MAXIMAL COHEN-MACAULAY MODULES OVER QUOTIENT SURFACE SINGULARITIES

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Abstract. In this note I discuss a relationship between the McKay Correspondence for two-dimensional quotient singularities and the theory of maximal Cohen-Macaulay modules.

1. McKay's observation

Let $G \subseteq \mathsf{SL}_2(\mathbb{C})$ be a finite group. Then one can attach to it the following pair of combinatorial objects.

First object. By Maschke's theorem, the category of finite dimensional representations of G over the field $\mathbb{C}$ is semi-simple. Let $\{V_0, V_1, \dots, V_n\}$ be the set of the isomorphy classes of the *irreducible* representations of G, where $V_0 = \mathbb{C}$ is the trivial representation, and $W = \mathbb{C}^2$ be the *fundamental* representation of G induced by the embedding $G \subseteq \mathsf{SL}_2(\mathbb{C})$. For any $0 \le i \le n$, we set $m_i = \dim_{\mathbb{C}}(V_i)$. For any $0 \le i \le n$ we have decompositions

$$V_i \otimes_{\mathbb{C}} W \cong \bigoplus_{j=0}^{n} V_j^{a_{ij}}.$$

One can show that $a_{ii} = 0$ and $a_{ij} = a_{ji}$ for all $0 \le i, j \le n$.

Definition 1. The McKay graph $\mathsf{MK}(G)$ of a finite group $G \subseteq \mathsf{SL}_2(\mathbb{C})$ is defined as follows.

(1) The set of vertices of $\mathsf{MK}(G)$ is $\{0, 1, \dots, n\}$.
(2) For any $0 \le i \ne j \le n$ the vertex i is connected with the vertex j by a_{ij} arrows.
(3) The vertex i has "weight" m_i.

Second object. Let $A = \mathbb{C}[\![x, y]\!]^G$ be the quotient singularity defined by G, $X = \mathsf{Spec}(A)$, $o \in X$ the closed point of X and $\widetilde{X} \xrightarrow{\pi} X$ a minimal resolution of singularities. Let $E = \pi^{-1}(o)$ be the exceptional divisor of the resolution. It is well-known that E is a tree of projective lines.

In 1978 John McKay made [8] the following striking

Observation. Let $G \subset \mathsf{SL}_2(\mathbb{C})$ be a finite subgroup. Then we have:

(1) The number of the irreducible components of E is equal to the number of non-trivial irreducible representations of G.
(2) Let $\mathsf{MK}(G)' = \mathsf{MK}(G) \setminus \{0\}$ be the graph obtained from $\mathsf{MK}(G)$ by excluding the vertex 0 and all arrows connected with it. Then $\mathsf{MK}(G)'$ is isomorphic to the dual intersection graph Γ_E of the curve E. In other words, there exists a labeling of the irreducible components $E_1, \dots, E_n$ such that for any $1 \le i \ne j \le n$ we have:
$$a_{ij} = \#(E_i \cap E_j) =: c_{ij}.$$
(3) The cycle $Z = \sum_{i=1}^{n} m_i [E_i] \in H_2(\widetilde{X}, \mathbb{Z})$ is the *fundamental cycle* of the resolution $\widetilde{X}$. This can be expressed in plain words as follows. For any $1 \le i \le n$ let $c_{ii} := -2 = E_i^2$ be the self-intersection index of E_i and $C = (c_{ij}) \in \mathsf{Mat}_{n \times n}(\mathbb{Z})$ be the *intersection matrix* of E.

Then Z is the smallest vector $\underline{l} = (l_1, l_2, \dots, l_n)$ with non-negative integral entries such that

$$\langle \underline{e}_i, \underline{l} \rangle_C := \langle \underline{e}_i^t \, C \, \underline{l} \rangle \leq 0$$

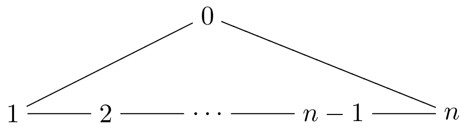
for all $1 \leq i \leq n$, where $e_i = (0, \dots, 0, 1, 0, \dots 0)$ is the i-th basic vector of $\mathbb{Z}^n$.

Example 2. Let $\mathbb{Z}/(n+1)\mathbb{Z} \cong G = \langle g \rangle \subset \mathsf{SL}_2(\mathbb{C})$ be a cyclic subgroup of order $n+1$ generated by the element

$$g = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix},$$

where ξ is a primitive $(n+1)$-st root of 1. Then we have:

- G has $n+1$ irreducible representations $\{V_0, V_1, \dots, V_n\}$, where all $V_i = \mathbb{C}$ and the action of g is given by the multiplication with ξ^i. It is easy to see that $W = V_1 \oplus V_n$ and the McKay's graph $\mathsf{MK}(G)$ is a cycle



- Next, we have:

$$A := \mathbb{C}[\![x, y]\!]^G = \mathbb{C}[\![x^{n+1}, xy, y^{n+1}]\!] \cong \mathbb{C}[\![u, v, w]\!]/(uw - v^{n+1})$$

  is a simple surface singularity of type A_n. It is well-known that the exceptional divisor of a minimal resolution of singularities of $\mathsf{Spec}(A)$ is a chain of n projective lines. Hence, the intersection matrix of the exceptional divisor is just

$$C = \begin{pmatrix} -2 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & -2 & 1 \\ 0 & \dots & 0 & 1 & -2 \end{pmatrix}.$$

It is easy to show that in this case the fundamental cycle Z is equal to $\sum_{i=1}^{n} [E_i] = (1, 1, \dots, 1)$, in a full accordance with McKay's observation.

Explanation. McKay himself has verified his observation using Klein's classification of finite subgroups in $\mathsf{SL}_2(\mathbb{C})$ by a tedious case-by-case analysis [8]. It turns out, however, that the McKay correspondence can be explained in a more conceptual way by introducing the third intermediate object: the *stable category of the maximal Cohen-Macaulay* A–modules $\underline{\mathsf{CM}}(A)$. Namely, there exist natural bijections

$$\mathsf{MK}(G)' \xleftarrow{\sim} \mathrm{ind}\big(\underline{\mathsf{CM}}(A)\big) \xrightarrow{\sim} \Gamma_E,$$

where $\mathrm{ind}\big(\underline{\mathsf{CM}}(A)\big)$ is the set of the isomorphy classes of indecomposable objects in $\underline{\mathsf{CM}}(A)$. The statement about the fundamental cycle and the dimensions of irreducible representations of G can be derived using the Auslander-Reiten theory of the category $\underline{\mathsf{CM}}(A)$.

2. Algebraic McKay Correspondence

Let $G \subset \mathsf{SL}_2(\mathbb{C})$ be a finite subgroup. Then the ring of invariants $A = \mathbb{C}[\![x, y]\!]^G$ is a normal surface singularity. Recall the following standard facts about Cohen-Macaulay modules over surface singularities.

Theorem 3. *Let $(A, \mathfrak{m})$ be a local Noetherian ring of Krull dimension two.*

- *A is normal if and only if it is Cohen-Macaulay and regular in codimension one.*

- *Assume A to be Cohen-Macaulay. Then for any maximal Cohen-Macaulay module M and any Noetherian module N the module $\mathsf{Hom}_A(N, M)$ is maximal Cohen-Macaulay.*
- *Assume additionally that A is Gorenstein in codimension one (for instance, A is a normal singularity). Then a Noetherian module M is maximal Cohen-Macaulay if and only if it is reflexive. Moreover, the functor $M \mapsto M^{\vee\vee}$ is left adjoint to the forgetful functor $\mathsf{CM}(A) \longrightarrow A - \mathsf{mod}$.*
- *Let $(A, \mathfrak{m}) \subseteq (B, \mathfrak{n})$ be a finite extension of Cohen-Macaulay surface singularities, which are Gorenstein in codimension one. Then for any Noetherian B–module M we have an isomorphism of A–modules $M^{\vee\vee_A} \cong M^{\vee\vee_B}$.*
- *Let A be regular. Then any maximal Cohen-Macaulay module over A is free.*

For a proof one may consult [3, Section 3] and references therein.

The following theorem of Herzog [7] was the starting point of an extensive study of maximal Cohen-Macaulay modules over surface singularities.

Theorem 4. *Let k be an algebraically closed field, $G \subset \mathsf{GL}_2(k)$ be a finite subgroup such that $\gcd\big(|G|, \mathrm{char}(k)\big) = 1$, $R = k[\![x, y]\!]$ and $A = R^G$. Then we have: $\mathsf{CM}(A) = \mathsf{add}_A(R)$. In other words, any maximal Cohen-Macaulay module over A is isomorphic to a direct sum of direct summands of R viewed as an A–module.*

Proof. The embedding $i : A \to R$ has a left inverse $p : R \to A$ given by the Reinold's operator

$$p(r) = \frac{1}{|G|} \sum_{g \in G} g(r).$$

It is easy to see that the map p is A-linear. Hence, we have an isomorphism $R \cong A \oplus A'$ in the category of A–modules. Next, for any Noetherian A–module M we have:

$$R \otimes_A M \cong M \oplus (A' \otimes_A M).$$

If M is maximal Cohen-Macaulay over A then there exists a positive integer t such that

$$R^t \cong (R \otimes_A M)^{\vee\vee_R} \cong (R \otimes_A M)^{\vee\vee_A} \cong M \oplus (A' \otimes_A M)^{\vee\vee_A}.$$

Hence, M is a direct summand of R^t as stated. □

From Herzog's result we get the following corollary.

Corollary 5. *Let $\Lambda = \mathsf{End}_A(R)$. Then the functor*

$$\mathsf{Hom}_A(R, -) : \mathsf{CM}(A) = \mathsf{add}_A(R) \longrightarrow \mathsf{pro}(\Lambda)$$

is an equivalence of categories, where $\mathsf{pro}(\Lambda)$ is the category of the finitely generated projective right Λ–modules.

The following result is due to Auslander [2].

Theorem 6. *Let k be an algebraically closed field and $G \subset \mathsf{GL}_2(k)$ be a small finite subgroup such that $\gcd\big(|G|, \mathrm{char}(k)\big) = 1$ (note that any subgroup in $\mathsf{SL}_2(k)$ is automatically small). Let $R = k[\![x, y]\!]$ and $A = R^G$. Then the algebra homomorphism*

$$\theta : R * G \longrightarrow \mathsf{End}_A(R), \quad t[g] \stackrel{\theta}{\mapsto} \big(r \mapsto tg(r)\big)$$

is an isomorphism of algebras.

As a corollary, we obtain the following "algebraic" version of the McKay Correspondence, which is due to Auslander [2], see also [10].

Theorem 7. *Let k be an algebraically closed field and $G \subset \mathsf{GL}_2(k)$ be a small finite subgroup such that $\gcd\big(|G|, \mathrm{char}(k)\big) = 1$, $R = k[\![x, y]\!]$ and $A = R^G$.*

(1) *The functor* $\mathsf{pro}(R*G) \to \mathsf{CM}(A)$ *assigning to a projective module* P *its* A*-submodule of invariants* P^G, *is an equivalence of categories quasi-inverse to the functor*

$$\mathsf{Hom}_A(R, -) : \mathsf{CM}(A) \longrightarrow \mathsf{pro}(R*G).$$

(2) *Since we have an isomorphism* $R*G/\mathsf{rad}(R*G) \cong k[G]$, *a bijection between the projective and the semi-simple right* $R*G$*–modules* $P \mapsto P/\mathsf{rad}(P)$ *yields a bijection between the isomorphy classes of irreducible representations of the group* G *and indecomposable projective right modules over* $R*G$. *If* V *is an irreducible representation of* G, *then the corresponding projective* $R*G$*–module is just* $R\otimes_k V$, *where the action of an element* $t[g] \in R*G$ *on a simple tensor* $r\otimes h \in R\otimes_k V$ *is given by:*

$$t[g]\circ(r\otimes h) = tg(r)\otimes gh.$$

(3) *The correspondence between the irreducible representations of* G *and the indecomposable maximal Cohen-Macaulay modules over* A *is given by the functor*

$$\mathsf{Rep}(G) \ni V \mapsto \bigl(R\otimes_k V\bigr)^G \in \mathsf{CM}(A).$$

In the notations of the above theorem, consider the Koszul resolution of the trivial representation $V_0 = k$ of the group G, viewed as an $R*G$–module:

$$0 \to R\otimes_k \wedge^2(W) \xrightarrow{\alpha} R\otimes_k W \xrightarrow{\beta} R \xrightarrow{\phi} k \to 0$$

where $\alpha\bigl(p\otimes(f_1\otimes f_2 - f_2\otimes f_1)\bigr) = p\tilde{f_1}\otimes f_2 - p\tilde{f_2}\otimes f_1$, $\beta(q\otimes f) = q\tilde{f}$ and $\phi(t) = t(0,0)$.

Remark 8. Let V be a non-trivial irreducible $k[G]$–module. Then its minimal free projective resolution in the category of $R*G$–modules is

$$0 \to R\otimes_k(\wedge^2(W)\otimes_k V) \longrightarrow R\otimes_k(W\otimes_k V) \longrightarrow R\otimes_k V \longrightarrow V \to 0. \tag{1}$$

Since the functor of taking G-invariants is exact, we obtain a short exact sequence of Cohen-Macaulay A–modules

$$0 \to \bigl(R\otimes_k(\wedge^2(W)\otimes_k V)\bigr)^G \longrightarrow \bigl(R\otimes_k(W\otimes_k V)\bigr)^G \longrightarrow \bigl(R\otimes_k V\bigr)^G \to 0, \tag{2}$$

which is precisely the Auslander-Reiten sequence sequence ending at the indecomposable Cohen-Macaulay module $\bigl(R\otimes_k V\bigr)^G$, see [2] and [10].

Corollary 9. *If* G *is a finite subgroup of* $\mathsf{SL}_2(k)$ *then we have:* $\wedge^2 W \cong V_0 = k$. *Hence, the Auslander-Reiten quiver of the category* $\mathsf{CM}(A)$ *is obtained from the McKay' graph* $\mathsf{MK}(G)$ *by "doubling" all the arrows.*

Example 10. Let $\mathbb{Z}/(n+1)\mathbb{Z} \cong G \subset \mathsf{SL}_2(\mathbb{C})$ be as in Example 2, $A = \mathbb{C}[\![x,y]\!]^G$ and $\{V_0, V_1, \dots, V_n\}$ be the set of the isomorphy classes of irreducible representations of G, where $V_i = \mathbb{C}$ and $g\cdot 1 = \xi^i$ for $0 \le i \le n$. Then the corresponding indecomposable Cohen-Macaulay A–modules are

$$\mathbb{C}[\![x,y]\!] \supseteq I_l := \bigl(\mathbb{C}[\![x,y]\!]\otimes_k V_l\bigr)^G = \Bigl\{\sum_{i,j=0}^{\infty} a_{ij}x^iy^j \Bigm| a_{ij}\in\mathbb{C},\ i-j \equiv l \bmod n\Bigr\}, \quad 0 \le l \le n.$$

The following result is due to Auslander [2].

Theorem 11. *Let* $(A,\mathfrak{m})$ *be a normal surface singularity with a canonical module* K.

- *Let* $\omega \in \mathsf{Ext}^2_A(k,K) \cong k$ *be a generator and*

$$0 \longrightarrow K \longrightarrow D \longrightarrow A \longrightarrow k \longrightarrow 0 \tag{3}$$

  be the corresponding extension class. Then the module D *is maximal Cohen-Macaulay.*
- *Let* $G \subset \mathsf{GL}_2(k)$ *be a finite subgroup,* $R = k[\![x,y]\!]$ *and* $A = R^G$ *be the corresponding quotient singularity. Then the sequence (3) is obtained from the sequence (1) by taking* G*-invariants. In particular, we have:* $K \cong (R\otimes_k \wedge^2 W)^G$ *(see* [9]*) and* $D \cong (R\otimes_k W)^G$.

- *For a non-regular indecomposable Cohen-Macaulay module M, the complex*

$$0 \longrightarrow (K \otimes_A M)^{\vee\vee} \longrightarrow (D \otimes_A M)^{\vee\vee} \longrightarrow M \longrightarrow 0 \tag{4}$$

induced by the short exact sequence (3), is exact. Moreover, it is an Auslander-Reiten sequence, ending at M.
- *For $G \subset \mathsf{SL}_2(k)$ holds: $D \cong (\Omega^1_A)^{\vee\vee}$, where Ω^1_A is the module of Kähler differentials of A.*

3. Geometric McKay Correspondence

Let $G \subset \mathsf{SL}_2(\mathbb{C})$ be a finite subgroup, $A = \mathbb{C}[\![x,y]\!]^G$, $X = \mathsf{Spec}(A)$ and $\widetilde{X} \xrightarrow{\pi} X$ be a minimal resolution of singularities. Let $E = \pi^{-1}(o)$ be the exceptional divisor of π. The following facts are well-known.

(1) $E = E_1 \cup \cdots \cup E_n$ is a tree of projective lines.
(2) We have: $H_2(\widetilde{X},\mathbb{Z}) = \cup_{i=1}^n \mathbb{Z}[E_i] \cong \mathbb{Z}^n$.
(3) For any $1 \le i \le n$ there exists a unique element $E_i^* \in H_2(\widetilde{X},\mathbb{Z})$ such that $E_i^* \cdot E_j = \delta_{ij}$ for all $1 \le j \le n$.

The following result is due to Artin and Verdier [1], see also [6] and [4].

Theorem 12. *Let M be a maximal Cohen-Macaulay module over A and $\widetilde{M} = \pi^*(M)/\mathsf{tor}$ be the corresponding torsion free sheaf on $\widetilde{X}$. Then we have:*

(1) *The torsion free coherent sheaf $\widetilde{M}$ is locally free.*
(2) *The isomorphy class of M is uniquely determined by the pair*

$$\big(\mathrm{rk}(\widetilde{M}), c_1(\widetilde{M})\big) \in \mathbb{Z}_+ \times H^2(\widetilde{X},\mathbb{Z}).$$

(3) *If M is indecomposable than either $M \cong A$ or there exists $1 \le i \le n$ such that $c_1(\widetilde{M}) = E_i^*$. In that case we have: $\mathrm{rk}(\widetilde{M}) = c_1(\widetilde{M}) \cdot Z$, where Z is the fundamental cycle of $\widetilde{X}$.*

Hence, combining the Theorem 11 and Theorem 12, we get a bijection between the set of the isomorphy classes of non-trivial irreducible representations of G, the set of indecomposable objects of the stable category of the maximal Cohen-Macaulay modules $\underline{\mathsf{CM}}(A)$ and the set of the irreducible components of the exceptional divisor E.

If V is a representation of G and $M = (\mathbb{C}[\![x,y]\!] \otimes V)^G$ is the corresponding Cohen-Macaulay module, then $\mathrm{rk}(\widetilde{M}) = \mathsf{dim}_{\mathbb{C}}(V)$. Thus, the last part of Theorem 12 implies that the fundamental cycle Z is equal to $\sum_{i=1}^n m_i[E_i]$, where $m_i = \mathsf{dim}_{\mathbb{C}}(V_i)$ for $1 \le i \le n$.

The following result is due to Esnault and Knörrer [5].

Theorem 13. *Let V be a non-trivial irreducible representation of G, $M = (\mathbb{C}[\![x,y]\!] \otimes_{\mathbb{C}} V)^G$ be the corresponding indecomposable Cohen-Macaulay module, $\widetilde{M} = \pi^*(M)/\mathsf{tor}$ the corresponding vector bundle on $\widetilde{X}$ and F the irreducible component of E such that $c_1(\widetilde{M}) = F^*$. Let $N = (M \otimes_A \Omega^1_A)^{\vee\vee}$ and $\widetilde{N}$ be the corresponding vector bundle on $\widetilde{X}$. Then we have:*

$$\det(\widetilde{N}) \cong \det(\widetilde{M})^{\otimes 2} \otimes \mathcal{O}_{\widetilde{X}}(F).$$

Using this result, the isomorphism of the McKay graph $\mathsf{MK}(G)'$ and the dual intersection graph Γ_E follows from Theorem 11 and Theorem 12.

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