Du Val Singularities
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1 Introduction

We consider quotient singularities $\mathbb{C}^2/G$, where $G \subseteq SL_2(\mathbb{C})$ is a finite subgroup. Since we have a finite ring extension $\mathbb{C}[[x, y]]^G \subseteq \mathbb{C}[[x, y]]$, the Krull dimension of $\mathbb{C}[[x, y]]^G$ is 2.

Note that $\mathbb{C}[[x, y]]^G = \mathbb{C}[[x + y, xy]] \cong \mathbb{C}[[x, y]]$. In order to get a kind of bijection between finite subgroups and quotient singularities we need the following definition.

Definition 1.1 Let $G \subseteq GL_n(\mathbb{C})$ be a finite subgroup. An element $g \in G$ is called pseudo-reflection if $g$ is conjugated to $\text{diag}(1, 1, \ldots, 1, \lambda)$, where $\lambda \neq 1$. A group $G$ is called small if it contains no pseudo-reflections.

Proposition 1.2 1. Let $G \subseteq GL_n(\mathbb{C})$ be a finite subgroup. Then

$$\mathbb{C}[[x_1, x_2, \ldots, x_n]]^G \cong \mathbb{C}[[x_1, x_2, \ldots, x_n]]^{G'},$$

where the group $G'$ is a certain small subgroup of $GL_n(\mathbb{C})$.

2. Let $G', G'' \subseteq GL_n(\mathbb{C})$ be two small subgroups. Then

$$\mathbb{C}[[x_1, x_2, \ldots, x_n]]^{G'} \cong \mathbb{C}[[x_1, x_2, \ldots, x_n]]^{G''}$$

if and only if $G'$ and $G''$ to are conjugated.

3. Let $G \subseteq GL_n(\mathbb{C})$ be a small finite subgroup. Then $\mathbb{C}[[x_1, x_2, \ldots, x_n]]^G$ is always Cohen-Macaulay.

4. Let $G \subseteq GL_n(\mathbb{C})$ be a small finite subgroup. Then $\mathbb{C}[[x_1, x_2, \ldots, x_n]]^G$ is Gorenstein if $G \subseteq SL_n(\mathbb{C})$.

Remark 1.3 Note that every subgroup of $SL_n(\mathbb{C})$ is small.

Now we want to answer the following question: what are finite subgroups of $SL_2(\mathbb{C})$ modulo conjugation?
2 Finite subgroups of $SL_2(\mathbb{C})$

Lemma 2.1 Every finite subgroup of $SL_n(\mathbb{C})$ ($GL_n(\mathbb{C})$) is conjugated to a subgroup of $SU(n)$ ($U(n)$).

Proof. Let $(\ , \ )$ be a hermitian inner product on $\mathbb{C}^n$. Define
\[ \langle u, v \rangle := \frac{1}{|G|} \sum_{g \in G} (gu, gv). \]
(Notice that in case $G \subseteq U(n)$ it holds $\langle \ , \ \rangle = (\ , \ )$. Then $\langle \ , \ \rangle$ is a new hermitian inner product on $\mathbb{C}^n$:

1. $\langle u, u \rangle \geq 0$.
2. $\langle u, u \rangle = 0$ implies $u = 0$.
3. $\langle u, v \rangle = \langle v, u \rangle$.

It holds moreover
\[ \langle hu, hv \rangle = \frac{1}{|G|} \sum_{g \in G} (ghu, ghv) = \langle u, v \rangle. \]

Hence $G$ is unitary with respect to $\langle \ , \ \rangle$. Moreover, $\langle \ , \ \rangle$ has an orthonormal basis and let
\[ S : (\mathbb{C}^n, (\ , \ )) \longrightarrow (\mathbb{C}^n, (\ , \ )) \]
be a map sending vectors of the chosen orthonormal basis of the left space to vectors of the canonical basis of the right one. It holds $(u, v) = (Su, Sv)$. We know that $\langle gu, gv \rangle = \langle u, v \rangle$ for all $g \in G$, $u, v \in \mathbb{C}^n$. Hence we get $(Su, Sv) = (Sgu, Sgv)$, or, setting $S^{-1}u$ and $S^{-1}v$ instead of $u$ and $v$
\[ (u, v) = (SgS^{-1}u, SgS^{-1}v) \]
for all $g \in G$ and $u, v \in \mathbb{C}^n$.

Now we want to describe all finite subgroups of $SU(2)$. Recall that
\[ SU(2) = \{ A \in SL_2(\mathbb{C}) | A^{-1} = A^* \} = \{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \ | |\alpha|^2 + |\beta|^2 = 1 \}. \]

So, from the topological point of view $SU(2) \cong S^3$.

Theorem 2.2 There is an exact sequence of group homomorphisms
\[ 1 \longrightarrow \mathbb{Z}_2 \longrightarrow SU(2) \longrightarrow \pi \longrightarrow SO(3) \longrightarrow 1. \]

More precisely,
\[ \ker(\pi) = \{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \}. \]

Topologically $\pi$ is the map $S^3 \xrightarrow{2:1} \mathbb{R}P^3$. 

2
From this theorem follows that there is a connection between finite subgroups of \( SU(2) \) and \( SO(3) \). The classification of finite isometry groups of \( \mathbb{R}^3 \) is a classical result of F. Klein (actually of Platon). There are the following finite subgroups of \( SO(3, \mathbb{R}) \):

1. A cyclic subgroup \( \mathbb{Z}_n \), generated, for instance, by
   \[
   \begin{pmatrix}
   \cos\left(\frac{2\pi}{n}\right) & -\sin\left(\frac{2\pi}{n}\right) & 0 \\
   \sin\left(\frac{2\pi}{n}\right) & \cos\left(\frac{2\pi}{n}\right) & 0 \\
   0 & 0 & 1
   \end{pmatrix}
   \]

2. Dihedral group \( D_n \), \( |D_n| = 2n \). It is the automorphism group of a prisma. It is generated by a rotation \( a \) and a reflection \( b \) which satisfy the relations
   \[ a^n = e, b^2 = e, (ab)^2 = e. \]
   Note that the last relation can be chosen as \( ba = a^{n-1}b \) or \( bab^{-1} = a^{-1} \).

3. Group of automorphisms of a regular tetrahedron \( T = A_4 \). Note that \( |T| = 12 \).
4. Group of automorphisms of a regular octahedron \( O = S_4 \), \( |O| = 24 \).
5. Group of automorphisms of a regular icosahedron \( I = A_5 \), \( |I| = 60 \).

Observe that all non-cyclic subgroups of \( SO(3) \) have even order.

**Lemma 2.3** \[
\begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}
\] is the only element of \( SU(2) \) of degree 2.

**Proof.** The proof is a simple computation.

\[
\begin{pmatrix}
\alpha & \beta \\
-\beta & \alpha
\end{pmatrix}^2 = \begin{pmatrix}
\alpha^2 - |\beta|^2 & \alpha\beta + \beta\bar{\alpha} \\
-\beta\alpha - \beta\bar{\alpha} & \bar{\alpha}^2 - |\beta|^2
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

implies \( \alpha^2 = \bar{\alpha}^2 \) or \( \alpha = \pm\bar{\alpha} \). Hence \( \alpha = x, x \in \mathbb{R} \) or \( \alpha = ix, x \in \mathbb{R} \). Moreover \( \alpha \neq 0 \) since otherwise must hold \( -|\beta|^2 = 1 \). In case \( \alpha \in \mathbb{R} \) we have \( \alpha\beta + \beta\bar{\alpha} = 2\alpha\beta \) hence \( \beta = 0 \) and \( \alpha = \pm 1 \). If \( a = ix \) is purely imaginary then \( \alpha^2 - |\beta|^2 = -|x|^2 - |\beta|^2 < 0 \).

Let \( G \subseteq SU(2) \) be a finite subgroup, \( \pi : SU(2) \to SO(3) \) the \( 2 : 1 \) surjection. Consider two cases.

1. \( |G| \) is odd. Then \( G \cap \mathbb{Z}_2 = \{e\} \) (no elements of order 2 in \( G \)). So \( ker(\pi) \cap G = \{e\} \) and \( G \to \pi(G) \) is an isomorphism. Hence \( G \) is cyclic.
2. \( |G| \) is even. The due to the Sylow’s theorem \( G \) contains a subgroup of order \( 2k \) and hence contains an element of order 2. But there is exactly one element of the second order in \( SU(2) \) (see the lemma above). Hence \( ker(\pi) \subseteq G \) and \( G = \pi^{-1}(\pi(G)) \). So, in this case \( G \) is the preimage of a finite subgroup of \( SO(3) \).
From what was said we get the full classification of finite subgroups of $SL_2(\mathbb{C})$ modulo conjugation.

1. A cyclic subgroup $\mathbb{Z}_k$. Let $g$ be its generator. $g^k = e$ implies
\[
g \sim \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix},
\]
where $\varepsilon$ is some primitive root of 1 of $k$-th order.

2. Binary dihedral group $D_n$, $|D_n| = 4n$. To find the generators of $D_n$ we have to know the explicit form of the map $\pi : SU(2) \to SO(3)$. Skipping all details we just write down the answer. $D_n = \langle a, b \rangle$ with relations
\[
\begin{cases}
a^n &= b^2 \\
b^4 &= e \\
b^{-1} &= a^{-1}
\end{cases}
\]
To be concrete,
\[
a = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}, \quad \varepsilon = \exp\left(\frac{\pi i}{n}\right), \quad b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

3. Binary tetrahedral group $T$, $|T| = 24$. $T = \langle \sigma, \tau, \mu \rangle$, where
\[
\sigma = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mu = \frac{1}{\sqrt{2}} \begin{pmatrix} \varepsilon^7 & \varepsilon^7 \\ \varepsilon^5 & \varepsilon \end{pmatrix}, \quad \varepsilon = \exp\left(\frac{2\pi i}{8}\right).
\]

4. Binary octahedral group $O$, $|O| = 48$. This group is generated by $\sigma, \tau, \mu$ as in the case of $T$ and by
\[
\kappa = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-7} \end{pmatrix}.
\]

5. Finally we have the binary icosahedral subgroup $I$, $|I| = 120$. $I = \langle \sigma, \tau \rangle$, where
\[
\sigma = -\begin{pmatrix} \varepsilon^3 & 0 \\ 0 & \varepsilon^2 \end{pmatrix}, \tau = \frac{1}{\sqrt{5}} \begin{pmatrix} -\varepsilon + \varepsilon^4 & \varepsilon^2 - \varepsilon^3 \\ \varepsilon^2 - \varepsilon^3 & \varepsilon - \varepsilon^4 \end{pmatrix}, \quad \varepsilon = \exp\left(\frac{2\pi i}{5}\right).
\]

**Remark 2.4** The problem of classification of finite subgroups of $GL_2(\mathbb{C})$ is much more complicated. Indeed, every finite subgroup $G \subseteq GL_2(\mathbb{C})$ can be embedded into $SL_3(\mathbb{C})$ via the group monomorphism
\[
GL_2(\mathbb{C}) \to SL_3(\mathbb{C}) \\
g \mapsto \begin{pmatrix} g & 0 \\ 0 & \omega(g) \end{pmatrix}.
\]
Finite subgroups of $GL_2(\mathbb{C})$ give main series of finite subgroups of $SL_3(\mathbb{C})$.

Now we can compute the corresponding invariant subrings.
3 Description of Du Val singularities

1. A cyclic subgroup $\mathbb{Z}_n = \langle g \rangle$, $g : x \mapsto \varepsilon x, y \mapsto \varepsilon^{-1} y, \varepsilon = \exp(\frac{2\pi i}{n})$. It is not difficult to see that $X = x^n, Y = y^n$ and $Z = xy$ generate the whole ring of invariants.

$$\mathbb{C}[[x, y]]_{\mathbb{Z}_n} = \mathbb{C}[[X, Y, Z]]/(XY - Z^n) \cong \mathbb{C}[[x, y, z]]/(x^2 + y^2 + z^n).$$

It is an equation of $A_{n-1}$-singularity.

2. Binary dihedral group $D_n$.

$$\sigma : \begin{cases} x \mapsto \varepsilon x \\ y \mapsto \varepsilon^{-1} y \end{cases}, \quad \tau : \begin{cases} x \mapsto -y \\ y \mapsto x \end{cases},$$

where $\varepsilon = \exp(\frac{2\pi i}{n})$. The set of invariant monomials is

$$F = x^{2n} + y^{2n}, H = xy(x^{2n} - y^{2n}), I = x^{2}y^{2}.$$  

They satisfy the relation

$$H^2 = x^{2}y^{2}(x^{4n} + y^{4n} - 2x^{2n}y^{2n}) = IF^2 - 4I^{2(n+1)}.$$ 

The standard computations show that

$$\mathbb{C}[[x, y]]_{D_n} \cong \mathbb{C}[[x, y, z]]/(x^2 + y^2 + z^{n+1}).$$

It is an equation of $D_{n+2}$-singularity.

3. It can be checked that for the groups $T, O, I$ we get singularities

(a) $E_6$: $x^2 + y^3 + z^4 = 0,$

(b) $E_7$: $x^2 + y^3 + yz^3 = 0,$

(c) $E_8$: $x^2 + y^3 + z^5 = 0.$

We get the following theorem:

**Theorem 3.1** Du Val singularities are precisely simple hypersurface singularities $A - D - E$.

We want now to answer our next question: what are minimal resolutions and dual graphs of Du Val singularities?

4 $A_1$-singularity

Consider the germ of $A_1$-singularity $S = V(x^2 + y^2 + z^2) \subset \mathbb{A}^3$. Consider the blow-up of this singularity.

$$\tilde{\mathbb{A}}^3 = \{((x, y, z), (u : v : w)) \in \mathbb{A}^3 \times \mathbb{P}^2 | xv = yu, xw = zu, yw = zu\}.$$
Take the chart $u \neq 0$, i.e. $u = 1$. We get

$$
\begin{align*}
    x &= x \\
    y &= xv \\
    z &= xw
\end{align*}
$$

What is $\tilde{S} = \pi^{-1}(S \setminus \{0\})$? Consider first $x \neq 0$ (it means that we are looking for $\pi^{-1}(S \setminus \{0\})$).

$$
    x^2 + x^2v^2 + x^2w^2 = 0, x \neq 0,
$$

or

$$
    1 + v^2 + w^2 = 0, x \neq 0.
$$

In order to get $\tilde{S}$ we should allow $x$ to be arbitrary. In this chart $\tilde{S}$ is a cylinder $V(1 + v^2 + w^2) \subset \mathbb{A}^3$. What is $\pi^{-1}(0)$? Obviously it is the intersection of $\tilde{S}$ with the exceptional plane $((0,0,0), (u : v : w))$. In this chart we just have to set $x = 0$ in addition to the equation of the surface $\tilde{S}$.

$$
\pi^{-1}(0) = \begin{cases} 
1 + v^2 + w^2 = 0 \\
    x = 0.
\end{cases}
$$

We see that $E = \pi^{-1}(0)$ is rational and since all 3 charts of $\tilde{S}$ are symmetric, we conclude that $E$ is smooth, so $E = \mathbb{P}^1$. Now we have to compute the selfintersection number $E^2$. We do it using the following trick.

Let $\pi : \tilde{X} \to X$ be the minimal resolution. It induces an isomorphism of fields of rational functions $\pi^* : \mathbb{C}(X) \to \mathbb{C}(\tilde{X})$. Let $f \in \mathbb{C}(\tilde{X})$ be a rational function. Then it holds:

$$(f).E = deg_E(\mathcal{O}_E \otimes \mathcal{O}_{\tilde{X}}(f)) = deg_E(\mathcal{O}_E) = 0.$$

In particular it holds for $f \in \mathfrak{m} \subseteq \mathcal{O}_{X,0}$:

$$(f \circ \pi).E = 0.$$  

Consider the function $y \in \mathcal{O}_{X,0}$. In the chart $u \neq 0$ we get $y \circ \pi = xu$. What is the vanishing set of $y$?

1. $x = 0$ is an equation of $E$.
2. $u = 0$ implies $v^2 + 1 = 0$ or $v = \pm i$.

Hence $(f) = E + C_1 + C_2$ and we have the following picture:
So, \((f).E = E^2 + C_1.E + C_2.E = E^2 + 2 = 0\). From it follows \(E^2 = -2\).

5 \(E_6\)-singularity

In this section we want to compute a minimal resolution and dual graph of \(E_6\)-singularity \(X = V(x^2 + y^3 + z^4) \subset \mathbb{A}^3\). Let \(\tilde{X} \xrightarrow{\pi} X\) be a minimal resolution, \(E = \bigcup E_i = \pi^{-1}(0)\) the exceptional divisor. In order to compute selfintersection numbers \(E_i^2\) we have to consider the map \(\pi^* : \mathbb{C}(X) \longrightarrow \mathbb{C}(\tilde{X})\). Let \(f \in \mathfrak{m}_X \subset \mathcal{O}_X\), then equalities \((f \circ \pi_i).E_i = 0\) will imply the selfintersection numbers of \(E_i\).

First step. Let \(X = V(x^2 + y^3 + z^4) \subset \mathbb{A}^3, f = x\). Consider the blow-up of \(\mathbb{A}^3\):

\[\tilde{\mathbb{A}}^3 = \{(x, y, z, (u : v : w)) \in \mathbb{A}^3 \times \mathbb{P}^2 | xv = yu, xw = zu, yw = zw\}\]

Take first the chart \(v \neq 0\) (i.e. \(v = 1\)). We get equations

\[
\begin{cases}
  x = yu \\
y = y \\
z = yw.
\end{cases}
\]

To get the equation of the strict transform of \(\tilde{X}_1\) we assume that \(y \neq 0\) and

\[y^2u^2 + y^3 + y^4w^4 = 0,
\]

or

\[u^2 + y + y^2w^4 = 0.
\]

In this chart \(\tilde{X}_1\) is smooth: Jacobi criterion implies

\[
\begin{cases}
  u = 0 \\
  1 + 2yw^4 = 0 \\
w^3y^2 = 0.
\end{cases}
\]

It is easy to see that this system has no solutions.

In the chart \(u = 1\) the strict transform \(\tilde{X}_1\) is again smooth. Consider finally the chart \(w = 1\).

\[
\begin{cases}
  x = zu \\
y = zv \\
z = z.
\end{cases}
\]
The strict transform is
\[ z^2 u^2 + z^3 v^3 + z^4 = 0, \]
or
\[ u^2 + z v^3 + z^2 = 0. \]

Jacobi criterium implies that this surface has a unique singular point \( u = 0, v = 0, z = 0 \), or in the global coordinates \(((0, 0, 0), (0 : 0 : 1))\). We see that this point indeed lies only in one of three affine charts of \( \tilde{X}_1 \).

We now need an equation of exceptional fibre. The exceptional fibre is by the definition the intersection \( \tilde{X}_1 \cap \{(0, 0, 0), (u : v : w)\} \). To get its local equation in the chart \( w = 1 \) we just have to set \( z = 0 \) in the equation of \( \tilde{X}_1 \).\n
\[ z = 0 \implies u = 0. \]
Hence we get
\[ E_0 = \{(0, 0, 0), (0 : v : 1)\} \cong \mathbb{A}^1. \]

Going to the other charts shows that \( E_0 \cong \mathbb{P}^1 \).
Finally, the function \( f \) in this chart gets the form \( f = z u \).

Agreement. Since the number of indices depends exponentially on the number of blowing-ups, we shall denote the local coordinates of all charts of all blowing-ups \( \tilde{X}_i \) by the letters \((x, y, z)\).

Second step. We have the following situation:
\[
\begin{align*}
\text{surface} & \quad \quad \quad \quad \quad \quad x^2 + zy^3 + z^2 = 0 \\
\text{function} & \quad \quad \quad \quad \quad \quad f = x z \\
\text{exceptional divisor} E_0 & \quad \quad \quad \quad \quad \quad x = 0, z = 0.
\end{align*}
\]

Consider again the blowing-up of this surface. It is easy to see that the only interesting chart is
\[
\begin{align*}
x & = y u \\
y & = y \\
z & = y v.
\end{align*}
\]

We get the strict transform
\[ y^2 u^2 + yv^3 + y^2 v^2 = 0, y \neq 0, \]
or
\[ u^2 + y^2 v + v^2 = 0. \]
Again \( y = 0, u = 0, v = 0 \) is the only singularity of the blown-up surface. The exceptional fibre of this blowing-up has two irreducible components: \( y = 0 \) implies \( u \pm iv = 0 \) (we call this components \( E_1^i \) and \( E_2^j \)).

What is the preimage (under preimage we mean its strict transform) of \( E_0 \)? \( x = 0, z = 0 \) implies \( u = 0, v = 0 \).
The function \( f = x z \) gets in this chart the form \( f = y^2 u v. \)
Third step. We have the following situation:

\[
\begin{align*}
\text{surface} & \quad x^2 + y^2 z + z^2 = 0 \\
\text{function} & \quad f = xy^2 z \\
\text{exceptional divisor } E_0 & \quad x = 0, z = 0 \\
\text{exceptional divisor } E_1 & \quad y = 0, x \pm i z = 0.
\end{align*}
\]

Let us consider the next blowing-up.

\[
\begin{align*}
x & = yu \\
y & = y \\
z & = yv.
\end{align*}
\]

The strict transform is

\[
y^2 u^2 + y^2 yv + y^2 v^2 = 0, y \neq 0,
\]

or

\[
u^2 + yv + v^2 = 0.
\]

It is an equation of $A_1$-singularity (and it means that we are almost done). The exceptional fibre consists again of two irreducible components $E'_1, E''_1$. They local equations are $y = 0, u \pm iv = 0$. The function $f$ is $uy^3v$. It is easy to see that the preimage of $E_0$ is $u = 0, v = 0$.

What about the preimage of $E_1$? Our surface lies in the affine chart $\mathbb{A}^3$ embedded into $\mathbb{A}^3 \times \mathbb{P}^2$ via the map $(y, u, v) \mapsto ((yu, y, yv), (u : 1 : v))$. But then the condition $y = 0$ would imply that the preimage of $E_1$ lies in the exceptional plane $((0, 0, 0)(u : 1 : v))$. But it can not be true! The solution of this paradox is that the preimage of $E_1$ lies in another coordinate chart.

Consider

\[
\begin{align*}
x & = x \\
y & = xu \\
z & = xv,
\end{align*}
\]
The strict transform is
\[ x^2 + x^2 u^2 x v + x^2 v^2 = 0, x \neq 0 \]
or
\[ 1 + x u^2 v + v^2 = 0. \]
The equations of the exceptional fibre \( E_2 \) in this chart are \( x = 0 \), what implies \( v = \pm i \). The preimage of \( E_1 \)
\[ \begin{cases} x \pm i z = 0 \\ y = 0 \end{cases} \]
is given by
\[ \begin{cases} x \pm i x v = 0 \\ x u = 0, x \neq 0, \end{cases} \]
hence
\[ \begin{cases} v = \pm i \\ u = 0 \\ x \quad \text{arbitrary} \end{cases} \]
In the picture it looks like:

\[ \begin{array}{c}
\text{\( E_1 \)}
\end{array} \]

It is easy to see that all intersections are transversal.

Fourth step. We have the following situation: there are two coordinate charts

\[ \begin{cases} \text{surface} & x^2 + yz + z^2 = 0 \\ \text{function} & f = xy^4 z \\ \text{exceptional divisor \( E_0 \)} & x = 0, z = 0 \\ \text{exceptional divisor \( E_2 \)} & y = 0, x \pm i z = 0. \end{cases} \]

\[ \begin{cases} \text{surface} & 1 + xy^2 z + z^2 = 0 \\ \text{function} & f = x^4 y^2 z \\ \text{exceptional divisor \( E_1 \)} & y = 0, z = \pm i \\ \text{exceptional divisor \( E_2 \)} & x = 0, z = \pm i. \end{cases} \]
Our next step is the blowing-up at the point \((0, 0, 0)\) in the first coordinate chart. Again, in order to get equations of the preimages of \(E_0\) and \(E_2\) we have to consider two coordinate charts.

\[
\begin{align*}
  x &= yu \\
  y &= y \\
  z &= yw.
\end{align*}
\]

The strict transform is a cylinder

\[
u^2 + v + v^2 = 0.
\]

The preimage of \(E_0\) is given by equations \(u = 0, v = 0\), the exceptional fibre \(E_3\) is given by \(u^2 + v + v^2 = 0, y = 0\), our function \(f = uy^6v\). In another chart we have

\[
\begin{align*}
  x &= x \\
  y &= xu \\
  z &= xv.
\end{align*}
\]

The strict transform is given by

\[
x^2 + x^2uv + x^2v^2 = 0, x \neq 0
\]

or

\[
1 + uv + v^2 = 0.
\]

The exceptional fibre \(E_3\) is given by \(1 + uv + v^2 = 0, x = 0\), the preimages of \(E'_2\) and \(E''_2\) are given by \(u = 0, v = \pm i, f = x^6u^4v\).

Hence our exceptional fibre \(E\) is given by the following configuration of projective lines:
The dual graph of this configuration is

Fifth step. We have to take into account three coordinate charts of a minimal resolution.

\[
\begin{align*}
\hat{X} : & \quad x^2 + z + z^2 = 0 \\
f : & \quad xy^4z \\
E_0 : & \quad x = 0, z = 0 \\
E_3 : & \quad y = 0, x^2 + yz + z^2 = 0 \\
\hat{X} : & \quad 1 + yz + z^2 = 0 \\
f : & \quad x^6y^2z \\
E_3 : & \quad x = 0, x^2 + yz + z^2 = 0 \\
E_2 : & \quad y = 0, z = \pm i \\
\hat{X} : & \quad 1 + xy^2z + z^2 = 0 \\
f : & \quad x^6y^2z \\
E_1 : & \quad y = 0, z = \pm i \\
E_2 : & \quad x = 0, z = \pm i.
\end{align*}
\]

Now we have to compute the divisor \( f \).

Let \( X \subset \mathbb{A}^3 \) be a normal surface, \( Y \subset X \) a closed curve, \( f \in \mathbb{C}(X) \) a rational function. Suppose that \( p \subset \mathbb{C}[X] \) is the prime ideal corresponding to \( Y \). Then \( \mathbb{C}[X]_p \) is a discrete valuation ring and

\[ \text{mult}_Y(f) = \text{val}_{\mathbb{C}[X]_p}(f). \]

1. Consider the first chart. Let \( f = xy^6z = 0 \). \( x = 0 \) implies \( z = 0 \) or \( z = -1 \). \( x = 0, z = 0 \) is an equation of \( E_0 \), \( x = 0, z = -1 \) is the strict transform \( C \) of the curve \( x = 0 \) in \( V(x^2 + y^3 + z^3) \subset \mathbb{A}^3 \).

What is the multiplicity of \( E_0 \)? The generator of the maximal ideal of the ring \( \mathbb{C}[x, y, z]/(x^2 + y^3 + z^3)_p \) is \( \hat{x} \) and \( \hat{x}^2 \sim \hat{z} \). Therefore \( \text{mult}_{E_0}(f) = 3 \).

\( y = 0 \) gives an equation of \( E_3 \). It is easy to see that \( \text{mult}_{E_3}(f) = 6 \). Note that the curve \( C \) has transversal intersection with \( E_3 \) at the point \( x = 0, y = 0, z = -1 \).

2. Consider the second chart. In this chart holds \( f = x^6y^4z = 0 \). \( z = 0 \) is impossible, \( x = 0 \) cut out the equation of \( E_3 \) and \( y = 0 \) equations of \( E''_3 \) and \( E''_2 \). The same computation as above shows that \( \text{mult}_{E_3}(f) = 6 \) (what is not surprise and makes us sure that we did not make a mistake in computations) and \( \text{mult}_{E'_2}(f) = \text{mult}_{E''_2}(f) = 4 \).
3. In the same way we obtain that \( \text{mult}_{E_1}(f) = \text{mult}_{E_1'}(f) = 2 \). Therefore we obtain:

\[
(f) = 6E_3 + 4(E_2' + E_2'') + 2(E_1' + E_1'') + 3E_0 + C.
\]

We have \( C.E_1 = 1 \), all other intersection numbers of \( C \) with irreducible components of \( E \) are zero. Intersection numbers of irreducible components are coded in the dual graph (which is \( E_6 \), see the picture above). The whole job was done in order to compute self-intersections.

\[
(f).E_0 = 6 + 3E_0^2 = 0 \implies E_0^2 = -2.
\]

In the same way we conclude that the other selfintersection numbers are \(-2\).

**Remark 5.1** Let \( X \) be a normal surface singularity, \( \pi : \hat{X} \rightarrow X \) its minimal resolution, \( E = \bigcup_{i=1}^n E_i = \pi^{-1}(0) \) the exceptional divisor. Suppose that \( \hat{X} \) is a good resolution, all \( E_i \cong \mathbb{P}^1 \) and \( E_i^2 = -2 \). Then \( X \) is a simple hypersurface singularity.

Indeed we know that the intersection matrix \( (E_i.E_j)_{i,j=1}^n \) is negatively definite. Let \( \Gamma \) be the dual graph of \( X \). Then the quadratic form given by intersection matrix coincide with the Tits form of the dual graph:

\[
Q(x_1, x_2, \ldots, x_n) = -2(\sum_{i=1}^n x_i^2 - \sum_{1 \leq i < j \leq n} a_{ij}x_ix_j),
\]

where \( a_{ij} \) is the number of arrows connecting vertices \( i \) and \( j \). From the theorem of Gabriel we know that \( Q \) is negatively definite (and quiver is representation finite) if and only if \( \Gamma = A - D - E \). Since our singularity is rational, it is taut and uniquely determined by its dual graph.

### 6 2-dimensional McKay correspondence

Recall that we defined Du Val singularities as quotient singularities \( \mathbb{C}[[x, y]]^G \), where \( G \subseteq SU(2) \) is some finite subgroup. A natural question is: are there any connections between the representation theory of \( G \) and geometry of the minimal resolution of a singularity? Let us recall some standard facts about representations of finite groups.

**Theorem 6.1 (Mashke)** Let \( G \) be a finite group. Then the category of \( \mathbb{C}[G] \)-modules is semi-simple.

This theorem means that any exact sequence of \( \mathbb{C}[G] \)-modules splits. In particular, every finite-dimensional \( \mathbb{C}[G] \)-module is injective and projective. But an
indecomposable projective module by a theorem of Krull-Schmidt is isomorphic to a direct summand of the regular module. Let

\[ C[G] \cong \bigoplus_{i=1}^{s} \Phi_i^{n_i} \]

be a direct sum decomposition of \( C[G] \). Then \( \Phi_1, \Phi_2, \ldots, \Phi_s \) the whole list of indecomposable \( C[G] \)-modules.

**Lemma 6.2** Let \( C[G] \cong \bigoplus_{i=1}^{s} \Phi_i^{n_i} \) be a decomposition of the regular module into a direct sum of indecomposable ones. Then it holds

\[ \dim_{\mathbb{C}}(\Phi_i) = n_i. \]

In particular, the following identity is true:

\[ \sum_{i=1}^{s} n_i^2 = |G|. \]

**Definition 6.3** Let \( G \) be a group, \((\Phi, V), \Phi : G \to \text{End}(V)\) its representation. The character of a representation \( \Phi \) is the function \( \chi_{\Phi} : G \to \mathbb{C} \) defined by the rule \( \chi_{\Phi}(g) = \text{Tr}(\Phi(g)) \).

**Remark 6.4**

1. It is easy to see that the character does not depend on the choice of a representative from the isomorphism class of a representation:

\[ \text{Tr}(\Phi(g)) = \text{Tr}(S^{-1}\Phi(g)S). \]

2. It holds:

\[ \chi_{\Phi \otimes \psi} = \chi_{\Phi} \chi_{\psi}. \]

\[ \chi_{\Phi \oplus \psi} = \chi_{\Phi} + \chi_{\psi}. \]

In other words, \( \chi \) defines a rings homomorphism from the Grothendieck ring of \( C[G] \) to \( \mathbb{C} \).

3. It holds:

\[ \chi_{\Phi}(h^{-1}gh) = \text{Tr}(\Phi(h^{-1}gh)) = \text{Tr}(\Phi(h)^{-1}\Phi(g)\Phi(h)) = \text{Tr}(\Phi(g)) = \chi_{\Phi}(g). \]

It means that \( \chi_{\Phi} \) is a central function, i.e. a function which is constant on conjugacy classes of \( G \).

**Theorem 6.5** A finite dimensional representation of a finite group \( G \) is uniquely determined by its character.
Idea of the proof. Let $\varphi, \psi$ be two central functions on $G$. Set

$$\langle \varphi, \psi \rangle := \frac{1}{|G|} \sum_{g \in G} \chi_{\varphi}(g)\overline{\chi_{\psi}(h)}.$$ 

It defines an hermitian inner product on the space of all central functions on $G$. The theorem follows from the fact that $\chi_{\varphi_1}, \chi_{\varphi_2}, \ldots, \chi_{\varphi_s}$ is an orthonormal basis of this vector space. Indeed, let $\Phi$ be any finite-dimensional representation of $G$. Then we know that

$$\Phi \cong \oplus \Phi_i^{m_i}.$$ 

Then it obviously holds $m_i = \langle \chi_{\varphi_i}, \chi_{\varphi_i} \rangle$.

**Corollary 6.6** The number of indecomposable representations of a finite group $G$ is equal to the number of its conjugacy classes.

**Definition 6.7 (McKay quiver)** Let $G \subseteq SU(2)$ be a finite subgroup, $\Phi_0, \Phi_1, \ldots, \Phi_s$ all indecomposable representations of $G$. Let $\Phi_0$ be the trivial representation, $\Phi_{\text{nat}}$ the natural representation (i.e. the representation given by the inclusion $G \subseteq SU(2)$). Define the McKay graph of $G$ as the following:

1. Vertices are indexed by $\Phi_1, \ldots, \Phi_s$ (we skip $\Phi_0$).
2. Let $\Phi_i \otimes \Phi_{\text{nat}} \cong \bigoplus_{j=0}^s \Phi_j^{a_{ij}}$. 

(or, the same

$$\chi_i \chi_{\text{nat}} = \sum_{i=0}^s a_{ij} \chi_j.$$ 

Then we connect vertices $\Phi_i$ and $\Phi_j$ by $a_{ij}$ vertices.

It is easy to see that $\Phi_0 \otimes \Phi_{\text{nat}} = \Phi_{\text{nat}}$.

**Remark 6.8** It holds $a_{ij} = a_{ji}$.

Indeed,

$$a_{ij} = \langle \chi_i \chi_{\text{nat}}, \chi_j \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_i(g)\overline{\chi_{\text{nat}}(g)}\overline{\chi_j(g)} = \sum_{g \in G} \chi_i(g)\chi_{\text{nat}}(g)\chi_j(g^{-1})$$

(here we use that $g^n = 1$ and hence $\Phi(g)^n = \text{id}$. From this follows $\Phi(g) \sim \text{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$ and $\Phi(g^{-1}) \sim \text{diag}(\varepsilon_1^{-1}, \varepsilon_2^{-1}, \ldots, \varepsilon_k^{-1}) = \text{diag}(\bar{\varepsilon}_1, \bar{\varepsilon}_2, \ldots, \bar{\varepsilon}_k)$.) Since $\Phi_{\text{nat}}$ is the natural representation, all $\Phi_{\text{nat}}(g) \in SU(2), g \in G$. Let $A \in SU(2)$. If $A \sim \text{diag}(a, b)$ then $A^{-1} \sim \text{diag}(b, a)$ ($ab = 1$) Therefore we have $\chi_{\text{nat}}(g) = \chi_{\text{nat}}(g^{-1})$. Then we can continue our equality:

$$\sum_{g \in G} \chi_i(g)\chi_{\text{nat}}(g)\chi_j(g^{-1}) = \sum_{g \in G} \chi_i(g)\chi_{\text{nat}}(g^{-1})\chi_j(g^{-1}) = \langle \chi_i, \chi_{\text{nat}}\chi_j \rangle = a_{ij}.$$
Example 6.9 Let $G = \mathbb{D}_3$ be a binary dihedral group. As we already know, $|\mathbb{D}_3| = 12$. The group $\mathbb{D}_3$ has two generators $a, b$ which satisfy the following relations:

\[
\begin{align*}
a^3 &= b^2 \\
b^4 &= e \\
aba &= b^{-1}.
\end{align*}
\]

The group $\mathbb{D}_3$ has 4 1-dimensional representations $a = 1, b = 1; a = 1, b = -1; a = -1, b = i$ and $a = -1, b = -i$. The natural representation is also known: it is just

\[a = \left( \begin{array}{cc}
\varepsilon & 0 \\
0 & \varepsilon^{-1}
\end{array} \right), \quad b = \left( \begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array} \right),\]

where $\varepsilon = \exp(\pi i / 6) = \frac{1}{2} + \frac{\sqrt{3}}{2} i$. There is also another one irreducible 2-dimensional representation:

\[a = \left( \begin{array}{cc}
\cos \frac{2\pi}{3} & i \sin \frac{2\pi}{3} \\
i \sin \frac{2\pi}{3} & \cos \frac{2\pi}{3}
\end{array} \right), \quad b = \left( \begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array} \right).\]

We have found all indecomposable representations of $G$: $1 + 1 + 1 + 4 + 4 = 12 = |\mathbb{D}_3|$. We can sum up the obtained information into the character table.

<table>
<thead>
<tr>
<th>$\chi(a)$</th>
<th>$\chi(b)$</th>
<th>dim</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
<td>$i$</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
<td>$-i$</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>-1</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

From this table we can derive the whole structure of the Grothendieck ring of $|\mathbb{D}_3|$. $\chi^2_{\text{nat}}$ can be only $\chi_0 + \chi_1 + \chi_5$. In the same way $\chi_5 \chi_{\text{nat}} = \chi_2 + \chi_3 + \chi_4$.

We get the McKay graph of $|\mathbb{D}_3|$:}

Observe that we obtained the dual graph of the $D_5$-singularity. Note that the fundamental cycle of the $D_5$-singularity is

\[Z_{\text{fund}} = E_1 + 2E_4 + 2E_5 + E_2 + E_3.\]

But the coefficients of this decomposition are the same as the dimensions of the representations corresponding to the vertices of the McKay quiver.
Theorem 6.10 (McKay observation) Let $G \subseteq SU(2)$ be a finite subgroup, $\mathbb{C}[[x,y]]^G$ the corresponding invariant subring. Then the McKay quiver of $G$ coincide with the dual graph of $\mathbb{C}[[x,y]]^G$, dimensions of the representation corresponding to a vertex of McKay quiver is equal to the multiplicity of the corresponding component of the exceptional fibre in the fundamental cycle.