MAXIMAL COHEN-MACAULAY MODULES OVER
QUOTIENT SURFACE SINGULARITIES

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Abstract. In this note I discuss a relationship between the McKay Correspondence for two-dimensional quotient singularities and the theory of maximal Cohen-Macaulay modules.

1. McKay’s observation

Let $G \subseteq \text{SL}_2(\mathbb{C})$ be a finite group. Then one can attach to it the following pair of combinatorial objects.

First object. By Maschke’s theorem, the category of finite dimensional representations of $G$ over the field $\mathbb{C}$ is semi-simple. Let $\{V_0, V_1, \ldots, V_n\}$ be the set of the isomorphy classes of the irreducible representations of $G$, where $V_0 = \mathbb{C}$ is the trivial representation, and $W = \mathbb{C}^2$ be the fundamental representation of $G$ induced by the embedding $G \subseteq \text{SL}_2(\mathbb{C})$. For any $0 \leq i \leq n$, we set $m_i = \dim_{\mathbb{C}}(V_i)$. For any $0 \leq i, j \leq n$ we have decompositions

$$V_i \otimes_{\mathbb{C}} W \cong \bigoplus_{j=0}^n V_j^{a_{ij}}.$$

One can show that $a_{ii} = 0$ and $a_{ij} = a_{ji}$ for all $0 \leq i, j \leq n$.

Definition 1. The McKay graph $\text{MK}(G)$ of a finite group $G \subseteq \text{SL}_2(\mathbb{C})$ is defined as follows.

1. The set of vertices of $\text{MK}(G)$ is $\{0, 1, \ldots, n\}$.
2. For any $0 \leq i \neq j \leq n$ the vertex $i$ is connected with the vertex $j$ by $a_{ij}$ arrows.
3. The vertex $i$ has “weight” $m_i$.

Second object. Let $A = \mathbb{C}[x, y]^G$ be the quotient singularity defined by $G$, $X = \text{Spec}(A)$, $o \in X$ the closed point of $X$ and $\tilde{X} \xrightarrow{\pi} X$ a minimal resolution of singularities. Let $E = \pi^{-1}(o)$ be the exceptional divisor of the resolution. It is well-known that $E$ is a tree of projective lines.

In 1978 John McKay made [8] the following striking observation. Let $G \subseteq \text{SL}_2(\mathbb{C})$ be a finite subgroup. Then we have:

1. The number of the irreducible components of $E$ is equal to the number of non-trivial irreducible representations of $G$.
2. Let $\text{MK}(G)' = \text{MK}(G) \setminus \{0\}$ be the graph obtained from $\text{MK}(G)$ by excluding the vertex 0 and all arrows connected with it. Then $\text{MK}(G)'$ is isomorphic to the dual intersection graph $\Gamma_E$ of the curve $E$. In other words, there exists a labeling of the irreducible components $E_1, \ldots, E_n$ such that for any $1 \leq i \neq j \leq n$ we have:

$$a_{ij} = \#(E_i \cap E_j) =: c_{ij}.$$

3. The cycle $Z = \sum_{i=1}^n m_i [E_i] \in H_2(\tilde{X}, \mathbb{Z})$ is the fundamental cycle of the resolution $\tilde{X}$. This can be expressed in plain words as follows. For any $1 \leq i \leq n$ let $c_{ii} := -2 = E_i^2$ be the self-intersection index of $E_i$ and $C = (c_{ij}) \in \text{Mat}_{n \times n}(\mathbb{Z})$ be the intersection matrix of $E$. 


Then $Z$ is the smallest vector $\mathbf{l} = (l_1, l_2, \ldots, l_n)$ with non-negative integral entries such that

$$\langle e_i, \mathbf{l} \rangle_C := \langle e^t_i, \mathbf{l} \rangle \leq 0$$

for all $1 \leq i \leq n$, where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the $i$-th basic vector of $\mathbb{Z}^n$.

**Example 2.** Let $Z/(n+1) \cong G = \langle g \rangle \subset SL_2(\mathbb{C})$ be a cyclic subgroup of order $n+1$ generated by the element

$$g = \left( \begin{array}{cc} \xi & 0 \\ 0 & \xi^{-1} \end{array} \right),$$

where $\xi$ is a primitive $(n+1)$-st root of 1. Then we have:

- $G$ has $n+1$ irreducible representations $\{V_0, V_1, \ldots, V_n\}$, where all $V_i = \mathbb{C}$ and the action of $g$ is given by the multiplication with $\xi^i$. It is easy to see that $W = V_1 \oplus V_n$ and the McKay’s graph $MK(G)$ is a cycle

$$\begin{array}{cccccc}
0 & 1 & 2 & \cdots & n-1 & n \\
\uparrow & & & & & \\
\downarrow & & & & & \\
1 & 2 & \cdots & n-1 & n & 0 \\
\end{array}$$

- Next, we have:

$$A := \mathbb{C}[x, y]^G = \mathbb{C}[x^{n+1}, xy, y^{n+1}] \cong \mathbb{C}[u, v, w]/(uw - v^{n+1})$$

is a simple surface singularity of type $A_n$. It is well-known that the exceptional divisor of a minimal resolution of singularities of $\text{Spec}(A)$ is a chain of $n$ projective lines. Hence, the intersection matrix of the exceptional divisor is just

$$C = \left( \begin{array}{cccc}
-2 & 1 & 0 & \ldots & 0 \\
1 & -2 & 1 & \ldots & 0 \\
0 & \ldots & 1 & -2 & 1 \\
0 & \ldots & 0 & 1 & -2 \\
\end{array} \right).$$

It is easy to show that in this case the fundamental cycle $Z$ is equal to $\sum_{i=1}^n [E_i] = (1, 1, \ldots, 1)$, in a full accordance with McKay’s observation.

**Explanation.** McKay himself has verified his observation using Klein’s classification of finite subgroups in $SL_2(\mathbb{C})$ by a tedious case-by-case analysis [8]. It turns out, however, that the McKay correspondence can be explained in a more conceptual way by introducing the third intermediate object: the stable category of the maximal Cohen-Macaulay $A$–modules $\text{CM}(A)$. Namely, there exist natural bijections

$$MK(G)^{\prime} \cong \text{ind(\text{CM}(A))} \cong \Gamma_E,$$

where $\text{ind(\text{CM}(A))}$ is the set of the isomorphy classes of indecomposable objects in $\text{CM}(A)$. The statement about the fundamental cycle and the dimensions of irreducible representations of $G$ can be derived using the Auslander-Reiten theory of the category $\text{CM}(A)$.

2. **Algebraic McKay Correspondence**

Let $G \subset SL_2(\mathbb{C})$ be a finite subgroup. Then the ring of invariants $A = \mathbb{C}[x, y]^G$ is a normal surface singularity. Recall the following standard facts about Cohen-Macaulay modules over surface singularities.

**Theorem 3.** Let $(A, m)$ be a local Noetherian ring of Krull dimension two.

- $A$ is normal if and only if it is Cohen-Macaulay and regular in codimension one.
Assume $A$ to be Cohen-Macaulay. Then for any maximal Cohen-Macaulay module $M$ and any Noetherian module $N$ the module $\text{Hom}_A(N, M)$ is maximal Cohen-Macaulay.

Assume additionally that $A$ is Gorenstein in codimension one (for instance, $A$ is a normal singularity). Then a Noetherian module $M$ is maximal Cohen-Macaulay if and only if it is reflexive. Moreover, the functor $M \mapsto M^{\vee A}$ is left adjoint to the forgetful functor $\text{CM}(A) \rightarrow A-\text{mod}$.

Let $(A, m) \subseteq (B, n)$ be a finite extension of Cohen-Macaulay surface singularities, which are Gorenstein in codimension one. Then for any Noetherian $B$–module $M$ we have an isomorphism of $A$–modules $M^{\vee B} \cong M^{\vee A}$.

Let $A$ be regular. Then any maximal Cohen-Macaulay module over $A$ is free.

For a proof one may consult [3, Section 3] and references therein.

The following theorem of Herzog [7] was the starting point of an extensive study of maximal Cohen-Macaulay modules over surface singularities.

**Theorem 4.** Let $k$ be an algebraically closed field, $G \subset \text{GL}_2(k)$ be a finite subgroup such that $\gcd(|G|, \text{char}(k)) = 1$, $R = k[x, y]$ and $A = R^G$. Then we have: $\text{CM}(A) = \text{add}_A(R)$. In other words, any maximal Cohen-Macaulay module over $A$ is isomorphic to a direct sum of direct summands of $R$ viewed as an $A$–module.

**Proof.** The embedding $i : A \rightarrow R$ has a left inverse $p : R \rightarrow A$ given by the Reinold’s operator

$$p(r) = \frac{1}{|G|} \sum_{g \in G} g(r).$$

It is easy to see that the map $p$ is $A$-linear. Hence, we have an isomorphism $R \cong A \oplus A'$ in the category of $A$–modules. Next, for any Noetherian $A$–module $M$ we have:

$$R \otimes_A M \cong M \oplus (A' \otimes_A M).$$

If $M$ is maximal Cohen-Macaulay over $A$ then there exists a positive integer $t$ such that

$$R^t \cong (R \otimes_A M)^{\vee K} \cong (R \otimes_A M)^{\vee A} \cong M \oplus (A' \otimes_A M)^{\vee A}.$$

Hence, $M$ is a direct summand of $R^t$ as stated. \[\square\]

From Herzog’s result we get the following corollary.

**Corollary 5.** Let $\Lambda = \text{End}_A(R)$. Then the functor

$$\text{Hom}_A(R, -): \text{CM}(A) = \text{add}_A(R) \rightarrow \text{pro}(\Lambda)$$

is an equivalence of categories, where $\text{pro}(\Lambda)$ is the category of the finitely generated projective right $\Lambda$–modules.

The following result is due to Auslander [2].

**Theorem 6.** Let $k$ be an algebraically closed field and $G \subset \text{GL}_2(k)$ be a small finite subgroup such that $\gcd(|G|, \text{char}(k)) = 1$ (note that any subgroup in $\text{SL}_2(k)$ is automatically small). Let $R = k[x, y]$ and $A = R^G$. Then the algebra homomorphism

$$\theta : R * G \rightarrow \text{End}_A(R), \quad t[g] \mapsto (r \mapsto t g(r))$$

is an isomorphism of algebras.

As a corollary, we obtain the following “algebraic” version of the McKay Correspondence, which is due to Auslander [2], see also [10].

**Theorem 7.** Let $k$ be an algebraically closed field and $G \subset \text{GL}_2(k)$ be a small finite subgroup such that $\gcd(|G|, \text{char}(k)) = 1$, $R = k[x, y]$ and $A = R^G$. 

(1) The functor \( \text{pro}(R \ast G) \rightarrow \text{CM}(A) \) assigning to a projective module \( P \) its \( A \)-submodule of invariants \( P^G \), is an equivalence of categories quasi-inverse to the functor
\[
\text{Hom}_A(R, -) : \text{CM}(A) \longrightarrow \text{pro}(R \ast G).
\]

(2) Since we have an isomorphism \( R \ast G / \text{rad}(R \ast G) \cong k[G] \), a bijection between the projective and the semi-simple right \( R \ast G \)-modules \( P \mapsto \text{P}/\text{rad}(P) \) yields a bijection between the isomorphism classes of irreducible representations of the group \( G \) and indecomposable projective right modules over \( R \ast G \). If \( V \) is an irreducible representation of \( G \), then the corresponding projective \( R \ast G \)-module is just \( R \otimes_k V \), where the action of an element \( [g] \in R \ast G \) on a simple tensor \( r \otimes h \in R \otimes_k V \) is given by:
\[
t[g] \circ (r \otimes h) = t g(r) \otimes g h.
\]

(3) The correspondence between the irreducible representations of \( G \) and the indecomposable maximal Cohen-Macaulay modules over \( A \) is given by the functor
\[
\text{Rep}(G \supseteq V \mapsto (R \otimes_k V)^G \in \text{CM}(A)).
\]

In the notations of the above theorem, consider the Koszul resolution of the trivial representation \( V_0 = k \) of the group \( G \), viewed as an \( R \ast G \)-module:
\[
0 \rightarrow R \otimes_k \wedge^2(W) \xrightarrow{\alpha} R \otimes_k W \xrightarrow{\beta} R \rightarrow k \rightarrow 0
\]
where \( \alpha(p \otimes (f_1 \otimes f_2 - f_2 \otimes f_1)) = p f_1 \otimes f_2 - p f_2 \otimes f_1 \), \( \beta(q \otimes f) = q f \) and \( \phi(t) = t(0,0) \).

**Remark 8.** Let \( V \) be a non-trivial irreducible \( k[G] \)-module. Then its minimal free projective resolution in the category of \( R \ast G \)-modules is
\[
0 \rightarrow R \otimes_k (\wedge^2(W) \otimes_k V) \rightarrow R \otimes_k (W \otimes_k V) \rightarrow R \otimes_k V \rightarrow V \rightarrow 0.
\]

Since the functor of taking \( G \)-invariants is exact, we obtain a short exact sequence of Cohen-Macaulay modules
\[
0 \rightarrow (R \otimes_k (\wedge^2(W) \otimes_k V))^G \rightarrow (R \otimes_k (W \otimes_k V))^G \rightarrow (R \otimes_k V)^G \rightarrow 0,
\]
which is precisely the Auslander-Reiten sequence sequence ending at the indecomposable Cohen-Macaulay module \( (R \otimes_k V)^G \), see [2] and [10].

**Corollary 9.** If \( G \) is a finite subgroup of \( SL_2(k) \) then we have: \( \wedge^2 W \cong V_0 = k \). Hence, the Auslander-Reiten quiver of the category \( \text{CM}(A) \) is obtained from the McKay’ graph \( MK(G) \) by “doubling” all the arrows.

**Example 10.** Let \( \mathbb{Z}/(n+1)\mathbb{Z} \cong G \subset SL_2(\mathbb{C}) \) be as in Example 2, \( A = \mathbb{C}[x,y]^G \) and \( \{V_0, V_1, \ldots, V_n\} \) be the set of the isomorphism classes of irreducible representations of \( G \), where \( V_i = \mathbb{C} \) and \( g \cdot 1 = \xi^i \) for \( 0 \leq i \leq n \). Then the corresponding indecomposable Cohen-Macaulay \( A \)-modules are
\[
\mathbb{C}[x,y] \supseteq I_l := (\mathbb{C}[x,y] \otimes_k V_l)^G = \left\{ \sum_{i,j=0}^\infty a_{ij}x^iy^j \mid a_{ij} \in \mathbb{C}, \; i - j \equiv l \mod n \right\}, \quad 0 \leq l \leq n.
\]

The following result is due to Auslander [2].

**Theorem 11.** Let \( (A, \mathfrak{m}) \) be a normal surface singularity with a canonical module \( K \).
- Let \( \omega \in \text{Ext}_A^2(k, K) \cong k \) be a generator and
\[
0 \longrightarrow K \longrightarrow D \longrightarrow A \longrightarrow k \longrightarrow 0
\]
be the corresponding extension class. Then the module \( D \) is maximal Cohen-Macaulay.
- Let \( G \subset GL_2(k) \) be a finite subgroup, \( R = k[x,y] \) and \( A = R^G \) be the corresponding quotient singularity. Then the sequence (3) is obtained from the sequence (1) by taking \( G \)-invariants. In particular, we have: \( K \cong (R \otimes_k \wedge^2 W)^G \) (see [9]) and \( D \cong (R \otimes_k W)^G \).
For a non-regular indecomposable Cohen-Macaulay module $M$, the complex
\[ 0 \rightarrow (K \otimes_A M)^{\vee} \rightarrow (D \otimes_A M)^{\vee} \rightarrow M \rightarrow 0 \]
induced by the short exact sequence (3), is exact. Moreover, it is an Auslander-Reiten sequence, ending at $M$.

- For $G \subset SL_2(k)$ holds: $D \cong (\Omega_A^1)^{\vee}$, where $\Omega_A^1$ is the module of Kähler differentials of $A$.

3. Geometric McKay Correspondence

Let $G \subset SL_2(C)$ be a finite subgroup, $A = \mathbb{C}[x, y]^G$, $X = \text{Spec}(A)$ and $\tilde{X} \to X$ be a minimal resolution of singularities. Let $E = \pi^{-1}(o)$ be the exceptional divisor of $\pi$. The following facts are well-known.

1. $E = E_1 \cup \cdots \cup E_n$ is a tree of projective lines.
2. We have: $H_2(\tilde{X}, \mathbb{Z}) = \bigcup_{i=1}^n \mathbb{Z}[E_i] \cong \mathbb{Z}^n$.
3. For any $1 \leq i \leq n$ there exists a unique element $E_i^* \in H_2(\tilde{X}, \mathbb{Z})$ such that $E_i^* \cdot E_j = \delta_{ij}$ for all $1 \leq j \leq n$.

The following result is due to Artin and Verdier [1], see also [6] and [4].

**Theorem 12.** Let $M$ be a maximal Cohen-Macaulay module over $A$ and $\tilde{M} = \pi^*(M)/\text{tor}$ be the corresponding torsion free sheaf on $\tilde{X}$. Then we have:

1. The torsion free coherent sheaf $\tilde{M}$ is locally free.
2. The isomorphism class of $M$ is uniquely determined by the pair $(\text{rk}(\tilde{M}), c_1(\tilde{M})) \in \mathbb{Z} \times H^2(\tilde{X}, \mathbb{Z})$.
3. If $M$ is indecomposable than either $M \cong A$ or there exists $1 \leq i \leq n$ such that $c_1(\tilde{M}) = E_i^*$.

In that case we have: $\text{rk}(M) = c_1(M) \cdot Z$, where $Z$ is the fundamental cycle of $\tilde{X}$.

Hence, combining the Theorem 11 and Theorem 12, we get a bijection between the set of the isomorphism classes of non-trivial irreducible representations of $G$, the set of indecomposable objects of the stable category of the maximal Cohen-Macaulay modules $\text{CM}(A)$ and the set of the irreducible components of the exceptional divisor $E$.

If $V$ is a representation of $G$ and $M = (\mathbb{C}[x, y] \otimes V)^G$ is the corresponding Cohen-Macaulay module, then $\text{rk}(\tilde{M}) = \dim_{\mathbb{C}}(V)$. Thus, the last part of Theorem 12 implies that the fundamental cycle $Z$ is equal to $\sum_{i=1}^n m_i [E_i]$, where $m_i = \dim_{\mathbb{C}}(V_i)$ for $1 \leq i \leq n$.

The following result is due to Esnault and Knörrer [5].

**Theorem 13.** Let $V$ be a non-trivial irreducible representation of $G$, $M = (\mathbb{C}[x, y] \otimes V)^G$ be the corresponding indecomposable Cohen-Macaulay module, $\tilde{M} = \pi^*(M)/\text{tor}$ the corresponding vector bundle on $\tilde{X}$, and $F$ the irreducible component of $E$ such that $c_1(\tilde{M}) = F^*$. Let $N = (M \otimes_A \Omega_A^1)^{\vee}$ and $\tilde{N}$ be the corresponding vector bundle on $\tilde{X}$. Then we have:

$\det(\tilde{N}) \cong \det(\tilde{M})^{\otimes 2} \otimes \mathcal{O}_{\tilde{X}}(F)$.

Using this result, the isomorphism of the McKay graph $\text{MK}(G)^*$ and the dual intersection graph $\Gamma_E$ follows from Theorem 11 and Theorem 12.

**References**


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