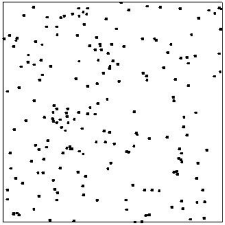
Rigidity phenomena in Strongly correlated random point fields and The emergence of forbidden regions

Subhro Ghosh
National University of Singapore

Key models



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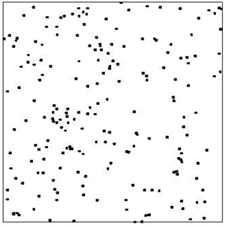

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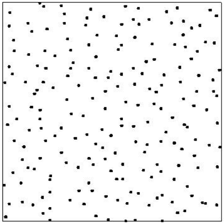

Gini-
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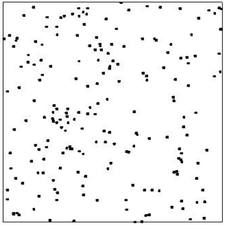
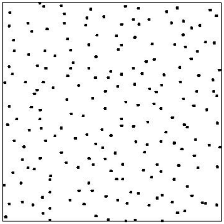
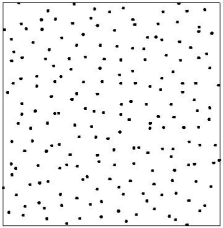


Poisson Process



Ginibre Eigenvalues



Gaussian Zeroes

Preliminaries

- We consider *point processes*, a.k.a. *random point fields*, living on a suitable ambient space, e.g. Euclidean spaces
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- We consider the conditional distribution $\rho_\omega(\zeta)$ of the points (denoted by ζ) inside a bounded open set $\mathcal{D}$ given the points outside $\mathcal{D}$ (denoted by ω)
- In Poisson point process, the points inside and outside $\mathcal{D}$ are independent of each other

Ginibre Ensemble

- Finite n: μ_n = Eigenvalues of $G_n = ((\xi_{ij}))_{1 \le i,j \le n}$, ξ_{ij} i.i.d $N_{\mathbb{C}}(0,1)$ (NO normalization by $\sqrt{n}$)

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- Introduced by the physicist J. Ginibre as a model for non-Hermitian matrix ensembles, motivated by Wigner's Hermitian matrix models of random Hamiltonians. Considered as a model for interacting particle systems in dimension 2, c.f. Forrester's "Log-gases and random matrices".

Gaussian Analytic Function

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- Arises in the study of quantum chaotic dynamics, eg. in the work of Bogomolnyi, Bohigas, Leboeuf.

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$\Sigma_{S(\omega)}$: constant sum hypersurface $\sum_{i=1}^{N(\omega)} \zeta_i = S(\omega)$ inside $\mathcal{D}^{N(\omega)}$

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- Natural point processes for Levels $k \geq 3$??

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- Phase transition in the rigidity behaviour in α at the values $\alpha = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$. (rigid at level k for $\alpha \in (\frac{1}{k}, \frac{1}{k-1}]$).

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- The standard planar GAF is Gaussian r.v. in the space of L^2 analytic functions w.r.t. complex Gaussian measure. The α-GAFs are Gaussian r.v. in the space of L^2 analytic functions w.r.t. other measures on $\mathbb{C}$ (roughly, density $\sim e^{-|z|^{2/\alpha}}$).

Theorem (Gangopadhyay-G.-Tan, Commun. Pure Appl. Math. '23)

For the general α-Gaussian zero ensembles, as well as the Ginibre ensemble, there are positive quantities $m(\omega), M(\omega)$ such that the conditional density f_ω satisfies, on its support $\Xi(\omega)$, the following :

$$m(\omega) \exp(2 \sum_{i \neq j} \log |\zeta_i - \zeta_j|) \mathbb{1}_{\Xi(\omega)} \leq f_\omega(\zeta),$$

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Approximate Gibbsian structures in point processes

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Some Interesting Results en route

Theorem (Reconstruction of Gaussian Analytic Function, G.-Peres)

*The zeroes of the GAF determine the function a.s. (up to a multiplicative factor of modulus 1). In other words, if ν denotes the zeroes of the GAF f, then $\exists$ an analytic function $g(z) = \sum_{k=0}^{\infty} a_k(\nu) z^k$ such that $f(z) = \gamma . g(z)$
Here γ follows $Unif(S^1)$ and is independent of ν.*

Mutual singularity of Palm measures:

Theorem (G., Elec. Comm. Probab. '16)

For a point process P on a background space $\Xi(\mathbb{R}$ or $\mathbb{C})$ which is rigid at level k, and point configurations $x \in \Xi^m, y \in \Xi^n$, the reduced Palm measures P_x and P_y are mutually absolutely continuous if and only if we have matching moments $m_i(x) = m_i(y)$ for $i \leq k$.

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- In particular, for the GAF zero process, the Palm measures ρ_z and ρ_w are mutually singular for a.e. z, w (in spite of translation invariance).
- Rigidity phenomena have subsequently seen extensive works by Bufetov, Dereudre, Leble, Maida, Leble, Shirai, Najnudel ....

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- For Gaussian random polynomials of high degree (growing as αR^2)The probability of this event is $\asymp e^{-cR^4}$.

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- For Gaussian random polynomials of high degree (growing as αR^2)The probability of this event is $\asymp e^{-cR^4}$.
- The constant c is explicitly known.

Question (Key questions)

- *Given that there is a hole of size R, how do the zeroes look like, even in expectation ?*

The emergence of *Forbidden Regions*

- Comparison with GUE eigenvalue process on $\mathbb{R}$: the conditional intensity exhibits a spike at R and subsequently decreases to the equilibrium intensity. (Majumdar et. al.)

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- Comparison with GUE eigenvalue process on $\mathbb{R}$: the conditional intensity exhibits a spike at R and subsequently decreases to the equilibrium intensity. (Majumdar et. al.)
- Comparison with Ginibre eigenvalue ensemble on $\mathbb{C}$: the conditional intensity exhibits a singular component at R and subsequently decreases to the equilibrium intensity. (Lebowitz et. al., Shirai)

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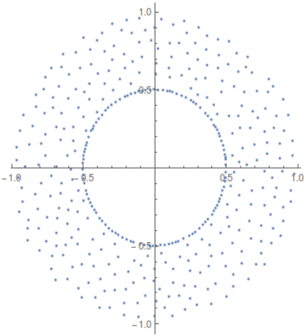
- *There is a singular component at the edge of the hole*
- *There is subsequent "forbidden region", namely, in the annulus* $R < r < \sqrt{e}R$, *the conditional intensity* $\to 0$ *as* $R \to \infty$.
- *Beyond* $\sqrt{e}R$, *the conditional intensity behaves, in the limit* $R \to \infty$, *like the equilibrium intensity.*

The emergence of *Forbidden Regions*

Theorem (with A. Nishry, Commun. Pure Appl. Math. '19)

The conditional intensity for zeroes of Gaussian random polynomials has the following behaviour:

- *There is a singular component at the edge of the hole*
- *There is subsequent "forbidden region", namely, in the annulus* $R < r < \sqrt{e}R$, *the conditional intensity* $\to 0$ *as* $R \to \infty$.
- *Beyond* $\sqrt{e}R$, *the conditional intensity behaves, in the limit* $R \to \infty$, *like the equilibrium intensity.*
- *Subsequent results on holes of more general shape (Nishry-Wenmann)*



Gaus-
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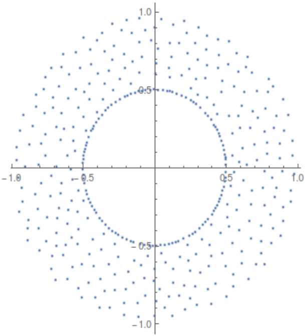
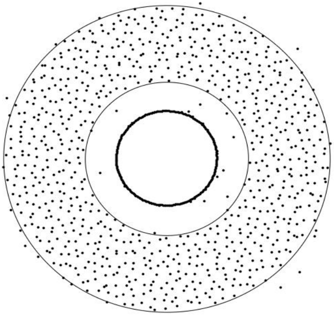
Matrices





Gaussian Matrices

Gaussian Polynomials

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- Explicit computations possible on sphere, torus ....

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- Know outside zeroes $\Rightarrow$ Know $\int_{\mathcal{D}_L \setminus \mathcal{D}} \varphi_L d\nu \Rightarrow$ Compute $n(\mathcal{D})$ approximately, now let $L \to \infty$

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Happy Birthday to Prof Dinh !!

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