FROBENIUS MORPHISM AND VECTOR BUNDLES ON CYCLES OF PROJECTIVE LINES

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Abstract. In this paper we describe the action of the Frobenius morphism on the indecomposable vector bundles on cycles of projective lines. This gives an answer on a question of Paul Monsky, which appeared in his study of the Hilbert–Kunz theory for plane cubic curves.

This note arose as a answer on a question posed by Paul Monsky in his study of the Hilbert–Kunz theory for plane cubic curves [5]. Let $\boldsymbol{k}$ be a field of characteristic $p > 0$ and E be a plane rational nodal curve or a cycle of projective lines over $\boldsymbol{k}$. Our goal is to describe the action of the Frobenius morphism on the set of indecomposable vector bundles on E.

We start with recalling the general technique used in a study of vector bundles on singular projective curves, see [3, 1, 2]. Let X be a reduced singular (projective) curve over $\boldsymbol{k}$, $\pi : \widetilde{X} \to X$ its normalization and $\mathcal{I} := \mathcal{H}om_{\mathcal{O}}\bigl(\pi_*(\mathcal{O}_{\widetilde{X}}), \mathcal{O}\bigr) = \mathcal{A}nn_{\mathcal{O}}\bigl(\pi_*(\mathcal{O}_{\widetilde{X}})/\mathcal{O}\bigr)$ the conductor ideal sheaf. Denote by $\eta : Z = V(\mathcal{I}) \longrightarrow X$ the closed artinian subspace defined by $\mathcal{I}$ (its topological support is precisely the singular locus of X) and by $\tilde{\eta} : \widetilde{Z} \longrightarrow \widetilde{X}$ its preimage in $\widetilde{X}$, defined by the Cartesian diagram

$$\begin{array}{ccc} \widetilde{Z} & \xrightarrow{\tilde{\eta}} & \widetilde{X} \\ {\scriptstyle\tilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ Z & \xrightarrow{\eta} & X. \end{array} \tag{1}$$

Definition 1. The category of triples $\mathsf{Tri}(X)$ is defined as follows.

- Its objects are triples $\bigl(\widetilde{\mathcal{F}}, \mathcal{V}, \widetilde{\mathsf{m}}\bigr)$, where $\widetilde{\mathcal{F}} \in \mathsf{VB}(\widetilde{X})$, $\mathcal{V} \in \mathsf{VB}(Z)$ and
$$\mathsf{m} : \tilde{\pi}^*\mathcal{V} \longrightarrow \tilde{\eta}^*\widetilde{\mathcal{F}}$$
is an isomorphism of $\mathcal{O}_{\widetilde{Z}}$-modules, called the *gluing map*.
- The set of morphisms $\mathsf{Hom}_{\mathsf{Tri}(X)}\bigl((\widetilde{\mathcal{F}}_1, \mathcal{V}_1, \mathsf{m}_1), (\widetilde{\mathcal{F}}_2, \mathcal{V}_2, \mathsf{m}_2)\bigr)$ consists of all pairs (F, f), where $F : \widetilde{\mathcal{F}}_1 \to \widetilde{\mathcal{F}}_2$ and $f : \mathcal{V}_1 \to \mathcal{V}_2$ are morphisms of vector bundles

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such that the following diagram is commutative

$$\begin{tikzcd} \tilde{\pi}^*\mathcal{V}_1 \arrow[r, "\mathsf{m}_1"] \arrow[d, "\tilde{\pi}^*(f)"'] & \tilde{\eta}^*\widetilde{\mathcal{F}}_1 \arrow[d, "\tilde{\eta}^*(F)"] \\ \tilde{\pi}^*\mathcal{V}_2 \arrow[r, "\mathsf{m}_2"] & \tilde{\eta}^*\widetilde{\mathcal{F}}_2. \end{tikzcd}$$

Theorem 2 (Lemma 2.4 in [3] and Theorem 1.3 in [2])**.** *Let X be a reduced curve. Then the functor $\mathbb{F} : \mathsf{VB}(X) \longrightarrow \mathsf{Tri}(X)$ assigning to a vector bundle $\mathcal{F}$ the triple $(\pi^*\mathcal{F}, \eta^*\mathcal{F}, \mathsf{m}_{\mathcal{F}})$, where $\mathsf{m}_{\mathcal{F}} : \tilde{\pi}^*(\eta^*\mathcal{F}) \longrightarrow \tilde{\eta}^*(\pi^*\mathcal{F})$ is the canonical isomorphism, is an equivalence of categories.*

Remark 3. In the case when X is a configuration of projective lines intersecting transversally, the above theorem also follows from a more general result of Lunts [4].

For a ringed space $(Y, \mathcal{O}_Y)$ over the field $\boldsymbol{k}$ we denote by φ_Y the Frobenius morphism $(Y, \mathcal{O}_Y) \to (Y, \mathcal{O}_Y)$. Then for an open set $U \subset Y$ the algebra homomorphism $\varphi_Y(U) : \mathcal{O}_Y(U) \to \mathcal{O}_Y(U)$ is given by the formula $\varphi_Y(f) = f^p$, $f \in \mathcal{O}_Y(U)$. For a sake of simplicity, we shall omit the subscript in the notation of the Frobenius map.

Definition 4. The endofunctor $\mathbb{P} : \mathsf{Tri}(X) \longrightarrow \mathsf{Tri}(X)$ is defined as follows. Let $\mathcal{T} = (\widetilde{\mathcal{F}}, \mathcal{V}, \mathsf{m})$ be an object of the category $\mathsf{Tri}(X)$. Then $\mathbb{P}(\mathcal{T}) := (\varphi^*\widetilde{\mathcal{F}}, \varphi^*\mathcal{V}, \mathsf{m}^{\varphi})$, where the gluing map m^{φ} is determined via the commutative diagram

$$\begin{tikzcd} \varphi^*\tilde{\pi}^*\mathcal{V} \arrow[r, "\varphi^*(\mathsf{m})"] \arrow[d, "\mathsf{can}"'] & \varphi^*\tilde{\eta}^*\widetilde{\mathcal{F}} \arrow[d, "\mathsf{can}"] \\ \tilde{\pi}^*\varphi^*\mathcal{V} \arrow[r, "\mathsf{m}^{\varphi}"] & \tilde{\eta}^*\varphi^*\widetilde{\mathcal{F}}, \end{tikzcd}$$

where the vertical maps are canonical isomorphisms.

Lemma 5. *Consider the following diagram of categories and functors:*

$$\begin{tikzcd} \mathsf{VB}(X) \arrow[r, "\mathbb{F}"] \arrow[d, "\mathbb{T}"'] & \mathsf{Tri}(X) \arrow[d, "\mathbb{P}"] \\ \mathsf{VB}(X) \arrow[r, "\mathbb{F}"] & \mathsf{Tri}(X), \end{tikzcd}$$

where for a vector bundle $\mathcal{F}$ on X we set: $\mathbb{T}(\mathcal{F}) = \varphi_X^*(\mathcal{F})$. Then there exists an isomorphism of functors $\mathbb{P} \circ \mathbb{F} \to \mathbb{F} \circ \mathbb{T}$.

Proof. Let $\mathcal{F}$ be a vector bundle on X. Then the canonical isomorphisms $\varphi^*\tilde{\eta}^*\mathcal{F} \to \tilde{\eta}^*\varphi^*\mathcal{F}$ and $\varphi^*\pi^*\mathcal{F} \to \pi^*\varphi^*\mathcal{F}$ induce the commutative diagram

$$\begin{array}{ccc} \tilde{\pi}^*\varphi^*\tilde{\eta}^*\mathcal{F} & \xrightarrow{\mathsf{m}^{\varphi}_{\mathcal{F}}} & \tilde{\eta}^*\varphi^*\pi^*\mathcal{F} \\ {\scriptstyle \mathsf{can}}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{can}} \\ \tilde{\pi}^*\tilde{\eta}^*\varphi^*\mathcal{F} & \xrightarrow{\mathsf{m}_{\varphi^*\mathcal{F}}} & \tilde{\eta}^*\pi^*\varphi^*\mathcal{F}, \end{array}$$

which yields the desired isomorphism of functors. □

Next, we need a description of the action of the Frobenius map on the vector bundles on a projective line. Let (z_0, z_1) be coordinates on $V = \mathbb{C}^2$. They induce homogeneous coordinates $(z_0 : z_1)$ on $\mathbb{P}^1 = \mathbb{P}^1(V) = \big(V \setminus \{0\}\big)/\sim$, where $v \sim \lambda v$ for all $v \in V$ and $\lambda \in \mathbb{C}^*$.

We set $U_0 = \big\{(z_0 : z_1) | z_0 \neq 0\big\}$ and $U_\infty = \big\{(z_0 : z_1) | z_1 \neq 0\big\}$ and put $0 := (1 : 0)$, $\infty := (0 : 1)$, $z = z_1/z_0$ and $w = z_0/z_1$. So, z is a coordinate in a neighbourhood of 0. If $U = U_0 \cap U_\infty$ and $w = 1/z$ is used as a coordinate on U_∞, then the transition function of the line bundle $\mathcal{O}_{\mathbb{P}^1}(n)$ is

$$U_0 \times \mathbb{C} \supset U \times \mathbb{C} \xrightarrow{(z,v)\mapsto\left(\frac{1}{z}, \frac{v}{z^n}\right)} U \times \mathbb{C} \subset U_\infty \times \mathbb{C}. \tag{2}$$

Using the formula (2), the proof of the following lemma is straightforward.

Lemma 6. *For any $n \in \mathbb{Z}$ we have: $\varphi^*\big(\mathcal{O}_{\mathbb{P}^1}(n)\big) \cong \mathcal{O}_{\mathbb{P}^1}(np)$.*

Next, recall the following classical result on vector bundles on a projective line.

Theorem 7 (Birkhoff–Grothendieck)**.** *Any vector bundle $\widetilde{\mathcal{F}}$ on $\mathbb{P}^1$ splits into a direct sum of line bundles:*

$$\widetilde{\mathcal{F}} \cong \bigoplus_{l \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^1}(l)^{m_l}. \tag{3}$$

Now assume that E is an *irreducible* plane nodal cubic curve. Theorem 7 implies that for an object $(\widetilde{\mathcal{F}}, \mathcal{V}, \widetilde{\mathsf{m}})$ of the category of triples $\mathsf{Tri}(E)$ with $\mathrm{rk}(\widetilde{\mathcal{F}}) = n$, we have

$$\widetilde{\mathcal{F}} = \bigoplus_{l \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^1}(l)^{m_l} \quad \text{and} \quad \mathcal{V} \cong \mathcal{O}_Z^n, \quad \text{where} \sum_{l \in \mathbb{Z}} m_l = n.$$

Note that $\mathcal{V}$ is in fact free, because Z is artinian. From now on we shall always fix a decomposition of $\widetilde{\mathcal{F}}$ as above. In order to describe the morphism $\widetilde{\mathsf{m}}$ in the terms of matrices, some additional choices have to be done.

Recall that the vector bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$ is isomorphic to the sheaf of sections of the so-called tautological line bundle

$$\big\{(l, v) |\, v \in l\big\} \subset \mathbb{P}^1(V) \times V = \mathcal{O}^2_{\mathbb{P}^1}.$$

The choice of coordinates on $\mathbb{P}^1$ fixes two distinguished elements, z_0 and z_1, in the space $\mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1}\big)$:

$$\begin{tikzcd}
\mathbb{P}^1 \times \mathbb{C}^2 \arrow[dr] & \mathcal{O}_{\mathbb{P}^1}(-1) \arrow[l, hook'] \arrow[r, "z_i"] \arrow[d] & \mathbb{P}^1 \times \mathbb{C} \arrow[dl] \\
& \mathbb{P}^1 &
\end{tikzcd}$$

where z_i maps $\big(l, (v_0, v_1)\big)$ to (l, v_i) for $i = 0, 1$. It is clear that the section z_0 vanishes at ∞ and z_1 vanishes at 0. After having made this choice, we may write

$$\mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{O}_{\mathbb{P}^1}(n), \mathcal{O}_{\mathbb{P}^1}(m)\big) = \mathbb{C}[z_0, z_1]_{m-n} := \big\langle z_0^{m-n}, z_0^{m-n-1} z_1, \dots, z_1^{m-n} \big\rangle_{\mathbb{C}}.$$

In what follows we shall assume that the coordinates on the normalization $\widetilde{E}$ are chosen in such a way that $\mathsf{Spec}(\boldsymbol{k} \times \boldsymbol{k}) \cong \widetilde{Z} = \pi^{-1}(Z) = \{0, \infty\}$.

Definition 8. For any $l \in \mathbb{Z}$ we define the isomorphism $\xi_l : \tilde{\eta}^*\big(\mathcal{O}_{\mathbb{P}^1}(l)\big) \to \mathcal{O}_{\widetilde{Z}}$ by the formula $\xi_l(p) = \big(\frac{p}{z_0^l}(0), \frac{p}{z_1^l}(\infty)\big)$, where p is an arbitrary local section of the line bundle $\mathcal{O}_{\mathbb{P}^1}(l)$. Hence, for any vector bundle $\widetilde{\mathcal{F}}$ of rank n on $\mathbb{P}^1$ given by the formula (3), we have the induced isomorphism $\xi_{\widetilde{\mathcal{F}}} : \tilde{\eta}^*\widetilde{\mathcal{F}} \to \mathcal{O}_{\widetilde{Z}}^n$.

Let $(\widetilde{\mathcal{F}}, \mathcal{O}_Z^n, \mathsf{m})$ be a object of the category of triples $\mathsf{Tri}(E)$. Then the morphism m can be presented by a matrix $M(\mathsf{m})$ via the following commutative diagram

$$\begin{tikzcd}
\tilde{\pi}^*\mathcal{O}_Z^n \arrow[r, "\mathsf{m}"] \arrow[d, "\mathsf{can}"'] & \tilde{\eta}^*\widetilde{\mathcal{F}} \arrow[d, "\xi_{\widetilde{\mathcal{F}}}"] \\
\mathcal{O}_{\widetilde{Z}}^n \arrow[r, "M(\mathsf{m})"] & \mathcal{O}_{\widetilde{Z}}^n,
\end{tikzcd} \tag{4}$$

where the first vertical map is the canonical isomorphism. Hence, the morphism $M(\mathsf{m})$ is given by a pair of invertible $(n \times n)$ matrices $M(0)$ and $M(\infty)$ over the field $\boldsymbol{k}$. Applying to (4) the functor φ^*, we get the following commutative diagram:

$$\begin{tikzcd}
\tilde{\pi}^*\varphi^*\mathcal{O}_Z^n \arrow[r, "\mathsf{m}^\varphi"] \arrow[ddd, bend right=60, "\mathsf{can}"'] & \tilde{\eta}^*\varphi^*\widetilde{\mathcal{F}} \arrow[ddd, bend left=60, "\xi_{\varphi^*\widetilde{\mathcal{F}}}"] \\
\varphi^*\tilde{\pi}^*\mathcal{O}_Z^n \arrow[r, "\varphi^*(\mathsf{m})"] \arrow[u, "\mathsf{can}"] \arrow[d, "\mathsf{can}"'] & \varphi^*\tilde{\eta}^*\widetilde{\mathcal{F}} \arrow[u, "\mathsf{can}"'] \arrow[d, "\varphi^*(\xi_{\widetilde{\mathcal{F}}})"] \\
\varphi^*\big(\mathcal{O}_{\widetilde{Z}}^n\big) \arrow[r, "\varphi^*(M(\mathsf{m}))"] \arrow[d, "\mathsf{can}"'] & \varphi^*\big(\mathcal{O}_{\widetilde{Z}}^n\big) \arrow[d, "\mathsf{can}"] \\
\mathcal{O}_{\widetilde{Z}}^n \arrow[r, "M(\mathsf{m}^\varphi)"] & \mathcal{O}_{\widetilde{Z}}^n.
\end{tikzcd} \tag{5}$$

Corollary 9. Let E be an irreducible nodal cubic curve over a field $\boldsymbol{k}$ of characteristic $p > 0$ and $\mathcal{F}$ be a vector bundle corresponding to the triple $(\widetilde{\mathcal{F}}, \mathcal{O}_Z^n, \mathsf{m})$, where $\widetilde{\mathcal{F}} \cong \oplus_{l\in\mathbb{Z}} \mathcal{O}_{\mathbb{P}^1}(l)^{m_l}$ and m is given by a pair of matrices

$$M(0) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \quad \text{and} \quad M(\infty) = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix}.$$

Then the vector bundle $\varphi^*\mathcal{F}$ is given by the triple $(\varphi^*\widetilde{\mathcal{F}}, \mathcal{O}_Z^n, \mathsf{m}^\varphi)$, where $\varphi^*\widetilde{\mathcal{F}} \cong \oplus_{l\in\mathbb{Z}} \mathcal{O}_{\mathbb{P}^1}(lp)^{m_l}$ and m^φ corresponds to the pair of matrices

$$\begin{pmatrix} a_{11}^p & a_{12}^p & \dots & a_{1n}^p \\ a_{21}^p & a_{22}^p & \dots & a_{2n}^p \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}^p & a_{n2}^p & \dots & a_{nn}^p \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b_{11}^p & b_{12}^p & \dots & b_{1n}^p \\ b_{21}^p & b_{22}^p & \dots & b_{2n}^p \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1}^p & b_{n2}^p & \dots & b_{nn}^p \end{pmatrix}.$$

Corollary 10. Indecomposable vector bundles on an irreducible nodal cubic curve E over an algebraically closed field $\boldsymbol{k}$ are described by a non-periodic sequence of integers $(d_1, \dots, d_l)$, a positive integer m and a continuous parameter $\lambda \in \boldsymbol{k}^*$, see [3, 1, 2]. The normalization of the corresponding vector bundle $\mathcal{B}\big((d_1, d_2, \dots, d_l), m, \lambda\big)$ is $\oplus_{i=1}^l \mathcal{O}_{\mathbb{P}^1}(d_i)^m$. The gluing map m is described by a pair of matrices $M(0)$ and $M(\infty)$, which are given by the explicit formulae written in [1, Section 3.1].

In these notations, the action of φ^* on the indecomposable vector bundles on E takes the following form:

$$\varphi^*\mathcal{B}\big((d_1, d_2, \dots, d_l), m, \lambda\big) \cong \mathcal{B}\big((pd_1, pd_2, \dots, pd_l), m, \lambda^p\big). \tag{6}$$

Remark 11. The same argument applies literally to the case, when E is a cycle of projective lines. In particular, the formula (6) holds in that case, too. However, we have to stress that the parameter $\lambda \in \boldsymbol{k}^*$ arising in the formula (6) has no intrinsic meaning as an invariant of a vector bundle and its value actually depends on the choice of the trivializations $\big\{\xi_l\big\}_{l\in\mathbb{Z}}$, fixed in Definition 8.

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