Du Val Singularities

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1 Introduction

We consider quotient singularities $\mathbb{C}^2/G$, where $G \subseteq SL_2(\mathbb{C})$ is a finite subgroup. Since we have a finite ring extension $\mathbb{C}[[x,y]]^G \subseteq \mathbb{C}[[x,y]]$, the Krull dimension of $\mathbb{C}[[x,y]]^G$ is 2.

Note that $\mathbb{C}[[x,y]]^{\Sigma_2} = \mathbb{C}[[(x+y, xy)]] \cong \mathbb{C}[[x,y]]$. In order to get a kind of bijection between finite subgroups and quotient singularities we need the following definition.

Definition 1.1 *Let $G \subseteq GL_n(\mathbb{C})$ be a finite subgroup. An element $g \in G$ is called* pseudo-reflection *if g is conjugated to* $\mathrm{diag}(1, 1, \ldots, 1, \lambda)$, *where $\lambda \neq 1$. A group G is called* small *if it contains no pseudo-reflections.*

Proposition 1.2

1. *Let $G \subseteq GL_n(\mathbb{C})$ be a finite subgroup. Then*

$$\mathbb{C}[[x_1, x_2, \ldots, x_n]]^G \cong \mathbb{C}[[x_1, x_2, \ldots, x_n]]^{G'},$$

where the group G' is a certain small subgroup of $GL_n(\mathbb{C})$.

2. *Let $G', G'' \subseteq GL_n(\mathbb{C})$ be two small subgroups. Then*

$$\mathbb{C}[[x_1, x_2, \ldots, x_n]]^{G'} \cong \mathbb{C}[[x_1, x_2, \ldots, x_n]]^{G''}$$

if and only if G' and G'' to are conjugated.

3. *Let $G \subseteq GL_n(\mathbb{C})$ be a small finite subgroup. Then $\mathbb{C}[[x_1, x_2, \ldots, x_n]]^G$ is always Cohen-Macaulay.*

4. *Let $G \subseteq GL_n(\mathbb{C})$ be a small finite subgroup. Then $\mathbb{C}[[x_1, x_2, \ldots, x_n]]^G$ is Gorenstein iff $G \subseteq SL_n(\mathbb{C})$.*

Remark 1.3 *Note that every subgroup of $SL_n(\mathbb{C})$ is small.*

Now we want to answer the following question: what are finite subgroups of $SL_2(\mathbb{C})$ modulo conjugation?

2 Finite subgroups of $SL_2(\mathbb{C})$

Lemma 2.1 *Every finite subgroup of $SL_n(\mathbb{C})$ ($GL_n(\mathbb{C})$) is conjugated to a subgroup of $SU(n)$ ($U(n)$).*

Proof. Let $(\quad,\quad)$ be a hermitian inner product on $\mathbb{C}^n$. Define

$$\langle u, v\rangle := \frac{1}{|G|}\sum_{g\in G}(gu, gv).$$

(Note that in case $G \subseteq U(n)$ it holds $\langle\quad,\quad\rangle = (\quad,\quad)$). Then $\langle\quad,\quad\rangle$ is a new hermitian inner product on $\mathbb{C}^n$:

1. $\langle u, u\rangle \geq 0$.
2. $\langle u, u\rangle = 0$ implies $u = 0$.
3. $\langle u, v\rangle = \overline{\langle v, u\rangle}$.

It holds moreover

$$\langle hu, hv\rangle = \frac{1}{|G|}\sum_{g\in G}(ghu, ghv) = \langle u, v\rangle.$$

Hence G is unitary with respect to $\langle\quad,\quad\rangle$. Moreover, $\langle\quad,\quad\rangle$ has an orthonormal basis and let

$$S : (\mathbb{C}^n, \langle\quad,\quad\rangle) \longrightarrow (\mathbb{C}^n, (\quad,\quad))$$

be a map sending vectors of the choosen orthonormal basis of the left space to vectors of the canonical basis of the right one. It holds $\langle u, v\rangle = (Su, Sv)$. We know that $\langle gu, gv\rangle = \langle u, v\rangle$ for all $g \in G$, $u, v \in \mathbb{C}^n$. Hence we get $(Su, Sv) = (Sgu, Sgv)$, or, setting $S^{-1}u$ and $S^{-1}v$ instead of u and v

$$(u, v) = (SgS^{-1}u, SgS^{-1}v)$$

for all $g \in G$ and $u, v \in \mathbb{C}^n$.

Now we want to describe all finite subgroups of $SU(2)$. Recall that

$$SU(2) = \{A \in SL_2(\mathbb{C}) | A^{-1} = A^*\} = \{\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} | |\alpha|^2 + |\beta|^2 = 1\}.$$

So, from the topological point of view $SU(2) \cong S^3$.

Theorem 2.2 *There is an exact sequence of group homomorphisms*

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow SU(2) \xrightarrow{\pi} SO(3) \longrightarrow 1.$$

More precisely,

$$ker(\pi) = \{\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}.$$

Topologically π is the map $S^3 \xrightarrow{2:1} \mathbb{RP}^3$.

From this theorem follows that there is a connection between finite subgroups of $SU(2)$ and $SO(3)$. The classification of finite isometry groups of $\mathbb{R}^3$ is a classical result of F.Klein (actually of Platon). There are the following finite subgroups of $SO(3,\mathbb{R})$:

1. A cyclic subgroup $\mathbb{Z}_n$, generated, for instance, by

$$\begin{pmatrix} \cos(\frac{2\pi}{n}) & -\sin(\frac{2\pi}{n}) & 0 \\ \sin(\frac{2\pi}{n}) & \cos(\frac{2\pi}{n}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2. Dihedral group D_n, $|D_n| = 2n$. It is the automorphism group of a prisma. It is generated by a rotation a and a reflection b which satisfy the relations $a^n = e, b^2 = e, (ab)^2 = e$. Note that the last relation can be choosen as $ba = a^{n-1}b$ or $bab^{-1} = a^{-1}$.

3. Group of automorphisms of a regular tetrahedron $T = A_4$. Note that $|T| = 12$.

4. Group of automorphisms of a regular octahedron $O = S_4$, $|O| = 24$.

5. Group of automorphisms of a regular icosahedron $I = A_5$, $|I| = 60$.

Observe that all non-cyclic subgroups of $SO(3)$ have even order.

Lemma 2.3 $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ *is the only element of* $SU(2)$ *of degree 2.*

Proof. The proof is a simple computation.

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}^2 = \begin{pmatrix} \alpha^2 - |\beta|^2 & \alpha\beta + \beta\bar{\alpha} \\ -\bar{\beta}\alpha - \bar{\beta}\bar{\alpha} & \bar{\alpha}^2 - |\beta|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

implies $\alpha^2 = \bar{\alpha}^2$ or $\alpha = \pm\bar{\alpha}$. Hence $\alpha = x, x \in \mathbb{R}$ or $\alpha = ix, x \in \mathbb{R}$. Moreover $\alpha \neq 0$ since otherwise must hold $-|\beta|^2 = 1$. In case $\alpha \in \mathbb{R}$ we have $\alpha\beta + \beta\bar{\alpha} = 2\alpha\beta$ hence $\beta = 0$ and $\alpha = \pm 1$. If $a = ix$ is purely imaginary then $\alpha^2 - |\beta|^2 = -|x|^2 - |\beta|^2 < 0$.

Let $G \subseteq SU(2)$ be a finite subgroup, $\pi : SU(2) \longrightarrow SO(3)$ the $2:1$ surjection. Consider two cases.

1. $|G|$ is odd. Then $G \cap \mathbb{Z}_2 = \{e\}$ (no elements of order 2 in G). So $ker(\pi) \cap G = \{e\}$ and $G \longrightarrow \pi(G)$ is an isomorphism. Hence G is cyclic.

2. $|G|$ is even. The due to the Sylow's theorem G contains a subgroup of order 2^k and hence contains an element of order 2. But there is exactly one element of the second order in $SU(2)$ (see the lemma above). Hence $ker(\pi) \subseteq G$ and $G = \pi^{-1}(\pi(G))$. So, in this case G is the preimage of a finite subgroup of $SO(3)$.

From what was said we get the full classification of finite subgroups of $SL_2(\mathbb{C})$ modulo conjugation.

1. A cyclic subgroup $\mathbb{Z}_k$. Let g be its generator. $g^k = e$ implies

$$g \sim \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix},$$

where ε is some primitive root of 1 of k-th order.

2. Binary dihedral group $\mathbb{D}_n$, $|\mathbb{D}_n| = 4n$. To find the generators of $\mathbb{D}_n$ we have to know the explicit form of the map $\pi : SU(2) \longrightarrow SO(3)$. Skipping all details we just write down the answer. $\mathbb{D}_n = \langle a, b \rangle$ with relations

$$\begin{cases} a^n & = & b^2 \\ b^4 & = & e \\ bab^{-1} & = & a^{-1} \end{cases}$$

To be concrete,

$$a = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}, \varepsilon = exp(\frac{\pi i}{n}), \quad b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

3. Binary tetrahedral group $\mathbb{T}$, $|\mathbb{T}| = 24$. $\mathbb{T} = \langle \sigma, \tau, \mu \rangle$, where

$$\sigma = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mu = \frac{1}{\sqrt{2}} \begin{pmatrix} \varepsilon^7 & \varepsilon^7 \\ \varepsilon^5 & \varepsilon \end{pmatrix}, \quad \varepsilon = exp(\frac{2\pi i}{8}).$$

4. Binary octahedral group $\mathbb{O}$, $|\mathbb{O}| = 48$. This group is generated by σ, τ, μ as in the case of $\mathbb{T}$ and by

$$\kappa = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^7 \end{pmatrix}.$$

5. Finally we have the binary icosahedral subgroup $\mathbb{I}$, $|\mathbb{I}| = 120$. $\mathbb{I} = \langle \sigma, \tau \rangle$, where

$$\sigma = -\begin{pmatrix} \varepsilon^3 & 0 \\ 0 & \varepsilon^2 \end{pmatrix}, \tau = \frac{1}{\sqrt{5}} \begin{pmatrix} -\varepsilon + \varepsilon^4) & \varepsilon^2 - \varepsilon^3 \\ \varepsilon^2 - \varepsilon^3 & \varepsilon - \varepsilon^4 \end{pmatrix}, \quad \varepsilon = exp(\frac{2\pi i}{5}).$$

Remark 2.4 *The problem of classification of finite subgroups of $GL_2(\mathbb{C})$ is much more complicated. Indeed, every finite subgroup $G \subseteq GL_2(\mathbb{C})$ can be embedded into $SL_3(\mathbb{C})$ via the group monomorphism*

$$GL_2(\mathbb{C}) \longrightarrow SL_3(\mathbb{C})$$

$$g \mapsto \begin{pmatrix} g & 0 \\ 0 & \frac{1}{det(g)} \end{pmatrix}.$$

Finite subgroups of $GL_2(\mathbb{C})$ give main series of finite subgroups of $SL_3(\mathbb{C})$.

Now we can compute the corresponding invariant subrings.

3 Description of Du Val singularities

1. A cyclic subgroup $\mathbb{Z}_n = \langle g \rangle$, $g : x \mapsto \varepsilon x, y \mapsto \varepsilon^{-1} y, \varepsilon = exp(\frac{2\pi i}{n})$. It is not difficult to see that $X = x^n, Y = y^n$ and $Z = xy$ generate the whole ring of invariants.

   $$\mathbb{C}[[x,y]]^{\mathbb{Z}_n} = \mathbb{C}[[X,Y,Z]]/(XY - Z^n) \cong \mathbb{C}[[x,y,z]]/(x^2 + y^2 + z^n).$$

   It is an equation of A_{n-1}-singularity.

2. Binary dihedral group $\mathbb{D}_n$.

   $$\sigma : \begin{cases} x & \mapsto & \varepsilon x \\ y & \mapsto & \varepsilon^{-1} y \end{cases} \qquad \tau : \begin{cases} x & \mapsto & -y \\ y & \mapsto & x \end{cases},$$

   where $\varepsilon = exp(\frac{\pi i}{n})$. The set of invariant monomials is

   $$F = x^{2n} + y^{2n}, H = xy(x^{2n} - y^{2n}), I = x^2 y^2.$$

   They satisfy the relation

   $$H^2 = x^2 y^2 (x^{4n} + y^{4n} - 2x^{2n} y^{2n}) = IF^2 - 4I^{2(n+1)}.$$

   The standart computations show that

   $$\mathbb{C}[[x,y]]^{\mathbb{D}_n} \cong \mathbb{C}[[x,y,z]]/(x^2 + yz^2 + z^{n+1}).$$

   It is an equation of D_{n+2}-singularity.

3. It can be checked that for the groups $\mathbb{T}, \mathbb{O}, \mathbb{I}$ we get singularities

   (a) E_6: $x^2 + y^3 + z^4 = 0$,

   (b) E_7: $x^2 + y^3 + yz^3 = 0$,

   (c) E_8: $x^2 + y^3 + z^5 = 0$.

We get the following theorem:

Theorem 3.1 *Du Val singularities are precisely simple hypersurface singularities $A - D - E$.*

We want now to answer our next question: what are minimal resolutions and dual graphs of Du Val singularities?

4 A_1-singularity

Consider the germ of A_1-singularity $S = V(x^2 + y^2 + z^2) \subset \mathbf{A}^3$. Consider the blow-up of this singularity.

$$\tilde{\mathbf{A}}^3 = \{((x,y,z),(u:v:w)) \in \mathbf{A}^3 \times \mathbf{P}^2 | xv = yu, xw = zu, yw = zu\}.$$

Take the chart $u \neq 0$, i.e. $u = 1$. We get

$$\begin{cases} x &= x \\ y &= xv \\ z &= xw \end{cases}$$

What is $\tilde{S} = \overline{\pi^{-1}(S \backslash \{0\})}$? Consider first $x \neq 0$ (it means that we are looking for $\pi^{-1}(S \backslash \{0\})$).

$$x^2 + x^2v^2 + x^2w^2 = 0, x \neq 0,$$

or

$$1 + v^2 + w^2 = 0, x \neq 0.$$

In order to get $\tilde{S}$ we should allow x to be arbitrary. In this chart $\tilde{S}$ is a cylinder $V(1 + v^2 + w^2) \subset \mathbf{A}^3$. What is $\pi^{-1}(0)$? Obviously it is the intersection of $\tilde{S}$ with the exceptional plane $((0,0,0),(u:v:w))$. In this chart we just have to set $x = 0$ in addition to the equation of the surface $\tilde{S}$.

$$\pi^{-1}(0) = \begin{cases} 1 + v^2 + w^2 &= 0 \\ x &= 0. \end{cases}$$

We see that $E = \pi^{-1}(0)$ is rational and since all 3 charts of $\tilde{S}$ are symmetric, we conclude that E is smooth, so $E = \mathbf{P}^1$. Now we have to compute the selfintersection number E^2. We do it using the following trick.

Let $\pi : \tilde{X} \longrightarrow X$ be the minimal resolution. It induces an isomorphism of fields of rational functions $\pi^* : \mathbb{C}(X) \longrightarrow \mathbb{C}(\tilde{X})$. Let $f \in \mathbb{C}(\tilde{X})$ be a rational function. Then it holds:

$$(f).E = deg_E(\mathcal{O}_E \otimes \mathcal{O}_{\tilde{X}}(f)) = deg_E(\mathcal{O}_E) = 0.$$

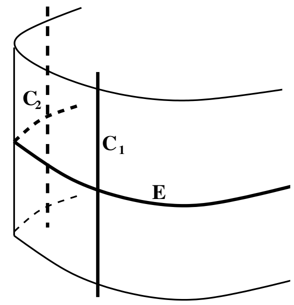
In particular it holds for $f \in \mathfrak{m} \subseteq \mathcal{O}_{X,0}$:

$$(f \circ \pi).E = 0.$$

Consider the function $y \in \mathcal{O}_{X,0}$. In the chart $u \neq 0$ we get $y \circ \pi = xu$. What is the vanishing set of y?

1. $x = 0$ is an equation of E.
2. $u = 0$ implies $v^2 + 1 = 0$ or $v = \pm i$.

Hence $(f) = E + C_1 + C_2$ and we have the following picture:



So, $(f).E = E^2 + C_1.E + C_2.E = E^2 + 2 = 0$. From it follows $E^2 = -2$.

5 E_6-singularity

In this section we want to compute a minimal resolution and dual graph of E_6-singularity $X = V(x^2 + y^3 + z^4) \subset \mathbf{A}^3$. Let $\tilde{X} \xrightarrow{\pi} X$ be a minimal resolution, $E = \cup E_i = \pi^{-1}(0)$ the exceptional divisor. In order to compute selfintersection numbers E_i^2 we have to consider the map $\pi^* : \mathbb{C}(X) \longrightarrow \mathbb{C}(\tilde{X})$. Let $f \in \mathfrak{m}_X \subset \mathcal{O}_X$, then equalities $(f \circ \pi).E_i = 0$ will imply the selfintersection numbers of E_i.
First step. Let $X = V(x^2 + y^3 + z^4) \subset \mathbf{A}^3$, $f = x$. Consider the blow-up of $\mathbf{A}^3$:

$$\tilde{\mathbf{A}}^3 = \{((x,y,z),(u:v:w)) \in \mathbf{A}^3 \times \mathbf{P}^2 | xv = yu, xw = zu, yw = zu\}.$$

Take first the chart $v \neq 0$ (i.e. $v = 1$). We get equations

$$\begin{cases} x &= yu \\ y &= y \\ z &= yw. \end{cases}$$

To get the equation of the strict transform of $\tilde{X}_1$ we assume that $y \neq 0$ and

$$y^2u^2 + y^3 + y^4w^4 = 0,$$

or

$$u^2 + y + y^2w^4 = 0.$$

In this chart $\tilde{X}_1$ is smooth: Jacobi criterium implies

$$\begin{cases} u &= 0 \\ 1 + 2yw^4 &= 0 \\ w^3y^2 &= 0. \end{cases}$$

It is easy to see that this system has no solutions.

In the chart $u = 1$ the strict transform $\tilde{X}_1$ is again smooth.
Consider finally the chart $w = 1$.

$$\begin{cases} x &= zu \\ y &= zv \\ z &= z. \end{cases}$$

The strict transform is

$$z^2u^2 + z^3v^3 + z^4 = 0,$$

or

$$u^2 + zv^3 + z^2 = 0.$$

Jacobi criterium implies that this surface has a unique singular point $u = 0, v = 0, z = 0$, or in the global coordinates $((0,0,0),(0:0:1))$. We see that this point indeed lies only in one of three affine charts of $\tilde{X}_1$.

We now need an equation of exceptional fibre. The exceptional fibre is by the definition the intersection $\tilde{X}_1 \cap \{((0,0,0),(u : v : w))\}$. To get its local equation in the chart $w = 1$ we just have to set $z = 0$ in the equation of $\tilde{X}_1$. $z = 0$ implies $u = 0$. Hence we get

$$E_0 = \{((0,0,0),(0 : v : 1))\} \cong \mathbf{A}^1.$$

Going to the other charts shows that $E_0 \cong \mathbf{P}^1$.
Finally, the function f in this chart gets the form $f = zu$.

Agreement. Since the number of indices depends exponentially on the number of blowing-ups, we shall denote the local coordinates of all charts of all blowing-ups $\tilde{X}_i$ by the letters (x, y, z).

Second step. We have the following situation:

$$\begin{cases} \text{surface} & x^2 + zy^3 + z^2 = 0 \\ \text{function} & f = xz \\ \text{exceptional divisor } E_0 & x = 0, z = 0. \end{cases}$$

Consider again the blowing-up of this surface. It is easy to see that the only interesting chart is

$$\begin{cases} x &= & yu \\ y &= & y \\ z &= & yv. \end{cases}$$

We get the strict transform

$$y^2u^2 + yvy^3 + y^2v^2 = 0, y \neq 0,$$

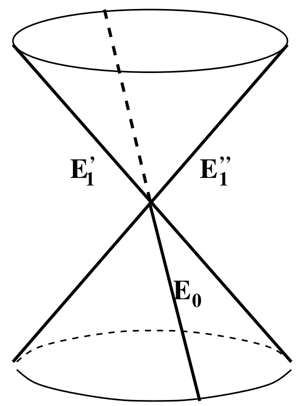
or

$$u^2 + y^2v + v^2 = 0.$$

Again $y = 0, u = 0, v = 0$ is the only singularity of the blown-up surface. The exceptional fibre of this blowing-up has two irreducible components: $y = 0$ implies $u \pm iv = 0$ (we call this components E_1' and E_2'').
What is the preimage (under preimage we mean its strict transform) of E_0? $x = 0, z = 0$ implies $u = 0, v = 0$.
The function $f = xz$ gets in this chart the form $f = y^2uv$.

Third step. We have the following situation:

$$\begin{cases} \text{surface} & x^2+y^2z+z^2=0 \\ \text{function} & f=xy^2z \\ \text{exceptional divisor } E_0 & x=0, z=0 \\ \text{exceptional divisor } E_1 & y=0, x\pm iz=0. \end{cases}$$



Let us consider the next blowing-up.

$$\begin{cases} x &=& yu \\ y &=& y \\ z &=& yv. \end{cases}$$

The strict transform is

$$y^2u^2+y^2yv+y^2v^2=0, y\neq 0,$$

or

$$u^2+yv+v^2=0.$$

It is an equation of A_1-singularity (and it means that we are almost done). The exceptional fibre consists again of two irreducible components E_2', E_2''. They local equations are $y=0, u\pm iv=0$. The function f is uy^4v. It is easy to see that the preimage of E_0 is $u=0, v=0$.
What about the preimage of E_1? Our surface lies in the affine chart $\mathbf{A}^3$ embedded into $\mathbf{A}^3\times\mathbf{P}^2$ via the map $(y,u,v)\mapsto((yu,y,yv),(u:1:v))$. But then the condition $y=0$ would imply that the preimage of E_1 lies in the exceptional plane $((0,0,0)(u:1:v))$. But it can not be true! The solution of this paradox is that the preimage of of E_1 lies in another coordinate chart.

Consider

$$\begin{cases} x &=& x \\ y &=& xu \\ z &=& xv, \end{cases}$$

The strict transform is

$$x^2 + x^2u^2xv + x^2v^2 = 0, x \neq 0$$

or

$$1 + xu^2v + v^2 = 0.$$

The equations of the exceptional fibre E_2 in this chart are $x = 0$, what implies $v = \pm i$. The preimage of E_1

$$\begin{cases} x \pm iz &= 0 \\ y &= 0 \end{cases}$$

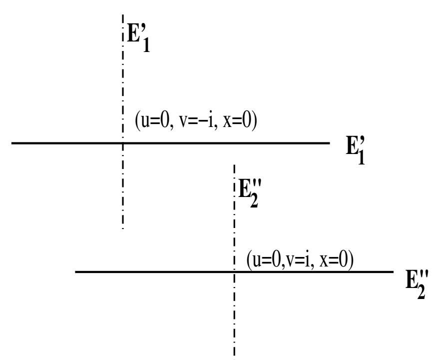
is given by

$$\begin{cases} x \pm ixv &= 0 \\ xu &= 0, x \neq 0, \end{cases}$$

hence

$$\begin{cases} v &= \pm i \\ u &= 0 \\ x & \text{arbitrary} \end{cases}$$

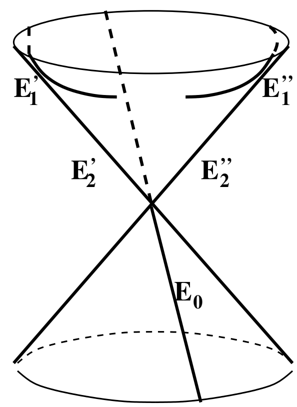
In the picture it looks like:



It is easy to see that all intersections are transversal.

Fourth step. We have the following situation: there are two coordinate charts

$$\begin{cases} \text{surface} & x^2 + yz + z^2 = 0 \\ \text{function} & f = xy^4z \\ \text{exceptional divisor } E_0 & x = 0, z = 0 \\ \text{exceptional divisor } E_2 & y = 0, x \pm iz = 0. \end{cases} \qquad \begin{cases} \text{surface} & 1 + xy^2z + z^2 = 0 \\ \text{function} & f = x^4y^2z \\ \text{exceptional divisor } E_1 & y = 0, z = \pm i \\ \text{exceptional divisor } E_2 & x = 0, z = \pm i. \end{cases}$$



Our next step is the blowing-up at the point $(0,0,0)$ in the first coordinate chart. Again, in order to get equations of the preimages of E_0 and E_2 we have to consider two coordinate charts.

$$\begin{cases} x &= & yu \\ y &= & y \\ z &= & yw. \end{cases}$$

The strict transform is a cylinder

$$u^2 + v + v^2 = 0.$$

The preimage of E_0 is given by equations $u = 0, v = 0$, the exceptional fibre E_3 is given by $u^2 + v + v^2 = 0, y = 0$, our function $f = uy^6v$. In another chart we have

$$\begin{cases} x &= & x \\ y &= & xu \\ z &= & xv. \end{cases}$$

The strict transform is given by

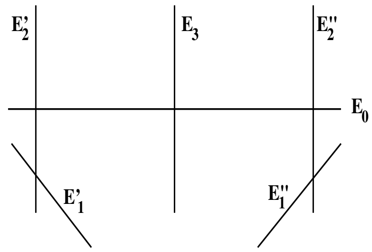
$$x^2 + x^2uv + x^2v^2 = 0, x \neq 0$$

or

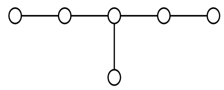
$$1 + uv + v^2 = 0.$$

The exceptional fibre E_3 is given by $1 + uv + v^2 = 0, x = 0$, the preimages of E_2' and E_2'' are given by $u = 0, v = \pm i$, $f = x^6u^4v$.

Hence our exceptional fibre E is given by the following configuration of projective lines:



The dual graph of this configuration is



Fifth step. We have to take into account three coordinate charts of a minimal resolution.

$$\begin{cases} \tilde{X}: & x^2+z+z^2=0 \\ f: & xy^4z \\ E_0: & x=0, z=0 \\ E_3: & y=0, x^2+yz+z^2=0 \end{cases} \quad \begin{cases} \tilde{X}: & 1+yz+z^2=0 \\ f: & x^6y^4z \\ E_3: & x=0, x^2+yz+z^2=0 \\ E_2: & y=0, z=\pm i \end{cases} \quad \begin{cases} \tilde{X}: & 1+xy^2z+z^2=0 \\ f: & x^6y^2z \\ E_1: & y=0, z=\pm i \\ E_2: & x=0, z=\pm i. \end{cases}$$

Now we have to compute the divisor (f).

Let $X \subset \mathbf{A}^3$ be a normal surface, $Y \subset X$ a closed curve, $f \in \mathbb{C}(X)$ a rational function. Suppose that $\mathfrak{p} \subset \mathbb{C}[X]$ is the prime ideal corresponding to Y. Then $\mathbb{C}[X]_{\mathfrak{p}}$ is a discrete valuation ring and

$$\mathrm{mult}_Y(f) = \mathrm{val}_{\mathbb{C}[X]_{\mathfrak{p}}}(f).$$

1. Consider the first chart. Let $f = xy^6z = 0$. $x = 0$ implies $z = 0$ or $z = -1$. $x = 0$, $z = 0$ is an equation of E_0, $x = 0, z = -1$ is the strict transform C of the curve $x = 0$ in $V(x^2 + y^3 + z^4) \subset \mathbf{A}^3$.

   What is the multiplicity of E_0? The generator of the maximal ideal of the ring $(\mathbb{C}[x, y, z]/(x^2 + z + z^2))_{\mathfrak{p}}$ is $\bar{x}$ and $\bar{x}^2 \sim \bar{z}$. Therefore $\mathrm{mult}_{E_0}(f) = 3$.

   $y = 0$ gives an equation of E_3. It is easy to see that $\mathrm{mult}_{E_3}(f) = 6$. Note that the curve C has transversal intersection with E_3 at the point $x = 0, y = 0, z = -1$.

2. Consider the second chart. In this chart holds $f = x^6y^4z = 0$. $z = 0$ is impossible, $x = 0$ cut out the equation of E_3 and $y = 0$ equations of E_2' and E_2''. The same computation as above shows that $\mathrm{mult}_{E_3}(f) = 6$ (what is not surprise and makes us sure that we did not make a mistake in computations) and $\mathrm{mult}_{E_2'}(f) = \mathrm{mult}_{E_2''}(f) = 4$.

3. In the same way we obtain that $\mathrm{mult}_{E_1'}(f) = \mathrm{mult}_{E_1''}(f) = 2$. Therefore we obtain:

$$(f) = 6E_3 + 4(E_2' + E_2'') + 2(E_1' + E_1'') + 3E_0 + C.$$

   We have $C.E_3 = 1$, all other intersection numbers of C with irreducible components of E are zero. Intersection numbers of irreducible components are coded in the dual graph (which is E_6, see the picture above). The whole job was done in order to compute self-intersections.

$$(f).E_0 = 6 + 3E_0^2 = 0 \Longrightarrow E_0^2 = -2.$$

   In the same way we conclude that the other selfintersection numbers are -2.

Remark 5.1 *Let X be a normal surface singularity, $\pi : \tilde{X} \longrightarrow X$ its minimal resolution, $E = \bigcup\limits_{i=1}^{n} E_i = \pi^{-1}(0)$ the exceptional divisor. Suppose that $\tilde{X}$ is a good resolution, all $E_i \cong \mathbf{P}^1$ and $E_i^2 = -2$. Then X is a simple hypersurface singularity.*

Indeed we know that the intersection matrix $(E_i.E_j)_{i,j=1}^n$ is negatively definite. Let Γ be the dual graph of X. Then the quadratic form given by intersection matrix coinside with the Tits form of the dual graph:

$$Q(x_1, x_2, \ldots, x_n) = -2(\sum_{i=1}^{n} x_i^2 - \sum_{1=i<j=n} a_{ij}x_ix_j),$$

where a_{ij} is the number of arrows connecting vertices i and j. From the theorem of Gabriel we know that Q is negatively definite (and quiver is representation finite) if and only if $\Gamma = A - D - E$. Since our singularity is rational, it is taut and uniquely determined by its dual graph.

6 2-dimensional McKay correspondence

Recall that we defined Du Val singularities as quotient singularities $\mathbb{C}[[x, y]]^G$, where $G \subseteq SU(2)$ is some finite subgroup. A natural question is: are there any connections between the representation theory of G and geometry of the minimal resolution of a singularity? Let us recall some standart facts about representations of finite groups.

Theorem 6.1 (Mashke) *Let G be a finite group. Then the category of $\mathbb{C}[G]$-modules is semi-simple.*

This theorem means that any exact sequence of $\mathbb{C}[G]$-modules splits. In particular, every finite-dimensional $\mathbb{C}[G]$-module is injective and projective. But an indecomposable projective module by a theorem of Krull-Schmidt is isomorphic to a direct summand of the regular module. Let

$$\mathbb{C}[G] \cong \bigoplus_{i=1}^{s} \Phi_i^{n_i}$$

be a direct sum decomposition of $\mathbb{C}[G]$. Then $\Phi_1, \Phi_2, \ldots, \Phi_s$ the whole list of indecomposable $\mathbb{C}[G]$-modules.

Lemma 6.2 *Let $\mathbb{C}[G] \cong \bigoplus\limits_{i=1}^{s} \Phi_i^{n_i}$ be a decomposition of the regular module into a direct sum of indecomposable ones. Then it holds*

$$dim_{\mathbb{C}}(\Phi_i) = n_i.$$

In particular, the following identity is true:

$$\sum_{i=1}^{s} n_i^2 = |G|.$$

Definition 6.3 *Let G be a group, (Φ, V), $\Phi : G \longrightarrow \mathrm{End}(V)$ its representation. The character of a representation Φ is the function $\chi_\Phi : G \longrightarrow \mathbb{C}$ defined by the rule $\chi_\Phi(g) = Tr(\Phi(g))$.*

Remark 6.4

1. *It is easy to see that the character does not depend on the choice of a representative from the isomorphism class of a representation:*

   $$Tr(\Phi(g)) = Tr(S^{-1}\Phi(g)S).$$

2. *It holds:*

   $$\chi_{\Phi\otimes\Psi} = \chi_\Phi \chi_\Psi,$$

   $$\chi_{\Phi\oplus\Psi} = \chi_\Phi + \chi_\Psi.$$

   In other words, χ defines a rings homomorphism from the Grothendieck ring of $\mathbb{C}[G]$ to $\mathbb{C}$.

3. *It holds:*

   $$\chi_\Phi(h^{-1}gh) = Tr(\Phi(h^{-1}gh)) = Tr(\Phi(h)^{-1}\Phi(g)\Phi(h)) = Tr(\Phi(g)) = \chi_\Phi(g).$$

   It means that χ_Φ is a central function*, i.e. a function which is is constant on conjugacy classes of G.*

Theorem 6.5 *A finite dimensional representation of a finite group G is uniquely determined by its character.*

Idea of the proof. Let φ, ψ be two central functions on G. Set

$$\langle \varphi, \psi \rangle := \frac{1}{|G|} \sum_{g \in G} \chi_\Phi(g) \overline{\chi_\Psi(h)}.$$

It defines an hermitian inner product on the space of all central functions on G. The theorem follows from the fact that $\chi_{\Phi_1}, \chi_{\Phi_2}, \dots, \chi_{\Phi_s}$ is an orthonormal basis of this vector space. Indeed, let Φ be any finite-dimensional representation of G. Then we know that

$$\Phi \cong \oplus \Phi_i^{m_i}.$$

Then it obviously holds $m_i = \langle \chi_\Phi, \chi_{\Phi_i} \rangle$.

Corollary 6.6 *The number of indecomposable representations of a finite group G is equal to the number of its conjugacy classes.*

Definition 6.7 (McKay quiver) *Let $G \subseteq SU(2)$ be a finite subgroup, $\Phi_0, \Phi_1, \dots, \Phi_s$ all indecomposable representations of G. Let Φ_0 be the trivial representation, Φ_{nat} the natural representation (i.e. the representation given by the inclusion $G \subseteq SU(2)$). Define the McKay graph of G as the following:*

1. *Vertices are indexed by $\Phi_1, \dots, \Phi_s$ (we skip Φ_0).*
2. *Let*
$$\Phi_i \otimes \Phi_{nat} \cong \bigoplus_{j=0}^{s} \Phi_j^{a_{ij}}.$$
(or, the same
$$\chi_i \chi_{nat} = \sum_{i=0}^{s} a_{ij} \chi_j.$$
Then we connect vertices Φ_i and Φ_j by a_{ij} vertices.

It is easy to see that $\Phi_0 \otimes \Phi_{\text{nat}} = \Phi_{\text{nat}}$.

Remark 6.8 *It holds $a_{ij} = a_{ji}$.*

Indeed,

$$a_{ij} = \langle \chi_i \chi_{\text{nat}}, \chi_j \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_i(g) \chi_{\text{nat}}(g) \overline{\chi_j(g)} = \sum_{g \in G} \chi_i(g) \chi_{\text{nat}}(g) \chi_j(g^{-1})$$

(here we use that $g^n = 1$ and hence $\Phi(g)^n = \text{id}$. From this follows $\Phi(g) \sim \text{diag}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k)$ and $\Phi(g^{-1}) \sim \text{diag}(\varepsilon_1^{-1}, \varepsilon_2^{-1}, \dots, \varepsilon_k^{-1}) = \text{diag}(\bar{\varepsilon}_1, \bar{\varepsilon}_2, \dots, \bar{\varepsilon}_k)$.) Since Φ_{nat} is the natural representation, all $\Phi_{\text{nat}}(g) \in SU(2), g \in G$. Let $A \in SU(2)$. If $A \sim \text{diag}(a, b)$ then $A^{-1} \sim \text{diag}(b, a)$ $(ab = 1.)$ Therefore we have $\chi_{\text{nat}}(g) = \chi_{\text{nat}}(g^{-1})$. Then we can continue our equality:

$$\sum_{g \in G} \chi_i(g) \chi_{\text{nat}}(g) \chi_j(g^{-1}) = \sum_{g \in G} \chi_i(g) \chi_{\text{nat}}(g^{-1}) \chi_j(g^{-1}) = \langle \chi_i, \chi_{\text{nat}} \chi_j \rangle = a_{ji}.$$

Example 6.9 Let $G = \mathbb{D}_3$ be a binary dihedral group. As we already know, $|\mathbb{D}_3| = 12$. The group $\mathbb{D}_3$ has two generators a, b which satisfy the following relations:

$$\begin{cases} a^3 &= b^2 \\ b^4 &= e \\ aba &= b^{-1}. \end{cases}$$

The group $\mathbb{D}_3$ has 4 1-dimensional representations $a = 1, b = 1$; $a = 1, b = -1$; $a = -1, b = i$ and $a = -1, b = -i$. The natural representation is also known: it is just

$$a = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

where $\varepsilon = exp(\frac{\pi i}{6}) = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$. There is also another one irreducible 2-dimensional representation:

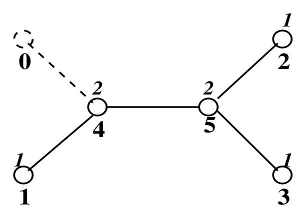
$$a = \begin{pmatrix} \cos\frac{2\pi}{3} & i\sin\frac{2\pi}{3} \\ i\sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We have found all indecomposable representations of G: $1+1+1+1+4+4 = 12 = |\mathbb{D}_3|$. We can sum up the obtained information into the character table.

	$\chi(a)$	$\chi(b)$	dim	
0	1	1	1	trivial
1	1	-1	1	
2	-1	i	1	
3	-1	$-i$	1	
4	1	0	2	natural
5	-1	0	2	

From this table we can derive the whole structure of the Grothendieck ring of $|\mathbb{D}_3|$. χ^2_{nat} can be only $\chi_0 + \chi_1 + \chi_5$. In the same way $\chi_5 \chi_{\text{nat}} = \chi_2 + \chi_3 + \chi_4$.

We get the McKay graph of $|\mathbb{D}_3|$:



Observe that we obtained the dual graph of the D_5-singularity. Note that the fundamental cycle of the D_5-singularity is

$$Z_{\text{fund}} = E_1 + 2E_4 + 2E_5 + E_2 + E_3.$$

But the coeffitients of this decomposition are the same as the dimensions of the representations corresponding to the vertices of the McKay quiver.

Theorem 6.10 (McKay observation) *Let $G \subseteq SU(2)$ be a finite subgroup, $\mathbb{C}[[x,y]]^G$ the corresponding invariant subring. Then the McKay quiver of G coinside with the dual graph of $\mathbb{C}[[x,y]]^G$, dimensions of the representation corresponding to a vertex of McKay quiver is equal to the multiplicity of the corresponding component of the exceptional fibre in the fundamental cycle.*