# Mass Equidistribution for Random Holomorphic Sections

Turgay Bayraktar

(Sabancı University)

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# Motivation

Let $(M, g)$ be a compact Riemannian manifold without boundary and $\Delta := \Delta_g$ denotes the Laplace-Beltrami operator associated with the metric $g$. We also let $dV$ be the corresponding volume element.

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Let $(M, g)$ be a compact Riemannian manifold without boundary and $\Delta := \Delta_g$ denotes the Laplace-Beltrami operator associated with the metric $g$. We also let $dV$ be the corresponding volume element. An eigenfunction $\phi : M \to \mathbb{R}$ of $\Delta$ has the property $-\Delta\phi = \lambda\phi$ for some $\lambda \geq 0$. For such a $\phi$ one can define a probability measure

$$\mu_\phi := \frac{|\phi(x)|^2}{\|\phi\|^2} dV$$

where

$$\|\phi\|^2 = \int_M |\phi(x)|^2 dV(x).$$

A natural question in quantum chaos is to understand limiting behavior of $\mu_\phi$ as $\lambda \to \infty$.

# Quantum Ergodicity



**Definition**

A weak limit $\mu_{\phi_\lambda} \to \nu$ is called a quantum limit.

**Fact**

*Any quantum limit is invariant under geodesic flow.*

# Quantum Ergodicity

On the other hand, there is a natural measure $\mu$ called Liouville measure which is invariant under geodesic flow.

Recall that geodesic flow is ergodic with respect to such a probability measure $\mu$ if the only flow invariant subsets are either of $\mu$-measure zero or one. In this case, by Birkhoff ergodic theorem $\mu$-almost all geodesics are $\mu$-equidistributed.

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**Theorem (Zelditch, Duke '87)**

*Let $(M, g)$ be a compact Riemannian manifold without boundary. Assume that the geodesic flow is $\mu$-ergodic. If $\{\phi_j\}_{j=0}^{\infty}$ is an ONB of eigenfunctions with $0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$ then there is a subsequence $j_k$ of density one (i.e. $\frac{j_k}{k} \to 1$) such that $\mu_{\phi_{j_k}} \to \mu$.*

**Question**

*Does there exists a quantum limit other than $\mu$?*

# Quantum Unique Ergodicity (QUE)



**Conjecture (Rudnick-Sarnak, CMP '94)**

*Let $(M, g)$ be a negatively curved compact Riemannian manifold (in particular, the Liouville measure is ergodic) then $\mu_{\phi_\lambda} \to \mu$ i.e. $\mu$ is the unique quantum limit.*

**Theorem (Lindenstrauss, Annals of Math '06)**

*Let $M$ be a compact arithmetic surface then the only quantum limit is the Liouville measure.*

**Theorem (Anantharaman, Annals of Math '08)**

*If $\nu$ is a quantum limit then the entropy $h(\nu) > 0$.*

# Complex Geometry Setting

Let $M$ be a projective manifold of complex dimension $m$ and $L \to M$ be an ample holomorphic line bundle endowed with a smooth (at least $\mathscr{C}^2$) Hermitian metric $h = e^{-\varphi}$ where $\varphi = \{\varphi_\alpha\}$ is a local weight of the metric.

The latter means that if $e_\alpha$ is a holomorphic frame for $L$ over an open set $U_\alpha$ then $|e_\alpha|_h = e^{-\varphi_\alpha}$ where $\varphi_\alpha \in \mathscr{C}^2(U_\alpha)$ such that $\varphi_\alpha = \varphi_\beta + \log|g_{\alpha\beta}|$ and $g_{\alpha\beta} := e_\beta/e_\alpha \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ are the transition functions for $L$.

# Equilibrium weight and measure

In this setting, we define global extremal weight $\varphi_e$ to be

$$\varphi_e := \sup\{\psi \text{ is a psh weight} : \psi \leq \varphi \text{ on } M\}.$$

Then $\varphi_e$ defines a singular Hermitian metric $h_e := e^{-\varphi_e}$ on $L$. We denote its curvature current by $dd^c\varphi_e := dd^c(\varphi_{e,\alpha})$ on $U_\alpha$.

**Theorem (Berman '09)**

*The metric is of class $\mathscr{C}^{1,1}$. Moreover, the equilibrium measure*

$$\mu_{\varphi_e} := (dd^c\varphi_e)^m/m!$$

*is supported on the compact set*

$$S_\varphi := M_\varphi(0) \cap D$$

*where $M_\varphi(0) := \{x \in M : dd^c\varphi(x) > 0\}$ and $D := \{x \in M : \varphi(x) = \varphi_e(x)\}$.*

# $L^2$-global holomorphic sections



The geometric data given above allow us to define a scalar inner product on the vector space of *global holomorphic sections* $H^0(M, L^{\otimes n})$ via

$$\langle s_1, s_2 \rangle := \int_X \langle s_1(x), s_2(x) \rangle_{h^{\otimes n}} dV$$

where $dV$ is a fixed smooth volume form on $M$. We also let

$$SH^0(M, L^{\otimes n}) := \{s \in H^0(M, L^{\otimes n}) : \|s\|_n = 1\}.$$

For a section $s_n \in SH^0(M, L^{\otimes n})$ we define its mass to be the probability measure

$$|s_n(x)|^2_{h^{\otimes n}} dV(x)$$

on $M$.

# Basic Example: SU(m+1) Polynomials



**Example**

Let $M = \mathbb{P}^m$ be complex projective space and $L = \mathcal{O}(1)$ hyperplane bundle. Then $H^0(\mathbb{P}^m, \mathcal{O}(N))$ can be identified with homogenous polynomials in $m+1$ variables of degree $N$. Letting $h = h_{FS}$ Fubini-Study metric, the sections

$$S_J = [\frac{(N+m)!}{m! j_0! \dots j_m!}]^{\frac{1}{2}} z^J, \;\; J = (j_0, \dots, j_m), |J| = N$$

form ONB for $H^0(\mathbb{P}^m, \mathcal{O}(N))$.

A random SU(m+1) Polynomial is of the form

$$f_N(z_0, \dots, z_m) = \sum_{|J|=N} a_J S_J$$

where $a_J$ are i.i.d. standard complex Gaussian.

# Mass asymptotics & Zeros

**Theorem (Nonnenmacher-Voros '98; Shiffman-Zelditch '99; B. '20)**

*Let* $s_n \in SH^0(M, L^{\otimes n})$. *Assume that*

$$|s_n(x)|^2_{h^{\otimes n}} dV(x) \to \mu_{\varphi_e}$$

*in weak* topology as* $n \to \infty$. *Then* $\frac{1}{n}[Z_{s_n}] \to dd^c \varphi_e$ *in the sense of currents.*

# Random holomorphic sections



Now, we fix an ONB $\{S_j^n\}_{j=1}^{d_n}$ for $H^0(M, L^{\otimes n})$ with respect to the inner product given earlier. Then a *a random holomorphic section* is of the form

$$s_n := \sum_{j=1}^{d_n} c_j^n S_j^n$$

where $c_j^n$ are iid (real or complex) random variables of mean zero and variance one i.e. $\mathbb{E}[|c_j^n|^2] = 1$.

This definition induces a $d_n$-fold product probability measure $Prob_n$ on the vector space $H^0(M, L^{\otimes n})$. We also consider the product probability space $\prod_{n=1}^{\infty} \left(H^0(M, L^{\otimes n}), Prob_n\right)$.

# Mass asymptotics of random holomorphic sections

We are interested in asymptotic distribution of masses of random holomorphic sections:

$$\nu_{s_n} := |s_n(x)|^2_{h^{\otimes n}} dV.$$

More precisely,

$$X_n : H^0(M, L^{\otimes n}) \to \mathcal{M}(X)$$

$$s_n \to \nu_{s_n}$$

defines a measure valued random variable. We wish to study

- asymptotics of $\mathbb{E}[X_n]$ or $Var[X_n]$ etc.
- a.e. limiting behavior of $X_n$
- linear statistics of $X_n$ etc.

# Mass asymptotics of random holomorphic sections

For a continous function $g : M \to \mathbb{R}$ consider the random variables

$$X_n^g : H^0(M, L^{\otimes n}) \to \mathbb{R}$$

$$\begin{aligned} X_n^g(s_n) &= \int_M g(z)|s_n(x)|^2_{h^{\otimes n}} dV \\ &= \langle T_n^g(s_n), s_n \rangle_n \end{aligned}$$

where $T_n = \Pi_n \circ M^g$ is the Toeplitz operator with multiplier $g$.

# Mass asymptotics of random holomorphic sections



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where $T_n = \Pi_n \circ M^g$ is the Toeplitz operator with multiplier $g$.On the other hand,

$$\begin{aligned} \mathbb{E}(X_n^g) &= \int_M g(z)\mathbb{E}|s_n(x)|^2_{h^{\otimes n}} dV \\ &= \int_M g(z) \sum_j |S_j^n(x)|^2_{h^{\otimes n}} dV \\ &= \int_M g(z) B_n(x) dV = Tr(T_n^g). \end{aligned}$$

# Bergman Kernel Asymptotics

**Theorem (Berman '09)**

*Let $L \to M$ be an ample line bundle endowed with a $\mathcal{C}^2$ metric $e^{-\varphi}$ and $dV$ be a smooth volume form. Then*

$$\frac{1}{d_n} B_n(x) dV \to \mu_{\varphi_e}$$

*in weak* topology on $M$. Moreover,*

$$\frac{1}{d_n} |K_n(x,y)|^2_{h^{\otimes n}} dV(x) \wedge dV(y) \to \Delta \wedge \mu_{\varphi_e}$$

Here $\Delta := [\{x = y\}]$ denotes the current of integration along the diagonal in $M \times M$ and for any bounded continuous function $\Psi$ we have

$$\int_{M\times M} \Psi(x,y)\Delta \wedge \mu_{\varphi_e} := \int_{S_\varphi} \Psi(x,x) d\mu_{\varphi_e}.$$

# Mass asymptotics of random holomorphic sections

**Corollary**

*In the above setting, we have*

$$\frac{1}{d_n}\mathbb{E}[X_n^g] \to \int_{\mathbb{C}^m} g(z)d\mu_{\varphi_e} \text{ as } n \to \infty.$$

*Moreover, if the random coefficients are i.i.d. Gaussian then*

$$Var[X_n^g] = O(d_n).$$

# Sub-Gaussian random variables



A real valued random variable $X : \Omega \to \mathbb{R}$ is called subgaussian with parameter $b > 0$ (or $b$-subgaussian) if the moment generating function (MGF) of $X$ is dominated by MGF of normalized Gaussian $N(0, b)$ that is

$$\mathbb{E}[e^{tX}] \leq e^{\frac{b^2 t^2}{2}} \text{ for all } t \in \mathbb{R}.$$

The classical examples of 1-subgaussian random variables are Standard Gaussian $N(0, 1)$, Bernoulli random variables $\mathbb{P}[X = \pm 1] = \frac{1}{2}$, and uniform distribution on $[-1, 1]$. Moreover, all bounded random variables of mean zero are subgaussian.

# Mass asymptotics of random holomorphic sections



**Theorem (B. '20)**

*Assume that the random coefficients $c_j$ are i.i.d. sub-Gaussians with mean zero and unit variance. Then for almost every sequence in $\prod_{n=1}^{\infty}\left(H^0(M, L^{\otimes n}), Prob_n\right)$ the masses*

$$\frac{1}{d_n}|s_n(z)|^2_{h^{\otimes n}}dV \to d\mu_{\varphi_e}$$

*in the weak-star sense on $S_\varphi$. In particular, almost surely in $\prod_{n=1}^{\infty}\left(H^0(M, L^{\otimes n}), Prob_n\right)$ the normalized currents of integration*

$$\frac{1}{n}[Z_{s_n}] \to dd^c\varphi_e$$

*in the sense of currents.*

This generalizes [Shiffman-Zelditch '99].

Thank you!