COHERENT SHEAVES ON AN ELLIPTIC CURVE

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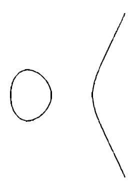
Abstract. These are notes by Kristian Brüning of a mini-course given by Igor Burban at the summer school "Derived categories in representation theory" held at Tsinghua University, Beijing: 22-26 August, 2005. The aim of this series of lectures is to give a classification of indecomposable coherent sheaves on an elliptic curve using a braid group action on the derived category.

Contents

1. Introduction

We want to study the category of coherent sheaves $\mathrm{Coh}(X)$ on an elliptic curve X. By definition, an elliptic curve is a smooth plain curve in $\mathbb{P}^2$ of degree three (e.g. $zy^2 = x^3 - z^2x$), and in the affine chart $z \neq 0$ it looks like:



This category $\mathrm{Coh}(X)$ has nice properties: it is an abelian, finite dimensional hereditary category and is therefore Krull-Schmidt [Ste75, VIII,4.3]. We shall classify the indecomposable coherent sheaves on X using a braid group action on the derived category $\mathcal{D}^b(\mathrm{Coh}(X))$. It yields a background for the study of the Hall algebra of the category $\mathrm{Coh}(X)$ [BS]. Before dealing with coherent sheaves on projective curves we first take a look at the affine case.

Fix an algebraically closed field k. Unless otherwise stated, all rings which we shall consider are commutative and are algebras over the field k.

Definition 1.1. Let $I = (f_1, ..., f_s)$ be an ideal in $k[X_1, ..., X_n]$. The corresponding *affine variety* $X = V(I) \subset k^n = \mathbb{A}_k^n$ is the set of solutions of the system of equations $\{f_1 = 0, ..., f_s = 0\}$. Denote by $k[X] := k[X_1, ..., X_n]/I$ the *coordinate ring* of X.

We have a contravariant functor

$$\begin{array}{ccc} (\text{affine varieties}) & \to & (\text{noetherian } k-\text{algebras}) \\ X & \mapsto & k[X]. \end{array}$$

Principle 1.2. Everything we want to know about X is contained in its coordinate ring $k[X]$.

Definition 1.3. For a noetherian ring R we denote by

$$\mathrm{Max}(R) = \{\text{maximal ideals in } R\}$$

the *maximal spectrum* of R.

As a first incarnation of our principle we have:

Theorem 1.4 (Hilbert Nullstellensatz)**.** *Let X be an affine variety. There is a one to one correspondence*

$$\{\text{points in } X\} \xleftrightarrow{1:1} \mathrm{Max}(k[X]).$$

Given a point $p \in X$, the corresponding maximal ideal $\mathfrak{m}_p = \{f \in k[X] | f(p) = 0\}$ gives rise to the short exact sequence:

$$\begin{array}{ccc} 0 \to \mathfrak{m}_p \to k[X] & \to & k \to 0 \\ f & \mapsto & f(p). \end{array}$$

Let $T_pX := (\mathfrak{m}_p/\mathfrak{m}_p^2)^*$ denote the *tangent space* of X at the point p.

Definition 1.5. Let A be a noetherian domain, and let $\mathfrak{p} \subset A$ be a prime ideal. The ring

$$A_\mathfrak{p} := \left\{ \frac{a}{b} | \ a \in A, \ b \in A \backslash \mathfrak{p} \right\}$$

is called the *localization of A at $\mathfrak{p}$*.

Definition 1.6. An affine variety is a *smooth curve* if for all points $p \in X$ the tangent space T_pX is 1-dimensional.

One can characterize smooth affine curves in the following way:

Lemma 1.7. *For an affine variety X the following conditions are equivalent:*

(i) *X is a smooth curve*
(ii) *$k[X]$ is a Dedekind domain*
(iii) *For all $\mathfrak{m} \in \mathrm{Max}(k[X])$ the localization $k[X]_\mathfrak{m}$ is a principal ideal domain.*

Proof. The equivalence $(ii) \Leftrightarrow (iii)$ is just a characterization of a Dedekind domain [AM69, Proposition 9.2, Theorem 9.3]. For the other equivalence, since $(T_pX)^* \cong \mathfrak{m}_p/\mathfrak{m}_p^2$ is one dimensional, this vector space is generated by the image of an element in $\mathfrak{m}_p$, say a. By Nakayama's Lemma [AM69, Proposition 2.8], the ideal $\mathfrak{m}_p$ is generated by a in $k[X]_{\mathfrak{m}_p}$. □

Definition 1.8. Let X be an affine curve. The category of *coherent sheaves* $\mathrm{Coh}(X)$ on X is the category $k[X]$-mod of noetherian $k[X]$-modules.

2. Modules over Dedekind domains

Let A be any Dedekind domain, for example the coordinate ring $k[X] = k[x,y]/(y^2 - x^3 + x)$ of the curve $X = V(y^2 - x^3 + x)$ or the ring of Gaussian integers $\mathbb{Z}[i]$. We shall investigate the module category of A which in the first example of a category of coherent sheaves.

Let M be a noetherian A-module.

Definition 2.1. The *torsion part* of M is defined as

$$T(M) = \{m \in M|\ \exists a \in A \setminus \{0\}|\ am = 0\}.$$

There is a short exact sequence:

$$0 \to T(M) \to M \to M/T(M) \to 0 \qquad (*)$$

where $T(M)$ and hence $M/T(M)$ are noetherian and $M/T(M)$ is by construction torsion free.

Proposition 2.2. *Every noetherian module M over A splits into a torsion part and a projective part:*

$$M \cong T(M) \oplus M/T(M).$$

Proof. Let $\mathfrak{m} \in \mathrm{Max}(A)$ be a maximal ideal. The module $(M/T(M))_{\mathfrak{m}}$ is torsion free. By the classification of modules over principal ideal domains, we know that $(M/T(M))_{\mathfrak{m}}$ is already free. Therefore

$$\mathrm{Ext}^1_A(M/T(M), T(M))_{\mathfrak{m}} = \mathrm{Ext}^1_{A_{\mathfrak{m}}}((M/T(M))_{\mathfrak{m}}, T(M)_{\mathfrak{m}}) = 0.$$

Since $\mathrm{Ext}^1_A(M/T(M), T(M))_{\mathfrak{m}} = 0$ for all maximal ideals we conclude that

$$\mathrm{Ext}^1_A(M/T(M), T(M)) = 0.$$

Therefore the short exact sequence $(*)$ splits. □

We collect some facts about the module category of A in the following:

Proposition 2.3. *Let A be a Dedekind domain.*

(i) *The category A-mod is hereditary.*

(ii) *If an A-module M is torsion free then there are ideals $I_1, \dots, I_s \subset A$ such that*

$$M \cong \bigoplus_{i=1}^{s} I_i.$$

(iii) *If I and J are ideals in A then*

$$I \oplus J \cong IJ \oplus A.$$

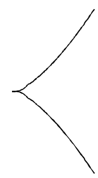
(iv) *If M is a noetherian A-module then there exist ideals $I \subset A$, $\mathfrak{m}_1, \dots, \mathfrak{m}_s \in \mathrm{Max}(A)$ and integers $m, p_1, \dots, p_s \in \mathbb{Z}_{>0}$, such that*

$$M \cong I \oplus A^m \oplus (\bigoplus_{i=1}^{s} A/\mathfrak{m}_i^{p_i}).$$

Definition 2.4. A k-linear category is called *finite dimensional* if all the Hom-spaces are finite dimensional as k-vector spaces.

Remark 2.5. Since $\mathrm{End}(A) = A$ the category of A-modules is infinite dimensional. The assertion (i) can be shown by a " local-global" argument similar to the proof of 2.2. The statement (ii) tells us that the indecomposable projective modules are ideals. The category A-mod is not a Krull-Schmidt category due to (iii).

Exercise 2.6. Let X be the cuspidal curve, i.e. $X = V(y^2 - x^3)$ and $A := k[x, y]/(y^2 - x^3)$.



- Show that the category A-mod is not hereditary. (Hint: find a module M with $\mathrm{Ext}^2(M, M) \neq 0$)
- Construct an indecomposable A-module which is neither torsion nor torsion free.

3. Coherent sheaves on projective varieties

In this section we shall introduce the category of coherent sheaves on a projective variety X and define some basic functors like the internal tensor product and the internal Hom-functor.

Definition 3.1. The n-dimensional *projective space* over the field k is defined as

$$\mathbb{P}_k^n := k^{n+1} \setminus \{0\}/k^*,$$

where two points x and x' in $k^{n+1} \setminus \{0\}$ are identified if there is a $\lambda \in k^*$ such that $x = \lambda x'$. For an $x = (x_0, \dots, x_n) \in k^{n+1} \setminus \{0\}$ let $(x_0 : \dots : x_n)$ denote its equivalence class which is called *homogeneous coordinates.*

If f is a homogeneous polynomial of degree d, i.e. $f(\lambda(x_0, \dots, x_n)) = \lambda^d f(x_0, \dots, x_n)$ for all $\lambda \neq 0$, then $f = 0$ defines a subset of $\mathbb{P}_k^n$. Recall that the radical $\sqrt{I}$ of an ideal I is $\{f \in k[X] \mid \exists\, n \in \mathbb{N} : \ f^n \in I\}$.

Definition 3.2. Let $I \subset k[X_0, \dots, X_n]$ be a homogeneous ideal, that is generated by homogeneous polynomials $f_0, \dots, f_s$ such that $\sqrt{I} \neq (X_0, \dots, X_n)$ then the corresponding *projective variety*

$$X = PV(I) \subset \mathbb{P}_k^n$$

is the set of solutions of the system of equations $\{f_0 = 0, \ldots, f_n = 0\}$.

Remark 3.3. The ideal $\langle X_0, \ldots, X_n \rangle$ does not define a projective variety and therefore is called the *irrelevant* ideal.

Like in the affine case let $k[X] = k[X_0, \ldots, X_n]/I$ denote the coordinate ring of the projective variety $X = PV(I)$. Observe that $R = k[X]$ is a graded ring:

$$R = \bigoplus_{i=0}^{\infty} R_i \qquad R_i \cdot R_j \subset R_{i+j},$$

where R_i is generated by the classes of homogeneous polynomials of degree i.

Definition 3.4. Let $\mathrm{grmod}(R)$ be the category of finitely generated R-modules $M = \bigoplus_{i=n_0}^{\infty} M_i$ satisfying $R_i \cdot M_j \subset M_{i+j}$.

Exercise 3.5. The category $\mathrm{grmod}(R)$ is finite dimensional.

For a graded ring R let $\mathrm{grMax}(R) = \{$homogeneous prime ideals of coheight 1 in $R\}$ be its graded maximal spectrum. Here, a homogeneous prime ideal $\mathfrak{p}$ has coheight 1 if the only proper homogeneous prime ideal $\mathfrak{m}$ containing $\mathfrak{p}$ is the irrelevant ideal. In analogy with the affine case we have a projective Hilbert Nullstellensatz:

Theorem 3.6 (Hilbert Nullstellensatz)**.** *Let X be a projective variety. There is a one to one correspondence*

$$\{\text{points in } X\} \xleftrightarrow{1:1} \mathrm{grMax}(k[X]).$$

The category $\mathrm{grmod}(k[X])$ has less nice properties than its affine analogon $k[X]$-mod. So what are the problems with $\mathrm{grmod}(k[X])$?

- If two projective varieties are isomorphic $X \cong Y$ then $k[X] \not\cong k[Y]$ in general.

Example 3.7. The conic $X_0^2 + X_1^2 + X_2^2 = 0$ in $\mathbb{P}_k^2$ is isomorphic to $\mathbb{P}_k^1$. But $k[X_0, X_1, X_2]/(X_0^2 + X_1^2 + X_2^2) \not\cong k[Y_0, Y_1]$.

- The category $\mathrm{grmod}(k[\mathbb{P}_k^1]) = \mathrm{grmod}(k[X_0, X_1])$ is not hereditary.

Definition 3.8. Let R be a graded ring. For $M \in \mathrm{grmod}(R)$, $n \in \mathbb{Z}$ define the *shift* $M(n)$ of M via $M(n)_i = M(n+i)$.

For $S = k[Y_0, Y_1]$ the sequence

$$0 \to S(-2) \xrightarrow{\binom{-Y_1}{Y_0}} S(-1)^2 \xrightarrow{(Y_0\ Y_1)} S \to k \to 0$$

is a minimal projective resolution of k. Hence, $\mathrm{gl.dim}(\mathrm{grmod}(S)) \geq 2$. It is actually equal to 2 due to Hilbert's Syzygy Theorem. Even worse: $\mathrm{gl.dim}(\mathrm{grmod}(k[X_0, X_1, X_2]/(X_0^2 + X_1^2 + X_2^2))) = \infty$. So we need to modify $\mathrm{grmod}(k[X])$ to get a nice category worth to call $\mathrm{Coh}(X)$.

Let $X \subseteq \mathbb{P}_k^n$ be a projective variety with coordinate ring $R = k[X]$ which is graded. Let $\mathcal{A} := \mathrm{grmod}(R)$ denote the category of finitely generated graded R-modules and $\mathcal{B} := \mathrm{grmod}_0(R)$ the category of finite dimensional graded R-modules. In order to construct the quotient $\mathcal{A}/\mathcal{B}$ the following is fact useful:

Observation 3.9. Let

$$0 \to M \to N \to K \to 0$$

be exact in $\mathcal{A}$. Then $N \in \mathcal{B}$ if and only if M and K are in $\mathcal{B}$.

Definition 3.10. Let $S_{\mathcal{B}} := \{f \in \mathrm{Mor}(\mathcal{A}) |\ \ker(f), \mathrm{coker}(f) \in \mathcal{B}\}$.

Exercise 3.11. The set $S_{\mathcal{B}}$ is multiplicatively closed (i.e. if $f, g \in S_{\mathcal{B}}$ are composable morphisms then the composition $f \circ g$ is in $S_{\mathcal{B}}$ and $S_{\mathcal{B}}$ contains the identity morphisms).

Having a multiplicatively closed set, we can form the localization with respect to that set: define the *Serre quotient*

$$\mathcal{A}/\mathcal{B} := \mathcal{A}[S_{\mathcal{B}}^{-1}].$$

We shall give a second definition of the category $\mathcal{A}/\mathcal{B}$ which has the advantage of a more concrete description of its morphisms. Let $\mathcal{A}$ and $\mathcal{B}$ be as above. The objects of the Serre quotient $\mathcal{A}/\mathcal{B}$ are just the objects of $\mathcal{A}$. For $M, N \in \mathcal{A}$ introduce the set:

$$I_{M,N} := \{((X, \varphi), (Y, \psi))|\ \ X \overset{\varphi}{\hookrightarrow} M\,,\, N \overset{\psi}{\twoheadrightarrow} Y\,,\ \ \mathrm{coker}(\varphi), \ker(\psi) \in \mathcal{B}\},$$

and notice that, given φ and ψ, there is a map

$$\mathrm{Hom}_{\mathcal{A}}(M, N) \to \mathrm{Hom}_{\mathcal{A}}(X, Y).$$

We endow the set $I_{M,N}$ with an ordering $\leq$:

$$((X, \varphi), (Y, \psi))) \leq ((X', \varphi'), (Y', \psi')))$$

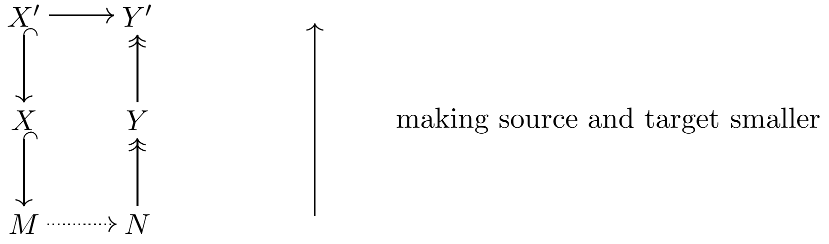
if there are maps f and g in $\mathcal{A}$ such that the following diagrams commute:

Exercise 3.12. Show that the partially ordered set $I_{M,N}$ is directed, that is for all $a, b \in I_{M,N}$ there is a $c \in I_{M,N}$ such that $a \leq c$ and $b \leq c$.

Now we can define the morphisms in our category $\mathcal{A}/\mathcal{B}$ as the direct limit of the inductive system $\{\mathrm{Hom}_{\mathcal{A}}(X, Y)\}_{((X,\varphi),(Y,\psi)) \in I_{M,N}}$ [AM69, 2, Exercise 14]:

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(M, N) := \varinjlim_{I_{M,N}} \mathrm{Hom}_{\mathcal{A}}(X, Y).$$

The effect of this definition is that a morphism $M \to N$ in $\mathcal{A}/\mathcal{B}$ is represented by a map $X \to Y$ with smaller source and target. In a picture:



With this definition we have [Gab62]:

Theorem 3.13 (Gabriel)**.** *The category $\mathcal{A}/\mathcal{B}$ is abelian and the canonical functor*

$$\mathcal{A} \to \mathcal{A}/\mathcal{B}$$

is exact.

For $M \in \mathcal{A}$ let $\widetilde{M}$ denote its image in the Serre quotient. Now we can give the definition of coherent sheaves in the projective setup:

Definition 3.14. The category of coherent sheaves on a projective variety $X \subseteq \mathbb{P}^n_k$ is defined as the Serre quotient:

$$\mathrm{Coh}(X) := \mathrm{grmod}(k[X]) / \mathrm{grmod}_0(k[X]).$$

Remark 3.15. The definition 3.14 of the category of coherent sheaves on a projective variety X is due to Serre [Ser55] and is different from the standard one, see e.g. [Har77]. This way to define $\mathrm{Coh}(X)$ via quotient categories does not require any special knowledges of the sheaf theory and plays a key role in the computer algebra approach to the study of coherent sheaves on projective varieties. This category has nice properties like in the affine case

Theorem 3.16 (Serre)**.** [Ser55] *Let $X \subseteq \mathbb{P}^n_k$ be a projective variety. The category* $\mathrm{Coh}(X)$ *is a finite dimensional, abelian, noetherian category. If X is smooth of dimension n then*

$$\mathrm{gl.\,dim}(\mathrm{Coh}(X)) = n.$$

Note that if X is a smooth projective curve then $\mathrm{Coh}(X)$ is hereditary. The category $\mathrm{Coh}(X)$ is Krull-Schmidt, because it is finite dimensional and abelian.

Remark 3.17. This theorem shows an important difference between projective and affine varieties: the category of coherent sheaves is always finite dimensional if the variety is projective but for an affine variety this is the case if and only if it has Krull dimension 0.

In the case $X = \mathbb{P}^1_k$ we can classify the indecomposable objects.

Example 3.18. Let $S = k[Y_0, Y_1]$ with k algebraically closed. Then the indecomposable objects of $\mathrm{Coh}(\mathbb{P}^1_k)$ are

- $\mathcal{O}_{\mathbb{P}^1_k}(n) := \widetilde{S(n)}$, $n \in \mathbb{Z}$

- $\widetilde{S/p^s}$, where $p = \lambda Y_0 - \mu Y_1$, $(\lambda : \mu) \in \mathbb{P}^1$ and $s \in \mathbb{Z}_{>0}$.

Compare the indecomposable objects of $\mathrm{Coh}(\mathbb{P}^1_k)$ and $k[T]$-mod.

Functors on $\mathrm{Coh}(X)$**.** In the following we shall introduce the internal Hom-functor and the internal tensor product on the category of $\mathrm{Coh}(X)$, that is for $\widetilde{M} \in \mathrm{Coh}(X)$ we shall define the functors

$$\widetilde{M} \otimes_{\mathcal{O}_X} - : \mathrm{Coh}(X) \to \mathrm{Coh}(X) \quad \text{and} \quad \mathcal{H}om(\widetilde{M}, -) : \mathrm{Coh}(X) \to \mathrm{Coh}(X)$$

which will be induced by the tensor product and Hom-functor on the category of graded modules over the coordinate ring $k[X]$.

So let $R = k[X]$, $\mathcal{A} = \mathrm{grmod}(R)$ the category of finitely generated graded R-modules and $\mathcal{B} = \mathrm{grmod}_0(R)$ the category of finite dimensional graded R-modules. If M is in $\mathcal{A}$ then there is a functor

$$M \otimes_R - : \mathcal{A} \to \mathcal{A}.$$

Lemma 3.19. *The tensor functor $M \otimes_R -$ maps the set $S_{\mathcal{B}}$ of morphisms in $\mathcal{A}$ whose kernel and cokernel belong to $\mathcal{B}$ into itself.*

Proof. Let $f : N \to K$ be a morphism of graded modules such that $\ker(f)$ and $\mathrm{coker}(f)$ are finite dimensional. We shall show that then for every finitely generated module M the modules $\ker(1_M \otimes f)$ and $\mathrm{coker}(1_M \otimes f)$ are finite dimensional.

The functor $M \otimes -$ is right exact, so $\mathrm{coker}(1_M \otimes f) \cong M \otimes \mathrm{coker}(f)$ is finite dimensional since the tensor product of a finite-dimensional module with a noetherian module is always finite dimensional. Consider the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
\ker(a) & \longrightarrow & M \otimes \ker(f) & \xrightarrow{a} & M \otimes N & \xrightarrow{1_M \otimes f} & M \otimes \mathrm{im}(f) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle s} & & \\
0 & \longrightarrow & \ker(1_M \otimes f) & \longrightarrow & M \otimes N & \xrightarrow{1_M \otimes f} & M \otimes K & \longrightarrow & M \otimes \mathrm{coker}(f).
\end{array}$$

Since $M \otimes \ker(f)$ is finite dimensional, the submodule $\ker(a)$ is finite dimensional, too. The map s enters in the short exact sequence:

$$\mathrm{Tor}^1(M, \mathrm{coker}(f)) \to M \otimes \mathrm{im}(f) \xrightarrow{s} M \otimes K \to M \otimes \mathrm{coker}(f) \to 0.$$

Note that a graded noetherian R-module L is finite dimensional if and only if for all $\mathfrak{m} \in \mathrm{grMax}(R)$ it holds $L_{\mathfrak{m}} = 0$. Since the localization functor is exact and commutes with tensor products, it also commutes with Tor^i and we may conclude that $\mathrm{Tor}^1(M, \mathrm{coker}(f))$ is finite dimensional. The canonical functor

$$\mathrm{grmod}(R) \to \mathrm{grmod}(R) / \mathrm{grmod}_0(R)$$

is exact by 3.13 , so we know that $s : M \otimes \mathrm{im}(f) \to M \otimes K$ is an isomorphism in $\mathrm{grmod}(R)/\mathrm{grmod}_0(R)$. Projecting the commutative diagram above to $\mathrm{grmod}(R)/\mathrm{grmod}_0(R)$ and using the 5-lemma, we conclude that t is an isomorphism in the Serre quotient. By [Gab62] it means that $\ker(t)$ and $\mathrm{coker}(t)$ are finite dimensional and hence $\ker(1_M \otimes f)$ is finite dimensional, too. □

Using the universal property of the Serre quotient $\mathcal{A}/\mathcal{B} = \mathcal{A}[S_{\mathcal{B}}^{-1}]$ we obtain:

Corollary 3.20. *There is a unique endofunctor* $\widetilde{M} \otimes_{\mathcal{O}_X} -$ *on* $\mathrm{Coh}(X)$ *making the following diagram commutative*

$$\begin{array}{ccc} \mathrm{grmod}(R) & \longrightarrow & \mathrm{Coh}(X) \\ {\scriptstyle M\otimes_R -}\downarrow & & \downarrow{\scriptstyle \widetilde{M}\otimes_{\mathcal{O}_X} -} \\ \mathrm{grmod}(R) & \longrightarrow & \mathrm{Coh}(X). \end{array}$$

In a similar way, for $M \in \mathcal{A}$ the functor

$$\begin{array}{rcl} \mathrm{gr}\,\mathrm{Hom}(-, M) : \mathcal{A}^{\mathrm{op}} & \to & \mathcal{A} \\ N & \mapsto & \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}(N, M(n)) \end{array}$$

induces $\mathcal{H}om(-, \widetilde{M}) : \mathrm{Coh}(X)^{\mathrm{op}} \to \mathrm{Coh}(X)$. Analogously one can construct $\mathcal{H}om(\widetilde{M}, -)$.

Remark 3.21. The functor $\mathcal{H}om(\widetilde{M}, -) : \mathrm{Coh}(X)^{\mathrm{op}} \to \mathrm{Coh}(X)$ should not be mixed with the categorical Hom-functor $\mathrm{Hom}_{\mathrm{Coh}(X)}(\tilde{M}, -) : \mathrm{Coh}(X) \to \mathrm{Vect}_k$.

At the end of this section on coherent sheaves on projective varieties, we take a closer look at localization and an important class of coherent sheaves, namely the locally free ones and discuss some examples.

Definition 3.22. Let R be a graded domain and $\mathfrak{m} \in \mathrm{grMax}(R)$ a maximal ideal. Define the localization

$$R_{\mathfrak{m}} := \left\{ \frac{f}{g} | f \in R, g \in R \setminus \mathfrak{m},\ \ deg(f) = deg(g) \right\}.$$

The localization $R_{\mathfrak{m}}$ is a local non-graded ring. We have a functor

$$\mathrm{grmod}(R) \to \mathrm{mod}(R_{\mathfrak{m}})$$

which induces a functor

$$\mathrm{Coh}(X) \to \mathrm{mod}(R_{\mathfrak{m}}) \qquad \mathcal{F} \mapsto \mathcal{F}_{\mathfrak{m}}.$$

Let $\mathfrak{m}_x \in \mathrm{grMax}(R)$ be the maximal ideal corresponding to the point $x \in X$ according to the Hilbert Nullstellensatz 3.6.

Definition 3.23. Let $\mathcal{F}$ be a coherent sheaf over a projective variety X.

- If $x \in X$ then the localized sheaf $\mathcal{F}_{\mathfrak{m}_x} =: \mathcal{F}_x$ is called the *stalk* of $\mathcal{F}$ at x.
- The sheaf $\mathcal{F}$ is called *locally free* if for all $x \in X$ the stalk $\mathcal{F}_x$ is a free $R_{\mathfrak{m}_x}$-module.

Examples of coherent sheaves. Let X be a projective curve, $X \subset \mathbb{P}_k^n$ and $R = R_X = k[X]$ its coordinate ring. Remember, that for an R-module M the image under

$$\mathrm{grmod}(k[X]) \to \mathrm{grmod}(k[X]) \big/ \mathrm{grmod}_0(k[X]) = \mathrm{Coh}(X)$$

is denoted by $\widetilde{M}$. There are examples of coherent sheaves:

- The structure sheaf $\mathcal{O}_X = \mathcal{O} = \widetilde{R}$
- Let $x \in X$ and $\mathfrak{m}_x$ be the corresponding maximal ideal in R. The structure sheaf at a point x is $k(x) := \widetilde{R/\mathfrak{m}_x}$. The ideal sheaf of x is defined as $\mathcal{J}_x := \widetilde{\mathfrak{m}_x}$. We have a short exact sequence in $\mathrm{Coh}(X)$

  $$0 \to \mathcal{J}_x \to \mathcal{O}_X \to k(x) \to 0.$$

  If X is a smooth curve then $\mathcal{O}(-x) := \mathcal{J}_x$ is locally free.
- The sheaf $\mathcal{O}(x) = \mathcal{H}om(\mathcal{O}(-x), \mathcal{O}_X)$ occurs as the middle term in a short exact sequence

  $$0 \to \mathcal{O} \to \mathcal{O}(x) \to k(x) \to 0.$$

4. Coherent sheaves on an elliptic curve

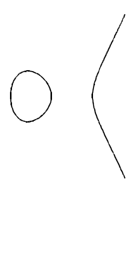
In this section we shall restrict ourselves to a special class of projective varieties called elliptic curves, and describe all indecomposable coherent sheaves on them. So let $X \subseteq \mathbb{P}_k^n$ be a projective variety, $R = k[X]$ its coordinate ring and $\mathrm{Coh}(X)$ the category of coherent sheaves over X. We know that $\mathrm{Coh}(X)$ is abelian, noetherian, finite dimensional Krull-Schmidt category. So in order to classify the coherent sheaves we can confine ourselves to a description of the indecomposable ones. If X is in addition a smooth curve we know by 3.16 that $\mathrm{Coh}(X)$ is hereditary. The Calabi-Yau property of $\mathrm{Coh}(X)$, if X is an elliptic curve will be essential for the classification of indecomposable objects.

Definition 4.1. An *elliptic curve* over a field k is a smooth plain projective curve of degree 3 given by a homogeneous polynomial

$$P_3(x, y, z) = 0 \subset \mathbb{P}_k^2$$

and having a solution over k.

We have seen examples of elliptic curves given by the equation $zy^2 = 4x^3 - axz^2 - bz^3$, where $\Delta := a^3 - 27b^2 \neq 0$, which looks in the affine chart $z \neq 0$ like:



Calabi-Yau property. If X is an elliptic curve, then there is an important isomorphism between the Hom and the Ext functors: let $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves over X, then there is a bifunctorial isomorphism

$$\mathrm{Hom}(\mathcal{F},\mathcal{G}) \cong \mathrm{Ext}^1(\mathcal{G},\mathcal{F})^*.$$

A hereditary abelian category $\mathcal{A}$ satisfying the property above is sometimes called *Calabi-Yau* category of dimension 1.

There is a classification result by Reiten and van den Bergh [RVdB02, Theorem C] that characterizes hereditary finite dimensional noetherian abelian categories with Serre-duality. In our situation it reduces to:

Theorem 4.2 (Reiten, van den Bergh)**.** *Let k be an algebraically closed field and $\mathcal{A}$ an abelian, noetherian, hereditary, indecomposable Calabi-Yau category with finite dimensional Hom-spaces. Then $\mathcal{A}$ is equivalent to the category of nilpotent representations of the quiver with one vertex and one loop:*

$$\mathcal{A} \simeq \mathrm{Nil}(\bullet \circlearrowleft)$$

or $\mathcal{A}$ is equivalent to the category of coherent sheaves on an elliptic curve X:

$$\mathcal{A} \simeq \mathrm{Coh}(X).$$

Let us now introduce some invariants of coherent sheaves on an elliptic curve that play a key role in the classification.

Definition 4.3. For $\mathcal{E}$ and $\mathcal{F}$ in $\mathrm{Coh}(X)$ the *Euler form* of $\mathcal{E}$ and $\mathcal{F}$ is defined as

$$\langle \mathcal{E},\mathcal{F}\rangle := \dim_k \mathrm{Hom}(\mathcal{E},\mathcal{F}) - \dim_k \mathrm{Ext}^1(\mathcal{E},\mathcal{F}).$$

The *Euler characteristic* of $\mathcal{F}$ is by the definition $\chi(\mathcal{F}) := \langle \mathcal{O},\mathcal{F}\rangle$.

We have the following properties:

- The Euler characteristic is additive in the following sense: if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is exact then:

$$\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'').$$

- Let $\mathcal{F} \in \mathrm{Coh}(X)$, $x \in X$ and $\mathfrak{m}_x \in \mathrm{grMax}(R)$ then the following functors are exact

$$\begin{array}{ccccc} \mathrm{Coh}(X) & \to & R_{\mathfrak{m}_x}\text{-mod} & \to & \mathrm{Vect}_{\mathrm{Quot}(R_{\mathfrak{m}_x})} \\ \mathcal{F} & \mapsto & \mathcal{F}_x & \mapsto & \mathcal{F}_x \otimes_{R_{\mathfrak{m}_x}} \mathrm{Quot}(R_{\mathfrak{m}_x}). \end{array}$$

Definition 4.4. The *rank* of a coherent sheaf $\mathcal{F}$ is defined as

$$\mathrm{rk}(\mathcal{F}) := \dim_{\mathrm{Quot}(R_{\mathfrak{m}_x})}(\mathcal{F}_x \otimes_{R_{\mathfrak{m}_x}} \mathrm{Quot}(R_{\mathfrak{m}_x})).$$

The following lemma assures that the rank is well-defined.

Lemma 4.5. *The rank of a coherent sheaf does not depend on the choice of the point $x \in X$.*

The rank is also additive with respect to short exact sequences (compare with the Euler characteristic). In fact we get

$$(\chi, \mathrm{rk}) : K_0(\mathrm{Coh}(X)) \to \mathbb{Z}^2,$$

where K_0 denotes the Grothendieck group. The quotient of the Euler characteristic by the rank will be useful in the classification of the coherent sheaves.

Definition 4.6. The *slope* of a coherent sheaf $\mathcal{F}$ is an element in $\mathbb{Q} \cup \infty$ defined as

$$\mu(\mathcal{F}) = \frac{\chi(\mathcal{F})}{\mathrm{rk}(\mathcal{F})}.$$

Lemma 4.7. *If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is exact then exactly one of the following assertions is true:*

- $\mu(\mathcal{F}') < \mu(\mathcal{F}) < \mu(\mathcal{F}'')$
- $\mu(\mathcal{F}') > \mu(\mathcal{F}) > \mu(\mathcal{F}'')$
- $\mu(\mathcal{F}') = \mu(\mathcal{F}) = \mu(\mathcal{F}'')$.

Definition 4.8. A coherent sheaf $\mathcal{F}$ is called *stable* (or *semi-stable*) if for any non-trivial exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ holds $\mu(\mathcal{F}') < \mu(\mathcal{F})$ ($\mu(\mathcal{F}') \leq \mu(\mathcal{F})$).

Proposition 4.9. *Let X be a smooth projective curve over a field k.*

- *If two semi-stable coherent sheaves $\mathcal{E}, \mathcal{F} \in \mathrm{Coh}(X)$ satisfy $\mu(\mathcal{E}) > \mu(\mathcal{F})$ then $\mathrm{Hom}(\mathcal{E}, \mathcal{F}) = 0$.*
- *If $\mathcal{F}$ is stable then $\mathrm{End}(\mathcal{F}) = K$ for a finite field extension $k \subset K$.*
- *Let $\mu \in \mathbb{Q} \cup \infty$ then the full subcategory*

  $$SS_X^{\mu} = \{\text{semi-stable coherent sheaves of slope } \mu\}$$

  is abelian and artinian.

Proof. We only show the first assertion: let $\mathcal{E}$ and $\mathcal{F}$ be coherent sheaves such that $\mu(\mathcal{E}) > \mu(\mathcal{F})$. Let $f : \mathcal{E} \to \mathcal{F}$ be a morphism which is not zero. There are exact sequences

$$\mathcal{E} \to \mathrm{im}(f) \to 0 \qquad 0 \to \mathrm{im}(f) \to \mathcal{F}.$$

By 4.7 we have

$$\mu(\mathcal{E}) \leq \mu(\mathrm{im}(f)) \leq \mu(\mathcal{F})$$

which contradicts the assumption. □

The following theorem will be used to derive that all indecomposable coherent sheaves on an elliptic curve are semi-stable.

Theorem 4.10 (Harder-Narasimhan, Rudakov)**.** [HN75] *Let X be a projective curve, then for a given $\mathcal{F} \in \mathrm{Coh}(X)$ there is a unique filtration:*

$$\mathcal{F} = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots \supset \mathcal{F}_n \supset \mathcal{F}_{n+1} = 0$$

such that

- *$\mathcal{A}_i := \mathcal{F}_i / \mathcal{F}_{i+1}$ for $0 \leq i \leq n$ are semi-stable and*
- *$\mu(\mathcal{A}_0) < \mu(\mathcal{A}_1) < \cdots < \mu(\mathcal{A}_n)$.*

In the sequel, we shall abbreviate the filtration of this theorem by HNF (Harder-Narasimhan-filtration) and call n the length of the filtration.

Corollary 4.11. *Let X be an elliptic curve and $\mathcal{F} \in \mathrm{Coh}(X)$ indecomposable. Then $\mathcal{F}$ is semi-stable.*

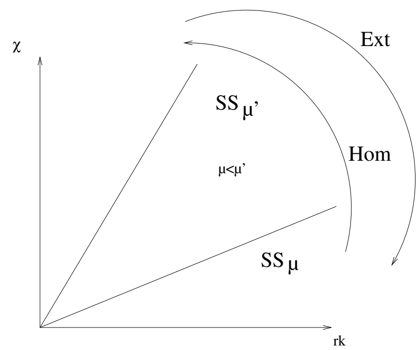
Proof. Assume $\mathcal{F}$ is indecomposable but not semi-stable. If the HNF has length 1 we get a short exact sequence

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{A}_0 \to 0$$

such that $\mathcal{F}_1$ and $\mathcal{A}_0$ are semi-stable. By the second property of the HNF we know that $\mu(\mathcal{A}_0) < \mu(\mathcal{F}_1)$. The Calabi-Yau property of $\mathrm{Coh}(X)$ yields that $\mathrm{Ext}^1(\mathcal{A}_0, \mathcal{F}_1) \cong \mathrm{Hom}(\mathcal{F}_1, \mathcal{A}_0)^*$. By 4.9 we know that $\mathrm{Hom}(\mathcal{F}_1, \mathcal{A}_0) = 0$. Hence $\mathrm{Ext}^1(\mathcal{A}_0, \mathcal{F}_1) = 0$ and the short exact sequence above splits. So, $\mathcal{F} \cong \mathcal{F}_1 \oplus \mathcal{A}_0$ which is a contradiction. The general argument is an induction on the length of the HNF. $\square$

Corollary 4.12. *If X is an elliptic curve then*

$$\mathrm{Coh}(X) = \bigcup_{\mu \in \mathbb{Q} \cup \infty} SS_X^{\mu}.$$



To sum up, we can decompose every coherent sheaf on an elliptic curve into a direct sum of semi-stable ones. So the classification boils down to a description of semi-stable sheaves.

We shall see that

$$SS_X^{\mu} \cong SS_X^{\mu'} \cong SS_X^{\infty} \qquad \forall \mu, \mu' \in \mathbb{Q},$$

where SS_X^{∞} is the category of torsion sheaves. In order to prove this fact we have to introduce some preliminaries.

Theorem 4.13 (Riemann-Roch formula)**.** *Let X be an elliptic curve and $\mathcal{E}$ and $\mathcal{F}$ coherent sheaves over X. Then*

$$\langle \mathcal{E}, \mathcal{F} \rangle = \chi(\mathcal{F}) \operatorname{rk}(\mathcal{E}) - \chi(\mathcal{E}) \operatorname{rk}(\mathcal{F}).$$

Remark 4.14. By the Calabi-Yau property we have

$$\langle \mathcal{E}, \mathcal{F} \rangle = -\langle \mathcal{F}, \mathcal{E} \rangle.$$

Recall that the *left radical* (respectively *right radical*) is the set

$$\operatorname{l.rad}\langle -,-\rangle = \{\mathcal{F} \in K_0(\operatorname{Coh}(X)) | \langle \mathcal{F}, -\rangle = 0\}$$

(respectively $\operatorname{r.rad}\langle -,-\rangle = \{\mathcal{F} \in K_0(\operatorname{Coh}(X)) | \langle -, \mathcal{F}\rangle = 0\}$).

Corollary 4.15. *From the Riemann-Roch formula it follows that the right and the left radical coincide and are both equal to:*

$$\operatorname{l.rad}\langle -,-\rangle = \operatorname{r.rad}\langle -,-\rangle = \{\mathcal{F} \in K_0(\operatorname{Coh}(X)) | \operatorname{rk}(\mathcal{F}) = \chi(\mathcal{F}) = 0\}.$$

Example 4.16. Let $x_1, x_2 \in X$. Then

$$0 \neq [k(x_1)] - [k(x_2)] \in \operatorname{l.rad}\langle -,-\rangle.$$

If the left and the right radical of a bilinear form coincide we shall call it just the radical. As a consequence of the Corollary 4.15 we get an isomorphism

$$Z := \begin{pmatrix} \operatorname{rk} \\ \chi \end{pmatrix} : K_0(\operatorname{Coh}(X))\big/_{\operatorname{rad}\langle -,-\rangle} \to \mathbb{Z}^2.$$

The *charge* of a sheaf $\mathcal{E}$ is defined to be $Z(\mathcal{E})$.

Remark 4.17. The map Z is surjective since

$$Z(\mathcal{O}_X) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad Z(k(x)) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The group of exact auto-equivalences $\operatorname{Aut}(\mathcal{D}^b(\operatorname{Coh}(X)))$ of the bounded derived category of coherent sheaves on X acts on $K_0(\operatorname{Coh}(X))$ by automorphisms, since K_0 is actually an invariant of the derived category. This action preserves the Euler form and its radical. Hence, we get a group homomorphism

$$\pi : \operatorname{Aut}(\mathcal{D}^b(\operatorname{Coh}(X))) \to \operatorname{SL}_2(\mathbb{Z})$$

which sends an auto-equivalence f to the upper map in the following commutative diagram:

$$\begin{array}{ccc} \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^2 \\ {\scriptstyle Z^{-1}}\downarrow\cong & & \cong\uparrow{\scriptstyle Z} \\ K_0(\operatorname{Coh}(X))\big/_{\operatorname{rad}\langle -,-\rangle} & \xrightarrow[\cong]{f_*} & K_0(\operatorname{Coh}(X))\big/_{\operatorname{rad}\langle -,-\rangle} \end{array}$$

which is in $\operatorname{SL}_2(\mathbb{Z})$. In the sequel we show that π is surjective by defining two derived equivalences $\mathbb{A}$ and $\mathbb{B}$ which are mapped under π to the matrices $A := \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ respectively $B := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

There is an evaluation map in $\mathcal{D}^b(\operatorname{Coh}(X))$

$$\operatorname{ev}_{\mathcal{F}} : \operatorname{R}\operatorname{Hom}(\mathcal{E}, \mathcal{F}) \otimes^{\operatorname{L}} \mathcal{E} \to \mathcal{F}.$$

Definition 4.18. For a coherent sheaf $\mathcal{E}$ on a projective variety X the *twist functor* is defined as

$$\begin{array}{rcl} T_{\mathcal{E}} : \mathcal{D}^b(\operatorname{Coh}(X)) & \to & \mathcal{D}^b(\operatorname{Coh}(X)) \\ \mathcal{F} & \mapsto & \operatorname{cone}(\operatorname{ev}_{\mathcal{F}}) \end{array}$$

Remark 4.19. If $\mathcal{E}$ and $\mathcal{F}$ are coherent sheaves on a projective curve X such that $\mathrm{Ext}^1(\mathcal{E},\mathcal{F})=0$ then $T_{\mathcal{E}}(\mathcal{F})$ is isomorphic to the complex

$$0 \to \mathrm{Hom}(\mathcal{E},\mathcal{F}) \overset{k}{\otimes} \mathcal{E} \overset{\mathrm{ev}_{\mathcal{F}}}{\to} \mathcal{F} \to 0.$$

In particular, if $\mathrm{ev}_{\mathcal{F}}$ is injective then $T_{\mathcal{E}}(\mathcal{F}) \cong \mathrm{coker}(\mathrm{ev}_{\mathcal{F}})$ and if $\mathrm{ev}_{\mathcal{F}}$ is surjective then $T_{\mathcal{E}}(\mathcal{F}) \cong \ker(\mathrm{ev}_{\mathcal{F}})[1]$

The following theorem shows that this notion is useful:

Theorem 4.20 (Seidel-Thomas)**.** [ST01, 2b] *The assignment $T_{\mathcal{E}}$ is a triangle functor.*

By definition there is a triangle

$$\mathrm{R}\,\mathrm{Hom}(\mathcal{E},\mathcal{F}) \otimes^{\mathrm{L}} \mathcal{E} \xrightarrow{ev} \mathcal{F} \to T_{\mathcal{E}}(\mathcal{F}) \to \Sigma(\mathrm{R}\,\mathrm{Hom}(\mathcal{E},\mathcal{F}) \otimes^{\mathrm{L}} \mathcal{E}).$$

Hence in $K_0(\mathrm{Coh}(X))$ it holds:

$$T_{\mathcal{E}} : [\mathcal{F}] \mapsto [\mathcal{F}] - \langle \mathcal{E},\mathcal{F}\rangle[\mathcal{E}].$$

The following theorem says when a twist functor is an equivalence.

Theorem 4.21 (Seidel-Thomas, Lenzing-Meltzer)**.** [ST01, Proposition 2.10] [LM00] *Let X be an elliptic curve. If a coherent sheaf $\mathcal{E}$ is endo-simple, i.e.* $\mathrm{End}(\mathcal{E}) = k$*, then the twist functor $T_{\mathcal{E}}$ is an exact equivalence of* $D^b(\mathrm{Coh}(X))$*.*

Example 4.22. The sheaves $\mathcal{O}_X$ and $k(x)$ for a k-point $x \in X$ are endo-simple.

Lemma 4.23. [ST01, formula 3.11] *There is an isomorphism of functors: $T_{k(x)} \cong \mathcal{O}(x) \otimes -$.*

So by the Theorem 4.21 we know that $\mathbb{A} := T_{\mathcal{O}}$ and $\mathbb{B} := T_{k(x)}$ are autoequivalences. Let $A := \pi(\mathbb{A})$ and $B := \pi(\mathbb{B})$ be the images in $\mathrm{SL}_2(\mathbb{Z})$. Now we shall determine the matrices A and B by looking at the action of maps $\mathbb{A}$ and $\mathbb{B}$ in the Grothendieck group:

$$T_{\mathcal{O}} : [\mathcal{O}] \mapsto [\mathcal{O}] - \langle \mathcal{O},\mathcal{O}\rangle[\mathcal{O}] = [\mathcal{O}]$$

and

$$T_{\mathcal{O}} : [k(x)] \mapsto [k(x)] - \langle \mathcal{O},k(x)\rangle[\mathcal{O}] = [k(x)] - [\mathcal{O}].$$

On the other hand

$$T_{k(x)} : [k(x)] \mapsto [k(x)] - \langle k(x),k(x)\rangle[k(x)] = [k(x)]$$

and

$$T_{k(x)} : [\mathcal{O}] \mapsto [\mathcal{O}] - \langle k(x),\mathcal{O}\rangle[k(x)] = [\mathcal{O}] + [k(x)].$$

Since $Z(\mathcal{O}_X) = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $Z(k(x)) = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$, it follows that $A = \left(\begin{smallmatrix}1 & -1\\0 & 1\end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix}1 & 0\\1 & 1\end{smallmatrix}\right)$. Since A and B generate $\mathrm{SL}_2(\mathbb{Z})$, we conclude that

$$\pi : \mathrm{Aut}(\mathcal{D}^b(\mathrm{Coh}(X))) \to \mathrm{SL}_2(\mathbb{Z})$$

is surjective.

Classification of the indecomposable coherent sheaves. Now we are able to classify the indecomposables in $\mathrm{Coh}(X)$.

Let $\mathcal{F}$ be indecomposable in $\mathrm{Coh}(X)$ of charge $Z(\mathcal{F}) = \binom{r}{d} \in \mathbb{Z}^2$ and if $d \neq 0$ let $n = \gcd(r, d)$ be the greatest common divisor. If $d \neq 0$ then there is a matrix $F \in \mathrm{SL}_2(\mathbb{Z})$ with $F\binom{r}{d} = \binom{0}{n}$. If $d = 0$ let F be the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbb{Z})$ which flips the coordinate axes. Since $\pi : \mathrm{Aut}(\mathcal{D}^b(\mathrm{Coh}(X))) \to \mathrm{SL}_2(\mathbb{Z})$ is surjective we can lift F to an auto-equivalence $\mathbb{F} \in \mathrm{Aut}(\mathcal{D}^b(\mathrm{Coh}(X)))$. Since $\mathbb{F}$ is an equivalence of categories the image $\mathbb{F}(\mathcal{F})$ is indecomposable again. The object $\mathbb{F}(\mathcal{F})$ is then a shift of a coherent sheaf of rank 0 because $\mathrm{Coh}(X)$ is hereditary. Therefore $\mathbb{F}(\mathcal{F})$ is a torsion sheaf $\mathcal{T}$ and $\mathcal{F} = \mathbb{F}^{-1}(\mathcal{T})$. It means that any indecomposable coherent sheaf can be obtained from a torsion sheaf by applying an exact auto-equivalence of the derived category $\mathcal{D}^b(\mathrm{Coh}(X))$.

The indecomposable torsion sheaves are known: if $\mathfrak{m} \in \mathrm{grMax}(R)$ and $s > 0$ then

$$\mathcal{F} = \mathbb{F}^{-1}(\widetilde{R_X/_{\mathfrak{m}^s}}).$$

So we have:

Theorem 4.24 (Atiyah)**.** [Ati57, Theorem 7] *If $X \subset \mathbb{P}^2_k$ is an elliptic curve then an indecomposable coherent sheaf on X is uniquely determined by its charge in $\mathbb{Z}^2$ and one continuous parameter $\mathfrak{m} \in \mathrm{grMax}(R)$.*

Corollary 4.25. *The category $\mathrm{Coh}(X)$ is tame.*

Summary. Let X be an elliptic curve.

- The indecomposable objects of $\mathrm{Coh}(X)$ are semi-stable:
$$\mathrm{Coh}(X) = \bigcup SS_X^{\mu}$$
where $SS_X^{\mu} = \{\text{semi-stable sheaves of slope } \mu\}$.
- For $\mu, \mu' \in \mathbb{Q} \cup \{\infty\}$ there is an equivalence of abelian categories $SS_X^{\mu} \cong SS_X^{\mu'}$ induced by an auto-equivalence of $\mathcal{D}^b(\mathrm{Coh}(X))$.
- An indecomposable object in $\mathrm{Coh}(X)$ is determined by its charge and a maximal ideal $\mathfrak{m} \in \mathrm{grMax}(R)$.
- Let $\mathcal{F}$ and $\mathcal{F}'$ be indecomposable such that $Z(\mathcal{F}) = (r, \chi)$ and $Z(\mathcal{F}') = (r', \chi')$. If $\frac{\chi}{r} > \frac{\chi'}{r'}$ then $\mathrm{Hom}(\mathcal{F}, \mathcal{F}') = 0$ and $\dim \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}') = \chi r' - \chi' r$.
If $\frac{\chi}{r} < \frac{\chi'}{r'}$ then $\dim \mathrm{Hom}(\mathcal{F}, \mathcal{F}') = \chi' r - \chi r'$ and $\mathrm{Ext}^1(\mathcal{F}, \mathcal{F}') = 0$.

In the rest of these notes we shall investigate the group $\mathrm{Aut}(\mathcal{D}^b(\mathrm{Coh}(X)))$ and indicate that it is closely related to the braid group B_3 acting on 3 strands. At the end we shall cite Burban's and Schiffmann's theorem on the structure of the Hall algebra of $\mathrm{Coh}(X)$ which uses the braid group action on $\mathcal{D}^b(\mathrm{Coh}(X)))$.

Recall that $\mathbb{A} := T_{\mathcal{O}}$ and $\mathbb{B} = T_{k(x)}$ and their images under π are $A := \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$ respectively $B := \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. Note the following equalities:

- $ABA = BAB$;
- $(AB)^3 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

It turns out that these relations can be lifted to the group $\mathrm{Aut}(\mathcal{D}^b(\mathrm{Coh}(X)))$:

Theorem 4.26 (Mukai, Seidel-Thomas)**.** [ST01, Proposition 2.13]

- $\mathbb{ABA} \cong \mathbb{BAB}$
- $(\mathbb{AB})^3 \cong i^*[1]$ *for an involution* $i : X \to X$*.*

This theorem implies that the group $\langle \mathbb{A}, \mathbb{B} \rangle$ generated by $\mathbb{A}$ and $\mathbb{B}$ is isomorphic to $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$ which is a central extension of $\mathrm{SL}_2(\mathbb{Z})$:

$$0 \to \mathbb{Z} \to \widetilde{\mathrm{SL}_2(\mathbb{Z})} \to \mathrm{SL}_2(\mathbb{Z}) \to 1.$$

We shall illustrate Theorem 4.26 by showing $\mathbb{ABA}(\mathcal{O}) \cong \mathbb{BAB}(\mathcal{O})$. Recall that the twist functor $T_{\mathcal{E}}$ fits by definition in the triangle:

$$\mathrm{R\,Hom}(\mathcal{E}, \mathcal{F}) \otimes^{\mathrm{L}} \mathcal{E} \xrightarrow{ev} \mathcal{F} \to T_{\mathcal{E}}(\mathcal{F}) \to \Sigma(\mathrm{R\,Hom}(\mathcal{E}, \mathcal{F}) \otimes^{\mathrm{L}} \mathcal{E}). \qquad (*)$$

To compute the action of $\mathbb{ABA}$ on $\mathcal{O}$ we have to compute the triangle $(*)$. The functor $\mathbb{A}$ maps $\mathcal{O}$ to itself since the triangle $(*)$ for $\mathcal{E} = \mathcal{F} = \mathcal{O}$ is isomorphic to

$$\mathcal{O} \oplus \mathcal{O}[-1] \xrightarrow{\binom{\mathrm{id}}{w}} \mathcal{O} \to \mathcal{O} \xrightarrow{+}$$

where $w \in \mathrm{Ext}^1(\mathcal{O}, \mathcal{O})$. The functor $\mathbb{B}$ is given by tensoring with the sheaf $\mathcal{O}(x)$ by 4.23. Hence, $\mathbb{B}(\mathcal{O}) = \mathcal{O}(x)$. Since there is a short exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(x) \to k(x) \to 0$$

and $\mathrm{Ext}^1(\mathcal{O}, \mathcal{O}(x)) = 0$ we find that $\mathbb{A}(\mathcal{O}(x)) \cong k(x)$. So we have shown:

$$\mathcal{O} \xmapsto{\mathbb{A}} \mathcal{O} \xmapsto{\mathbb{B}} \mathcal{O}(x) \xmapsto{\mathbb{A}} k(x).$$

Similar computations show that

$$\mathcal{O} \xmapsto{\mathbb{B}} \mathcal{O}(x) \xmapsto{\mathbb{A}} k(x) \xmapsto{\mathbb{B}} k(x).$$

Hence $\mathbb{ABA}(\mathcal{O}) \cong \mathbb{BAB}(\mathcal{O})$.

Remark 4.27. Some versions of twist functors appeared in the literature under different names:

name	appearance
shrinking functors	Ringel, tubular algebras [Rin84]
reflection functors	Mukai, $K3$-surfaces
tubular mutations	Lenzing-Meltzer [LM00]
monodromy transformations	mirror symmetry
Fourier-Mukai transformations	algebraic geometry

Let X be an elliptic curve over a finite field $k = F_q$. As we have seen, the category of coherent sheaves Coh(X) is a finite dimensional abelian hereditary category over k. By a general construction of Ringel [Rin90] one can attach to X an associative algebra $H(\mathrm{Coh}(X))$ over the field $K = Q(\sqrt{q})$. As a K- vector space

$$H(\mathrm{Coh}(X)) = \bigoplus_{F \in \mathrm{I}so(\mathrm{Coh}(X))} K[F].$$

and for $F, G \in \mathrm{Iso}(\mathrm{Coh}(X))$ the product $*$ is defined as

$$[F] * [G] = \sqrt{\frac{|\mathrm{Hom}(F,G)|}{|\mathrm{Ext}^1(F,G)|}} \sum_{H \in \mathrm{Iso}(\rho A)} P^H_{F,G}\,[H],$$

where $P^H_{F,G} = \frac{1}{a_F \cdot a_G} |\{0 \to G \to H \to F \to 0\}|$ is the "orbifold" number of short exact sequences with end terms F and G and the middle term H, and a_X denotes the order of the automorphism ring of the object X.

However, for applications it is more convenient to consider the extended Hall algebra

$$\widetilde{H}(\mathrm{Coh}(X) := H(\mathrm{Coh}(\mathbb{X})) \otimes K[Z^2]$$

where $Z^2 = K(X)/\operatorname{rad}\langle -, - \rangle$. The main reason to enlarge the Hall algebra $H(\mathrm{Coh}(X))$ to $\widetilde{H}(\mathrm{Coh}(X))$ is that due to a work of Green [Gre94] the algebra $\widetilde{H}(\mathrm{Coh}(X))$ carries a natural bialgebra structure.

It turns out that our approach to coherent sheaves on elliptic curves via derived categories gives a proper tool to study properties of the extended Hall algebra $\widetilde{H}(\mathrm{Coh}(X))$.

Theorem 4.28 (Burban-Schiffmann)**.** [BS] *Let X be an elliptic curve over the field $k = \mathbb{F}_q$. Then the group $\mathrm{Aut}(\mathcal{D}^b(\mathrm{Coh}(X)))$ acts on the reduced Drinfeld double of the extended Hall algebra $\widetilde{H}(\mathrm{Coh}(X))$ by algebra homomorphisms.*

This symmetry allows to construct a certain natural subalgebra $U(X) \subset D\widetilde{H}(\mathrm{Coh}(X))$ and show that $U(X)$ is a flat deformation of the ring

$$Q(\sqrt{q})[s^{\pm 1}, t^{\pm 1}][x_1^{\pm 1}, \ldots, x_n^{\pm 1}, \ldots, y_1^{\pm 1}, \ldots, y_m^{\pm 1}, \ldots]^{Sym_\infty}$$

of symmetric Laurent series.

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