof only of the first one. By assumption, we have for any k

$$\sum_{i,j} a_j^{(1)} (c_k^{(1)})_i^{(2)} \big((c_k^{(1)})_i^{(1)}, a_j^{(2)}\big) = \sum_{i,j} (c_k^{(1)})_i^{(1)} a_j^{(2)} \big((c_k^{(1)})_i^{(2)}, a_j^{(1)}\big), \tag{B.1}$$

$$\sum_{i,j} b_j^{(1)} (c_k^{(2)})_i^{(2)} \big((c_k^{(2)})_i^{(1)}, b_j^{(2)}\big) = \sum_{i,j} (c_k^{(2)})_i^{(1)} b_j^{(2)} \big((c_k^{(2)})_i^{(2)}, b_j^{(1)}\big). \tag{B.2}$$

Note that all sums above are in fact finite. Now, we compute

$$\begin{aligned} \sum_{k,j} (ab)_j^{(1)} c_k^{(2)} \big(c_k^{(1)}, (ab)_j^{(2)}\big) &= \sum_{i,j,k} a_i^{(1)} b_j^{(1)} c_k^{(2)} \big(c_k^{(1)}, a_i^{(2)} b_j^{(2)}\big) \\ &= \sum_{i,j,k,l} a_i^{(1)} b_j^{(1)} c_k^{(2)} \big((c_k^{(1)})_l^{(1)}, a_i^{(2)}\big)\big((c_k^{(1)})_l^{(2)}, b_j^{(2)}\big), \end{aligned} \tag{B.3}$$

where we used the Hopf property of the pairing (,) and Proposition 2.2. Next, by coassociativity, we have $\sum_{k,l} (c_k^{(1)})_l^{(1)} \otimes (c_k^{(1)})_l^{(2)} \otimes c_k^{(2)} = \sum_{k,l} c_k^{(1)} \otimes (c_k^{(2)})_l^{(1)} \otimes (c_k^{(2)})_l^{(2)}$, and substituting in (B3), we obtain

$$\begin{aligned}(B.4)\qquad &\sum_{i,j,k,l} a_i^{(1)}b_j^{(1)}c_k^{(2)}\big((c_k^{(1)})_l^{(1)},a_i^{(2)}\big)\big((c_k^{(1)})_l^{(2)},b_j^{(2)}\big)\\ &= \sum_{i,j,k,l} a_i^{(1)}b_j^{(1)}(c_k^{(2)})_l^{(2)}(c_k^{(1)},a_i^{(2)})\big((c_k^{(2)})_l^{(1)},b_j^{(2)}\big)\\ &= \sum_{i,j,k,l} a_i^{(1)}(c_k^{(2)})_l^{(1)}b_j^{(2)}(c_k^{(1)},a_i^{(2)})\big((c_k^{(2)})_l^{(2)},b_j^{(1)}\big),\end{aligned}$$

where we made use of (B.2). In the same way, coassociativity and (B.1) allow us to transform the last expression into

$$\begin{aligned}&\sum_{i,j,k,l} a_i^{(1)}(c_k^{(1)})_l^{(2)}b_j^{(2)}\big((c_k^{(1)})_l^{(1)},a_i^{(2)}\big)(c_k^{(2)},b_j^{(1)})\\ &= \sum_{i,j,k,l} (c_k^{(1)})_l^{(1)}a_i^{(2)}b_j^{(2)}\big((c_k^{(1)})_l^{(2)},a_i^{(1)}\big)(c_k^{(2)},b_j^{(1)})\\ &= \sum_{i,j,k,l} c_k^{(1)}a_i^{(2)}b_j^{(2)}\big((c_k^{(2)})_l^{(1)},a_i^{(1)}\big)\big((c_k^{(2)})_l^{(2)},b_j^{(1)}\big).\end{aligned}$$

Finally, using the Proposition 2.2 and the Hopf property of (,) again, we can rewrite the last term as

$$\sum_{i,j,k} c_k^{(1)}a_i^{(2)}b_j^{(2)}\big(c_k^{(2)},a_i^{(1)}b_j^{(1)}\big) = \sum_{i,k} c_k^{(1)}(ab)_i^{(2)}\big(c_k^{(2)},(ab)_i^{(1)}\big).$$

All together, we see that $R(ab,c)$ is a consequence of relations (B.1) and (B.2). The Lemma is proved. ✓

The remaining part of Appendix B is devoted to the

Proof of Proposition 3.4. Recall that by the definition of Δ we have

$$\Delta\big([\mathcal{F}]\big) = \sum_{\mathcal{K}\rightarrowtail\mathcal{F}} v^{-\langle\mathcal{F}/\mathcal{K},\mathcal{K}\rangle}\frac{P^{\mathcal{F}}_{\mathcal{F}/\mathcal{K},\mathcal{K}}}{a_{\mathcal{F}}}[\mathcal{F}/\mathcal{K}]\otimes\mathcal{K}.$$

Iterating this formula we have

$$\Delta^2\big([\mathcal{F}]\big) = (1\otimes\Delta)\Delta\big([\mathcal{F}]\big) = \sum_{\mathcal{L}\rightarrowtail\mathcal{K}\rightarrowtail\mathcal{F}} c^{\mathcal{F}}_{\mathcal{F}/\mathcal{K},\mathcal{K}/\mathcal{L},\mathcal{L}}[\mathcal{F}/\mathcal{K}]\otimes[\mathcal{K}/\mathcal{L}]\otimes[\mathcal{L}],$$

where $c^{\mathcal{F}}_{\mathcal{F}/\mathcal{K},\mathcal{K}/\mathcal{L},\mathcal{L}} = P^{\mathcal{F}}_{\mathcal{F}/\mathcal{K},\mathcal{K}}P^{\mathcal{K}}_{\mathcal{K}/\mathcal{L},\mathcal{L}}\dfrac{v^{-\langle\mathcal{K}/\mathcal{K},\mathcal{K}\rangle-\langle\mathcal{K}/\mathcal{L},\mathcal{L}\rangle}}{a_{\mathcal{F}}}$. So, in general we can write

$$(B.5)\qquad \Delta^n\big([\mathcal{F}]\big) = \sum_{\mathcal{F}_n\rightarrowtail\mathcal{F}_{n-1}\rightarrowtail\cdots\rightarrowtail\mathcal{F}_1\rightarrowtail\mathcal{F}} c^{\mathcal{F}}_{\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_{n+1}}[A_1]\otimes[\mathcal{A}_2]\otimes\cdots\otimes[\mathcal{A}_{n+1}],$$

where $\mathcal{A}_i = \mathcal{F}_{i-1}/\mathcal{F}_i$, $\mathcal{A}_{n+1} = \mathcal{F}_n$ and

$$c^{\mathcal{F}}_{\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_{n+1}} = P^{\mathcal{F}}_{\mathcal{A}_1,\mathcal{F}_1}P^{\mathcal{F}_1}_{\mathcal{A}_2,\mathcal{F}_2}\ldots P^{\mathcal{F}_{n-1}}_{\mathcal{A}_n,\mathcal{F}_n}\frac{v^{-\sum_{i=1}^n\langle\mathcal{A}_i,\mathcal{F}_i\rangle}}{a_{\mathcal{F}}}.$$

We can also write in a dual way:

$$\Delta^n\big([\mathcal{F}]\big) = \sum_{\mathcal{F}\twoheadrightarrow\mathcal{F}_1\twoheadrightarrow\mathcal{F}_2\twoheadrightarrow\cdots\twoheadrightarrow\mathcal{F}_n} d^{\mathcal{F}}_{\mathcal{B}_1,\mathcal{B}_2,\ldots,\mathcal{B}_{n+1}}[\mathcal{B}_{n+1}]\otimes[\mathcal{B}_n]\otimes\cdots\otimes[\mathcal{B}_1],$$

where $\mathcal{B}_{n+1} = \mathcal{F}_n$, $\mathcal{B}_i = \ker(\mathcal{F}_{i-1}\longrightarrow\mathcal{F}_i)$ and

$$(B.6)\qquad d^{\mathcal{F}}_{\mathcal{B}_1,\mathcal{B}_2,\ldots,\mathcal{B}_{n+1}} = P^{\mathcal{F}}_{\mathcal{F}_1,\mathcal{B}_1}P^{\mathcal{F}_1}_{\mathcal{F}_2,\mathcal{B}_2}\ldots P^{\mathcal{F}_{n-1}}_{\mathcal{F}_n,\mathcal{B}_n}\frac{v^{-\sum_{i=1}^n\langle\mathcal{F}_i,\mathcal{B}_i\rangle}}{a_{\mathcal{F}}}.$$

Definition B.1. For $\alpha\in(\mathbb{Z}^2)^+$ define an operator $T:\mathbf{H}_X[\alpha]\longrightarrow\widehat{\mathbf{H}}_X[\alpha]$ by the following formulas:

(1) $T\big([0]\big) = T(1) = 1$.
(2) Let $\alpha \neq (0,0)$ and $\mathcal{F}$ be a coherent sheaf of class α. Then

$$\text{(B.7)}\quad T\big([\mathcal{F}]\big) = -[\mathcal{F}] + \sum_{n=1}^{\infty}(-1)^n \sum_{\mathcal{F}_n \overset{\neq}{\rightarrowtail} \cdots \overset{\neq}{\rightarrowtail} \mathcal{F}_1 \overset{\neq}{\rightarrowtail} \mathcal{F}} c^{\mathcal{F}}_{\mathcal{A}_1,\mathcal{A}_2,\dots,\mathcal{A}_{n+1}} [\mathcal{A}_{n+1}] \otimes \cdots \otimes [\mathcal{A}_1],$$

where $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_{n+1}$ and $c^{\mathcal{F}}_{\mathcal{A}_1,\mathcal{A}_2,\dots,\mathcal{A}_{n+1}}$ are the same as in (B.5).

In order to see that the operator T is well-defined we introduce one more definition:

Definition B.2. We call two coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ of the same rank and degree *swept-equivalent*, if there exist two filtrations $0 = \mathcal{F}_{n+1} \subset \mathcal{F}_n \subset \mathcal{F}_{n-1} \subset \cdots \subset \mathcal{F}_1 \subset \mathcal{F}_0 = \mathcal{F}$ and $0 = \mathcal{G}_{n+1} \subset \mathcal{G}_n \subset \mathcal{G}_{n-1} \subset \cdots \subset \mathcal{G}_1 \subset \mathcal{G}_0 = \mathcal{G}$ with quotients $\mathcal{K}_i := \mathcal{F}_{i-1}/\mathcal{F}_i \cong \mathcal{G}_{n-i+1}/\mathcal{G}_{n-i+2}$, $1 \le i \le n+1$. Two such filtrations are called *admissible* filtrations associated to the swept-equivalent pair $(\mathcal{F},\mathcal{G})$.

Lemma B.3. *For given two coherent sheaves $\mathcal{F}$ and $\mathcal{G}$*

- *there are finitely many swept-equivalent pairs of coherent sheaves $(\mathcal{F}',\mathcal{G}')$ such that $\mathcal{F}' \rightarrowtail \mathcal{F}$ and $\mathcal{G} \twoheadrightarrow \mathcal{G}'$.*
- *If $\mathcal{F}$ and $\mathcal{G}$ are themselves swept-equivalent, then there exist only finitely many admissible filtrations associated with $(\mathcal{F},\mathcal{G})$.*

Proof. Let us first deal with the second part. Denote by $\tau(\mathcal{H})$ the torsion part of the sheaf $\mathcal{H}$. We argue by induction on the pair $\big(\mathsf{rank}(\mathcal{G}), \mathsf{deg}(\tau(\mathcal{F}))\big)$, where the order is lexicographic. The Lemma is obvious if $\mathsf{rank}(\mathcal{G}) = 0$. Now we fix $\mathcal{F},\mathcal{G} \in Coh(X)$ and assume we have an admissible filtration associated with the pair $(\mathcal{F},\mathcal{G})$ and having the quotients $\mathcal{K}_1, \mathcal{K}_2, \dots, \mathcal{K}_{n+1}$.

Note that $\mathcal{K}_{n+1}$ is both a subsheaf of $\mathcal{F}$ and a quotient of $\mathcal{G}$. Hence there are only finitely many possibilities for $\mathcal{K}_{n+1}$, and for each such $\mathcal{K}_{n+1}$, only finitely many embeddings $\phi : \mathcal{K}_{n+1} \rightarrowtail \mathcal{F}$ and quotients $\psi : \mathcal{G} \twoheadrightarrow \mathcal{K}_{n+1}$. For fixed ϕ and ψ there is a bijection between admissible filtrations of $(\mathcal{F},\mathcal{G})$ with quotients $\mathcal{K}_1,\dots,\mathcal{K}_{n+1}$ and admissible filtrations of $(\mathrm{coker}(\phi), \ker(\psi))$ with quotients $\mathcal{K}_1,\dots,\mathcal{K}_n$. But $\big(\mathsf{rank}(\ker(\psi)), \mathsf{deg}(\tau(\mathrm{coker}(\phi)))\big) < \big(\mathsf{rank}(\mathcal{G}), \mathsf{deg}(\tau(\mathcal{F}))\big)$, so the induction hypothesis allows us to conclude.

To prove the first part note that there is a bijection between the sequences of inclusions $0 = \mathcal{H}'_{n+1} \rightarrowtail \mathcal{H}'_n \rightarrowtail \mathcal{H}'_{n-1} \rightarrowtail \cdots \rightarrowtail \mathcal{H}'_0 = \mathcal{H}$ and sequences of projections $\mathcal{H} = \mathcal{H}''_0 \twoheadrightarrow \mathcal{H}''_1 \twoheadrightarrow \dots \mathcal{H}''_n \twoheadrightarrow \mathcal{H}''_{n+1} = 0$ such that $\mathrm{coker}(\mathcal{H}'_{i+1} \rightarrowtail \mathcal{H}'_i) \cong \ker(\mathcal{H}''_{n-i} \twoheadrightarrow \mathcal{H}''_{n-i+1})$ (we can simply put $\mathcal{H}''_i := \mathcal{H}/\mathcal{H}'_{n-i+1}$).

Therefore, existence of a swept-equivalent pair $(\mathcal{F}',\mathcal{G}')$, where $\mathcal{F}' \rightarrowtail \mathcal{F}$, $\mathcal{G} \twoheadrightarrow \mathcal{G}'$ is equivalent to existence of a sequence of inclusions $0 \rightarrowtail \mathcal{F}'_n \rightarrowtail \cdots \rightarrowtail \mathcal{F}'_1 \rightarrowtail \mathcal{F}'_0 = \mathcal{F}' \rightarrowtail \mathcal{F}$ and a sequence of surjections $\mathcal{G} \twoheadrightarrow \mathcal{G}' = \mathcal{G}'_0 \twoheadrightarrow \mathcal{G}'_1 \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{G}'_n \twoheadrightarrow 0$ such that $\mathrm{coker}(\mathcal{F}'_{i+1} \longrightarrow \mathcal{F}'_i) \cong \ker(\mathcal{G}'_i \longrightarrow \mathcal{G}'_{i+1})$. But obviously, such sequences stand in a bijection with sequences associated with the pair $\big(\mathcal{F}/\mathcal{F}'_n, \ker(\mathcal{G} \longrightarrow \mathcal{G}'_n)\big)$. This implies the first part. The lemma is proved. ✓

From this lemma it follows that the operator $T : \mathbf{H}_X[\alpha] \longrightarrow \widehat{\mathbf{H}}_X[\alpha]$ is well-defined, i.e. the series (B.7) for $T\big([\mathcal{F}]\big)$ is convergent. Indeed, for any coherent sheaf $\mathcal{G}$ of class α there exist only finitely many admissible filtrations associated with $(\mathcal{F},\mathcal{G})$, what means that each term $[\mathcal{G}] \in \mathbf{H}_X[\alpha]$ appears the expansion of $T\big([\mathcal{F}]\big)$ finitely many times.

For $\alpha \in (\mathbb{Z}^2)^+$ and let Δ^n_* be the composition of Δ^n and the canonical projection

$$\prod_{\substack{\alpha_1+\cdots+\alpha_n=\alpha \\ \alpha_i \in (\mathbb{Z}^2)^+}} \mathbf{H}_X[\alpha_1]\widehat{\otimes} \dots \widehat{\otimes}\mathbf{H}_X[\alpha_n] \longrightarrow \prod_{\substack{\alpha_1+\cdots+\alpha_n=\alpha \\ \alpha_i \in (\mathbb{Z}^2)^+, \alpha_i \neq 0}} \mathbf{H}_X[\alpha_1]\widehat{\otimes} \dots \widehat{\otimes}\mathbf{H}_X[\alpha_n]$$

then we denote

$$\Delta_*(a) = \sum_{\overline{a_i^{(1)}}\neq 0,\cdots,\overline{a_i^{(l+1)}}\neq 0} a_i^{(1)} \otimes a_i^{(2)} \otimes \cdots \otimes a_i^{(n)}.$$

Using this notations, we may write the operator $T : \mathbf{H}_X[\alpha] \longrightarrow \widehat{\mathbf{H}}_X[\alpha]$:

$$T(a) = \left(-a + \sum_{l=1}^{\infty} (-1)^l \sum_{\overline{a_i^{(1)}}\neq 0,\cdots,\overline{a_i^{(l+1)}}\neq 0} a_i^{(l+1)} \cdots a_i^{(1)} \right).$$

Note that in the case of the Hall algebra of the category of representations of a finite quiver the map T is the inverse of the antipode.

Lemma B.4. *Let $\mathcal{F}$ be a coherent sheaf and $\Delta\big([\mathcal{F}]\big) = \sum_i \mathcal{F}_i^{(1)} \otimes \mathcal{F}_i^{(2)} \in \mathbf{H}_X \widehat{\otimes} \mathbf{H}_X$. Then we have $\sum_i [\mathcal{F}_i^{(2)}] T\big([\mathcal{F}_i^{(1)}]\big) = \varepsilon\big([\mathcal{F}]\big)1$ and $\sum_i T\big([\mathcal{F}_i^{(2)}]\big)[\mathcal{F}_i^{(1)}] = \varepsilon\big([\mathcal{F}]\big)1$, where both equalities are taken in $\widehat{\mathbf{H}}_X$.*

Proof. We shall prove, following Theorem 1.6.3 in [K1] only the first statement, the proof of the second is dual. Since the assertion trivially holds for $\mathcal{F} = 0$, assume we have a coherent sheaf $\mathcal{F}$ of class $\alpha \neq 0$. First of all let us check the convergence of the series $\sum_i [\mathcal{F}_i^{(2)}] T\big([\mathcal{F}_i^{(1)}]\big)$ in $\widehat{\mathbf{H}}_X[\alpha]$. It is clear that each $[\mathcal{G}] \in \mathbf{H}_X[\alpha]$ gets non-zero contributions only from finitely many summands $[\mathcal{F}_i^{(2)}] T\big([\mathcal{F}_i^{(1)}]\big)$. Indeed, by Lemma B.3 there are only finitely many exact sequences $0 \to \mathcal{F}_i^{(2)} \to \mathcal{F} \to \mathcal{F}_i^{(1)} \to 0$ and $0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{F}_i^{(2)} \to 0$ such that $\mathcal{G}'$ and $\mathcal{F}_i^{(1)}$ are swept-equivalent. Now note that

$$\sum_i [\mathcal{F}_i^{(2)}] T\big([\mathcal{F}_i^{(1)}]\big) = \sum_{n=1}^{\infty} (-1)^n \sum_{\mathcal{F} \overset{\varphi_1}{\twoheadrightarrow} \mathcal{F}_1 \overset{\varphi_2}{\twoheadrightarrow} \mathcal{F}_2 \twoheadrightarrow \cdots \overset{\varphi_n}{\twoheadrightarrow} \mathcal{F}_n} d^{\mathcal{F}}_{\mathcal{B}_1,\ldots,\mathcal{B}_{n+1}} [\mathcal{B}_{n+1}] \otimes \cdots \otimes [\mathcal{B}_1],$$

where $\mathcal{B}_{n+1} = \mathcal{F}_n$, $\mathcal{B}_i = \ker(\mathcal{F}_{i-1} \xrightarrow{\varphi_i} \mathcal{F}_i)$ and the sum is taken in such a way that the epimorphisms $\varphi_2, \ldots \varphi_n$ are strict and φ_1 is arbitrary. Now note that each term $[\mathcal{B}_{n+1}] \otimes \cdots \otimes [\mathcal{B}_1]$ occurs exactly twice in the sum with two different signs: one comes from the sequence $\mathcal{F} \overset{\varphi_1}{\twoheadrightarrow} \mathcal{F}_1 \overset{\varphi_2}{\twoheadrightarrow} \mathcal{F}_2 \twoheadrightarrow \cdots \overset{\varphi_n}{\twoheadrightarrow} \mathcal{F}_n$ where all epimorphisms φ_i are strict and the second comes from $\mathcal{F} \xrightarrow{id} \mathcal{F} \overset{\varphi_1}{\twoheadrightarrow} \mathcal{F}_1 \overset{\varphi_2}{\twoheadrightarrow} \mathcal{F}_2 \twoheadrightarrow \cdots \overset{\varphi_n}{\twoheadrightarrow} \mathcal{F}_n$. This shows the lemma. ✓

Lemma B.5. *Let $\alpha, \beta \in (\mathbb{Z}^2)^+$, $a \in \mathbf{H}_X[\alpha]$ and $b \in \mathbf{H}_X[\beta]$. Then we have $T(ab) = T(b)T(a)$ in $\widehat{\mathbf{H}}_X[\alpha + \beta]$.*

Proof. Let $\Delta^2(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)} \otimes a_i^{(3)}$ and $\Delta^2(b) = \sum_j b_j^{(1)} \otimes b_j^{(2)} \otimes b_j^{(3)}$. Consider an expression

$$c = \sum_{i,j} T\big(b_j^{(3)}\big) T(a_i^{(3)}) a_i^{(2)} b_j^{(2)} T\big(a_i^{(1)} b_j^{(1)}\big)$$

in $\widehat{\mathbf{H}}_X[\alpha + \beta]$. To see that this sum converges, assume that both a and b are classes of coherent sheaves. A coherent sheaf $\mathcal{F}$ of class $\alpha + \beta$ enters in the sum c if and only if there is a filtration $0 \rightarrowtail \mathcal{F}_4 \overset{\varphi_3}{\rightarrowtail} \mathcal{F}_3 \overset{\varphi_2}{\rightarrowtail} \mathcal{F}_2 \overset{\varphi_1}{\rightarrowtail} \mathcal{F}_1 \overset{\varphi_0}{\rightarrowtail} \mathcal{F}_0 = \mathcal{F}$ such that $\operatorname{coker}(\varphi_0)$ is swept-equivalent to $b_j^{(3)}$, $\operatorname{coker}(\varphi_1)$ is swept-equivalent to $a_i^{(3)}$, $\operatorname{coker}(\varphi_2)$ is isomorphic to $a_i^{(2)}$, $\operatorname{coker}(\varphi_3)$ is isomorphic to $b_j^{(2)}$ and finally $\mathcal{F}_4$ is swept-equivalent to $a_i^{(1)} b_j^{(1)}$.

Since $b_j^{(3)}$ is a subsheaf of b and is swept-equivalent to $\mathrm{coker}(\varphi_0)$, by Lemma B.3 there are finitely many choices for $b_j^{(3)}$ and φ_0, and therefore finitely many contributions of $T(b_j^{(3)})$ to $\mathcal{F}$. Assuming that $b_j^{(3)}$ and $\mathcal{F}_1 \overset{\varphi_0}{\rightarrowtail} \mathcal{F}$ are fixed, by the same argument we see that there are finitely many subobjects $a_i^{(3)}$ of a and finitely many inclusions $\mathcal{F}_2 \overset{\varphi_1}{\rightarrowtail} \mathcal{F}_1$ such that $a_i^{(3)}$ and $\mathrm{coker}(\varphi_1)$ are swept-equivalent. Next, there are finitely many inclusions $\mathcal{F}_3 \overset{\varphi_2}{\rightarrowtail} \mathcal{F}_2$ such that there is a subobject $a_i^{(2)}$ of $a/a_i^{(3)}$ isomorphic to $\mathrm{coker}(\varphi_2)$. In the same way, we have only finitely many inclusions $\mathcal{F}_4 \overset{\varphi_3}{\rightarrowtail} \mathcal{F}_3$ such that $\mathrm{coker}(\varphi_3)$ is isomorphic to a subobject of $b/b_j^{(3)}$. But choices of $a_i^{(2)}$ and $a_i^{(3)}$ also determine $a_i^{(1)}$, the same holds for $b_j^{(2)}$ and $b_j^{(3)}$, hence there are finitely many subobjects of $\mathcal{F}_4$ swept-equivalent to some summand of $a_i^{(1)}b_j^{(1)}$. Gathering all together we conclude, that the element c is correctly defined in $\widehat{\mathbf{H}}_X[\alpha+\beta]$.

Using Lemma B.4 we can transform the series c in two different ways. From the one side we have

$$c = \sum_j T(b_j^{(3)})\varepsilon(a_i^{(2)})b_j^{(2)}T(a_i^{(1)}b_j^{(1)}) = \sum_j T(b_j^{(3)})b_j^{(2)}T(ab_j^{(1)})$$
$$= \sum_j \varepsilon(b_j^{(2)})T(ab_j^{(1)}) = T(ab)$$

and from another side,

$$c = \sum_{i,j,k,l} T(b_j^{(2)})T(a_i^{(2)})(a_i^{(1)})_l^{(2)}(b_j^{(1)})_k^{(2)}T\big((a_i^{(1)})_l^{(1)}(b_j^{(1)})_k^{(1)}\big)$$
$$= \sum_{i,j,k} T(b_j^{(2)})T(a_i^{(2)})(a_i^{(1)}b_j^{(1)})_k^{(2)}T\big((a_i^{(1)}b_j^{(1)})_k^{(1)}\big)$$
$$= \sum_{i,j} T(b_j^{(2)})T(a_i^{(2)})\varepsilon(a_i^{(1)}b_j^{(1)}) = T(b)T(a).$$

✓

Lemma B.6. *Let $a \in \mathbf{H}_X^-$ and $b \in \mathbf{H}_X^+$, then the following equation holds in the Drinfeld double $\mathbf{DH}_X$:*

$$ab = \sum_{i,j} \big(a_i^{(1)}, b_j^{(3)}\big) b_j^{(2)} a_i^{(2)} \big(T(a_i^{(3)}), b_j^{(1)}\big). \tag{B.8}$$

Proof. First of all note, that the right-hand side of the equation (B.8) is finite by Lemma B.3. The relation $R(a,b)$ in the Drinfeld double implies

$$ab = -\sum_{\substack{i,j \\ a_i^{(2)} \neq 0}} a_i^{(1)} b_j^{(2)} \big(a_i^{(2)}, b_j^{(1)}\big) + \sum_{i,j} b_j^{(1)} a_i^{(2)} \big(a_i^{(1)}, b_j^{(2)}\big).$$

Now use this equality to rewrite each term $a_i^{(1)} b_j^{(2)}$:

$$a_i^{(1)} b_j^{(2)} = -\sum_{\substack{k,l \\ (a_i^{(2)})_k^{(2)} \neq 0}} (a_i^{(1)})_k^{(1)} (b_j^{(2)})_l^{(2)} \big((a_i^{(1)})_k^{(2)}, (b_j^{(2)})_l^{(1)}\big) +$$
$$+ \sum_{k,l} (b_j^{(2)})_l^{(1)} (a_i^{(1)})_k^{(2)} \big((a_i^{(1)})_k^{(1)}, (b_j^{(2)})_l^{(2)}\big).$$

Combining these two equations we obtain

$$ab = \sum_{\substack{i,j \\ a_i^{(3)}\neq 0, a_i^{(2)}\neq 0}} a_i^{(1)} b_j^{(3)} (a_i^{(2)}, b_j^{(2)})(a_i^{(3)}, b_j^{(1)}) - \sum_{\substack{i,j \\ a_i^{(3)}\neq 0}} b_j^{(2)} a_i^{(2)} (a_i^{(1)}, b_j^{(3)})(a_i^{(3)}, b_j^{(1)}) + \\ + \sum_{i,j} b_j^{(1)} a_i^{(2)} (a_i^{(1)}, b_j^{(2)}).$$

But note that

$$\sum_{\substack{i,j \\ a_i^{(3)}\neq 0, a_i^{(2)}\neq 0}} a_i^{(1)} b_j^{(3)} (a_i^{(2)}, b_j^{(2)})(a_i^{(3)}, b_j^{(1)}) = \sum_{\substack{i,j \\ a_i^{(3)}\neq 0, a_i^{(2)}\neq 0}} a_i^{(1)} b_j^{(2)} (a_i^{(3)} a_i^{(2)}, b_j^{(1)}).$$

Moreover, we have

$$\sum_{i,j} b_j^{(1)} a_i^{(2)} (a_i^{(1)}, b_j^{(2)}) = \sum_{\substack{i,j \\ a_i^{(3)}=0}} (a_i^{(1)}, b_j^{(3)}) b_j^{(2)} a_i^{(2)} (a_i^{(3)}, b_j^{(1)}).$$

Summing everything up, we get

$$ab = \sum_{\substack{i,j \\ a_i^{(3)}=0}} (a_i^{(1)}, b_j^{(3)}) b_j^{(2)} a_i^{(2)} (a_i^{(3)}, b_j^{(1)}) - \sum_{\substack{i,j \\ a_i^{(3)}\neq 0}} b_j^{(2)} a_i^{(2)} (a_i^{(1)}, b_j^{(3)})(a_i^{(3)}, b_j^{(1)}) + \\ + \sum_{\substack{i,j \\ a_i^{(3)}\neq 0, a_i^{(2)}\neq 0}} a_i^{(1)} b_j^{(2)} (a_i^{(3)} a_i^{(2)}, b_j^{(1)}).$$

Iterating this procedure, we get, for each $k > 0$,

$$ab = \sum_{i,j} (a_i^{(1)}, b_j^{(3)}) b_j^{(2)} a_i^{(2)} \big(T^k(a_i^{(3)}), b_j^{(1)}\big) + \\ + (-1)^k \sum_{\substack{i,j \\ a_i^{(2)}\neq 0, \ldots, a_i^{(k+1)}\neq 0}} a_i^{(1)} b_j^{(2)} (a_i^{(k+1)} \cdots a_i^{(2)}, b_j^{(1)}), \tag{B.9}$$

where

$$T_k(a) = \left(-a + \sum_{l=1}^{k-2} (-1)^l \sum_{a_i^{(1)}\neq 0, \cdots, a_i^{(l+1)}\neq 0} a_i^{(l+1)} \cdots a_i^{(1)} \right).$$

It follows from the first part of Lemma B.3 that the second term in (B.9) vanishes for $k \gg 0$ and the operators T^k converge pointwise as $k \to \infty$ to operators $T : \mathbf{H}_X[\alpha] \to \widehat{\mathbf{H}}_X[\alpha]$ yielding the equation (B.8) as wanted. ✓

This lemma shows that the map $\mathbf{H}_X^+ \otimes \mathbf{H}_X^- \xrightarrow{m} \mathbf{DH}_X$ is surjective. Next, we define an associative algebra structure on $\mathbf{H}_X^+ \otimes \mathbf{H}_X^-$ by setting

$$(a \otimes a') \cdot (b \otimes b') = (m \otimes m)(a \otimes L(a', b) \otimes b'),$$

where

$$L(x, y) = \sum_{i,j} \big(x_1^{(i)}, y_j^{(3)}\big) y_j^{(2)} x_i^{(2)} \big(T(x_i^{(3)}), y_j^{(1)}\big).$$

A proof of associativity of this product is based on Proposition 2.2, Remark 2.5 and Lemma B.5 and can be shown along the same lines as in in [J, 3.2.4] using similar calculations as in the proof of Lemma B.6.

Now we can construct the inverse map $n : \mathbf{DH}_X \longrightarrow \mathbf{H}_X^+ \otimes \mathbf{H}_X^-$ by putting $n(a) = 1 \otimes a$, $n(b) = b \otimes 1$ for $a \in \mathbf{H}_X^- \subset \mathbf{DH}_X$ and $b \in \mathbf{H}_X^+ \subset \mathbf{DH}_X$. To see that we get a well-defined map we need to check that all relations $R(a,b)$ are preserved in the Drinfeld double. But indeed,

$$\sum_{i,j} a_i^{(1)} b_j^{(2)} (a_i^{(2)}, b_j^{(1)}) = \sum_{i,j} (a_i^{(1)}, b_j^{(4)}) a_i^{(2)} b_j^{(3)} \bigl(T(a_i^{(3)}), b_j^{(2)}\bigr)(a_i^{(4)}, b_j^{(1)}) =$$

$$\sum_{i,j} (a_i^{(1)}, b_j^{(3)}) a_i^{(2)} b_j^{(2)} \bigl(T(a_i^{(3)}) a_i^{(4)}, b_j^{(1)}\bigr) = \sum_{i,j} (a_i^{(1)}, b_j^{(3)}) a_i^{(2)} b_j^{(2)} \bigl(\varepsilon(a_i^{(3)})1, b_j^{(1)}\bigr) =$$

$$= \sum_{i,j} (a_i^{(1)}, b_j^{(2)}) a_i^{(2)} b_j^{(1)}.$$

This concludes the proof of injectivity and surjectivity of the linear map $m : \mathbf{H}_X^+ \otimes \mathbf{H}_X^- \longrightarrow \mathbf{DH}_X$. Proposition 3.4 is proven. ✓

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