Restricted spaces of holomorphic sections vanishing along subvarieties

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Geometric and Topological Properties of Random Algebraic Varieties
Köln, October 4-6, 2023

Joint work with George Marinescu and Viêt-Anh Nguyên

Acknowledgement: Work supported by the National Science Foundation (DMS-2154273)

Plan of the talk:

1. Preliminaries and notation
2. Holomorphic sections vanishing along subvarieties
3. Restricted spaces of holomorphic sections vanishing along subvarieties
4. Restricted partial Bergman kernels and Fubini-Study currents
5. Zeros of random sequences of holomorphic sections

1. Preliminaries and notation

X compact complex manifold, $\dim X = n$, ω Hermitian form on X

If α is a smooth real closed $(1,1)$-form on X, we let ($dd^c = \frac{i}{\pi}\partial\overline{\partial}$):

$$\mathrm{PSH}(X,\alpha) = \{\varphi : X \to \mathbb{R} \cup \{-\infty\} : \varphi \text{ quasi-psh}, \ \alpha + dd^c\varphi \geq 0\}.$$

$\{\alpha\}_{\partial\overline{\partial}} \in H^{1,1}_{\partial\overline{\partial}}(X,\mathbb{R})$ is *big* if it contains a *Kähler current* T, i.e. $T \geq \varepsilon\omega$.
Set $E_+(T) = \{x \in X : \nu(T,x) > 0\}$.

Non-Kähler locus of $\{\alpha\}_{\partial\overline{\partial}}$ (Boucksom):

$$E_{nK}(\alpha) = \bigcap \big\{E_+(T) : \ T \in \{\alpha\}_{\partial\overline{\partial}} \text{ Kähler current}\big\}.$$

Then there exists a Kähler current $T_0 \in \{\alpha\}_{\partial\overline{\partial}}$ with *almost algebraic singularities* such that

$$E_{nK}(\alpha) = E_+(T_0).$$

Singular Hermitian metric h on a holomorphic line bundle $L \longrightarrow X$:

Let $X = \bigcup U_\alpha$, U_α open, $e_\alpha : U_\alpha \to L$, $g_{\alpha\beta} = e_\beta / e_\alpha \in \mathcal{O}^*_X(U_\alpha \cap U_\beta)$,

$$\{\varphi_\alpha \in L^1_{loc}(U_\alpha, \omega^n)\}_\alpha,\ |e_\alpha|_h = e^{-\varphi_\alpha},\ \varphi_\alpha = \varphi_\beta + \log|g_{\alpha\beta}| \text{ on } U_\alpha \cap U_\beta.$$

The curvature current $c_1(L,h)$ of h:

$$c_1(L,h)|_{U_\alpha} = dd^c \varphi_\alpha\,,\ \ c_1(L,h) \in c_1(L).$$

$L^p := L^{\otimes p}$, $\quad H^0(X, L^p) =$ space of global holomorphic sections of L^p

Siegel's Lemma: $\exists\, C > 0$ *such that* $\dim H^0(X, L^p) \leq Cp^n$ *for all* $p \geq 1$.

A line bundle L is called *big* if $\mathrm{Vol}(L) := \limsup_{p\to\infty} \dfrac{n!}{p^n} \dim H^0(X, L^p) > 0$.

Ji-Shiffman: *L is big if and only if there exists a singular Hermitian metric h on L such that $c_1(L,h)$ is a Kähler current.*

2. Holomorphic sections vanishing along subvarieties

(A) X is a compact, irreducible, normal complex space, $\dim X = n$.

(B) L is a holomorphic line bundle on X.

(C) $\Sigma = (\Sigma_1, \dots, \Sigma_\ell)$, $\Sigma_j \not\subset X_{\mathrm{sing}}$, are distinct irreducible proper analytic subsets of X. We set

$$\Sigma^\cup = \bigcup_{j=1}^{\ell} \Sigma_j .$$

(D) $\tau = (\tau_1, \dots, \tau_\ell)$, $\tau_j \in (0, +\infty)$, and $\tau_j > \tau_k$ if $\Sigma_j \subset \Sigma_k$.

$H^0_0(X, L^p)$ space of sections vanishing to order $\geq \tau_j p$ along Σ_j, $1 \leq j \leq \ell$:

$$t_{j,p} := \tau_j p \text{ if } \tau_j p \in \mathbb{N}, \;\; t_{j,p} := \lfloor \tau_j p \rfloor + 1 \text{ if } \tau_j p \notin \mathbb{N}, \;\; 1 \leq j \leq \ell, \; p \geq 1.$$

$$H^0_0(X, L^p) = H^0_0(X, L^p, \Sigma, \tau) := \left\{ S \in H^0(X, L^p) : \operatorname{ord}(S, \Sigma_j) \geq t_{j,p} \right\}$$

We say that **the triplet** (L, Σ, τ) **is big** if

$$\mathrm{Vol}_{X,\Sigma,\tau}(L) := \limsup_{p\to\infty} \frac{n!}{p^n} \dim H^0_0(X, L^p) > 0.$$

Characterization when X is a complex manifold and $\dim \Sigma_j = n-1$:

Theorem 1

Let L be a holomorphic line bundle on a compact complex manifold X, $\dim X = n$, and $\Sigma = (\Sigma_1, \ldots, \Sigma_\ell)$, $\tau = (\tau_1, \ldots, \tau_\ell)$, where Σ_j are distinct irreducible complex hypersurfaces in X, $\tau_j \in (0, +\infty)$. TFAE:

(i) (L, Σ, τ) is big;

(ii) There exists a singular metric h on L such that $c_1(L, h) - \sum_{j=1}^{\ell} \tau_j[\Sigma_j]$ is a Kähler current on X, where $[\Sigma_j]$ is the current of integration along Σ_j;

(iii) $\exists\, c > 0$, $p_0 \geq 1$, such that $\dim H^0_0(X, L^p) \geq cp^n$ for all $p \geq p_0$.

Proposition 2

Let X, Σ *verify (A), (C). Then there exist a compact complex manifold* $\widetilde{X}$*,* $\dim \widetilde{X} = n$*, and a surjective holomorphic map* $\pi : \widetilde{X} \to X$*, given as the composition of finitely many blow-ups with smooth center, such that:*

(i) $\exists\, X_\pi \subset X$ *analytic subset such that* $\dim X_\pi \leq n-2$, $X_{\rm sing} \subset X_\pi$, $\Sigma_j \subset X_\pi$ *if* $\dim \Sigma_j \leq n-2$, $X_\pi \subset X_{\rm sing} \cup \Sigma^{\cup}$, $E_\pi = \pi^{-1}(X_\pi)$ *is a normal crossings divisor in* $\widetilde{X}$*, and* $\pi : \widetilde{X} \setminus E_\pi \to X \setminus X_\pi$ *is a biholomorphism.*

(ii) There exist smooth complex hypersurfaces $\widetilde{\Sigma}_1, \ldots, \widetilde{\Sigma}_\ell$ *in* $\widetilde{X}$ *such that* $\pi(\widetilde{\Sigma}_j) = \Sigma_j$*. If* $\dim \Sigma_j = n-1$ *then* $\widetilde{\Sigma}_j$ *is the final strict transform of* Σ_j*, and if* $\dim \Sigma_j \leq n-2$ *then* $\widetilde{\Sigma}_j$ *is an irreducible component of* E_π*.*

(iii) If $F \to X$ *is a holomorphic line bundle and* $S \in H^0(X, F)$ *then* $\operatorname{ord}(S, \Sigma_j) = \operatorname{ord}(\pi^\star S, \widetilde{\Sigma}_j)$*, for all* $j = 1, \ldots, \ell$*.*

If $\widetilde{X}$, π, $\widetilde{\Sigma} := (\widetilde{\Sigma}_1, \ldots, \widetilde{\Sigma}_\ell)$, verify the conclusions of Proposition 2, we say that $(\widetilde{X}, \pi, \widetilde{\Sigma})$ is a **divisorization** of (X, Σ). Then

$$H^0_0(X, L^p, \Sigma, \tau) \cong H^0_0(\widetilde{X}, \pi^\star L^p, \widetilde{\Sigma}, \tau)\,, \ \forall\, p \geq 1.$$

Theorem 3

Let X, L, Σ, τ verify assumptions (A)-(D). The following are equivalent:

(i) (L, Σ, τ) is big;

(ii) $\forall\, (\widetilde{X}, \pi, \widetilde{\Sigma})$ divisorization of (X, Σ), $\exists\, h^\star$ singular metric on $\pi^\star L$ such that $c_1(\pi^\star L, h^\star) - \sum_{j=1}^{\ell} \tau_j [\widetilde{\Sigma}_j]$ is a Kähler current on $\widetilde{X}$;

(iii) Assertion (ii) holds for some divisorization $(\widetilde{X}, \pi, \widetilde{\Sigma})$ of (X, Σ);

(iv) $\exists\, c > 0$, $p_0 \geq 1$, such that $\dim H^0_0(X, L^p) \geq c p^n$ for all $p \geq p_0$.

Given a triplet (L, Σ, τ) and a divisorization $(\widetilde{X}, \pi, \widetilde{\Sigma})$ of (X, Σ), we consider the cohomology class

$$\Theta_\pi = \Theta_{\pi,L,\Sigma,\tau} := c_1(\pi^\star L) - \sum_{j=1}^{\ell} \tau_j \{\widetilde{\Sigma}_j\}_{\partial\overline{\partial}}\,,$$

where $\{\widetilde{\Sigma}_j\}_{\partial\overline{\partial}} \in H^{1,1}_{\partial\overline{\partial}}(\widetilde{X}, \mathbb{R})$ denotes the class of the current $[\widetilde{\Sigma}_j]$.

Note that the following are equivalent:

(i) There exists a singular Hermitian metric $h^\star$ on $\pi^\star L$ such that $c_1(\pi^\star L, h^\star) - \sum_{j=1}^{\ell} \tau_j[\widetilde{\Sigma}_j]$ is a Kähler current on $\widetilde{X}$.

(ii) The class $\Theta_\pi \in H^{1,1}_{\partial\overline{\partial}}(\widetilde{X}, \mathbb{R})$ is big.

3. Restricted spaces of holomorphic sections vanishing along subvarieties

Hisamoto (2012): L big line bundle over a projective manifold X,
$Y \subset X$ complex submanifold, $\dim Y = m$,
$Y \not\subset E_{nK}\big(c_1(L)\big)$ (augmented base locus of L).

Restricted volume: $\mathrm{Vol}_{X|Y}(L) := \limsup_{p\to\infty} \dfrac{m!}{p^m} \dim \big\{ S|_Y : \, S \in H^0(X, L^p) \big\}$

Assume that X, L, Σ, τ verify (A)-(D), (L, Σ, τ) is big, and that

(E) Y is an irreducible analytic subset of X, $\dim Y = m$, such that

$$Y \not\subset X_{\mathrm{sing}} \cup \Sigma^{\cup} \cup A\,,$$

where

$$A := \bigcap \big\{ \pi\big(E_{nK}\big(\Theta_\pi\big)\big) : \, (\widetilde{X}, \pi, \widetilde{\Sigma}) \text{ is a divisorization of } (X, \Sigma) \big\}.$$

We consider here the subspace $H^0_0(X|Y, L^p) \subset H^0(Y, L^p|_Y)$,

$$H^0_0(X|Y, L^p) = H^0_0(X|Y, L^p, \Sigma, \tau) := \{S|_Y : S \in H^0_0(X, L^p)\}.$$

Restricted volume of L relative to Y with vanishing along (Σ, τ):

$$\mathrm{Vol}_{Y,\Sigma,\tau}(L) := \limsup_{p\to\infty} \frac{m!}{p^m} \dim H^0_0(X|Y, L^p).$$

Theorem 4

If X, L, Σ, τ verify (A)-(D), (L, Σ, τ) is big and X is a Kähler space, then

$$\mathrm{Vol}_{Y,\Sigma,\tau}(L) > 0$$

for any analytic subset $Y \subset X$ that verifies (E). In fact, if Y verifies (E) then there exist constants $C > 0, p_0 \in \mathbb{N}$ such that

$$\dim H^0_0(X|Y, L^p) \geq Cp^m\,, \ \forall\, p > p_0.$$

The proof uses Bonavero's singular holomorphic Morse inequalities, and an Ohsawa-Takegoshi-Manivel L^2 extension theorem due to Hisamoto.

By (E), fix a divisorization $(\widetilde{X}, \pi, \widetilde{\Sigma})$ of (X, Σ) such that

$$Y \not\subset X_{\rm sing} \cup \Sigma^{\cup} \cup \pi\big(E_{nK}(\Theta_\pi)\big).$$

Let $\widetilde{Y}$ be the strict transform of Y and E_π be the final exceptional divisor.

$\exists\, \widetilde{\pi} : \widehat{X} \to \widetilde{X}$ a composition of finitely many blow-ups with smooth center, and final exceptional divisor $\widehat{E}$, such that the strict transform $\widehat{Y}$ of $\widetilde{Y}$ is smooth and $\widehat{Y}, \widehat{E}_\pi, \widehat{E}$ have simultaneously only normal crossings.

Let $\widehat{\Sigma}_j$ be the strict transform of $\widetilde{\Sigma}_j$ under $\widetilde{\pi}$, $\widehat{\Sigma}^{\cup} := \bigcup_{j=1}^{\ell} \widehat{\Sigma}_j$. Set

$$\widehat{\pi} := \pi \circ \widetilde{\pi} : \widehat{X} \to X\,,\ \ \widehat{L} := \widehat{\pi}^\star L\,,\ \ \widehat{\Theta} := c_1(\widehat{L}) - \sum_{j=1}^{\ell} \tau_j \{\widehat{\Sigma}_j\} \in H^{1,1}(\widehat{X}, \mathbb{R}).$$

We show that $\widehat{\Theta}$ is big and $\widehat{Y} \not\subset E_{nK}\big(\widehat{\Theta}\big) \cup \widehat{\Sigma}^{\cup}$.

4. Restricted partial Bergman kernels and Fubini-Study currents

Assume: X, Y, L, Σ, τ verify (A)-(E)

ω is a Kähler form on X

h is a singular metric on L

If h_0 is a smooth Hermitian metric on L then

$$\alpha := c_1(L, h_0)\,,\ \ h = h_0 e^{-2\varphi}\,,\ \text{ so } c_1(L, h) = \alpha + dd^c\varphi.$$

$\varphi \in L^1(X, \omega^n)$ is called the *(global) weight of h relative to h_0*.

h is called continuous, resp. Hölder continuous, if φ is as such on X.

$H^0_{(2)}(X, L^p) =$ *Bergman space of L^2-holomorphic sections of L^p* relative to the metric $h^p := h^{\otimes p}$ on L^p and volume ω^n on X

$$(S, S')_p := \int_X \langle S, S'\rangle_{h^p} \frac{\omega^n}{n!}\,,\ \ \|S\|_p^2 := (S, S)_p$$

If h is **continuous** then:

$$H^0_0(X, L^p) \subset H^0(X, L^p) = H^0_{(2)}(X, L^p),$$

$h|_Y = h_0|_Y e^{-2\varphi|_Y}$ is a well defined singular metric on $L|_Y$,

$$H^0_0(X|Y, L^p) \subset H^0_{(2)}(Y, L^p|_Y, h^p|_Y, \omega^m|_Y).$$

Notation when $H^0_0(X|Y, L^p)$ is considered as a Bergman space:

$$H^0_{0,(2)}(X|Y, L^p) = H^0_{0,(2)}(X|Y, L^p, \Sigma, \tau, h^p, \omega^m) := H^0_0(X|Y, L^p)$$

$d_p + 1 = \dim H^0_{0,(2)}(X|Y, L^p)$, $\{S^p_0, \ldots, S^p_{d_p}\}$ o.n. basis of $H^0_{0,(2)}(X|Y, L^p)$

(Restricted partial) Bergman kernel of $H^0_{0,(2)}(X|Y, L^p)$:

$$P^Y_p(x) = \sum_{j=0}^{d_p} |S^p_j(x)|^2_{h^p}, \; x \in Y$$

e_U holomorphic frame of L on $U \subset Y$ open,

$|e_U|_h = e^{-\varphi_U}$, $S_j^p = s_j^p e_U^{\otimes p}$, $\varphi_U \in \mathscr{C}(U)$, $s_j^p \in \mathscr{O}_Y(U)$

Fubini-Study current of $H^0_{0,(2)}(X|Y, L^p)$:

$$\gamma_p^Y|_U = \frac{1}{2}\, dd^c \log\Big(\sum_{j=0}^{d_p} |s_j^p|^2\Big)$$

Have: $\log P_p^Y \,|_U = \log\Big(\sum_{j=0}^{d_p} |s_j^p|^2\Big) - 2p\varphi_U$, $\log P_p^Y \in L^1(Y, \omega^m|_Y)$

$$\tfrac{1}{p}\,\gamma_p^Y = c_1(L,h)|_Y + \tfrac{1}{2p}\, dd^c \log P_p^Y = \alpha|_Y + dd^c \varphi_p^Y,$$

$$\varphi_p^Y = \varphi|_Y + \tfrac{1}{2p} \log P_p^Y = \textit{global Fubini-Study potential of } \gamma_p^Y/p.$$

Note that: $\gamma_p^Y \geq 0$ has local psh potentials, $\varphi_p^Y \in PSH(Y, \alpha|_Y)$.

Theorem 5

Let X, Y, L, Σ, τ verify (A)-(E). Assume that (L, Σ, τ) is big, ω is a Kähler form on X and h is a continuous Hermitian metric on L. Then there exists a weakly $\alpha|_Y$-psh function φ^Y_{eq} on Y such that, as $p \to \infty$,

$$\int_Y |\varphi^Y_p - \varphi^Y_{\mathrm{eq}}|\,\omega^m \to 0\,,\ \ \frac{1}{p}\gamma^Y_p = \alpha|_Y + dd^c\varphi^Y_p \to T^Y_{\mathrm{eq}} := \alpha|_Y + dd^c\varphi^Y_{\mathrm{eq}}\,,$$

weakly on Y. If h is Hölder continuous then $\exists\, C > 0$, $p_0 \in \mathbb{N}$, such that

$$\int_Y |\varphi^Y_p - \varphi^Y_{\mathrm{eq}}|\,\omega^m \le C\,\frac{\log p}{p}\,,\ \ \forall\, p \ge p_0\,.$$

Definition 6

The current T^Y_{eq} from Theorem 5 is called the *equilibrium current associated to* (Y, L, h, Σ, τ).

Related results. *Consider the following setting:*

(L, h) *singular Hermitian holomorphic line bundle on a compact normal Kähler space* (X, ω), $\dim X = n$

Q_p, $\delta_p =$ *(full) Bergman kernel function, resp. Fubini-Study current, of the Bergman space* $H^0_{(2)}(X, L^p, h^p, \omega^n)$

Tian (1990): If X is a projective manifold and (L, h) is *positive* ($c_1(L, h)$ is a Kähler form), then $\frac{1}{p}\,\delta_p \to c_1(L, h)$ in the $\mathscr{C}^\infty$ topology on X.

In the above singular setting:

If $c_1(L, h)$ is a Kähler current then $\frac{1}{p}\,\delta_p \to c_1(L, h)$ in the weak sense of currents on X (Coman-Ma-Marinescu 2017).

Our Theorem 6 applies with $\Sigma = \emptyset$, $Y = X$, and shows that $\frac{1}{p}\,\delta_p \to T_{\mathrm{eq}}$ in the weak sense of currents on X.

Proof of Theorem 5:

Recall: $(\widetilde{X}, \pi, \widetilde{\Sigma})$ divisorization with $Y \not\subset X_{\rm sing} \cup \Sigma^{\cup} \cup \pi\big(E_{nK}(\Theta_\pi)\big)$,
$\widehat{\pi} := \pi \circ \widetilde{\pi} : \widehat{X} \to X$, the strict transform $\widehat{Y}$ of Y is smooth,
$\widehat{\Sigma}_j$ is the strict transform of $\widetilde{\Sigma}_j$ under $\widetilde{\pi}$, $\widehat{\Sigma} := (\widehat{\Sigma}_1, \ldots, \widehat{\Sigma}_\ell)$.

Let $\widehat{\omega}$ be a Kähler form on $\widehat{X}$ such that $\widehat{\omega} \geq \widehat{\pi}^\star\omega$, and set

$$\widehat{L} = \widehat{\pi}^\star L\,,\ \widehat{h}_0 = \widehat{\pi}^\star h_0\,,\ \widehat{\alpha} = \widehat{\pi}^\star \alpha = c_1(\widehat{L}, \widehat{h}_0)\,,\ \widehat{h} = \widehat{\pi}^\star h\,,\ \widehat{\varphi} = \varphi \circ \widehat{\pi}.$$

The map $\widehat{\pi}^\star$ is an isometry,

$$S \in H^0_{0,(2)}(X|Y, L^p) \longrightarrow \widehat{\pi}^\star S \in H^0_{0,(2)}(\widehat{X}|\widehat{Y}, \widehat{L}^p) = H^0_{0,(2)}(\widehat{X}|\widehat{Y}, \widehat{L}^p, \widehat{\Sigma}, \tau, \widehat{h}^p, \widehat{\pi}^\star\omega^m).$$

$$\widehat{P}_p^{\widehat{Y}} = P_p^Y \circ \widehat{\pi}\,,\ \widehat{\gamma}_p^{\widehat{Y}} = \widehat{\pi}^\star \gamma_p^Y\,,\ p^{-1}\,\widehat{\gamma}_p^{\widehat{Y}} = \widehat{\alpha}|_{\widehat{Y}} + dd^c \widehat{\varphi}_p^{\widehat{Y}}\,,\ \widehat{\varphi}_p^{\widehat{Y}} = \varphi_p^Y \circ \widehat{\pi},$$

are the Bergman kernel, resp. Fubini-Study current, of $H^0_{0,(2)}(\widehat{X}|\widehat{Y}, \widehat{L}^p)$.

Equilibrium envelope of $(\widehat{\alpha}, \widehat{Y}, \widehat{\Sigma}, \tau, \widehat{\varphi})$:

$$\widehat{\varphi}_{\mathrm{eq}}^{\widehat{Y}}(x) = \sup\{\psi(x) : \psi \in \mathcal{L}(\widehat{X}, \widehat{\alpha}, \widehat{\Sigma}, \tau), \ \psi \le \widehat{\varphi} \text{ on } \widehat{Y}\}, \ x \in \widehat{Y},$$

$$\mathcal{L}(\widehat{X}, \widehat{\alpha}, \widehat{\Sigma}, \tau) = \big\{\psi \in \mathrm{PSH}(\widehat{X}, \widehat{\alpha}) : \nu(\psi, x) \ge \tau_j, \forall\, x \in \widehat{\Sigma}_j, 1 \le j \le \ell\big\}.$$

Theorem 7

In the setting of Theorem 5, we have $\widehat{\varphi}_p^{\widehat{Y}} \to \big(\widehat{\varphi}_{\mathrm{eq}}^{\widehat{Y}}\big)^\star$ *in* $L^1\big(\widehat{Y}, \widehat{\omega}^m|_{\widehat{Y}}\big)$ *as* $p \to \infty$*. If* φ *is Hölder continuous on* Y *then* $\exists\, C > 0$, $p_0 \in \mathbb{N}$*, such that*

$$\int_{\widehat{Y}} \big|\widehat{\varphi}_p^{\widehat{Y}} - \big(\widehat{\varphi}_{\mathrm{eq}}^{\widehat{Y}}\big)^\star\big|\, \widehat{\omega}^m \le C\, \frac{\log p}{p}, \ \forall\, p \ge p_0.$$

Note $\widehat{\pi} : \widehat{X} \setminus \widehat{\pi}^{-1}(Z) \to X \setminus Z$ is biholomorphism, $Y \not\subset Z$. Define

$$\varphi_{\mathrm{eq}}^{Y} := \big(\widehat{\varphi}_{\mathrm{eq}}^{\widehat{Y}}\big)^\star \circ \widehat{\pi}^{-1} \in PSH(Y_{\mathrm{reg}} \setminus Z, \alpha|_Y).$$

Since it is upper bounded, it extends to a $\alpha|_Y$-psh function on Y_{reg}.

5. Zeros of random sequences of holomorphic sections

Projectivization of spaces of holomorphic sections

$$\mathbb{X}_p^Y := \mathbb{P}H^0_{0,(2)}(X|Y, L^p)\,,\ \ d_p = \dim \mathbb{X}_p^Y\,,\ \ \sigma_p := \omega_{\mathrm{FS}}^{d_p}\,.$$

Product probability space $\quad (\mathbb{X}_\infty^Y, \sigma_\infty) := \prod_{p=1}^{\infty} (\mathbb{X}_p^Y, \sigma_p)$

Using the Dinh-Sibony equidistribution theorem for meromorphic transforms we obtain the following theorems:

Theorem 8

Let X, Y, L, Σ, τ verify (A)-(E). Assume that (L, Σ, τ) is big, ω is a Kähler form on X and h is a continuous Hermitian metric on L. Then

$$\frac{1}{p}\,[s_p = 0] \to T_{\mathrm{eq}}^Y\,,\ \textit{as } p \to \infty,$$

in the weak sense of currents on Y, for σ_∞-a.e. $\{s_p\}_{p\geq 1} \in \mathbb{X}_\infty^Y$.

Theorem 9

Let X, Y, L, Σ, τ verify (A)-(E). Assume that (L, Σ, τ) is big, ω is a Kähler form on X and h is a Hölder continuous Hermitian metric on L. Then there exists a constant $c > 0$ with the following property:

For any sequence $\lambda_p > 0$, $p \geq 1$, such that $\liminf_{p\to\infty} \frac{\lambda_p}{\log p} > (1+m)c$, there exist subsets $E_p \subset \mathbb{X}_p^Y$ such that, for all p sufficiently large,

(a) $\sigma_p(E_p) \leq c p^m \exp(-\lambda_p / c)$,

(b) if $s_p \in \mathbb{X}_p^Y \setminus E_p$ we have

$$\left| \left\langle \frac{1}{p} [s_p = 0] - T_{\rm eq}^Y, \phi \right\rangle \right| \leq \frac{c\lambda_p}{p} \|\phi\|_{\mathscr{C}^2}, \ \ \forall \, \phi \in \mathscr{C}^2_{m-1,m-1}(Y).$$

In particular, the estimate from (b) holds for σ_∞-a.e. $\{s_p\}_{p\geq 1} \in \mathbb{X}_\infty^Y$ provided that p is large enough.