TWO DESCRIPTIONS OF THE QUANTUM AFFINE ALGEBRA $U_v(\widehat{\mathfrak{sl}}_2)$ VIA HALL ALGEBRA APPROACH

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ABSTRACT. We compare the reduced Drinfeld doubles of the composition subalgebras of the category of representations of the Kronecker quiver $\overrightarrow{Q}$ and of the category of coherent sheaves on $\mathbb{P}^1$. Using this approach, we show that the Drinfeld–Beck isomorphism for the quantized enveloping algebra $U_v(\widehat{\mathfrak{sl}}_2)$ is a corollary of an equivalence between the derived categories $D^b\bigl(\mathsf{Rep}(\overrightarrow{Q})\bigr)$ and $D^b\bigl(\mathsf{Coh}(\mathbb{P}^1)\bigr)$. This technique allows to reprove several results on the integral form of $U_v(\widehat{\mathfrak{sl}}_2)$.

1. INTRODUCTION

In this article, we study the relation between the composition algebra of the category of representations of the Kronecker quiver

$$\overrightarrow{Q} = \bullet \rightrightarrows \bullet$$

and the composition algebra of the category of coherent sheaves on the projective line $\mathbb{P}^1$. As it was shown by Ringel [33] and Green [19], the generic composition algebra of the category of representations of $\overrightarrow{Q}$ is isomorphic to the positive part of the quantum affine algebra $U_v(\widehat{\mathfrak{sl}}_2)$ written in the terms of Drinfeld–Jimbo generators.

On the other side, as it was discovered by Kapranov [23] and extended by Baumann and Kassel [2], the Hall algebra of the category of coherent sheaves on a projective line $\mathbb{P}^1$ is closely related with Drinfeld's new realization $U_v\bigl(\mathfrak{L}\mathfrak{sl}_2\bigr)$ of the quantized enveloping algebra of $\widehat{\mathfrak{sl}}_2$ [16]. Since then, this subject drew attention of many authors, see for example [45, 40, 26, 44, 37, 38].

In this article, we work out this important observation a step further and show that the Drinfeld–Beck isomorphism $U_v\bigl(\widehat{\mathfrak{sl}}_2\bigr) \to U_v\bigl(\mathfrak{L}\mathfrak{sl}_2\bigr)$ (see [16, 3, 14, 28]) can be viewed as a corollary of the derived equivalence $D^b\bigl(\mathsf{Rep}(\overrightarrow{Q})\bigr) \to D^b\bigl(\mathsf{Coh}(\mathbb{P}^1)\bigr)$. The understanding of this isomorphism is of great importance for the representation theory of $U_v\bigl(\widehat{\mathfrak{sl}}_2\bigr)$ and its applications in mathematical physics, see for example [27]. Indeed, since $U_v\bigl(\widehat{\mathfrak{sl}}_2\bigr)$ is a Hopf algebra, the category of its finite-dimensional representations has a structure of a tensor category. However, in order to describe such representations themselves, it is frequently more convenient to work with Drinfeld's new realization $U_v\bigl(\mathfrak{L}\mathfrak{sl}_2\bigr)$. Using the fact that an equivalence of triangulated categories commutes with Serre functors, we also show that the Drinfeld-Beck isomorphism $U_v\bigl(\widehat{\mathfrak{sl}}_2\bigr) \to U_v\bigl(\mathfrak{L}\mathfrak{sl}_2\bigr)$ commutes with the Coxeter transformation acting on both sides. Finally, applying the technique of stability conditions, we reprove several known technical statements on the integral form of $U_v\bigl(\mathfrak{L}\mathfrak{sl}_2\bigr)$.

Notation. Throughout the paper, $k = \mathbb{F}_q$ is a finite field with q elements and $\widetilde{\mathbb{Q}} = \widetilde{\mathbb{Q}}_q = \mathbb{Q}[v, v^{-1}]/(v^{-2} - q) \cong \mathbb{Q}[\sqrt{q}]$. Next, $\mathcal{P}$ is the set of all integers of the form p^t, where p is a prime number and $t \in \mathbb{Z}_+$. For a positive integer n we set $[n] = [n]_v = \dfrac{v^n - v^{-n}}{v - v^{-1}}$ and $[n]! = [1]\dots[n]$.

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We denote by R the subring of the field $\mathbb{Q}(v)$ consisting of the rational functions having poles only at 0 or at roots of 1. For the affine Lie algebra $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$ we denote by $U_v(\mathfrak{g})$ its quantized enveloping algebra over the ring R, whereas $U_q(\mathfrak{g}) = U_v(\mathfrak{g}) \otimes_R \widetilde{\mathbb{Q}}$.

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2. Hereditary categories, their Hall algebras and Drinfeld doubles

Let A be an essentially small hereditary abelian k-linear category such that for all objects $M, N \in \mathrm{Ob}(\mathsf{A})$ the k–vector spaces $\mathsf{Hom}_{\mathsf{A}}(M,N)$ and $\mathsf{Ext}^1_{\mathsf{A}}(M,N)$ are finite dimensional. In what follows, we shall call such a category *finitary.* Let $\mathsf{J} = \mathsf{J}_{\mathsf{A}} := \bigl(\mathrm{Ob}(\mathsf{A})/\cong\bigr)$ be the set of isomorphy classes of objects in A. For an object $X \in \mathrm{Ob}(\mathsf{A})$, we denote by $[X]$ its image in J. Fix the following notations.

- For any object $X \in \mathrm{Ob}(\mathsf{A})$ we set $a_X = \bigl|\mathsf{Aut}_{\mathsf{A}}(X)\bigr|$.
- For any three objects $X, Y, Z \in \mathrm{Ob}(\mathsf{A})$ we denote

$$P^Z_{X,Y} = \left|\left\{(f,g) \in \mathsf{Hom}_{\mathsf{A}}(Y,Z) \times \mathsf{Hom}_{\mathsf{A}}(Z,X) \;\middle|\; 0 \to Y \xrightarrow{f} Z \xrightarrow{g} X \to 0 \quad \text{is exact}\right\}\right|.$$

- Finally, we put $F^Z_{X,Y} = \dfrac{P^Z_{X,Y}}{a_X \cdot a_Y}$.

Note that the numbers a_Z, $P^Z_{X,Y}$, $F^Z_{X,Y}$ and $\dfrac{P^Z_{X,Y}}{a_Z}$ depend only on the isomorphy classes of X, Y, Z and are integers.

Let $K = K_0(\mathsf{A})$ be the K–group of A. For an object $X \in \mathrm{Ob}(\mathsf{A})$, we denote by $\bar{X}$ its image in K. Next, let $\langle -, - \rangle : K \times K \to \mathbb{Z}$ be the Euler form:

$$\langle \bar{X}, \bar{Y} \rangle = \mathsf{dim}_k \mathsf{Hom}_{\mathsf{A}}(X,Y) - \mathsf{dim}_k \mathsf{Ext}^1_{\mathsf{A}}(X,Y)$$

and $(-, -) : K \times K \to \mathbb{Z}$ its symmetrization:

$$(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle, \quad \alpha, \beta \in K.$$

Following Ringel [33], one can attach to a finitary hereditary category A an associative algebra $H(\mathsf{A})$ called the extended twisted *Hall algebra* of A, defined over the field $\widetilde{\mathbb{Q}}$. As a vector space over $\widetilde{\mathbb{Q}}$, we have

$$\bar{H}(\mathsf{A}) := \bigoplus_{[Z] \in \mathsf{J}} \widetilde{\mathbb{Q}}[Z] \quad \text{and} \quad H(\mathsf{A}) := \bar{H}(\mathsf{A}) \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K].$$

For a class $\alpha \in K$ we denote by K_α the corresponding element in the group algebra $\widetilde{\mathbb{Q}}[K]$. Then we have: $K_\alpha \circ K_\beta = K_{\alpha+\beta}$. Next, for $[X], [Y] \in \mathsf{J}$ the product $\circ$ is defined to be

$$[X] \circ [Y] = \sqrt{\frac{|\mathrm{Hom}(X,Y)|}{|\,\mathsf{Ext}^1(X,Y)|}} \sum_{[Z] \in \mathsf{J}} F^Z_{X,Y}\,[Z] = v^{-\langle \bar{X}, \bar{Y} \rangle} \sum_{[Z] \in \mathsf{J}} F^Z_{X,Y}\,[Z].$$

Finally, for any $\alpha \in K$ and $[X] \in \mathsf{J}$ we have

$$K_\alpha \circ [X] = v^{-(\alpha, \bar{X})}[X] \circ K_\alpha.$$

As it was shown in [33], the product $\circ$ is associative and the element $1 := [0] \otimes K_0$ is the unit element. In what follows, we shall use the notation $[X]K_\alpha$ for the element $[X] \otimes K_\alpha \in H(\mathsf{A})$.

Let A be a finitary *finite length* hereditary category over k. By a result of Green [19], the Hall algebra $H(\mathsf{A})$ has a natural bialgebra structure, see also [35]. The comultiplication $\Delta : H(\mathsf{A}) \to H(\mathsf{A}) \otimes_{\widetilde{\mathbb{Q}}} H(\mathsf{A})$ and the counit $\eta : H(\mathsf{A}) \to \widetilde{\mathbb{Q}}$ are given by the following formulae:

$$\Delta\bigl([Z]K_\alpha\bigr) = \sum_{[X],\,[Y]\in \mathsf{J}} v^{-\langle \bar{X}, \bar{Y}\rangle} \frac{P^Z_{X,Y}}{a_Z} [X]K_{\bar{Y}+\alpha} \otimes [Y]K_\alpha \quad \text{and} \quad \eta\bigl([Z]K_\alpha\bigr) = \delta_{Z,0}.$$

Moreover, as it was shown by Xiao [42], the Hall algebra $H(\mathsf{A})$ is also a Hopf algebra. Finally, there is a pairing $(\,-\,,\,-\,) : H(\mathsf{A}) \times H(\mathsf{A}) \to \widetilde{\mathbb{Q}}$ introduced by Green [19], given by the expression

$$\Bigl([X]K_\alpha, [Y]K_\beta\Bigr) = v^{-(\alpha,\beta)} \frac{\delta_{X,Y}}{a_X}.$$

This pairing is non-degenerate on $\bar{H}(\mathsf{A})$ and symmetric. Next, it satisfies the following properties:

$$(a \circ b, c) = \bigl(a \otimes b, \Delta(c)\bigr) \quad \text{and} \quad (a, 1) = \eta(a)$$

for any $a, b, c \in H(\mathsf{A})$. In other words, it is a bialgebra pairing.

Remark 2.1. If A is not a category of finite length (for instance, if it is the category of coherent sheaves on a projective curve) then the Green's pairing $(\,-\,,\,-\,) : H(\mathsf{A}) \times H(\mathsf{A}) \to \widetilde{\mathbb{Q}}$ is still a well-defined symmetric bilinear pairing. However, the comultiplication $\Delta([X])$ is possibly an infinite sum. Nevertheless, it is possible to introduce a completed tensor product $H(\mathsf{A})\widehat{\otimes}H(\mathsf{A})$ (which is a $\widetilde{\mathbb{Q}}$–algebra) such that $\Delta : H(\mathsf{A}) \to H(\mathsf{A})\widehat{\otimes}H(\mathsf{A})$ is an algebra homomorphism and $(\Delta \otimes \mathbb{1}) \circ \Delta = (\mathbb{1} \otimes \Delta) \circ \Delta$. Moreover, for any elements $a, b, c \in H(\mathsf{A})$ the expression $\bigl(a \otimes b, \Delta(c)\bigr)$ takes a finite value and the equalities $(a \circ b,\, c) = \bigl(a \otimes b, \Delta(c)\bigr)$, $(a, 1) = \eta(a)$ are fulfilled. In such a situation we say that $H(\mathsf{A})$ is a *topological bialgebra*, see [9, Appendix B] for further details.

From now on, let A be an arbitrary k–linear hereditary finitary abelian category. Consider the root category $\mathsf{R}(\mathsf{A}) = D^b(\mathsf{A})/[2]$. Note that $\mathsf{R}(\mathsf{A})$ has a structure of a triangulated category such that the canonical functor $D^b(\mathsf{A}) \to \mathsf{R}(\mathsf{A})$ is exact, see [29, Section 7]. Moreover, any object of $\mathsf{R}(\mathsf{A})$ splits into a direct sum $X^+ \oplus X^-$, where $X^+ \in \mathrm{Ob}(\mathsf{A})$ and $X^- \in \mathrm{Ob}(\mathsf{A})[1]$.

Our next goal is to introduce the reduced Drinfeld double of the topological bialgebra $H(\mathsf{A})$. Roughly speaking (although, not completely correctly), it is an analogue of the Hall algebra, attached to the triangulated category $\mathsf{R}(\mathsf{A})$. To define it, consider the pair of algebras $H^\pm(\mathsf{A})$, where we use the notation

$$H^+(\mathsf{A}) = \bigoplus_{[Z]\in \mathsf{J}} \widetilde{\mathbb{Q}}[Z]^+ \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K] \text{ and } H^-(\mathsf{A}) = \bigoplus_{[Z]\in \mathsf{J}} \widetilde{\mathbb{Q}}[Z]^- \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K].$$

and $H^\pm(\mathsf{A}) = H(\mathsf{A})$ as $\widetilde{\mathbb{Q}}$-algebras. Let $a = [Z]K_\gamma$ and

$$\Delta(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)} = \sum_{[X],\,[Y]\in \mathsf{J}} v^{-\langle \bar{X}, \bar{Y}\rangle} \frac{P^Z_{X,Y}}{a_Z} [X]K_{\bar{Y}+\gamma} \otimes [Y]K_\gamma.$$

Then we denote

$$\Delta(a^\pm) = \sum_i a_i^{(1)\pm} \otimes a_i^{(2)\pm} = \sum_{[X],\,[Y]\in \mathsf{J}} v^{-\langle \bar{X}, \bar{Y}\rangle} \frac{P^Z_{X,Y}}{a_Z} [X]^\pm K_{\pm\bar{Y}+\gamma} \otimes [Y]^\pm K_\gamma.$$

Definition 2.2. The Drinfeld double of $H(\mathsf{A})$ with respect to the Green's pairing $(\,-\,,\,-\,)$ is the associative algebra $\widetilde{D}H(\mathsf{A})$, which is the free product of algebras $H^+(\mathsf{A})$ and $H^-(\mathsf{A})$ subject to the following relations $D(a, b)$ for all $a, b \in H(\mathsf{A})$:

$$\sum_{i,j} a_i^{(1)-} b_j^{(2)+} \bigl(a_i^{(2)}, b_j^{(1)}\bigr) = \sum_{i,j} b_j^{(1)+} a_i^{(2)-} \bigl(a_i^{(1)}, b_j^{(2)}\bigr).$$

The following proposition is well-known, see for example [22, Section 3.2] for the case of Hopf algebras and [9, Appendix B] for the case of topological bialgebras.

Theorem 2.3. *The multiplication morphism* $\mathrm{mult}: H^+(\mathsf{A})\otimes_{\widetilde{\mathbb{Q}}} H^-(\mathsf{A}) \to \widetilde{D}H(\mathsf{A})$ *is a isomorphism of* $\widetilde{\mathbb{Q}}$*–vector spaces. Moreover, if* A *is an abelian category of finite length, then* $\widetilde{D}H(\mathsf{A})$ *is also a Hopf algebra such that the above morphism* $H^+(\mathsf{A}) \to \widetilde{D}H(\mathsf{A})$, $a \mapsto a\otimes \mathbb{1}^-$ *is an injective homomorphisms of Hopf algebras.*

The following definition is due to Xiao [42].

Definition 2.4. Let A be a k–linear finitary hereditary category. The *reduced* Drinfeld double $DH(\mathsf{A})$ is the quotient of $\widetilde{D}H(\mathsf{A})$ by the two-sided ideal

$$I = \bigl\langle K^+_\alpha \otimes K^-_{-\alpha} - \mathbb{1}^+ \otimes \mathbb{1}^- \mid \alpha \in K \bigr\rangle.$$

Note for a finite length abelian category A, I is a Hopf ideal and $DH(\mathsf{A})$ is also a Hopf algebra.

Corollary 2.5. *We have an isomorphism of* $\widetilde{\mathbb{Q}}$*–vector spaces*

$$\mathrm{mult}: \bar{H}^+(\mathsf{A}) \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K] \otimes_{\widetilde{\mathbb{Q}}} \bar{H}^-(\mathsf{A}) \longrightarrow DH(\mathsf{A}).$$

Next, we shall need the following statement, a first version of which dates back to Dold [15].

Theorem 2.6. *Let* A *be a hereditary abelian category. Then any indecomposable object of the derived category* $D^b(\mathsf{A})$ *is isomorphic to* $X[n]$*, where* X *is an indecomposable object of* A.

Remark 2.7. In what follows, we shall identify an object $X \in \mathrm{Ob}(\mathsf{A})$ with its image under the canonical functor $\mathsf{A} \to D^b(\mathsf{A})$.

The following important theorem was recently proven by Cramer [13].

Theorem 2.8. *Let* A *and* B *be two* k*-linear finitary hereditary categories. Assume one of them is artinian and there is an equivalence of triangulated categories* $D^b(\mathsf{A}) \xrightarrow{\mathbb{F}} D^b(\mathsf{B})$*. Then there is an algebra isomorphism*

$$\mathbb{F}: DH(\mathsf{A}) \longrightarrow DH(\mathsf{B})$$

uniquely determined by the following properties. For any object $X \in \mathrm{Ob}(\mathsf{A})$ *such that* $\mathbb{F}(X) \cong \widehat{X}[n]$ *with* $\widehat{X} \in \mathrm{Ob}(\mathsf{B})$ *and* $n \in \mathbb{Z}$ *we have:*

$$\mathbb{F}\bigl([X]^\pm\bigr) = v^{-n\langle \bar{X}, \bar{X}\rangle}\bigl[\widehat{X}\bigr]^{\pm\varepsilon(n)}\bigl(K_{\widehat{X}}^{\pm\varepsilon(n)}\bigr)^n,$$

where $\varepsilon(n) = (-1)^n$*. For* $\alpha \in K$ *we have:* $\mathbb{F}(K_\alpha) = K_{\mathbb{F}(\alpha)}$.

3. Composition algebra of the Kronecker quiver

In this section, we study properties of the composition algebra of the Kronecker quiver

$$\overrightarrow{Q} = \begin{tikzcd} 1 \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & 2 \end{tikzcd}$$

and the reduced Drinfeld double of its Hall algebra.

Definition 3.1. Consider the pair of reflection functors $\mathbb{S}^\pm : \mathsf{Rep}(\overrightarrow{Q}) \to \mathsf{Rep}(\overrightarrow{Q})$ defined as follows, see [6, Section 1]. For a representation

$$X = \begin{tikzcd} V \arrow[r, shift left, "A"] \arrow[r, shift right, "B"'] & W \end{tikzcd}$$

consider the short exact sequences

$$0 \longrightarrow U' \xrightarrow{\binom{C'}{D'}} V \oplus V \xrightarrow{(A,B)} W \quad \text{and} \quad V \xrightarrow{\binom{A}{B}} W \oplus W \xrightarrow{(C'',\, D'')} U'' \longrightarrow 0.$$

Then we have:

$$\mathbb{S}^+(X) = \Big(U' \underset{D'}{\overset{C'}{\rightrightarrows}} V \Big) \quad \text{and} \quad \mathbb{S}^-(X) = \Big(W \underset{D''}{\overset{C''}{\rightrightarrows}} U'' \Big).$$

The action of $\mathbb{S}^\pm$ on morphisms is defined using the universal property of kernels and cokernels. Note that the functor $\mathbb{S}^+$ is left exact whereas $\mathbb{S}^-$ is right exact.

The following theorem summarizes main properties of the functors $\mathbb{S}^\pm$.

Theorem 3.2. *Let $\overrightarrow{Q}$ be the Kronecker quiver, $A = k\overrightarrow{Q}$ be its path algebra and $\mathsf{A} = \mathsf{Rep}(\overrightarrow{Q}) = A - \mathsf{mod}$. Then the following properties hold:*

(1) *The functors $\mathbb{S}^+$ and $\mathbb{S}^-$ are adjoint, i.e. for any $X, Y \in \mathrm{Ob}(\mathsf{A})$ we have:*
$$\mathsf{Hom}_{\mathsf{A}}\big(\mathbb{S}^-(X), Y\big) \cong \mathsf{Hom}_{\mathsf{A}}\big(X, \mathbb{S}^+(Y)\big).$$
(2) *The derived functors $\mathbb{RS}^+$ and $\mathbb{LS}^-$ are also adjoint. Moreover, they are mutually inverse auto-equivalences of the derived category $D^b(\mathsf{A})$.*

(3) *Let $X \in \mathrm{Ob}(\mathsf{A})$ be an indecomposable object. Then we have:*
$$\mathbb{R}^0\mathbb{S}^+(X) = \begin{cases} \mathbb{S}^+(X) & \text{if} \quad X \ncong S_2 \\ 0 & \text{if} \quad X \cong S_2 \end{cases} \quad \text{and} \quad \mathbb{R}^1\mathbb{S}^+(X) = \begin{cases} 0 & \text{if} \quad X \ncong S_2 \\ S_1 & \text{if} \quad X \cong S_2. \end{cases}$$
Similarly, we have:
$$\mathbb{L}^0\mathbb{S}^-(Y) = \begin{cases} \mathbb{S}^-(Y) & \text{if} \quad Y \ncong S_1 \\ 0 & \text{if} \quad Y \cong S_1 \end{cases} \quad \text{and} \quad \mathbb{L}^{-1}\mathbb{S}^-(Y) = \begin{cases} 0 & \text{if} \quad Y \ncong S_1 \\ S_2 & \text{if} \quad Y \cong S_1. \end{cases}$$
(4) *In particular, the reflection functors $\mathbb{S}^-$ and $\mathbb{S}^+$ yield mutually inverse equivalences between the categories $\mathsf{Rep}\big(\overrightarrow{Q}\big)^1$ and $\mathsf{Rep}\big(\overrightarrow{Q}\big)^2$, which are the full subcategories of $\mathsf{Rep}\big(\overrightarrow{Q}\big)$ consisting of objects without direct summands isomorphic to S_1 and S_2 respectively.*

(5) *Let $\nu = \mathbb{D}\big(\mathsf{Hom}_A(-\,, A)\big) : A - \mathsf{mod} \to A - \mathsf{mod}$ be the Nakayama functor and $\mathbb{S} := \mathbb{L}\nu : D^b(\mathsf{A}) \to D^b(\mathsf{A})$ be its derived functor. Then we have an isomorphism of vector spaces*
$$\mathsf{Hom}_{D^b(\mathsf{A})}\big(X,\, \mathbb{S}(Y)\big) \longrightarrow \mathbb{D}\,\mathsf{Hom}_{D^b(\mathsf{A})}(Y,\, X),$$
functorial in both arguments, where $\mathbb{D}$ is the duality over k. In other words, $\mathbb{S}$ is the Serre functor of the triangulated category $D^b(\mathsf{A})$ in the sense of [7].

(6) *The functors $\mathbb{S}$ and $\mathbb{S}^+$ are related by an isomorphism: $\mathbb{S} \cong (\mathbb{RS}^+)^2[1]$.*

Proof. The first part was essentially proven in [6, Section 1]. There the authors construct two natural transformations of functors $\imath : \mathbb{S}^-\mathbb{S}^+ \to \mathbb{1}$ and $\jmath : \mathbb{1} \to \mathbb{S}^+\mathbb{S}^-$. It can be easily shown that they define mutually inverse bijections
$$\mathsf{Hom}_{\mathsf{A}}\big(\mathbb{S}^-(X), Y\big) \longleftrightarrow \mathsf{Hom}_{\mathsf{A}}\big(X, \mathbb{S}^+(Y)\big).$$
See also [1, Section VII.5] for a proof using tilting functors.

The fact that the derived functors $\mathbb{RS}^+$ and $\mathbb{LS}^-$ are adjoint, is a general property of an adjoint pair, see for example [24, Lemma 15.6]. For the proof that $\mathbb{RS}^+$ and $\mathbb{LS}^-$ are equivalences of categories, see for example [1, Section VII.5].

By Theorem 2.6, the complexes $\mathbb{RS}^+(X)$ and $\mathbb{LS}^-(X)$ have exactly one non-vanishing cohomology for an indecomposable object $X \in \mathrm{Ob}(\mathsf{A})$. This proves the formulae listed in the third item. The fourth statement is proven in [6, Section 1] and for the fifth we refer to [20, Section 4.6].

For a proof of the last statement, first note that the Auslander-Reiten functor

$$\tau = \mathbb{D}\,\mathsf{Ext}^1_A(\,-\,, A) : \ \mathsf{Rep}(\overrightarrow{Q}) \longrightarrow \mathsf{Rep}(\overrightarrow{Q})$$

is isomorphic to the Coxeter functor $\mathbb{A}^+ := (\mathbb{S}^+)^2$, see [17, Section 5.3] and [41, Proposition II.3.2]. Next, the canonical transformation of functors $\mathbb{R}\mathbb{A}^+ \to (\mathbb{R}\mathbb{S}^+)^2$ is an isomorphism on the indecomposable injective modules $I(1)$ and $I(2)$, hence it is an isomorphism. Finally, by [21, Proposition I.7.4] we know that the derived functors $\mathbb{R}\tau[1]$ and $\mathbb{L}\nu$ are isomorphic. □

Remark 3.3. Let $i \in Q_0 = \{1, 2\}$ be a vertex and $P(1) = Ae_i$ be the indecomposable projective module, which is the projective cover of the simple module S_i. Then $\mathbb{L}\nu\big(P(i)\big) = \nu\big(P(i)\big) = I(i)$, where $I(i)$ is the injective envelope of S_i.

Let X be an indecomposable object of A and $B = \mathsf{End}_{\mathsf{A}}(X)$. By Serre duality, we have a canonical isomorphism of B-bimodules

$$\mathsf{Hom}_{\mathsf{A}}(X, X) \longrightarrow \mathbb{D}\Big(\mathsf{Hom}_{D^b(\mathsf{A})}\big(X, \mathbb{S}(X)\big)\Big).$$

Let w be a non-zero element of the socle of $\mathsf{Hom}_{D^b(\mathsf{A})}\big(X, \mathbb{S}(X)\big)$ viewed as the right B-module.

Lemma 3.4. *Consider a distinguished triangle*

$$\mathbb{S}[-1](X) \xrightarrow{u} Y \xrightarrow{v} X \xrightarrow{w} \mathbb{S}(X)$$

given by the morphism w. Then this triangle is almost split. Moreover, if X is non-projective, then $H^i\big(\mathbb{S}(X)\big) = 0$ for $i \ne 1$ and the short exact sequence

$$0 \longrightarrow \tau(X) \xrightarrow{u} Y \xrightarrow{v} X \longrightarrow 0$$

is almost split, where $\tau(X) = \mathbb{D}\,\mathsf{Ext}^1_A(X, A) \cong H^1\big(\mathbb{S}(X)\big)$.

Proof. For the first part of the statement, see the proof of [31, Proposition I.2.3]. For the second, see [20, Section 4.7]. □

Definition 3.5. An object $X \in \mathrm{Ob}(\mathsf{A})$ is called

(1) pre-projective if there exists a projective object P and $m \ge 0$ such that $X \cong \tau^{-m}(P)$,
(2) pre-injective if there exists an injective object I and $m \ge 0$ such that $X \cong \tau^{m}(I)$.

Recall the classification of the indecomposable objects of the category of representations of $\overrightarrow{Q}$ over an arbitrary field k, see for example [32, Section 3.2].

Theorem 3.6. *The indecomposable representations of the Kronecker quiver $\overrightarrow{Q}$ over an arbitrary field k are the following.*

(1) *Indecomposable pre-projective objects*

$$P_n = \ k^n \overset{A_n^{\mathrm{pro}}}{\underset{B_n^{\mathrm{pro}}}{\rightrightarrows}} k^{n+1}\ , \quad n \ge 0,$$

where

$$A_n^{\mathrm{pro}} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{pmatrix} \quad \textit{and} \quad B_n^{\mathrm{pro}} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

In particular, P_0 and P_1 are the indecomposable projective objects.

(2) *Indecomposable pre-injective objects*

$$I_n = \; k^{n+1} \overset{A_n^{\mathrm{inj}}}{\underset{B_n^{\mathrm{inj}}}{\rightrightarrows}} k^n \;, \quad n \geq 0,$$

where

$$A_n^{\mathrm{inj}} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \quad \text{and} \quad B_n^{\mathrm{inj}} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

In particular, I_0 and I_1 are the indecomposable injective objects.

(3) *Tubes*

$$T_{n,\pi} = \; k^{nl} \overset{A_{n,\pi}^{\mathrm{tub}}}{\underset{B_{n,\pi}^{\mathrm{tub}}}{\rightrightarrows}} k^{nl} \;, \quad n \geq 1,$$

where $\pi(x,y) \in k[x,y]$ is an irreducible homogeneous polynomial of degree l. For $(\pi) \neq (x)$ we have: $A_{n,\pi}^{\mathrm{tub}} = I_{nl}$ and $B_{n,\pi}^{\mathrm{tub}} = F\big(\pi(1,y)^n\big)$ is the Frobenius normal form defined by the polynomial $\pi(1,y)^n$. For $(\pi) = (x)$ we set $A_{n,\pi}^{\mathrm{tub}} = F(x^n)$ and $B_{n,\pi}^{\mathrm{tub}} = I_n$.

Moreover, the Auslander–Reiten quiver of $\mathsf{Rep}(\overrightarrow{Q})$ has the following form:

$$\begin{tikzcd}
 & & & & & \cdots \arrow[d, shift right] & & & & & \\
 & P_1 \arrow[dr, shift left] \arrow[dr, shift right] & & P_3 \arrow[ll, dotted, "\tau"'] \arrow[dr, shift left] \arrow[dr, shift right] & & T_{2,\pi} \arrow[u, shift right] \arrow[d, shift right] \arrow[loop right, dotted, "\tau"] & & I_3 \arrow[dr, shift left] \arrow[dr, shift right] & & I_1 \arrow[ll, dotted, "\tau"'] \arrow[dr, shift left] \arrow[dr, shift right] & \\
P_0 \arrow[ur, shift left] \arrow[ur, shift right] & & P_2 \arrow[ll, dotted, "\tau"'] \arrow[ur, shift left] \arrow[ur, shift right] & & \cdots & T_{1,\pi} \arrow[u, shift right] \arrow[loop right, dotted, "\tau"] & \arrow[ur, shift left] \arrow[ur, shift right] & \cdots & I_2 \arrow[ur, shift left] \arrow[ur, shift right] & & I_0 \arrow[ll, dotted, "\tau"']
\end{tikzcd}$$

whereas the Auslander–Reiten quiver of the derived category $D^b\big(\mathsf{Rep}(\overrightarrow{Q})\big)$ is the union of components obtained by applying the shift functor to

$$\begin{tikzcd}
 & & & & & & & & & \cdots \arrow[d, shift right] \\
 & I_2[-1] \arrow[dr, shift left] \arrow[dr, shift right] & & I_0[-1] \arrow[ll, dotted, "\tau"'] \arrow[dr, shift left] \arrow[dr, shift right] & & P_1 \arrow[ll, dotted, "\tau"'] \arrow[dr, shift left] \arrow[dr, shift right] & & \cdots \arrow[ll, dotted, "\tau"'] & & T_{2,\pi} \arrow[u, shift right] \arrow[d, shift right] \arrow[loop right, dotted, "\tau"] \\
\cdots \arrow[ur, shift left] \arrow[ur, shift right] & & I_1[-1] \arrow[ll, dotted, "\tau"'] \arrow[ur, shift left] \arrow[ur, shift right] & & P_0 \arrow[ll, dotted, "\tau"'] \arrow[ur, shift left] \arrow[ur, shift right] & & P_2 \arrow[ll, dotted, "\tau"'] \arrow[ur, shift left] \arrow[ur, shift right] & & & T_{1,\pi} \arrow[u, shift right] \arrow[loop right, dotted, "\tau"]
\end{tikzcd}$$

Now we return to our study of Hall algebras of quiver. Applying Theorem 2.8, we get the following corollary, which is due to Sevenhant and van den Bergh [39], see also [43].

Corollary 3.7. *The derived reflection functor $\mathbb{RS}^+$ induces an algebra isomorphism of the Drinfeld doubles: $\mathbb{S}^+ : DH(\overrightarrow{Q}) \to DH(\overrightarrow{Q})$, whose inverse is induced by the adjoint functor $\mathbb{LS}^-$.*

Definition 3.8. Let S_1 and S_2 be the simple objects of A. Consider the subalgebra $\bar{C}(\overrightarrow{Q})$ of the Hall algebra $H(\mathsf{A})$ generated by $[S_1]$ and $[S_2]$. The composition subalgebra is defined as follows:

$$C(\overrightarrow{Q}) := \bar{C}(\overrightarrow{Q}) \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K].$$

Note that $C(\overrightarrow{Q})$ is a Hopf subalgebra of $H(\mathsf{A})$.

Our next goal is to show the automorphisms $\mathbb{S}^{\pm}$ of $DH(\overrightarrow{Q})$ map the reduced Drinfeld double of the composition algebra $C(\overrightarrow{Q})$ to itself. In other words, one has to check that both simple modules $S_i \in \mathrm{Ob}(\mathsf{A})$, $i = 1, 2$ we have: $\mathbb{S}^{\pm}\big([S_i]\big) \in DC(\overrightarrow{Q})$. Note that in the *notations of Theorem 3.6* we have: $\mathbb{S}^{+}(S_1) = I_1$ is an indecomposable injective module and $\mathbb{S}^{-}(S_2) = P_1$ is an indecomposable projective module. Hence, it is sufficient to check the following lemma.

Lemma 3.9. *The elements $[P_1]$ and $[I_1]$ belong to the composition algebra $C(\overrightarrow{Q})$.*

Proof. Using a straightforward calculation, we get the following explicit formulae for the classes of the non-simple indecomposable projective and injective modules:

$$[P_1] = \sum_{a+b=2} (-1)^a v^{-b} [S_2]^{(a)} \circ [S_1] \circ [S_2]^{(b)} \qquad [I_1] = \sum_{a+b=2} (-1)^a v^{-b} [S_1]^{(a)} \circ [S_2] \circ [S_1]^{(b)},$$

where $[X]^{(n)} = \dfrac{[X]^n}{[n]_v!}$ for an object X of the category A. □

Definition 3.10. Let $C = \left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$ be the Cartan matrix of the affine Lie algebra $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$. The Hopf algebra $U_v\big(\widehat{\mathfrak{sl}}_2\big)$ is generated over R by the elements $E_1, E_2,\ F_1, F_2,\ K_1^{\pm}, K_2^{\pm}$ subject to the relations

(1) the elements $Z^{\pm} := K_1^{\pm} K_2^{\pm}$ are central;
(2) $K_i^{\pm} K_i^{\mp} = 1 = K_i^{\mp} K_i^{\pm},\ i = 1, 2$;
(3) the subalgebra $\mathbb{C}\langle K_1^{\pm}, K_2^{\pm}\rangle$ is commutative;
(4) $K_i E_j = v^{-c_{ij}} E_j K_i$ and $K_i F_j = v^{c_{ij}} F_j K_i,\ i, j = 1, 2$;
(5) $[E_i, F_j] = \delta_{ij} v \dfrac{K_i - K_i^{-1}}{v - v^{-1}},\ i = 1, 2$;
(6) $\sum_{k=0}^{3} (-1)^k E_i^{(k)} E_j E_i^{(3-k)} = 0$ for $1 \le i \ne j \le 2$;
(7) $\sum_{k=0}^{3} (-1)^k F_i^{(k)} F_j F_i^{(3-k)} = 0$ for $1 \le i \ne j \le 2$.

The Hopf algebra structure is given by the following formulae:

(1) $\Delta(E_i) = E_i \otimes \mathbb{1} + K_i \otimes E_i$, $\Delta(F_i) = F_i \otimes K_i^{-1} + \mathbb{1} \otimes F_i$ and $\Delta(K_i) = K_i \otimes K_i$;
(2) $\eta(E_i) = 0 = \eta(F_i)$, $\eta(K_i) = 1$;
(3) $S(E_i) = -K_i^{-1} E_i$, $S(F_i) = -F_i K_i$, $S(K_i) = K_i^{-1}$ for all $i = 1, 2$.

The following result is a special case of a more general statement, which is essentially due to Ringel [33] and Green [19].

Theorem 3.11. *The $\widetilde{\mathbb{Q}}$–linear morphism $U_q(\mathfrak{g}) := U_v(\mathfrak{g}) \otimes_R \widetilde{\mathbb{Q}} \xrightarrow{\mathsf{ev}_q} DC(\overrightarrow{Q})$ mapping E_i to $[S_i]^+$, F_i to $[S_i]^-$ and K_i to $K_{\bar{S}_i}$ for $i = 1, 2$, is an isomorphism of Hopf algebras. Moreover, if we set*

$$DC_{\mathrm{gen}}(\overrightarrow{Q}) := \prod_{q \in \mathcal{P}} DC\big(\mathsf{Rep}(\mathbb{F}_q \overrightarrow{Q})\big)$$

then the R–linear map $\mathsf{ev} = \prod \mathsf{ev}_q : U_v(\mathfrak{g}) \to DC_{\mathrm{gen}}(\overrightarrow{Q})$ is injective. The same applies to the subalgebra $U_v(\mathfrak{g}^+ \oplus \mathfrak{h})$ and the algebra $C_{\mathrm{gen}}(\overrightarrow{Q}) := \prod_{q \in \mathcal{P}} C\big(\mathsf{Rep}(\mathbb{F}_q \overrightarrow{Q})\big)$.

Corollary 3.12. *The derived functors $\mathbb{RS}^+$ and $\mathbb{LS}^-$ induce a pair of mutually inverse automorphisms $\mathbb{S}^\pm$ of the algebra $U_v(\mathfrak{g})$ such that the following diagrams are commutative:*

$$\begin{tikzcd} U_v(\mathfrak{g}) \arrow[r, "\mathbb{S}^+"] \arrow[d, "\mathsf{ev}_q"'] & U_v(\mathfrak{g}) \arrow[d, "\mathsf{ev}_q"] \\ DC(\overrightarrow{Q}) \arrow[r, "\mathbb{S}^+"] & DC(\overrightarrow{Q}) \end{tikzcd} \qquad \begin{tikzcd} U_v(\mathfrak{g}) \arrow[r, "\mathbb{S}^-"] \arrow[d, "\mathsf{ev}_q"'] & U_v(\mathfrak{g}) \arrow[d, "\mathsf{ev}_q"] \\ DC(\overrightarrow{Q}) \arrow[r, "\mathbb{S}^-"] & DC(\overrightarrow{Q}). \end{tikzcd}$$

Their action on the generators is given by the following formulae:

$E_1 \xrightarrow{\mathbb{S}^+} \sum\limits_{a+b=2} (-1)^a v^{-b} E_1^{(a)} E_2 E_1^{(b)}$	$E_2 \xrightarrow{\mathbb{S}^-} \sum\limits_{a+b=2} (-1)^a v^{-b} E_2^{(a)} E_1 E_2^{(b)}$
$F_1 \xrightarrow{\mathbb{S}^+} \sum\limits_{a+b=2} (-1)^a v^{-b} F_1^{(a)} F_2 F_1^{(b)}$	$F_2 \xrightarrow{\mathbb{S}^-} \sum\limits_{a+b=2} (-1)^a v^{-b} F_2^{(a)} F_1 F_2^{(b)}$
$E_2 \xrightarrow{\mathbb{S}^+} v^{-1} F_1 K_1, \quad F_2 \xrightarrow{\mathbb{S}^+} v^{-1} E_1 K_1^{-1}$	$E_1 \xrightarrow{\mathbb{S}^-} v F_2 K_2^{-1}, \quad F_1 \xrightarrow{\mathbb{S}^-} v E_2 K_2$
$K_1 \xrightarrow{\mathbb{S}^+} K_1^2 K_2, \quad K_2 \xrightarrow{\mathbb{S}^+} K_1^{-1}$	$K_1 \xrightarrow{\mathbb{S}^-} K_2^{-1}, \quad K_2 \xrightarrow{\mathbb{S}^-} K_1 K_2^2$

As it was explained in [39, Theorem 13.1]*, in the conventions of Remark 3.13, these automorphisms are the symmetries discovered by Lusztig* [25] *(strictly speaking, Lusztig's symmetries are obtained by composing the automorphisms $\mathbb{S}^\pm$ with a flip sending E_1 to E_2, F_1 to F_2 and K_1 to K_2. This happens because we wish to consider the functors $\mathbb{S}^\pm$ as* endofunctors *of $\mathsf{Rep}(\overrightarrow{Q})$ whereas in* [39] *one interchanges the direction of the arrows).*

Remark 3.13. Note that the relations of the quantum affine algebra $U_v(\widehat{\mathfrak{sl}}_2)$ given in Definition 3.10, slightly differ from the classical ones as defined for instance in [12, 25, 27]. Namely, we impose the commutation relation $[E_i, F_i] = v\dfrac{K_i - K_i^{-1}}{v - v^{-1}}$, whereas the conventional form would be $[E_i, F_i] = \dfrac{K_i - K_i^{-1}}{v - v^{-1}}$, $i = 1, 2$. However, one can easily pass to the conventional form replacing at the *first step* v by v^{-1} and then applying the Hopf algebra automorphism sending E_i to E_i, K_i to K_i and F_i to $-v^{-1}F_i$ for $i = 1, 2$ at the *second step.*

In order to get the relations of the Drinfeld double, which are closer to the conventional ones, one can alternatively redefine Green's form by setting

$$\Big([X]K_\alpha, [Y]K_\beta\Big)_{\text{new}} = v^{-(\alpha,\beta) - \mathsf{dim}(X)} \frac{\delta_{X,Y}}{a_X},$$

where $\mathsf{dim}(X)$ is the dimension of X over k viewed as an $k\overrightarrow{Q}$–module, and take the reduced Drinfeld double with respect to $(-, -)_{\text{new}}$.

Nevertheless, we prefer to follow the form of the relations as stated in Definition 3.10, because they seem to be more natural from the point of view of Hall algebras.

The following automorphism plays an important role in our study of the algebra $U_v(\widehat{\mathfrak{sl}}_2)$.

Definition 3.14. The automorphism $\mathbb{A} = \left(\mathbb{S}^+\right)^2 : DC(\overrightarrow{Q}) \to DC(\overrightarrow{Q})$ is called *Coxeter automorphism* of $DC(\overrightarrow{Q})$. Using Corollary 3.12, we also obtain the corresponding automorphism of the algebra $U_v(\mathfrak{g})$, given by the commutative diagram

$$\begin{tikzcd} U_v(\mathfrak{g}) \arrow[r, "\mathbb{A}^+"] \arrow[d, "\mathsf{ev}"'] & U_v(\mathfrak{g}) \arrow[d, "\mathsf{ev}"] \\ DC_{\mathrm{gen}}(\overrightarrow{Q}) \arrow[r, "(\mathbb{S}^+)^2"] & DC_{\mathrm{gen}}(\overrightarrow{Q}). \end{tikzcd}$$

The inverse automorphism $\mathbb{A}^- = \bigl(\mathbb{S}^-\bigr)^2$ is defined in a similar way.

Using this technique, we can give a new proof of the following well-known result, see [45, 40].

Lemma 3.15. *The composition algebra $C(\overrightarrow{Q})$ contains all the elements $[X]$, where X is either an indecomposable pre-injective module or an indecomposable pre-projective module.*

Proof. Let X be either pre-projective or pre-injective indecomposable representation of $\overrightarrow{Q}$. Since the automorphism $\mathbb{A}^+$ acts on the Drinfeld double $DC(\overrightarrow{Q})$, we know that $\mathbb{A}^+\bigl([X]\bigr)$ belongs to $DC(\overrightarrow{Q})$. Moreover, $\mathbb{A}^+\bigl([X]\bigr)$ has the form $[Y]^{\pm}K_\alpha$ for some indecomposable pre-projective or pre-injective representation Y and some $\alpha \in K$. As in Corollary 2.5, we have a triangular decomposition

$$DC(\overrightarrow{Q}) = \bar{C}(\overrightarrow{Q})^+ \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K] \otimes_{\widetilde{\mathbb{Q}}} \bar{C}(\overrightarrow{Q})^-.$$

From this fact it follows that $DC(\overrightarrow{Q}) \cap H(\overrightarrow{Q})^{\pm} = C(\overrightarrow{Q})^{\pm} := \bar{C}(\overrightarrow{Q})^+ \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K]$. Hence, the element $\mathbb{A}^+\bigl([X]\bigr)$ has to belong to one of the aisles $C(\overrightarrow{Q})^{\pm}$. The proof for $\mathbb{A}^-\bigl([X]\bigr)$ is analogous. Since any indecomposable pre-projective or pre-injective object X is (up to a shift) of the form $\mathbb{A}^m(S)$, where $m \in \mathbb{Z}$ and S is a simple object in $\mathsf{Rep}(\overrightarrow{Q})$, this implies the claim. $\square$

4. Composition algebra of the category of coherent sheaves on $\mathbb{P}^1$

In this subsection, we consider the composition subalgebra of the category of coherent sheaves on the projective line $\mathbb{P}^1$.

First note that the maps $\mathrm{Pic}(\mathbb{P}^1) \xrightarrow{\deg} \mathbb{Z}$ and $K_0\bigl(\mathsf{Coh}(\mathbb{P}^1)\bigr) \xrightarrow{(\mathrm{rk},\deg)} \mathbb{Z}^2$ are isomorphisms of abelian groups. Next, recall some well-known facts on coherent sheaves on $\mathbb{P}^1$.

Theorem 4.1. *The indecomposable objects of the category $\mathsf{Coh}(\mathbb{P}^1)$ are:*

(1) *line bundles $\mathcal{O}_{\mathbb{P}^1}(n)$, $n \in \mathbb{Z}$;*
(2) *torsion sheaves $\mathcal{T}_{t,x} := \mathcal{O}_{\mathbb{P}^1}/\mathfrak{m}_x^t$, where $x \in \mathbb{P}^1$ is a closed point and $t \in \mathbb{Z}_{>0}$.*

Definition 4.2. Let $\mathsf{Tor}(\mathbb{P}^1)$ be the abelian category of torsion coherent sheaves on $\mathbb{P}^1$ and $H(\mathbb{P}^1)_{\mathrm{tor}} \subseteq H(\mathbb{P}^1)$ be its Hall algebra. For any integer $r \geq 1$ consider the element

$$\mathbb{1}_{(0,r)} := \sum_{\mathcal{T} \in \mathsf{Tor}(\mathbb{P}^1):\, \bar{\mathcal{T}}=(0,r)} [\mathcal{T}] \in H(\mathbb{P}^1)_{\mathrm{tor}}.$$

Next, consider the family $\bigl\{T_r\bigr\}_{r\geq 1}$ determined by $\bigl\{\mathbb{1}_{(0,r)}\bigr\}_{r\geq 1}$ using the generating series

$$1 + \sum_{r=1}^{\infty} \mathbb{1}_{(0,r)} t^r = \exp\Bigl(\sum_{r=1}^{\infty} \frac{T_r}{[r]_v} t^r\Bigr).$$

Finally, the elements $\bigl\{\Theta_r\bigr\}_{r\geq 1}$ are defined by the generating series

$$1 + \sum_{r=1}^{\infty} \Theta_r t^r = \exp\bigl((v^{-1} - v)\sum_{r=1}^{\infty} T_r t^r\bigr).$$

In what follows, we set $\mathbb{1}_{(0,0)} = T_0 = \Theta_0 = [0] = \mathbb{1}$.

Proposition 4.3. *In the notations as above we have:*
- *Any of three families* $\{\mathbb{1}_{(0,r)}\}_{r\geq 1}$, $\{T_r\}_{r\geq 1}$ *and* $\{\Theta_r\}_{r\geq 1}$ *introduced in Definition 4.2, generates the same subalgebra* $U(\mathbb{P}^1)_{\mathrm{tor}}$ *of the Hall algebra* $H(\mathbb{P}^1)_{\mathrm{tor}}$*;*
- *For any* $r, s \geq 1$ *we have the equalities:*

$$\Delta(T_r) = T_r \otimes \mathbb{1} + K_{(0,r)} \otimes T_r \quad \text{and} \quad \bigl(T_r,\, T_s\bigr) = \delta_{r,s}\frac{[2r]}{r(v^{-1}-v)}.$$

Proof. The first part of this proposition is trivial, a proof of the second can be found in [38]. □

Using this proposition, we get the following statement.

Lemma 4.4. *In the notation as above we have:* $(\Theta_r, T_r) = \frac{[2r]}{r}$ *for any* $r \in \mathbb{Z}_{>0}$.

Proof. For a sequence of non-negative integers $\underline{c} = (c_r)_{r\in\mathbb{Z}_{>0}}$ such that all but finitely many entries are zero, we set:

$$T_{\underline{c}} := \prod_{r=1}^{\infty} \frac{T_r^{c_r}}{c_r!} \quad \text{and} \quad c = \sum_{r=1}^{\infty} r c_r.$$

Then we have:

$$\Delta(T_{\underline{c}}) = \sum_{\underline{a}+\underline{b}=\underline{c}} T_{\underline{a}} K_{(0,b)} \otimes T_{\underline{b}}.$$

In particular, by induction we obtain:

$$\bigl(T_c, T_{\underline{c}}\bigr) = \begin{cases} \dfrac{[2c]}{c(v^{-1}-v)} & \text{if} \quad \underline{c} = (0,\dots,0,\underset{c-\text{th pl}}{1},0,\dots), \\ 0 & \text{otherwise.} \end{cases}$$

Using the formula $\Theta_r = (v^{-1}-v)T_r +$ monomials of length ≥ 2 in T_s, the statement follows. □

Summing up, the first family of generators $\{\mathbb{1}_{(0,r)}\}_{r\geq 0}$ of the algebra $U(\mathbb{P}^1)_{\mathrm{tor}}$ has a clear algebro–geometric meaning. The second set $\{T_r\}_{r\geq 0}$ has a good behavior with respect to the bialgebra structure: all the generators T_r are primitive and orthogonal with respect to Green's pairing. The role of the third family $\{\Theta_r\}_{r\geq 0}$ is explained by the following proposition.

Proposition 4.5. *For any* $n \in \mathbb{Z}$ *we have:*

$$\Delta\bigl([\mathcal{O}_{\mathbb{P}^1}(n)]\bigr) = \bigl[\mathcal{O}_{\mathbb{P}^1}(n)\bigr] \otimes \mathbb{1} + \sum_{r=0}^{\infty} \Theta_r K_{(1,n-r)} \otimes \bigl[\mathcal{O}_{\mathbb{P}^1}(n-r)\bigr].$$

Proof. We refer to [23, Theorem 3.3] or [37, Section 12.2] for a proof of this result. □

Remark 4.6. In the next section, we shall need another description of the elements Θ_r, see for example [38, Example 4.12]

$$\Theta_r = v^{-r} \sum_{\substack{x_1,\dots,x_m\in\mathbb{P}^1;\, x_i\neq x_j\ 1\leq i\neq j\leq m \\ t_1,\dots,t_m\,:\,\sum_{i=1}^m t_i \deg(x_i)=r}} \prod_{i=1}^{m} \bigl(1 - v^{2\deg(x_i)}\bigr)\bigl[\mathcal{T}_{t_i,x_i}\bigr].$$

Definition 4.7. The composition algebra $U(\mathbb{P}^1)$ is the subalgebra of the Hall algebra $H\bigl(\mathsf{Coh}(\mathbb{P}^1)\bigr)$ generated by the elements $L_n := \bigl[\mathcal{O}_{\mathbb{P}^1}(n)\bigr]$, T_r and K_α, where $n \in \mathbb{Z}$, $r \geq 1$ and $\alpha \in K_0\bigl(\mathsf{Coh}(\mathbb{P}^1)\bigr) \cong \mathbb{Z}^2$. We also use the notations: $\delta = (0,1) \in K_0(\mathbb{P}^1)$, $C = K_\delta$ and $K = K_{(1,0)}$.

A complete list of relations between the generators of the composition algebra $U(\mathbb{P}^1)$ was obtained by Kapranov [23] and Baumann–Kassel [2], see also [38, Section 4.3].

Theorem 4.8. *The elements L_n, T_r, K and C satisfy the following relations:*

(1) *C is central;*
(2) *$[K, T_n] = 0 = [T_n, T_m]$ for all $m, n \in \mathbb{Z}_{>0}$;*
(3) *$KL_n = v^{-2}L_nK$ for all $n \in \mathbb{Z}$;*
(4) *$\bigl[T_r, L_n\bigr] = \frac{[2r]}{r}L_{n+r}$ for all $n \in \mathbb{Z}$ and $r \in \mathbb{Z}_{>0}$;*
(5) *$L_mL_{n+1} + L_nL_{m+1} = v^2\bigl(L_{n+1}L_m + L_{m+1}L_n\bigr)$ for all $m, n \in \mathbb{Z}$.*

Let $U(\mathbb{P}^1)_{\mathrm{vec}}$ be the subalgebra of $U(\mathbb{P}^1)$ generated by the elements L_n, $n \in \mathbb{Z}$. Then the $\widetilde{\mathbb{Q}}$–linear map $U(\mathbb{P}^1)_{\mathrm{vec}} \otimes_{\widetilde{\mathbb{Q}}} U(\mathbb{P}^1)_{\mathrm{tor}} \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K] \xrightarrow{\mathsf{mult}} U(\mathbb{P}^1)$ is an isomorphism. In particular, the elements

$$B_{\underline{m},\,\underline{l},\,a,\,b} = \prod_{n\in\mathbb{Z}} L_n^{m_n} \circ \prod_{r\in\mathbb{Z}^+} T_r^{l_r} \circ K^aC^b,$$

where $a, b \in \mathbb{Z}$, $\underline{m} = (m_n)_{n\in\mathbb{Z}}$ and $\underline{l} = (l_r)_{r\in\mathbb{Z}_{>0}}$ are sequences of non-negative integers such that all but finitely entries are zero, form a basis of $U(\mathbb{P}^1)$.

It turns out that in order to relate the reduced Drinfeld double $DU(\mathbb{P}^1)$ with Drinfeld's new presentation of $U_v\bigl(\widehat{\mathfrak{sl}}_2\bigr)$ [16], one has to modify the definition of the generators $T_r^{\pm}$ and $\Theta_r^{\pm}$. Similarly to Corollary 2.5, we have a triangular decomposition

$$DU(\mathbb{P}^1) = \bar{U}(\mathbb{P}^1)^+ \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K] \otimes_{\widetilde{\mathbb{Q}}} \bar{U}(\mathbb{P}^1)^-,$$

where $K = K_0(\mathbb{P}^1) \cong \mathbb{Z}^2$ is the K–group and the element $C = K_{(0,1)}$ is central, see also [9]. Consider the group $K' := \mathbb{Z} \oplus \frac{1}{2}\mathbb{Z}$ and the algebra $\widetilde{DU}(\mathbb{P}^1)$ obtained from $DU(\mathbb{P}^1)$ by adding two central generators $C^{\pm\frac{1}{2}} = K_{(0,\pm\frac{1}{2})}$ such that $C^{\frac{1}{2}}C^{-\frac{1}{2}} = \mathbb{1} = C^{-\frac{1}{2}}C^{\frac{1}{2}}$ and $\bigl(C^{\frac{1}{2}}\bigr)^2 = C$. For any $r \in \mathbb{Z}_{>0}$ we set: $\widetilde{T}_r^{\pm} = T_r^{\pm} \cdot C^{\mp\frac{r}{2}}$ and $\widetilde{\Theta}_r^{\pm} = \Theta_r^{\pm} \cdot C^{\mp\frac{r}{2}}$. Then we have:

$$\Delta\bigl(\widetilde{T}_r^+\bigr) = \widetilde{T}_r^+ \otimes C^{-\frac{r}{2}} + C^{\frac{r}{2}} \otimes \widetilde{T}_r^+, \quad r \in \mathbb{Z}_{>0}$$

and

$$\Delta(L_n^+) = L_n^+ \otimes \mathbb{1} + KC^n \otimes L_n^+ + \sum_{r=1}^{\infty} \widetilde{\Theta}_r^+ KC^{n-\frac{r}{2}} \otimes L_{n-r}^+, \quad n \in \mathbb{Z}.$$

Note that by Lemma 4.4 we have: $\bigl(\widetilde{\Theta}_r, \widetilde{T}_r\bigr) = \bigl(\Theta_r, T_r\bigr) = \frac{[2r]}{r}$. Using the relations of the Drinfeld double and rewriting the relations of Theorem 4.8, we obtain:

(1) $\bigl[K, \widetilde{T}_n^{\pm}\bigr] = 0 = \bigl[\widetilde{T}_n^{\pm}, \widetilde{T}_m^{\pm}\bigr]$ for all $m, n \in \mathbb{Z}_{>0}$;
(2) $KL_n^{\pm} = v^{\mp 2}L_n^{\pm}K$ for all $n \in \mathbb{Z}$;
(3) $\bigl[\widetilde{T}_r^{\pm}, L_n^{\pm}\bigr] = \frac{[2r]}{r}L_{n+r}^{\pm}C^{\mp\frac{r}{2}}$ for all $n \in \mathbb{Z}$ and $r \in \mathbb{Z}_{>0}$;
(4) $\bigl[L_n^{\pm}, \widetilde{T}_r^{\mp}\bigr] = \frac{[2r]}{r}L_{n-r}^{\pm}C^{\mp\frac{r}{2}}$ for all $n \in \mathbb{Z}$ and $r \in \mathbb{Z}_{>0}$;
(5) $\bigl[\widetilde{T}_r^+, \widetilde{T}_s^-\bigr] = \delta_{r,s}\frac{[2r]}{r}\frac{C^{-r} - C^r}{v^{-1} - v}$, where $r, s \in \mathbb{Z}_{>0}$;
(6) Finally, we have:

$$\bigl[L_n^+, L_m^-\bigr] = \begin{cases} \dfrac{v}{v - v^{-1}}\,\widetilde{\Theta}_{n-m}^+ KC^{\frac{m+n}{2}} & \text{if} \quad n > m, \\ 0 & \text{if} \quad n = m, \\ \dfrac{v}{v^{-1} - v}\,\widetilde{\Theta}_{m-n}^- K^{-1}C^{-\frac{m+n}{2}} & \text{if} \quad n < m. \end{cases}$$

Definition 4.9. Consider the R–algebra $U_v(\mathfrak{Lsl}_2)$ generated by the elements $X_n^{\pm}$ $(n \in \mathbb{Z})$, H_r $(r \in \mathbb{Z} \setminus \{0\})$, $C^{\pm\frac{1}{2}}$ and $K^{\pm}$ subject to the following relations:

(1) $C^{\frac{1}{2}}$ is central;

(2) $K^{\pm}K^{\mp} = 1 = K^{\mp}K^{\pm}$, $C^{\frac{1}{2}}C^{-\frac{1}{2}} = 1 = C^{-\frac{1}{2}}C^{\frac{1}{2}}$;
(3) $\big[K, H_r\big] = 0$ for all $r \in \mathbb{Z} \setminus \{0\}$, $KX_n^{\pm} = v^{\mp 2}X_n^{\pm}K$ for all $n \in \mathbb{Z}$;
(4) We have Heisenberg-type relations

$$\big[H_m, H_n\big] = \delta_{m+n,0}\frac{[2m]}{m}\frac{C^n - C^{-n}}{v - v^{-1}}$$

for all $m, n \in \mathbb{Z} \setminus \{0\}$;
(5) We have Hecke-type relations

$$\big[H_r, X_n^{\pm}\big] = \pm\frac{[2r]}{r}X_{n+r}^{\pm}C^{\mp\frac{|r|}{2}}$$

for all $n \in \mathbb{Z}$ and $r \in \mathbb{Z} \setminus \{0\}$;
(6) $X_m^{\pm}X_{n+1}^{\pm} + X_n^{\pm}X_{m+1}^{\pm} = v^{\pm 2}\big(X_{n+1}^{\pm}X_m^{\pm} + X_{m+1}^{\pm}X_n^{\pm}\big)$ for all $m, n \in \mathbb{Z}$;
(7) Finally, for all $m, n \in \mathbb{Z}$ we have:

$$\big[X_m^{+}, X_n^{-}\big] = \frac{v}{v - v^{-1}}\big(\Psi_{m+n}^{+}C^{\frac{m-n}{2}} - \Psi_{m+n}^{-}C^{\frac{n-m}{2}}\big)K^{\mathsf{sign}(m+n)},$$

where $\Psi_{\pm r}^{\pm} (r \geq 1)$ are given by the generating series

$$1 + \sum_{r=1}^{\infty}\Psi_{\pm r}^{\pm}t^r = \exp\big(\pm(v^{-1} - v)\sum_{r=1}^{\infty}H_{\pm r}t^r\big)$$

and $\Psi_{\pm r}^{\pm} = 0$ for $r < 0$.

Remark 4.10. Similarly to the case of $U_v(\widehat{\mathfrak{sl}}_2)$, our presentation of $U_v(\mathfrak{Lsl}_2)$ slightly differs from the conventional one, as used in [25, 12, 27]. To pass to their notation, one has to replace v by v^{-1} at the first step and then replace vX_n^{-} by $-X_n^{-}$ for all $n \in \mathbb{Z}$ at the second, see also Remark 3.13.

Proposition 4.11. *Let $U_q(\mathfrak{Lsl}_2) = U_v(\mathfrak{Lsl}_2) \otimes_R \widetilde{\mathbb{Q}}_q$. Then the map $\mathsf{ev}_q : U_q(\mathfrak{Lsl}_2) \to \widetilde{DU}(\mathbb{P}^1)$ given by the rule: $X_n^{+} \mapsto L_n^{+}$, $X_n^{-} \mapsto L_{-n}^{-}$ for $n \in \mathbb{Z}$, $H_r \mapsto \widetilde{T}_r^{+}$ for $r \in \mathbb{Z}_{>0}$ and $H_r \mapsto -\widetilde{T}_{-r}^{-}$ for $r \in \mathbb{Z}_{<0}$, $\Psi_r^{+} \mapsto \widetilde{\Theta}_r^{+}$ for $r \in \mathbb{Z}_{>0}$ and $\Psi_r^{-} \mapsto \widetilde{\Theta}_{-r}^{-}$ for $r \in \mathbb{Z}_{<0}$, $K \mapsto K$ and $C^{\frac{1}{2}} \mapsto C^{\frac{1}{2}}$, is an isomorphism of algebras.*

Proof. From the list of relations of Theorem 4.8 it follows that the morphism ev_q is well-defined. Next, consider the elements

$$B_{\underline{m}', \underline{l}', a, b, \underline{m}'', \underline{l}''} = \prod_{n\in\mathbb{Z}}[X_n^{+}]^{(m_n')} \circ \prod_{r\in\mathbb{Z}_{>0}}(H_r)^{l_r'} \circ K^a C^{\frac{b}{2}} \circ \prod_{n\in\mathbb{Z}}[X_n^{-}]^{(m_n'')} \circ \prod_{r\in\mathbb{Z}_{<0}}(H_r)^{l_r''}$$

of $U_v(\mathfrak{Lsl}_2)$, where $(m_n')_{n\in\mathbb{Z}}, (m_n'')_{n\in\mathbb{Z}}, (l_r')_{r\in\mathbb{Z}_{>0}}, (l_r'')_{r\in\mathbb{Z}_{>0}}$ run through the set of all sequences of non-negative integers such that all but finitely many entries are zero, and $a, b \in \mathbb{Z}$. Using the defining relations it is not difficult to show that all the elements $B_{\underline{m}', \underline{l}', a, b, \underline{m}'', \underline{l}''}$ generate $U_v(\mathfrak{Lsl}_2)$ as R-module. Indeed, observe first that any element of $U_v(\mathfrak{Lsl}_2)$ can be written as an R–linear combination of elements of the form $A_+ \cdot Z \cdot A_-$, where $Z \in \big\langle C^{\pm\frac{1}{2}}, K^{\pm}\big\rangle$ and $A_\pm \in \big\langle X_n^{\pm}, H_{\pm r} \big|\, n \in \mathbb{Z}, r \in \mathbb{Z}_{>0}\big\rangle$. Next, using Hecke-type relations of Definition 4.9, we can write any element $A \in \big\langle X_n, H_r, C^{\pm\frac{1}{2}} \big|\, n \in \mathbb{Z}, r \in \mathbb{Z}_{>0}\big\rangle$ as an R–linear combination of elements of the form $V \cdot T$, where $V \in \big\langle X_n \big|\, n \in \mathbb{Z}\big\rangle$ and $T \in \big\langle H_r, C^{\pm\frac{1}{2}} \big|\, r \in \mathbb{Z}_{>0}\big\rangle$. Finally, it remains to note that for all $m, n \in \mathbb{Z}$ such that $m > n$ we can write $X_m X_n$ as an R–linear combination of the elements $X_n X_m, X_{n+1}X_{m-1}, \dots, X_{n+l}X_{m-l}$, where l is the entier of $\frac{m-n}{2}$.

Moreover, for any $q \in \mathcal{P}$ the elements $\mathsf{ev}_q\big(B_{\underline{m}', \underline{l}', a, b, \underline{m}'', \underline{l}''}\big)$ are linearly independent in $\widetilde{DU}(\mathbb{P}^1)$. Hence, $U_v(\mathfrak{Lsl}_2)$ is free as an R–module and $B_{\underline{m}', \underline{l}', a, b, \underline{m}'', \underline{l}''}$ is its basis over R. In particular, the morphism ev_q is an isomorphism for any $q \in \mathcal{P}$. □

Corollary 4.12. *Let* $\widetilde{DU}_{\mathrm{gen}}(\mathbb{P}^1) := \prod_{q\in\mathcal{P}} DU\big(\mathbb{P}^1(\mathbb{F}_q)\big)$ *then the* R*–linear map*

$$\mathsf{ev} = \prod_{q\in\mathcal{P}} \mathsf{ev}_q : U_v(\mathfrak{Lsl}_2) \longrightarrow \widetilde{DU}_{\mathrm{gen}}(\mathbb{P}^1)$$

is injective. Moreover, the elements $B_{\underline{m}',\,\underline{l}',\,a,\,b,\,\underline{m}'',\,\underline{l}''}$ *form a Poincaré-Birkhoff-Witt (PBW) basis of* $U_v(\mathfrak{Lsl}_2)$ *viewed as an* R*–module.*

Remark 4.13. The Hall algebra approach to a construction of PBW bases of quantum groups is due to Ringel [34]. The case of $U_v^+(\mathfrak{Lsl}_2)$ was considered by Baumann and Kassel [2], see also [45]. PBW bases of quantum loop algebras were studied by Beck [3] and Beck, Chari and Pressley [4].

Remark 4.14. Consider the functor $\mathbb{A} = \mathcal{O}_{\mathbb{P}^1}(-2)\otimes - :\ D^b\big(\mathsf{Coh}(\mathbb{P}^1)\big) \to D^b\big(\mathsf{Coh}(\mathbb{P}^1)\big)$. Then it induces an automorphism of the algebra $\widetilde{DU}(\mathbb{P}^1)$ preserving the subalgebra $DU(\mathbb{P}^1)$ and given by:

$$L_n^{\pm} \xrightarrow{\mathbb{A}} L_{n\mp 2}^{\pm},\ T_r^{\pm} \xrightarrow{\mathbb{A}} T_r^{\pm},\ \Theta_r^{\pm} \xrightarrow{\mathbb{A}} \Theta_r^{\pm},\ \widetilde{T}_r^{\pm} \xrightarrow{\mathbb{A}} \widetilde{T}_r^{\pm},\ \widetilde{\Theta}_r^{\pm} \xrightarrow{\mathbb{A}} \widetilde{\Theta}_r^{\pm},\ C^{\frac{1}{2}} \xrightarrow{\mathbb{A}} C^{\frac{1}{2}},\ K \xrightarrow{\mathbb{A}} KC^{-2}.$$

In particular, it corresponds to the algebra automorphism $\mathbb{A}$ of $U_v(\mathfrak{Lsl}_2)$ such that

$$X_n^{\pm} \xrightarrow{\mathbb{A}} X_{n\mp 2}^{\pm},\ H_r \xrightarrow{\mathbb{A}} H_r,\ \Psi_r \xrightarrow{\mathbb{A}} \Psi_r,\ C^{\frac{1}{2}} \xrightarrow{\mathbb{A}} C^{\frac{1}{2}},\ K \xrightarrow{\mathbb{A}} KC^{-2}.$$

Moreover, the following diagram is commutative:

$$\begin{tikzcd} U_v(\mathfrak{Lsl}_2) \arrow[r, "\mathbb{A}"] \arrow[d, "\mathsf{ev}"'] & U_v(\mathfrak{Lsl}_2) \arrow[d, "\mathsf{ev}"] \\ \widetilde{DU}_{\mathrm{gen}}(\mathbb{P}^1) \arrow[r, "\mathbb{A}"] & \widetilde{DU}_{\mathrm{gen}}(\mathbb{P}^1). \end{tikzcd}$$

5. Categorification of Drinfeld-Beck isomorphism for $U_v\big(\widehat{\mathfrak{sl}}_2\big)$

Next, we elaborate a connection between the reduced Drinfeld doubles $DC(\overrightarrow{Q})$ and $DU(\mathbb{P}^1)$.

Theorem 5.1. *Let* $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ *and* $B = \mathsf{End}_{\mathbb{P}^1}(\mathcal{F})$*. Then*

$$\mathbb{F} := \mathbb{R}\,\mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{F}, -\big) : D^b\big(\mathsf{Coh}(\mathbb{P}^1)\big) \longrightarrow D^b\big(\mathsf{mod} - B\big)$$

is an equivalence of triangulated categories. Identifying the category of right B*–modules with the category of representations of the Kronecker quiver* $\overrightarrow{Q}$ *we have the following statements.*

(1) *In the diagram of categories and functors*

$$\begin{tikzcd} D^b\big(\mathsf{Coh}(\mathbb{P}^1)\big) \arrow[r, "\mathbb{F}"] \arrow[d, "\mathcal{O}_{\mathbb{P}^1}(\mp 2)\otimes -"'] & D^b\big(\mathsf{Rep}(\overrightarrow{Q})\big) \arrow[d, "(\mathbb{RS}^{\pm})^2"] \\ D^b\big(\mathsf{Coh}(\mathbb{P}^1)\big) \arrow[r, "\mathbb{F}"] & D^b\big(\mathsf{Rep}(\overrightarrow{Q})\big) \end{tikzcd}$$

both compositions are isomorphic.

(2) $\mathbb{F}\big(\mathcal{O}_{\mathbb{P}^1}(n)\big) \cong P_n$ *if* $n \ge 0$ *and* $I_{-n-1}[-1]$ *if* $n < 0$.

(3) $\mathbb{F}$ *induces an equivalence between the category* $\mathsf{Tor}(\mathbb{P}^1)$ *of torsion coherent sheaves on* $\mathbb{P}^1$ *and the subcategory* $\mathsf{Tub}(\overrightarrow{Q})$ *of* $\mathsf{Rep}(\overrightarrow{Q})$*, which is the additive closure of the category of modules lying in the tubes.*

Proof. The result that $\mathbb{F}$ is an equivalence of categories is due to Beilinson [5], see also [18]. Next, the functor $\mathbb{A} = \mathcal{O}_{\mathbb{P}^1}(-2)\otimes -$ is the Auslander-Reiten translate in $D^b\bigl(\mathsf{Coh}(\mathbb{P}^1)\bigr)$, see [7, 31]. On the other hand, by Theorem 3.2 the functor $(\mathbb{RS}^{\pm})^2$ is the Auslander-Reiten translate in $D^b\bigl(\mathsf{Rep}(\overrightarrow{Q})\bigr)$. Since $\mathbb{F}$ is an equivalence, we have: $\mathbb{F}\circ\mathbb{A} \cong (\mathbb{RS}^{+})^2\circ\mathbb{F}$, see for example [31, Proposition I.2.3].

It is clear that the algebra $B = \mathsf{End}_{\mathbb{P}^1}(\mathcal{F})$ is *isomorphic* to the path algebra of the Kronecker quiver. However, in order to describe the images of the indecomposable objects of $D^b\bigl(\mathsf{Coh}(\mathbb{P}^1)\bigr)$ in a precise way, we have to specify this isomorphism. Recall that a choice of homogeneous coordinates $(x : y)$ on $\mathbb{P}^1$ fixes two distinguished sections $x, y \in \mathsf{Hom}_{\mathbb{P}^1}\bigl(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1)\bigr)$ vanishing at $(0 : 1)$ and $(1 : 0)$ respectively. Let e_1 and e_2 be the primitive idempotents of $A := B^{\mathsf{op}}$ corresponding to the identity endomorphisms of $\mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(-1)$ respectively. Then we have: $\mathbb{F}\bigl(\mathcal{O}_{\mathbb{P}^1}\bigr) \cong Ae_1$ and $\mathbb{F}\bigl(\mathcal{O}_{\mathbb{P}^1}(-1)\bigr) \cong Ae_2$. We identify the path algebra of the Kronecker quiver $\overrightarrow{Q}$ with the algebra A in such a way that the sections x and y got identified with the upper and lower arrows of $\overrightarrow{Q}$ under

$$\mathsf{Hom}_{\mathbb{P}^1}\bigl(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1)\bigr) \xrightarrow{\mathbb{G}} \mathsf{Hom}_A\bigl(Ae_2, Ae_1\bigr) = e_2Ae_1 \cong e_2 \cdot k\overrightarrow{Q} \cdot e_1.$$

Note that for all $n \in \mathbb{Z}$ the triangle

$$\mathcal{O}_{\mathbb{P}^1}(n-1) \xrightarrow{(-x\;y)} \mathcal{O}_{\mathbb{P}^1}(n)^{\oplus 2} \xrightarrow{\binom{y}{x}} \mathcal{O}_{\mathbb{P}^1}(n+1) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(n-1)[1]$$

is almost split in $D^b\bigl(\mathsf{Coh}(\mathbb{P}^1)\bigr)$. Since $\mathbb{F}$ maps almost split triangles to almost split triangles, Theorem 3.6 implies the formula for the images of the lines bundles under the functor $\mathbb{F}$. Applying Theorem 2.6, Theorem 3.6 and Theorem 4.1, one can easily deduce that $\mathbb{F}$ restricts to an equivalence of abelian categories $\mathsf{Tor}(\mathbb{P}^1)$ and $\mathsf{Tub}(\overrightarrow{Q})$.

This correspondence can be made more precise. Let $\pi = (a : b) \in \mathbb{P}^1$ be a closed point of *degree one* given by the homogeneous form $p_\pi(x, y) = ay - bx$. Then the unique simple torsion sheaf $\mathcal{T}_{1,\pi}$ supported at π has a locally free resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{p_\pi} \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{T}_{1,\pi} \longrightarrow 0.$$

Since $\mathbb{G}\bigl(\mathcal{O}_{\mathbb{P}^1}(-1)\bigr) = P_0$ and $\mathbb{G}\bigl(\mathcal{O}_{\mathbb{P}^1}\bigr) = P_1$, we conclude that $\mathbb{G}\bigl(\mathcal{T}_{1,\pi}\bigr) \cong T_{1,\pi}$. It implies that for any $n \in \mathbb{Z}_{>0}$ we have: $\mathbb{G}\bigl(\mathcal{T}_{n,\pi}\bigr) \cong T_{n,\pi}$. The general case can be treated similarly. □

Applying Cramer's Theorem 2.8 we obtain:

Corollary 5.2. *The assignment $DH(\mathbb{P}^1) \xrightarrow{\mathbb{F}} DH(\overrightarrow{Q})$ is an isomorphism of algebras.*

Our next goal is to show this isomorphism restricts on the isomorphism between the reduced Drinfeld doubles of the composition subalgebras $DU(\mathbb{P}^1) \to DC(\overrightarrow{Q})$. For this it is convenient to consider a functor $\mathbb{G} : D^b\bigl(\mathsf{Rep}(\overrightarrow{Q})\bigr) \to D^b\bigl(\mathsf{Coh}(\mathbb{P}^1)\bigr)$, which is quasi-inverse to $\mathbb{F}$.

Theorem 5.3. *The algebra isomorphism $\mathbb{G} : DH(\overrightarrow{Q}) \to DH(\mathbb{P}^1)$ restricts on the isomorphism of the reduced Drinfeld doubles of the composition subalgebras $\mathbb{G} : DC(\overrightarrow{Q}) \to DU(\mathbb{P}^1)$.*

Proof. We use the notation $E_i = [S_i]^+$, $F_i = [S_i]^-$ and $K_i = K_{\bar{S}_i}$, $i = 1, 2$ for the elements of the algebra $DC(\overrightarrow{Q})$. Then we have:

$$E_1 \xrightarrow{\mathbb{G}} v^{-1}L^-_{-1}K^{-1}C,\; E_2 \xrightarrow{\mathbb{G}} L^+_0,\; F_1 \xrightarrow{\mathbb{G}} v^{-1}L^+_{-1}KC^{-1},\; F_2 \xrightarrow{\mathbb{G}} L^-_0, K_1 \xrightarrow{\mathbb{G}} K^{-1}C, K_2 \xrightarrow{\mathbb{G}} K.$$

This implies that the image of the subalgebra $DC(\overrightarrow{Q})$ is contained in $DU(\mathbb{P}^1)$. Moreover, from the commutativity of the diagram

$$\begin{tikzcd} DC(\overrightarrow{Q}) \arrow[r, "\mathbb{G}"] \arrow[d, hook] & DU(\mathbb{P}^1) \arrow[d, hook] \\ DH(\overrightarrow{Q}) \arrow[r, "\mathbb{G}"] & DH(\mathbb{P}^1) \end{tikzcd}$$

we derive that the algebra homomorphism $\mathbb{G} : DC(\overrightarrow{Q}) \to DU(\mathbb{P}^1)$ is injective. To show it is also surjective, we have to prove that all the elements $T_r^{\pm}$ and $\Theta_r^{\pm}$ are in the image of $\mathbb{G}$ for all $r \geq 1$. Note that for any pair of integers $n > m$ we have the following relation in $DU(\mathbb{P}^1)$:

$$[L_n^+, L_m^-] = \frac{v}{v - v^{-1}} \Theta_{n-m}^+ K C^m.$$

This implies that $\Theta_r^{\pm}$ belong to $\mathbb{G}\bigl(DC(\overrightarrow{Q})\bigr)$ for all $r > 0$. Hence, the elements $T_r^{\pm}$ belong to $\mathbb{G}\bigl(DC(\overrightarrow{Q})\bigr)$ for all $r > 0$. This shows the surjectivity of the map $\mathbb{G} : DC(\overrightarrow{Q}) \to DU(\mathbb{P}^1)$. □

As an application of the developed technique, we get a shorter and (on our mind) more conceptual proof of the following formula, obtained for the first time by Szántó [40, Theorem 4.3].

Theorem 5.4. *In the Hall algebra of the Kronecker quiver $\overrightarrow{Q}$ we have for any $m, n \in \mathbb{Z}_{\geq 0}$:*

$$[I_m] \cdot [P_n] - v^2 [P_n] \cdot [I_m] = \frac{v^{-(m+n+1)}}{v^{-1} - v} \sum_{\substack{\pi_1, \dots, \pi_l \in \mathcal{Q};\, \pi_i \neq \pi_j\ 1 \leq i \neq j \leq l \\ t_1, \dots, t_l \,:\, \sum_{i=1}^l t_i \deg(\pi_i) = m+n+1}} \prod_{i=1}^{l} \bigl(1 - v^{2\deg(\pi_i)}\bigr) \bigl[T_{t_i, \pi_i}\bigr],$$

where we sum over the set $\mathcal{Q}$ of all homogeneous prime ideals of height one in the ring $k[x,y]$. In particular the left-hand side of this formula depends only on the sum $m + n$.

Proof. Let m and n be non-negative integers such that $m + n + 1 = r$. Then we have the following identity in the reduced Drinfeld double $DU(\mathbb{P}^1)$:

$$L_{-m-1}^- L_n^+ - L_n^+ L_{-m-1}^- = \frac{1}{q-1} \Theta_r^+ K C^{-m-1}.$$

Note that the algebra homomorphism $\mathbb{F} : DU(\mathbb{P}^1) \to DC(\overrightarrow{Q})$ acts as follows:

$$L_{-m-1}^- \mapsto v[I_m]^+ K_1^{-m-1} K_2^{-m}, \quad L_n^+ \mapsto [P_n]^+, \quad K_{(1,-t)} = K C^{-t} \mapsto K_1^{-t} K_2^{-t+1},$$

where $m, n \in \mathbb{Z}_{\geq 0}$. It remains to observe that by Remark 4.6 we have:

$$\overline{\Theta}_r := \mathbb{F}(\Theta_r) = v^{-r} \sum_{\substack{t_1, \dots, t_l \,:\, \sum_{i=1}^l t_i \deg(\pi_i) = r \\ \pi_1, \dots, \pi_l \in \mathcal{Q};\, \pi_i \neq \pi_j\, 1 \leq i \neq j \leq l}} \prod_{i=1}^{l} \bigl(1 - v^{2\deg(\pi_i)}\bigr) \bigl[T_{t_i, \pi_i}\bigr].$$

In particular, we get the equality:

$$v[I_m] K_1^{-m-1} K_2^{-m} [P_n] - v[P_n][I_m] K_1^{-m-1} K_2^{-m} = \frac{1}{q-1} \overline{\Theta}_r K_1^{-m-1} K_2^{-m}.$$

Taking into account the fact that $K_1 K_2$ is central and $K_1^{-1}[P_n] = v^{-2}[P_n] K_1^{-1}$, we end up precisely with Szántó's formula. Our proof explains the conceptional meaning of this equality: this formula in the composition subalgebra of $\mathsf{Rep}(\overrightarrow{Q})$ is a translation of a "canonical" relation in the reduced Drinfeld double of $U(\mathbb{P}^1)$. □

Another application of our approach is the following important result, which was stated by Drinfeld [16] and proven by Beck [3], see also [14] and [28].

Theorem 5.5. *We have an injective homomorphism of R–algebras $\mathbb{G} : U_v\big(\widehat{\mathfrak{sl}}_2\big) \longrightarrow U_v(\mathfrak{L}\mathfrak{sl}_2)$ given by the following formulae:*

$$E_1 \xrightarrow{\mathbb{G}} v^{-1}X_1^- K^{-1}C,\ E_2 \xrightarrow{\mathbb{G}} X_0^+,\ F_1 \xrightarrow{\mathbb{G}} v^{-1}X_{-1}^+ KC^{-1},\ F_2 \xrightarrow{\mathbb{G}} X_0^-, K_1 \xrightarrow{\mathbb{G}} K^{-1}C, K_2 \xrightarrow{\mathbb{G}} K.$$

Its image is the subalgebra of $U_v(\mathfrak{L}\mathfrak{sl}_2)$ generated by the elements $X_n^\pm, H_r, C^\pm$ and $K^\pm$. Moreover, the following diagram is commutative for any $m \in \mathbb{Z}$:

$$\begin{tikzcd}
U_v\big(\widehat{\mathfrak{sl}}_2\big) \arrow[r, "\mathbb{A}^m"] \arrow[d, "\mathbb{G}"'] & U_v\big(\widehat{\mathfrak{sl}}_2\big) \arrow[d, "\mathbb{G}"] \\
U_v(\mathfrak{L}\mathfrak{sl}_2) \arrow[r, "\mathbb{A}^m"] & U_v(\mathfrak{L}\mathfrak{sl}_2).
\end{tikzcd}$$

Proof. This result follows directly from the commutativity of the following diagram:

$$\begin{tikzcd}
U_v\big(\widehat{\mathfrak{sl}}_2\big) \arrow[rrr, "\mathbb{A}^\pm"] \arrow[dr, "\mathsf{ev}"] \arrow[ddd, "\mathbb{G}"'] & & & U_v\big(\widehat{\mathfrak{sl}}_2\big) \arrow[dl, "\mathsf{ev}"'] \arrow[ddd, "\mathbb{G}"] \\
& DC_{\mathrm{gen}}(\overrightarrow{Q}) \arrow[r, "\mathbb{A}^\pm"] \arrow[d, "\mathbb{G}"'] & DC_{\mathrm{gen}}(\overrightarrow{Q}) \arrow[d, "\mathbb{G}"] & \\
& \widetilde{DU}_{\mathrm{gen}}(\mathbb{P}^1) \arrow[r, "\mathbb{A}^\pm"] & \widetilde{DU}_{\mathrm{gen}}(\mathbb{P}^1) & \\
U_v(\mathfrak{L}\mathfrak{sl}_2) \arrow[rrr, "\mathbb{A}^\pm"] \arrow[ur, "\mathsf{ev}"] & & & U_v(\mathfrak{L}\mathfrak{sl}_2), \arrow[ul, "\mathsf{ev}"']
\end{tikzcd}$$

which is obtained by patching together the diagrams, constructed in Corollary 3.12, Remark 4.14 and Theorem 5.3. □

Remark 5.6. The algebra homomorphism $\mathbb{G} : U_v\big(\widehat{\mathfrak{sl}}_2\big) \to U_v(\mathfrak{L}\mathfrak{sl}_2)$ is *not surjective* because the elements $C^{\pm\frac{1}{2}}$ do not belong to the image of $\mathbb{G}$.

Remark 5.7. In order to pass to the "conventional form" of the Drinfeld-Beck (iso)morphism, one has to apply the identifications of $U_v(\widehat{\mathfrak{sl}}_2)$ and $U_v(\mathfrak{L}\mathfrak{sl}_2)$ described in Remark 3.13 and Remark 4.10. In that terms, the "categorical isomorphism" $\mathbb{G}$ of Theorem 5.5 is equal to the composition of the conventional one (as can be found for instance in [27]) combined with the automorphism of $U_v(\widehat{\mathfrak{sl}}_2)$ given by the rule $E_1 \mapsto -C^{-1}E_1$, $F_1 \mapsto -CF_1$ and leaving the remaining generators unchanged.

As a final application of the technique of Hall algebras to the study of the quantum affine algebra $U_v\big(\widehat{\mathfrak{sl}}_2\big)$, we shall reprove several known results on its integral form.

Definition 5.8. The integral form $U_v^{\mathrm{int}}\big(\widehat{\mathfrak{sl}}_2\big)$ of the quantum affine algebra $U_v\big(\widehat{\mathfrak{sl}}_2\big)$ is the $\mathbb{Q}[v, v^{-1}]$ subalgebra of $U_v\big(\widehat{\mathfrak{sl}}_2\big)$ generated by $E_i^{(n)} = \dfrac{E_i^n}{[n]!}$, $F_i^{(n)} = \dfrac{F_i^n}{[n]!}$ for all $n \in \mathbb{Z}_{\geq 0}$, $i = 1, 2$, K_1 and K_2.

The following result is well-known, see for example [25].

Theorem 5.9. *The algebra $U_v^{\mathrm{int}}\big(\widehat{\mathfrak{sl}}_2\big)$ is a Hopf algebra over $\mathbb{Q}[v, v^{-1}]$ and we have:*

$$U_v^{\mathrm{int}}\big(\widehat{\mathfrak{sl}}_2\big) \otimes_{\mathbb{Q}[v,v^{-1}]} R = U_v\big(\widehat{\mathfrak{sl}}_2\big).$$

Let $U_v^{\mathrm{int}}\big(\mathfrak{Lsl}_2\big) := \mathbb{G}\big(U_v^{\mathrm{int}}\big(\widehat{\mathfrak{sl}}_2\big)\big)$. As an application of our approach, we show that certain elements of $U_v\big(\mathfrak{Lsl}_2\big)$ actually belong to $U_v^{\mathrm{int}}\big(\mathfrak{Lsl}_2\big)$. First note the following well-known fact, see e.g. [11].

Lemma 5.10. *The elements $X_n^{\pm(m)} \in U_v\big(\mathfrak{Lsl}_2\big)$ belong to the integral form $U_v^{\mathrm{int}}\big(\mathfrak{Lsl}_2\big)$ for all $n \in \mathbb{Z}$ and $m \in \mathbb{Z}_{>0}$.*

Proof. Let $\mathsf{A} = \mathsf{Rep}(\overrightarrow{Q})$ and $X \in \mathrm{Ob}(\mathsf{A})$ be an object such that $\mathsf{End}_{\mathsf{A}}(X) = k$ and $\mathsf{Ext}^1_{\mathsf{A}}(X, X) = 0$. Then for any $n \in \mathbb{Z}_{\geq 0}$ we have the following equality in the Hall algebra $H(\mathsf{A})$:

$$\big[X^{\oplus n}\big] = v^{n(n-1)} \frac{[X]^n}{[n]!} = v^{n(n-1)}[X]^{(n)}.$$

In particular, for any $i \in \{1, 2\}$, $n \in \mathbb{Z}_{>0}$ and and $q \in \mathcal{P}$ we have: $\mathsf{ev}_q\big(E_i^{(n)}\big) = v^{n(1-n)}\big[S_i^{\oplus n}\big]$. Our next aim is to show the automorphisms $\mathbb{S}^{\pm}$ of the algebra $U_v\big(\widehat{\mathfrak{sl}}_2\big)$ preserve the subalgebra $U_v^{\mathrm{int}}\big(\widehat{\mathfrak{sl}}_2\big)$. By Corollary 3.12 it is sufficient to check that for any $n \in \mathbb{Z}_{>0}$ we have: $\mathsf{ev}_q^{-1}\big([P_1^{\oplus n}]\big)$ and $\mathsf{ev}_q^{-1}\big([I_1^{\oplus n}]\big)$ belong to the subalgebra $U_v^{\mathrm{int},+}\big(\widehat{\mathfrak{sl}}_2\big)$. To show this, we use the following trick. For a representation $X = U \overset{A}{\underset{B}{\rightrightarrows}} V$ of the Kronecker quiver $\overrightarrow{Q}$ we denote $r(X) := \mathsf{dim}_k\big(\mathsf{Im}(A) + \mathsf{Im}(B)\big)$. For any $n \in \mathbb{Z}_{>0}$ and $0 \leq r \leq 2n$ consider

$$\widetilde{\mathbb{1}}^{(r)}_{(n,2n)} := \sum_{\substack{[X] \in J:\ \underline{\mathsf{dim}}(X) = (n,2n) \\ r(X) = r}} [X] \in H(\mathsf{A}).$$

Then $[P_1^{\oplus n}] = \widetilde{\mathbb{1}}^{(2n)}_{(n,2n)}$ and for all $a, b \in \mathbb{Z}_{\geq 0}$ such that $a + b = 2n$ we have the following identity:

$$[S_2^{\oplus a}] \circ [S_1^{\oplus n}] \circ [S_2^{\oplus b}] = v^{-ab+2nb} \sum_{r=0}^{b} \Big|\mathsf{Gr}_k\big(b - r, 2n - r\big)\Big| \, \widetilde{\mathbb{1}}^{(r)}_{(n,2n)},$$

where $\Big|\mathsf{Gr}_k\big(b-r, 2n-r\big)\Big| = v^{(b-2n)(b-r)} \dfrac{[2n-r]!}{[b-r]![2n-b]!}$ is the number of points of the Grassmanian $\mathsf{Gr}_k\big(b - r, 2n - r\big)$. From this formula one can deduce that

$$[P_1]^{(n)} = \sum_{a+b=2n} (-1)^a v^{-b} [S_2]^{(a)} \circ [S_1]^{(n)} \circ [S_2]^{(b)}.$$

In a similar way, one can show that

$$[I_1]^{(n)} = \sum_{a+b=2n} (-1)^a v^{-b} [S_1]^{(a)} \circ [S_2]^{(n)} \circ [S_1]^{(b)}.$$

From the invariance of $U_v^{\mathrm{int}}\big(\widehat{\mathfrak{sl}}_2\big)$ under the action of $\mathbb{S}^{\pm}$ it also follows that for any indecomposable pre-projective or pre-injective object $X \in \mathrm{Ob}(\mathsf{A})$ and $n \in \mathbb{Z}_{>0}$, the element $[X^{\oplus n}]$ belongs to $C_{\mathrm{gen}}(\overrightarrow{Q})$ and lies in the image of the homomorphism $\mathsf{ev} : U_v^{\mathrm{int},+}(\widehat{\mathfrak{sl}}_2) \to C_{\mathrm{gen}}(\overrightarrow{Q})$. Since for any $c \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$ the vector bundle $\mathcal{O}_{\mathbb{P}^1}(c)^{\oplus n}$ is isomorphic to $\mathbb{G}(X^{\oplus n})$ for an appropriate shift of an indecomposable pre-projective or pre-injective module X, Theorem 5.5 yields the claim. $\square$

Lemma 5.11. *For any pair of non-negative integers (a,b), the element*

$$\mathbb{1}_{(a,b)} = \sum_{[X]\in \mathsf{J}\,:\,\bar{X}=(a,b)} [X] \in H(\overrightarrow{Q})$$

belongs to the composition subalgebra $C(\overrightarrow{Q})$.

Proof. It follows from the equality $\mathbb{1}_{(a,b)} = v^{-2ab}[S_1^{\oplus a}] \circ [S_2^{\oplus b}]$. □

Corollary 5.12. *For any pair of non-negative integers (a,b) we have a well-defined element $\mathbb{1}_{(a,b)} \in C_{\mathrm{gen}}(\overrightarrow{Q})$ belonging to the image of* $\mathsf{ev} : U_v^{\mathrm{int},+}(\widehat{\mathfrak{sl}}_2) \to C_{\mathrm{gen}}(\overrightarrow{Q})$.

Lemma 5.13. *Let $\mathsf{A} = \mathsf{Rep}(\overrightarrow{Q})$, $\mathbb{H} = \{a \in \mathbb{C}\,|\,\mathrm{Im}(a) > 0\}$ and $Z : K_0(\mathsf{A}) \to \mathbb{C}$ be any additive group homomorphism such that for any non-zero object X of A we have: $Z(\bar{X}) \in \mathbb{H}$. For any $\alpha \in K_0(\mathsf{A})$ denote*

$$\mathbb{1}_{\alpha}^{\mathrm{ss}} = \mathbb{1}_{\alpha,Z}^{\mathrm{ss}} := \sum_{[X]\in \mathsf{J}:\, X\in \mathsf{A}_{\alpha}^{\mathrm{ss}}} [X],$$

where $\mathsf{A}_{\alpha}^{\mathrm{ss}}$ is the category of semi-stable objects of class α with respect to the stability condition Z, see [36] *for the definition. Then we have: $\mathbb{1}_{\alpha}^{\mathrm{ss}} \in C(\overrightarrow{Q})$.*

Proof. First note that the existence and uniqueness of the Harder-Narasimhan filtration [36] of an object of our abelian category A implies the following identity for an arbitrary class $\alpha \in K_0(\mathsf{A})$ and a given stability condition Z:

$$\mathbb{1}_{\alpha} = \mathbb{1}_{\alpha}^{\mathrm{ss}} + \sum_{t\ge 2} \sum_{\substack{\alpha_1+\dots+\alpha_t=\alpha \\ \mu(\alpha_1)\ge\dots\ge\mu(\alpha_t)}} v^{\sum_{i<j}\langle \alpha_i \alpha_j\rangle} \mathbb{1}_{\alpha_1}^{\mathrm{ss}} \circ \dots \circ \mathbb{1}_{\alpha_t}^{\mathrm{ss}}.$$

Since the expression on the right-hand side is a finite sum, by induction we obtain that for all classes $\alpha \in K_0(\mathsf{A})$ the element $\mathbb{1}_{\alpha}^{\mathrm{ss}}$ belongs to the subalgebra of $H(\overrightarrow{Q})$ generated by all the elements $\{\mathbb{1}_{\beta}\}_{\beta\in K_0(\mathsf{A})}$. According to Lemma 5.11, this algebra coincides with the composition subalgebra $C(\overrightarrow{Q})$, what implies the claim. □

Remark 5.14. A result of Reineke [30, Theorem 5.1] provides an explicit formula expressing the elements $\{\mathbb{1}_{\alpha}^{\mathrm{ss}}\}$ via $\{\mathbb{1}_{\beta}\}$ for an arbitrary stability function Z:

$$\mathbb{1}_{\alpha,Z}^{\mathrm{ss}} = \mathbb{1}_{\alpha} + \sum_{t\ge 2} (-1)^{t-1} \sum_{\substack{\alpha_1+\dots+\alpha_t=\alpha:\,\forall\, 1\le s\le t-1 \\ \mu(\alpha_1+\dots+\alpha_s)>\mu(\alpha)}} v^{\sum_{i<j}\langle \alpha_i \alpha_j\rangle} \mathbb{1}_{\alpha_1} \circ \dots \circ \mathbb{1}_{\alpha_t}.$$

Hence, the element $\mathbb{1}_{\alpha}^{\mathrm{ss}} = \mathbb{1}_{\alpha,Z}^{\mathrm{ss}} \in C_{\mathrm{gen}}(\overrightarrow{Q})$ belongs to the image of the algebra homomorphism $\mathsf{ev} : U_v^{\mathrm{int},+}(\widehat{\mathfrak{sl}}_2) \to C_{\mathrm{gen}}(\overrightarrow{Q})$ for any class $\alpha \in K_0(\mathsf{A})$ and a given stability condition $Z : K_0(\mathsf{A}) \to \mathbb{H}$.

Lemma 5.15. *For any $r \in \mathbb{Z}_{>0}$ the following element of the Hall algebra $H(\overrightarrow{Q})$*

$$\widetilde{\mathbb{1}}_{(r,r)} = \sum_{X\in \mathsf{Tub}(\overrightarrow{Q})\,:\,\bar{X}=(r,r)} [X]$$

belongs to the composition algebra $C(\overrightarrow{Q})$. Moreover, it determines an element of $C_{\mathrm{gen}}(\overrightarrow{Q})$ belonging to the image of the homomorphism $\mathsf{ev} : U_v^{\mathrm{int},+}(\widehat{\mathfrak{sl}}_2) \to C_{\mathrm{gen}}(\overrightarrow{Q})$.

Proof. Consider the stability condition on the category $\mathsf{Rep}(\overrightarrow{Q})$ defined by the function $Z : K_0\big(\mathsf{Rep}(\overrightarrow{Q})\big) \to \mathbb{R}^2$ given by the rule $Z(m\bar{S}_1 + n\bar{S}_2) = (m-n, m+n)$. Then the class of a pre-projective objects has the form $(-1,l)$, $l \in \mathbb{Z}_{>0}$, the class of a pre-injective representations has the form $(1,l)$, $l \in \mathbb{Z}_{>0}$, whereas the classes of the tubes have the form $(0,l)$, $l \in \mathbb{Z}_{>0}$. Recall that

- an object of an abelian category is semi-stable if and only if all its direct summands are semi-stable with the same slope;
- any non-semi-stable object can be destabilized by an indecomposable one;
- there are no morphisms from a pre-injective object of $\mathsf{Rep}(\overrightarrow{Q})$ to an object from a tube.

Hence, $\widetilde{\mathbb{1}}_{(r,r)} = \mathbb{1}^{\mathrm{ss}}_{(r,r)}$ for all $r \in \mathbb{Z}_{>0}$. Applying Lemma 5.13 and Remark 5.14, we get the claim. $\square$

Remark 5.16. Consider the standard stability condition on the category $\mathsf{Coh}(\mathbb{P}^1)$ given by the function $Z = (\mathrm{rk}, -\deg)$. Then it determines a stability condition on the derived category $D^b\bigl(\mathsf{Coh}(\mathbb{P}^1)\bigr)$ in the sense of Bridgeland [8] such that any indecomposable object of $D^b\bigl(\mathsf{Coh}(\mathbb{P}^1)\bigr)$ is Z-semi-stable. In particular, all objects of the category $\mathsf{Tor}(\mathbb{P}^1)$ are semi-stable of slope 0. The stability condition on $\mathsf{Rep}(\overrightarrow{Q})$ used in the proof of Lemma 5.15 induces a stability condition on the derived category $D^b\bigl(\mathsf{Rep}(\overrightarrow{Q})\bigr)$. In the notations of [8], this stability condition is $\widetilde{\mathsf{GL}}(2,\mathbb{R})^+$-equivalent to the standard stability condition on $D^b\bigl(\mathsf{Coh}(\mathbb{P}^1)\bigr)$.

The technique of stability conditions also plays a key role in our subsequent paper [10] on the composition Hall algebra of a weighted projective line.

We conclude this section with a new proof of the following proposition, which was obtained for the first time by Chari and Pressley in [11].

Proposition 5.17. *Consider the family of elements $\bigl\{P_r\bigr\}_{r\ge 1}$ of $U_v\bigl(\mathfrak{Lsl}_2\bigr)$ defined by the following generating series:*

$$1 + \sum_{r=1}^{\infty} P_r C^{-\frac{r}{2}} t^r = \exp\Bigl(\sum_{r=1}^{\infty} \frac{\Psi_r}{[r]} t^r\Bigr).$$

Then P_r belong to the algebra $U_v^{\mathrm{int}}\bigl(\mathfrak{Lsl}_2\bigr)$ for all $r \in \mathbb{Z}_{>0}$.

Proof. The elements P_r have a clear meaning in the language of the Hall algebra $H(\mathbb{P}^1)$. Indeed, by Definition 4.2 we have: $\mathsf{ev}_q(P_r) = \mathbb{1}_{r\delta}$ for any $q \in \mathcal{P}$.

On the other side, Theorem 5.1 implies that $\mathbb{1}_{r\delta} = \mathbb{G}\bigl(\widetilde{\mathbb{1}}_{(r,r)}\bigr)$ for all $r \in \mathbb{Z}_{>0}$. By Lemma 5.15 we know that $\widetilde{\mathbb{1}}_{(r,r)}$ belongs to the image of the algebra homomorphism $\mathsf{ev} : U_v^{\mathrm{int},+}(\widehat{\mathfrak{sl}}_2) \to DC_{\mathrm{gen}}(\overrightarrow{Q})$. Hence, $\mathbb{1}_{r\delta}$ belongs to the image of the algebra homomorphism $\mathsf{ev} : U_v^{\mathrm{int}}(\mathfrak{Lsl}_2) \to DU_{\mathrm{gen}}(\mathbb{P}^1)$. By Theorem 5.5, the element P_r belongs to the algebra $U_v^{\mathrm{int}}(\mathfrak{Lsl}_2)$ for all $r \in \mathbb{Z}_{>0}$, too. $\square$

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