The generalized Lelong numbers and Intersection theory

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Day in honour of Prof. Dinh's birthday

Plan of the talk: (8 sections)

1. Preliminaries, notation and known results and motivations
2. New spaces of currents, strongly admissible maps and the generalized Lelong numbers
3. Statement of the first main results

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3. Statement of the first main results
4. Lelong-Jensen formula for holomorphic vector bundle
5. Sketchy proof
6. Horizontal dimension and Siu's upper-semicontinuity

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- T is called *positive* and write $T \geq 0$ if $T \wedge \varphi := \langle T, \varphi \rangle \geq 0$ for any smooth positive test form φ of bidim (p,p) [Lelong 1957]

Consider the differentiel operators acting on the space of currents on X:

$$d = \partial + \overline{\partial}, \quad d^c = \frac{1}{2\pi i}(\partial - \overline{\partial}), \quad dd^c = \frac{i}{\pi}\partial\overline{\partial}$$

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$$\{\text{currents of integrations on complex subvarieties of codim } p\} \subsetneq \mathrm{CL}^p \subsetneq \mathrm{PH}^p \subsetneq \mathrm{SH}^p$$

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$$\nu(T,x) := \lim_{r\to 0} \frac{\sigma_T(\mathbb{B}(0,r))}{(2\pi)^{k-p} r^{2k-2p}}, \quad \text{where} \quad \sigma_T := \frac{1}{(k-p)!}\, T\wedge\left(\frac{i}{2}\partial\overline{\partial}\|z\|^2\right)^{k-p}$$

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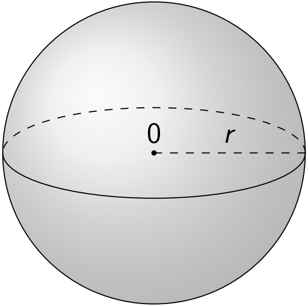

Figure: An illustrations of a ball $\mathbb{B}(x, r)$ with center $x = 0$ and radius r in $\mathbb{C}^k$.

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[Skoda 1982]: The result of [Lelong 1957] holds for $T \in \mathrm{SH}^p$

Logarithmic definition of Lelong number

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First interpretation: to regularize the current T (e.g. a standard convolution), $\exists\ (T_n)_{n=1}^{\infty} \subset \mathrm{SH}^p \cap \mathcal{C}^\infty(\mathbb{B}(0, r+\epsilon))$ for some $\epsilon > 0$ such that $T_n \to T$.

$$I_r := \lim_{n\to\infty} \int_{\mathbb{B}(0,r)} T_n(z) \wedge (dd^c \log(\|z\|^2))^{k-p}.$$

The integral on RHS is meaningful by Fornæss-Sibony, Demailly etc.

Second interpretation: to regularize the integral kernel $(dd^c \log(\|z\|^2))^{k-p}$ in a canonical way:

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Definition of logarithmic pointed Lelong number

If $T\in \mathrm{CL}^p$, $x\in X$ then

$$\lim_{r\to 0} I^\bullet_r = 0, \quad \text{where} \quad I^\bullet_r := \int_{\mathbb{B}(0,r)\setminus\{0\}} T(z)\wedge (dd^c\log(\|z\|^2))^{k-p}.$$

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The class of $\mathbb{T}_\infty$ (resp. of T_∞) in the de Rham cohomology of $\mathbb{P}^{k-1}$ (resp., of $\mathbb{P}^k$) is equal to $\nu(T, x)$ times the class of a linear subspace.

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[Kiselman 1991]: In general, the tangent current T_∞ is not unique

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Dinh-Sibony's idea: a softer notion: *the admissible maps*
Let τ be a diffeomorphism between a neighbourhood of V in X and a neighbourhood of V in $\mathbb{E}$ whose restriction to V is identity. Assume that τ is admissible in the sense that the endomorphism of $\mathbb{E}$ induced by the differential of τ is the identity map from $\mathbb{E}$ to $\mathbb{E}$.

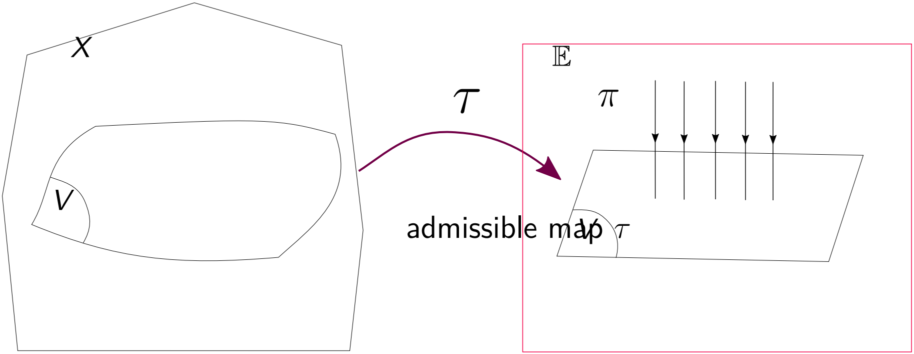

Figure: In the approach of Dinh and Sibony, admissible maps replace holomorphic changes of coordinates.

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(3) Let $-h_{\overline{\mathbb{E}}}$ *denote the tautological class of the bundle* $\overline{\pi} : \overline{\mathbb{E}} \to V$. *Then*

$$\{\mathbf{c}^{\mathrm{DS}}(T)\} = \sum_{j=\underline{\mathrm{m}}}^{\overline{\mathrm{m}}} \overline{\pi}^*(\mathbf{c}_j^{\mathrm{DS}}(T)) \smile h_{\overline{\mathbb{E}}}^{j-l+p}, \quad \textit{where} \quad \mathbf{c}_j^{\mathrm{DS}}(T) \in H_c^{2l-2j}(V, \mathbb{C}).$$

Remark. When V has positive dimension $l \geq 1$, according to Dinh and Sibony, the notion of Lelong number of the current T at a single point should be replaced by the family of cohomology classes $\{\mathbf{c}_j^{\mathrm{DS}}(T) : \ \underline{\mathrm{m}} \leq j \leq \overline{\mathrm{m}}\}$

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Complex geometry, pluripotential theory: Dinh-Ng., Huynh-Vu, Kaufmann-Vu, Huynh-Kaufmann-Vu, Vu etc.

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$$\nu_{\mathrm{AB}}(T, B) := \lim_{r \to 0+} \int_{\mathrm{Tube}(B,r)} T(z, w) \wedge \Big(\frac{dd^c \|z\|^2}{r^2}\Big)^{k-l-p} \wedge (dd^c \|w\|^2)^l,$$

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2. *There exist an open neighborhood* W *of* 0 *in* L, $W \subset \Omega$, *and a nonnegative plurisubharmonic function* f *on* W *such that*

$$\nu_{\mathrm{AB}}(T, B) = \int_B f(w)(dd^c \|w\|^2)^l$$

   for every open ball B *in* V *with* $B \Subset W$.

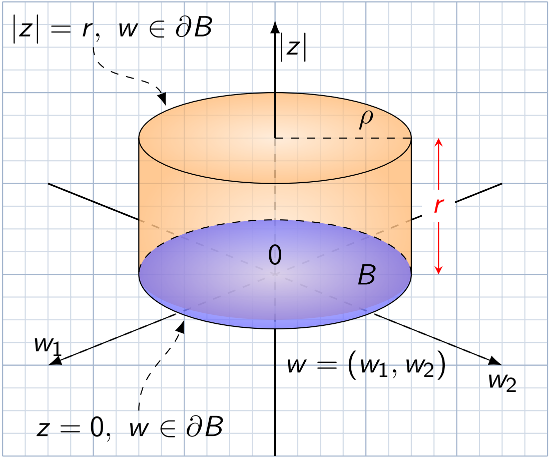

Figure: An illustrations of a tube $\mathrm{Tube}(B, r)$ in $\mathbb{C}^3$ with coordinates $(z, w) \in \mathbb{C} \times \mathbb{C}^2$, where the base B is a ball with center $0 \in \mathbb{C}^2$ and radius ρ in the plane $V = \{(0, w) : \; w = (w_1, w_2) \in \mathbb{C}^2\} \simeq \mathbb{C}^2$.

Theorem [Alessandrini–Bassanelli 1996] *Under the assumption of the previous theorem,* $\nu_{\mathrm{AB}}(T,B)$ *has a geometric meaning in the sense of Siu: There is a suitable blow-up model to a suitable Grassmannian manifold* $\Pi_p:\ \mathbb{X}_p\to\mathbb{C}^{k-l}\times\mathbb{C}^l$ *with center of blow-up* $V:=\{0\}\times\mathbb{C}^l$ *such that* $\nu_{\mathrm{AB}}(T,B)$ *is the mass of the cut-off current on the exceptional fiber of the weak limit* T_∞ *of the sequence* $\Pi_p^*T_n$*, where* (T_n) *is a sequence of approximating smooth forms of* T*. In other words,*

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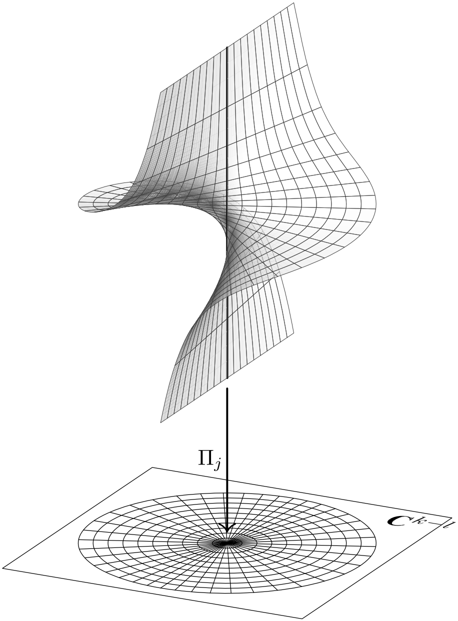
Theorem ([Siu 1974] for positive closed currents, [Alessandrini–Bassanelli 1996] for positive plurisubharmonic currents) *Let* $F:\ \Omega\to\Omega'$ *be a biholomorphic map between open subsets of* $\mathbb{C}^k$*. If* T *is a positive plurisubharmonic* (p,p)*-current on* Ω *and* $x\in\Omega$*, then*

$$\nu(T,x)=\nu(F_*T,F(x)).$$


Π_j
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2. To generalize the notion and the result of [Alessandrini–Bassanelli 1996] on Lelong numbers, and the results of [Siu 1974] and of [Alessandrini–Bassanelli 1996] on geometric characterizations of Lelong numbers to the above contexts.

2. New spaces of currents, strongly admissible maps and the generalized Lelong numbers

New spaces of currents

Let $m, m' \in \mathbb{N}$ with $m \geq m'$. Let $W \subset U \subset X$ be two open subsets. Let T be a positive (p, p)-current defined on an open set containing U. Let $\mathscr{F} \in \{\mathrm{CL}, \mathrm{PH}, \mathrm{SH}\}$.

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(1) We say that T is *approximable on U by $\mathcal{C}^m$-smooth $\mathcal{F}$-forms* and write $T \in \mathcal{F}^{p;m}(U)$ if there is a sequence of $\mathcal{C}^m$-smooth (p,p)-forms $(T_n)_{n=1}^{\infty} \subset \mathcal{F}$ defined on U such that

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If moreover, the following condition is fulfilled:

(iii-a) the restrictions of the forms T_n on W are of uniformly bounded $\mathscr{C}^{m'}$-norm;

then we say that T is *approximable on U by $\mathscr{C}^m$-smooth $\mathscr{F}$-forms with $\mathscr{C}^{m'}$-control on W*, and write $T \in \mathscr{F}^{p;m,m'}(U,W)$.

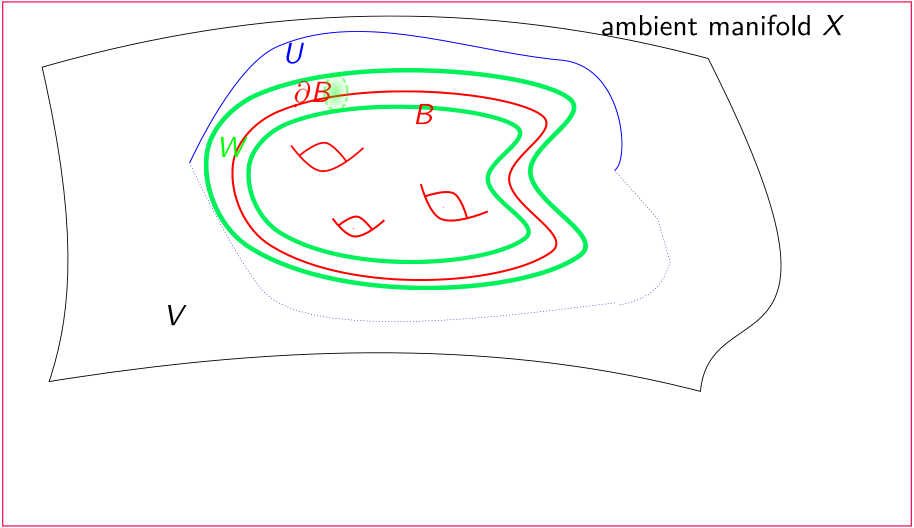

Figure: The current T is defined on $U \subset X$ (in blue) which is a neighborhood of $\overline{B}$ (the outer closed curve in red) in the ambient manifold X (in black).

If moreover, the following condition is fulfilled:

(iii-b) $\mathrm{supp}(T_n) \cap W = \varnothing$ for $n \geq 1$;

then we say that T is *approximable on* U *by* $\mathscr{C}^m$*-smooth* $\mathscr{F}$*-forms with support outside* W, and write $T \in \mathrm{SH}^{p;m}(U, W, \mathrm{comp})$.

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We say that $(T_n)_{n=1}^{\infty}$ is a *sequence of approximating forms* for T as an element of $\mathscr{F}^{p;m}(U)$ in the first case (resp. as an element of $\mathscr{F}^{p;m,m'}(U, W)$ in the second case, resp. as an element of $\mathscr{F}^{p;m}(U, W, \mathrm{comp})$ in the third case).

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Let $B \subset V$ be an open subset. We write $T \in \mathscr{F}^{p;m}(B)$ (resp. $T \in \mathscr{F}^{p;m,m'}(B)$) (resp. $T \in \mathscr{F}^{p;m}(B, \mathrm{comp})$) if there is an open neighborhood U of $\overline{B}$ in X such that $T \in \mathscr{F}^{p;m}(U)$ (resp. and there is an open neighborhood W of ∂B in U such that $T \in \mathscr{F}^{p;m,m'}(U, W)$) (resp. such that $T \in \mathscr{F}^{p;m}(U, W, \mathrm{comp})$)

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  Let B be a relatively compact nonempty open subset of V. A *strongly admissible* map along B is a $\mathcal{C}^2$-smooth diffeomorphism τ from an open neighborhood U of $\overline{B}$ in X onto an open neighborhood of $V \cap U$ in $\mathbb{E}$ such that for every point $x \in V \cap U$, for every local chart $y = (z, w) \in \mathbb{C}^{k-l} \times \mathbb{C}^l$ on a neighborhood W of x in U with $V \cap W = \{z = 0\}$, if we write $\tau(z, w) = (z', w') \in \mathbb{C}^{k-l} \times \mathbb{C}^l$, then

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- **Remarks** Holomorphic admissible maps are strongly admissible
  When X is Kähler, there exists a strongly admissible map along B

Function φ and forms α and β and tubes

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For for all $0 \le s < r < \infty$, define also the corona tube

$$\mathrm{Tube}(B, s, r) := \{y \in \mathbb{E} : \ \pi(y) \in B \quad \text{and} \quad s < \|y\| < r\}.$$

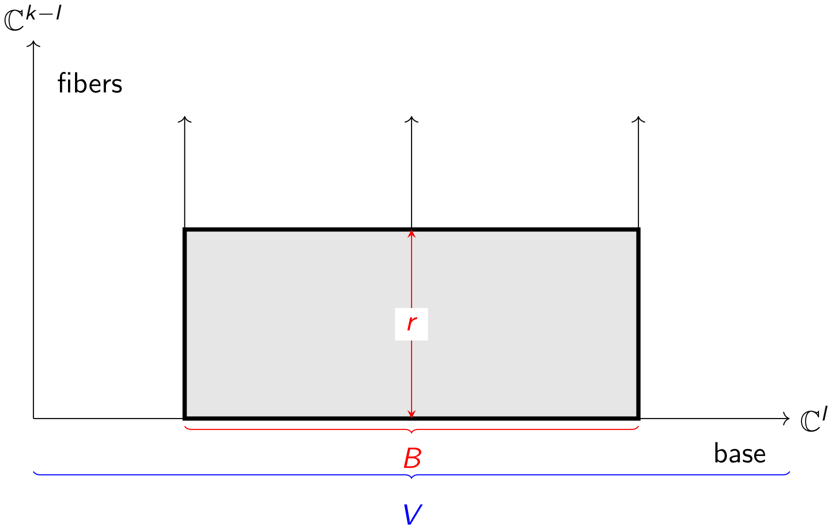

Figure: An illustrations of a tube $\mathrm{Tube}(B, r)$ with base B and radius r.

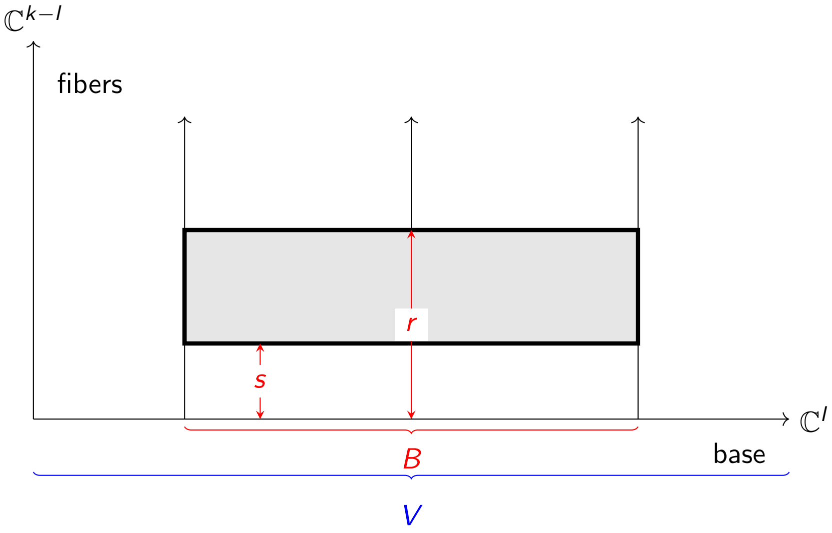

Figure: An illustrations of a corona tube $\mathrm{Tube}(B, s, r)$ with base B and smaller radius s and bigger radius r.

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Ng. 2021 For $0 \le j \le \overline{\mathrm{m}}$ and $0 < r \le \mathbf{r}$, consider

$$(1) \quad \nu_j(T, B, \omega, r, \tau, h) := \frac{1}{r^{2(k-p-j)}} \int_{\mathrm{Tube}(B,r)} (\tau_* T) \wedge \pi^*(\omega^j) \wedge \beta^{k-p-j}.$$

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$$(3) \qquad \begin{aligned} \kappa_j(T, B, \omega, r, \tau, h) &:= \int_{\mathrm{Tube}(B,r)} (\tau_* T) \wedge \pi^*(\omega^j) \wedge \alpha^{k-p-j}, \\ \kappa_j^\bullet(T, B, \omega, r, \tau, h) &:= \int_{\mathrm{Tube}(B,0,r)} (\tau_* T) \wedge \pi^*(\omega^j) \wedge \alpha^{k-p-j}. \end{aligned}$$

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$$(2) \quad \kappa_j(T, B, \omega, s, r, \tau, h) := \int_{\mathrm{Tube}(B,s,r)} (\tau_* T) \wedge \pi^*(\omega^j) \wedge \alpha^{k-p-j}.$$

We also consider

$$(3) \quad \begin{aligned} \kappa_j(T, B, \omega, r, \tau, h) &:= \int_{\mathrm{Tube}(B,r)} (\tau_* T) \wedge \pi^*(\omega^j) \wedge \alpha^{k-p-j}, \\ \kappa_j^\bullet(T, B, \omega, r, \tau, h) &:= \int_{\mathrm{Tube}(B,0,r)} (\tau_* T) \wedge \pi^*(\omega^j) \wedge \alpha^{k-p-j}. \end{aligned}$$

Remark. We can replace ω^j by an arbitrary closed smooth (j,j)-form $\omega^{(j)}$ on V_0 in order to obtain $\nu(T, B, \omega^{(j)}, r, \tau, h)$, $\kappa(T, B, \omega^{(j)}, r, \tau, h) \ldots$

- First interpretation: assume that $T = T^+ - T^-$ in an open neighborhood of $\overline{B}$ in X and $T^\pm \in \mathscr{F}^{m,m'}(B)$ for a suitable $\mathscr{F} \in \{\mathrm{CL}^p, \mathrm{PH}^p, \mathrm{SH}^p\}$ and for suitable $m, m' \in \mathbb{N}$. Let $(T_n^\pm)$ be a sequence of approximating forms for $T^\pm$. Then the RHS of (3) is

$$\lim_{n\to\infty} \kappa_j(T_n^+, B, r, \tau, h) - \lim_{n\to\infty} \kappa_j(T_n^-, B, r, \tau, h).$$

- Second interpretation: the RHS of (3) is

$$\lim_{\epsilon\to 0+} \int_{\mathrm{Tube}(B,r)} (\tau_* T) \wedge \pi^*(\omega^j) \wedge \alpha_\epsilon^{k-p-j}.$$

Here, α_ϵ is the smooth form on $\mathbb{E}$ defined by

$$\alpha_\epsilon := dd^c \varphi_\epsilon \qquad \text{and} \qquad \varphi_\epsilon := \varphi + \epsilon^2.$$

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Let T be a (p,p)-current of order 0 defined on an open neighborhood U of 0 in $\mathbb{C}^k$. We use the coordinates $(z,w) \in \mathbb{C}^{k-l} \times \mathbb{C}^l$. We may assume that U has the form $U = U' \times U''$. So $V = \{z = 0\} = U''$ and let $\mathbf{r} > 0$ such that $\{\|z\| < \mathbf{r}\} \times B \Subset U$. Consider the trivial vector bundle $\pi : \mathbb{E} \to U''$. For $\lambda \in \mathbb{C}^*$, let $a_\lambda : \mathbb{E} \to \mathbb{E}$ be the multiplication by λ on fibers, that is, $a_\lambda(z,w) := (\lambda z, w)$ for $(z,w) \in \mathbb{E}$. Admissible map τ is the identity id, $\|\cdot\|$ is Euclidean metric.

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Consider the positive closed $(1,1)$-forms:

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Let $\underline{\mathrm{m}} \leq j \leq \overline{\mathrm{m}}$. For $0 < r < \mathbf{r}$, consider the quantity

$$(4) \qquad \nu_j(T, B, r, \mathrm{id}, \|\cdot\|) := \frac{1}{r^{2(k-p-j)}} \int_{\|z\|<r,\ w\in B} T \wedge \omega_w^j \wedge \omega_z^{k-p-j}.$$

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$$(5) \qquad \kappa_j(T, B, s, r, \mathrm{id}, \|\cdot\|) := \int_{s<\|z\|<r,\ w\in B} T \wedge \omega_w^j \wedge \Upsilon_z^{k-p-j}.$$

Tangent Theorem I (for SH and PH currents) [Ng. 2021, 2023]

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1. *For every $\underline{\mathrm{m}} \le j \le \overline{\mathrm{m}}$, the following limit exists and is finite*

$$\nu(T, B, \omega^{(j)}) := \lim_{r \to 0+} \nu(T, B, \omega^{(j)}, r, \tau, h)$$

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3. *The following equalities hold*

$$\lim_{r\to 0+} \kappa(T,B,\omega^{(j)},r,\tau,h) = \nu(T,B,\omega^{(j)}),$$

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Remark: [Vu 2019]'s condition: there is a Hermitian metric $\hat{\omega}$ on X for which $dd^c\hat{\omega}^j = 0$ on V for $1 \le j \le k - p - 1$.

Existence of strongly admissible maps and approximability of SH, PH, CL-currents

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   - (i) *T is of class $\mathscr{C}^{m'}$ near ∂B;*
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   Then T can be written in an open neighborhood of $\overline{B}$ in X as $T = T^+ - T^-$ for some $T^\pm \in \mathrm{SH}^{p;m,m'}(B)$ (resp. $T^\pm \in \mathrm{PH}^{p;m,m'}(\overline{B})$, $T^\pm \in \mathrm{CL}^{p;m,m'}(B)$).

4. Lelong-Jensen formula for holomorphic vector bundle [Ng. 2021]

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Let V be a complex manifold of dimension l. Let $\mathbb{E}$ be a holomorphic bundle of rank $k - l$ over V. So $\mathbb{E}$ is a complex manifold of dimension k. Denote by $\pi : \mathbb{E} \to V$ the canonical projection. Let B be a relatively compact open set of V with piecewise $\mathcal{C}^2$-smooth boundary. Let $\mathbb{U}$ be an open neighborhood of $\overline{B}$ in $\mathbb{E}$.

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- $\varphi(y)=r_0^2$ for $y\in\mathbb{U}\cap V$ and $\varphi(y)>r_0^2$ for $y\in\mathbb{U}\setminus V$;
- for every $r\in(r_0,\mathbf{r}]$, the set $\{y\in\mathbb{U}:\ \varphi(y)=r^2\}$ is a connected nonsingular real hypersurface of $\mathbb{U}$ which intersects the real hypersurface $\pi^{-1}(\partial B)\subset\mathbb{E}$ transversally.

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Consider also the following closed $(1,1)$-forms on $\mathbb{U}$

$$\alpha := dd^c \log \varphi \quad \text{and} \quad \beta := dd^c \varphi. \tag{7}$$

Let $r > 0$ and $B \Subset V$ an open set. Consider the following *tube with base B and radius r*

(8) $$\mathrm{Tube}(B, r) := \left\{ y \in \mathbb{E} : \ \varphi(y) < r^2 \right\}.$$

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For all $r_0 \le r < s \le \mathbf{r}$, define

$$\mathrm{Tube}(B, r, s) := \left\{ y \in \mathbb{E} : \ \pi(y) \in B \quad \text{and} \quad r^2 < \varphi(y) < s^2 \right\}. \tag{9}$$

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Note that the boundary $\partial \mathrm{Tube}(B, r)$ can be decomposed as the disjoint union of the *vertical boundary* $\partial_{\mathrm{ver}} \mathrm{Tube}(B, r)$ and the *horizontal boundary* $\partial_{\mathrm{hor}} \mathrm{Tube}(B, r)$, where

$$\begin{aligned} \partial_{\mathrm{ver}} \mathrm{Tube}(B, r) &:= \left\{ y \in \mathbb{E} : \ \pi(y) \in \partial B \quad \text{and} \quad \varphi(y) \le r^2 \right\}, \\ \partial_{\mathrm{hor}} \mathrm{Tube}(B, r) &:= \left\{ y \in \mathbb{E} : \ \pi(y) \in B \quad \text{and} \quad \varphi(y) = r^2 \right\}. \end{aligned}$$

Under the above assumption on φ, we see that $\mathrm{Tube}(B, r)$ is a manifold with piecewise $\mathcal{C}^2$-smooth boundary for every $r \in [r_0, \mathbf{r}]$. When $\partial B = \varnothing$, we have $\partial_{\mathrm{ver}} \mathrm{Tube}(B, r) = \varnothing$.

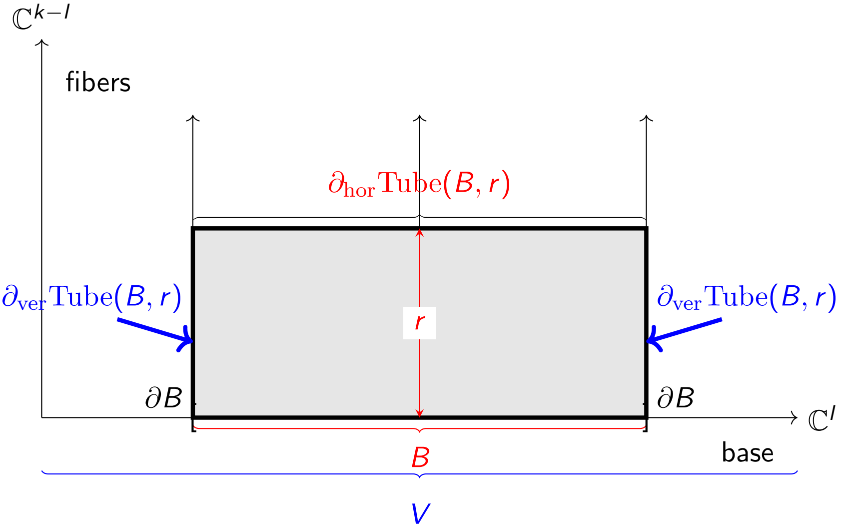

Figure: Illustrations of a Tube $\mathrm{Tube}(B,r)$ with base B and radius r, its horizontal boundary $\partial_{\rm hor}\mathrm{Tube}(B,r)$ and its vertical boundary $\partial_{\rm ver}\mathrm{Tube}(B,r)$.

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Let $r \in (r_0, \boldsymbol{r}]$ *and* $B \Subset V$ *a relatively compact open set with piecewice* $\mathcal{C}^2$*-smooth boundary. Let* S *be a real current of dimension* $2q$ *and of order* 0 *on a neighborhood of* $\overline{\mathrm{Tube}}(B, r)$ *such that* S *is suitably approximable by* $\mathcal{C}^2$*-smooth forms. Then, for all* $r_1, r_2 \in (r_0, r]$ *with* $r_1 < r_2$ *except for a countable set of values, we have that*

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$$\frac{1}{r_2^{2q}} \int_{\mathrm{Tube}(B,r_2)} S \wedge \beta^q - \frac{1}{r_1^{2q}} \int_{\mathrm{Tube}(B,r_1)} S \wedge \beta^q = \mathcal{V}(S, r_1, r_2)$$
$$+ \int_{\mathrm{Tube}(B,r_1,r_2)} S \wedge \alpha^q + \int_{r_1}^{r_2} \Big(\frac{1}{t^{2q}} - \frac{1}{r_2^{2q}}\Big) 2t dt \int_{\mathrm{Tube}(B,t)} dd^c S \wedge \beta^{q-1}$$
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$$- \int_{\partial_{\rm ver}{\rm Tube}(B,r_1,r_2)} d^c \log\varphi \wedge S^\sharp \wedge \alpha^{q-1}.$$

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- in the context of [Alessandrini–Bassanelli 1996] for top degree: $V = B \subset \mathbb{C}^l$, $\mathbb{E} = V \times \mathbb{C}^{k-l}$, $p < k - l$,
  write $y = (z, w) \in \mathbb{C}^l \times \mathbb{C}^{k-l}$, $\varphi(z) = \|w\|^2$ Euclidean metric on $\mathbb{C}^{k-l}$, S is full in bidegree $\{dw, d\bar{w}\}$. This assumption of fullness is essential for their method.

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  - We use the closedness and the positivity (when possible) of the basic $(1,1)$-forms $\pi^*\omega$, α and β on $\mathbb{E}$
- **Initital difficulty:**
  1. Since τ is not holomorphic, $dd^c(\tau_* T) \neq \tau_*(dd^c T)$.
  2. Both α and β are in general not positive.
  3. We need to control the boundary vertical terms appearing in the Lelong–Jensen formulas for vector bundles

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- **Final idea:**

  We develop a technique to control the positivity of the main parts of $\tilde{\tau}_\ell^*(\hat{\alpha}')$, $\tilde{\tau}_\ell^*(\hat{\beta})$ and $\tilde{\tau}_\ell^*(\pi^*\omega)$. We make an essential use of the strong admissibility of τ. Here, $\tilde{\tau}_\ell := \tau \circ \tau_\ell^{-1} : \ \mathbb{U}_\ell \to \tau(U_\ell)$.

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- *Let* $T_n, T \in \mathrm{CL}^p(X)$, $T_n \to T$. *Let* $\hbar$ *be the horizontal dimension of* T *along* V. *Then*
  1. *If* $j > \hbar$, *then* $\mathbf{c}_j^{\mathrm{DS}}(T_n) \to 0$.
  2. *If* $\mathbf{c}_\hbar$ *is a limit class of* $\mathbf{c}_\hbar^{\mathrm{DS}}(T_n)$, *then* $\mathbf{c}_\hbar$ *and* $\mathbf{c}_\hbar^{\mathrm{DS}}(T) - \mathbf{c}_\hbar$ *are pseudo-effective.*

Let $T \in \mathrm{SH}^{p;3,3}(B)$, (V, ω) Kähler. Let T_∞ be a tangent current to T along V, that is, $T_\infty = \lim_{n\to\infty} T_{\lambda_n}$ for some $(\lambda_n) \nearrow \infty$, where $T_\lambda := (A_\lambda)_* \tau_*(T)$. By Theorem 1, $T_\infty \wedge \pi^*\omega^{\underline{\mathrm{m}}}$ is V-conic pluriharmonic.

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  1. *If $j > \hbar$, then $\nu_j(T_n,B,\omega) \to 0$.*
  2. $\liminf_{n\to\infty} \nu_\hbar(T_n,B,\omega) \geq 0$ *and* $\nu_\hbar(T,B,\omega) - \limsup_{n\to\infty} \nu_\hbar(T_n,B,\omega) \geq 0$.

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Let (X, ω) *compact Kähler and* $T \in \mathrm{CL}^p(X)$.
Recall that $\overline{\mathrm{m}} := \min(l, k-p)$ *and* $\underline{\mathrm{m}} := \max(0, l-p)$.
For $\underline{\mathrm{m}} \leq j \leq \overline{\mathrm{m}}$, *let* $\omega^{(j)}$ *be a closed smooth* (j, j)*-form on* V_0, *e.g.* $\omega^{(j)} = \omega^j|_{V_0}$.

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Corollary 7 [Ng. 2023]

In the context of Dinh-Sibony, knowing Dinh-Sibony cohomology classes of T is **equivalent** *to knowing the generalized Lelong numbers of T.*

Indeed, we use, for $\underline{\mathrm{m}} \le j \le \overline{\mathrm{m}}$, several forms $\omega_s^{(j)}$ such that the classes $\{\omega_s^{(j)}\}$'s span $H^{j,j}(V)$.

8. Intersection theory and an effective criterion

Let (X, ω) compact Kähler and $T_j \in \mathrm{CL}^{p_j}(X)$ for $1 \le j \le m$ with $p := p_1 + \ldots + p_m \le k = \dim(X)$.

Consider $\mathbb{T} := T_1 \otimes \ldots \otimes T_m \in \mathrm{CL}^p(X^m)$.

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Theorem [Dinh–Sibony 2018]

Suppose that

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- Classical wedge-product of $(1,1)$-currents: [Bedford-Taylor 1987], [Fornæss-Sibony 1995], [Demailly 1990s] etc. More recent ones: [Dinh-Sibony 2009, 2010], algebraic flavor [Andersson-Wulcan 2014].
- [Huynh-Kaufmann-Vu 2019, 2023] prove that Dinh–Sibony wedge-product holds in many interesting situations.

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Theorem 8 [Ng. 2023]

Suppose that

1. $\kappa_j^\bullet(-\log \operatorname{dist}(\cdot, \Delta) \cdot \mathbb{T}, \Delta, \boldsymbol{r}, h) < \infty$ *for some* $\boldsymbol{r} > 0$ *and for all* $k - p \leq j \leq k - \max_{1 \leq i \leq m} p_i$;
2. $\nu_j(\mathbb{T}, \Delta) = 0$ *for all* $k - p < j \leq k - \max_{1 \leq i \leq m} p_i$.

Then $T_1 \curlywedge \ldots \curlywedge T_m$ *exists in the sense of Dinh-Sibony.*

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- [Dinh-Ng.-Vu 2018] for $m = 2$: If the superpotential of T_1 is continuous, then $T_1 \curlywedge T_2$ exists in the sense of Dinh-Sibony for all $T_2 \in \mathrm{CL}(X)$

Thank you !

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I wish Tien-Cuong a very successful and happy life !