Volume of random real algebraic submanifolds

Martin Puchol (Université Paris-Saclay)

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Joint work with Thomas Letendre

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Outline

Random real algebraic submanifolds

Kostlan–Shub–Smale polynomials

Notations

For $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{N}^{n+1}$, we write:

- $|\alpha| = \alpha_0 + \cdots + \alpha_n$,
- $X^\alpha = X_0^{\alpha_0} \cdots X_n^{\alpha_n}$,
- $\alpha! = \alpha_0! \cdots \alpha_n!$,
- if $|\alpha| = d$, $\binom{d}{\alpha} = \frac{d!}{\alpha!}$.

Definition

A Kostlan–Shub–Smale polynomial is a random $P \in \mathbb{R}_d^{\mathrm{hom}}[X_0, \dots, X_n]$ that can be decomposed as:

$$P = \sum_{|\alpha|=d} a_\alpha \sqrt{\binom{d}{\alpha}} X^\alpha,$$

where the $(a_\alpha)_{|\alpha|=d}$ are i.i.d real variable with law $\mathcal{N}(0,1)$.

Algebraic submanifolds of $\mathbb{S}^n$

Let us fix d, n and $r \in \{1, \ldots, n\}$.

Definition

Let $P_1, \ldots, P_r \in \mathbb{R}_d^{\text{hom}}[X_0, \ldots, X_n]$ be independent Kostlan–Shub–Smale polynomials, we set:

$$Z_d = \left(\bigcap_{i=1}^{r} P_i^{-1}(0) \right) \cap \mathbb{S}^n.$$

Lemma

Z_d is almost surely (empty or) a smooth submanifold of $\mathbb{S}^n$ of codimension r.

Theorem (Kostlan, 1993)

For all n, r and d, we have: $\mathbb{E}[\text{Vol}(Z_d)] = d^{\frac{r}{2}} \text{Vol}(\mathbb{S}^{n-r})$.

General setting

$\mathcal{X}$ compact (complex) projective manifold of dimension n,
$(\mathcal{E}, h^{\mathcal{E}}) \to \mathcal{X}$ rank r Hermitian vector bundle,
$(\mathcal{L}, h^{\mathcal{L}}) \to \mathcal{X}$ positive Hermitian line bundle.

Assume $\mathcal{X}$, $\mathcal{L}$ and $\mathcal{E}$ are equipped with compatible real structures: $c_{\mathcal{X}}$, $c_{\mathcal{L}}$ and $c_{\mathcal{E}}$.

Let $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ denote the space of global holomorphic sections of $\mathcal{E} \otimes \mathcal{L}^d \to \mathcal{X}$ invariant with respect to the real structures, that is:

$$(c_{\mathcal{E}} \otimes c_{\mathcal{L}^d}) \circ s \circ c_{\mathcal{X}} = s.$$

Let $M = \mathrm{Fix}(c_{\mathcal{X}})$ denote the real locus of $\mathcal{X}$. We assume $M \neq \emptyset$.
M is a smooth compact manifold without boundary of dimension n.

Zeros of random sections and random measures

$\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is finite dimensional and equipped with a natural L^2 inner product.

Definition

Let $s_d \sim \mathcal{N}(0, \mathrm{Id})$ be a random section in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, we set $Z_d = s_d^{-1}(0) \cap M$.

Lemma

For d large enough, Z_d is a.s. a codimension r smooth submanifold of M.

Example : if $\mathcal{X} = \mathbb{CP}^n$, $\mathcal{L} = \mathcal{O}(1)$ and $\mathcal{E}$ is trivial, we get $M = \mathbb{RP}^n$ and

$$\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) = \left(\mathbb{R}_d^{\mathrm{hom}}[X_0, \ldots, X_n]\right)^r.$$

Thus s_d corresponds to r independent Kostlan–Shub–Smale polynomials.

Random Radon measures

$\mathcal{L} \to \mathcal{X}$ induces a Riemannian metric on $\mathcal{X}$, hence on M and Z_d.

Let $|\mathrm{d}V_M|$ be the Riemannian volume measure on M
Let $|\mathrm{d}V_d|$ be the Riemannian volume measure on Z_d.

Z_d can also be viewed as a continuous linear form on $\left(\mathcal{C}^0(M), \|\cdot\|_\infty\right)$ by:

$$\forall \phi \in \mathcal{C}^0(M), \qquad (Z_d, \phi) = \int_{Z_d} \phi \, |\mathrm{d}V_d| \,.$$

Example: If $\phi \equiv 1$, $(Z_d, \phi) = \mathrm{Vol}(Z_d)$.

General question

What can be said about Z_d ?

Here we cannot hope to have result for all d, we will instead take $d \to \infty$.

Example : Gayet-Welschinger, Ancona, have studied topological properties of Z_d.

Expectation and variance of the volume

Expected volume

Let $s_d \sim \mathcal{N}(0, \mathrm{Id})$ in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ and let Z_d denote its real zero set.

Theorem (Letendre)

For all $\phi \in \mathcal{C}^0(M)$ we have:

$$\mathbb{E}[(Z_d, \phi)] = d^{\frac{r}{2}} \left(\int_M \phi \, |\mathrm{d}V_M| \right) \frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^n)} + \|\phi\|_\infty \, O\Big(d^{\frac{r}{2}-1}\Big),$$

where the error term does not depend on ϕ.

Corollary (Equidistribution of the mean)

We have:

$$d^{-\frac{r}{2}} \mathbb{E}[Z_d] \xrightarrow[d \to +\infty]{} \frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^n)} \, |\mathrm{d}V_M|$$

as continuous linear forms.

Variance of the volume

Theorem (Letendre-P.)

1. *For all $\phi \in \mathcal{C}^0(M)$ we have:*

$$\mathrm{Var}((Z_d,\phi)) = d^{r-\frac{n}{2}} \left(\int_M \phi^2 \, |\mathrm{d}V_M| \right) \mathcal{C}_{n,r} + o\Big(d^{r-\frac{n}{2}}\Big),$$

where $\mathcal{C}_{n,r}$ is an explicit constant depending only on n and r, and $0 \leqslant \mathcal{C}_{n,r} < +\infty$.

2. *In fact $\mathcal{C}_{n,r} > 0$.*

Corollary 1 (concentration in probability)

If $1 \leqslant r < n$ and $\frac{n}{2} + \alpha > 0$, then for all $\phi \in \mathcal{C}^0(M)$ we have:

$$\mathbb{P}\left(d^{-\frac{r}{2}} \left| \langle Z_d - \mathbb{E}[Z_d]\,, \phi \rangle \right| \geqslant d^{\frac{\alpha}{2}} \right) = O\Big(d^{-(\frac{n}{2}+\alpha)}\Big).$$

Equidistribution results

Corollary 2 (asymptotic density in probability)

If $1 \leqslant r < n$, then for any open subset $U \subset M$ we have:

$$\mathbb{P}\left(Z_d \cap U = \emptyset\right) = O\left(d^{-\frac{n}{2}}\right).$$

Corollary 3 (almost sure equidistribution)

If $n \geqslant 3$, then for almost every random sequence $(P_d)_{d \geqslant 1}$ we have:

$$\forall \phi \in \mathcal{C}^0(M), \qquad \frac{1}{\sqrt{d}} \left\langle Z_{P_d}, \phi \right\rangle \xrightarrow[d \to +\infty]{} \frac{\mathrm{Vol}\left(\mathbb{S}^{n-1}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n}\right)} \int_M \phi \left|\mathrm{d}V_M\right|.$$

Related works

- Kostlan: $n = r = 1$;
- Letendre: point 1 of Theorem for $r < n$;
- Dalmao: CLT for roots of KSS polynomials for $n = r = 1$;
- Dalmao, Armentano-Azaïs-Dalmao-Leon: point 2 of Theorem for $n = r$;
- Letendre-Ancona: for $n = r = 1$, computation of the higher moments of Z_d and CLT for linear statistics;
- ...

About the proofs of the main theorem

Correlation function

A random section $s_d \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ defines a centered Gaussian process $(s_d(x))_{x \in M}$ with correlation function $e_d : (x, y) \mapsto \mathbb{E}[s_d(x) s_d(y)]$.

Remark

Taking partial derivatives, we get: $\frac{\partial e_d}{\partial x_i}(x, y) = \mathbb{E}\left[\frac{\partial s_d}{\partial x_i}(x) s_d(y)\right]$.

For instance, for KSS polynomials,

$$\begin{aligned} e_d(x, y) &= \sum_{|\alpha|=d=|\beta|} \mathbb{E}[a_\alpha a_\beta] \sqrt{\binom{d}{\alpha}} \sqrt{\binom{d}{\beta}} x^\alpha y^\beta \\ &= \sum_{|\alpha|=d} \binom{d}{\alpha} x^\alpha y^\alpha = (\langle x \, , y \rangle)^d \\ &= \cos\left(\rho(x, y)\right)^d , \end{aligned}$$

where ρ is the geodesic distance on $\mathbb{S}^n$.

The Bergman kernel

Fact: e_d is the restriction on $M \times M$ of Bergman kernel of $\mathcal{E} \otimes \mathcal{L}^d$, which is the Schwatrz kernel of the projection $\mathcal{C}^\infty(\mathcal{X}, \mathcal{L}^d \otimes \mathcal{E}) \to H^0(\mathcal{X}, \mathcal{L}^d \otimes \mathcal{E})$.

Theorem (Dai–Liu–Ma)

The Bergman kernel e_d has a universal scaling limit:

$$e_d(x,y) \simeq \exp\left(-\frac{d}{2}\|x-y\|^2\right),$$

uniformly for (x,y) such that $\rho(x,y) \leqslant K\frac{\log d}{\sqrt{d}}$.

Theorem (Ma–Marinescu)

There exists $C > 0$ such that, for any $k \in \mathbb{N}$, uniformly on $M \times M$

$$\|e_d(x,y)\|_{\mathcal{C}^k} = O\Big(d^{\frac{k}{2}} \exp\left(-C\sqrt{d}\rho(x,y)\right)\Big).$$

Kac–Rice formula

Kac-Rice formula

For any $\phi \in \mathcal{C}^0(M)$, we have:

$$\mathbb{E}\left[\int_{Z_d} \phi \, |\mathrm{d}V_d|\right] = \frac{1}{\sqrt{2\pi}} \int_{x \in M} \phi(x) \frac{\mathbb{E}\left[\|d_x P\| \,\middle|\, P(x) = 0\right]}{\sqrt{e_d(x,x)}} \, |\mathrm{d}V_M| \, .$$

Note that by the above results, $x \mapsto e_d(x,x)$ does not vanish for d large enough.

We need to estimate $\dfrac{\mathbb{E}\left[\|d_x s_d\| \,\middle|\, s_d(x) = 0\right]}{\sqrt{e_d(x,x)}}$ for a given $x \in M$.

Asymptotic of the expectation

Fact: $(s_d(x), d_x s_d)$ is a centered Gaussian vector with variance

$$\Lambda = \begin{pmatrix} e_d(x,x) & \partial_{y_1} e_d(x,x) & \cdots & \partial_{y_n} e_d(x,x) \\ \partial_{x_1} e_d(x,x) & \partial_{x_1}\partial_{y_1} e_d(x,x) & \cdots & \partial_{x_1}\partial_{y_n} e_d(x,x) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{x_n} e_d(x,x) & \partial_{x_n}\partial_{y_1} e_d(x,x) & \cdots & \partial_{x_n}\partial_{y_n} e_d(x,x) \end{pmatrix}.$$

The distribution of $d_x s_d$ given that $s_d(x) = 0$ is thus also a centered Gaussian and its variance only depends on e_d and its derivatives at (x,x).

We get a universal asymptotic for $\dfrac{\mathbb{E}\left[\|d_x s_d\| \,\middle|\, s_d(x) = 0\right]}{\sqrt{e_d(x,x)}}$, using the results of Dai-Liu-Ma and Ma-Marinescu.

A formula for the variance

$$\begin{aligned}\mathrm{Var}((Z_d,\phi)) &= \mathbb{E}\left[(Z_d,\phi)^2\right] - \mathbb{E}[(Z_d,\phi)]^2\\ &= \mathbb{E}\left[\int_{x,y\in Z_d}\phi(x)\phi(y)\,|\mathrm{d}V_d|^2\right] - \mathbb{E}\left[\int_{x\in Z_d}\phi(x)\,|\mathrm{d}V_d|\right]^2.\end{aligned}$$

By Kac–Rice type formulas, we get:

$$\mathrm{Var}((Z_d,\phi)) = \int_{x,y\in M}\phi(x)\phi(y)\mathcal{D}_d(x,y)\,|\mathrm{d}V_M|^2,$$

where $\mathcal{D}_d(x,y)$ only depends on e_d and its derivatives at (x,x), (x,y), (y,x) and (y,y).

Main problem

$\mathcal{D}_d$ is singular on the diagonal in $M\times M$.

Behaviour of the density $\mathcal{D}_d$

Far from the diagonal

For a good choice of $K > 0$, we have $\mathcal{D}_d(x,y) = O\left(d^{r-\frac{n}{2}-1}\right)$ uniformly on:

$$\left\{(x,y) \in M \times M \;\middle|\; \rho(x,y) \geqslant K\frac{\log d}{\sqrt{d}}\right\}.$$

Near the diagonal

On $\left\{(x,y) \in M \times M \;\middle|\; \rho(x,y) < K\frac{\log d}{\sqrt{d}}\right\}$, we have the following universal scaling limit:

$$\mathcal{D}_d\left(x, x + \frac{z}{\sqrt{d}}\right) \simeq d^r \mathcal{D}(\|z\|),$$

where $\|z\| < K \log d$.

Asymptotic of the variance

$$\mathrm{Var}((Z_d,\phi)) \simeq \int_{x\in M}\int_{y\in B(x,K\frac{\log d}{\sqrt{d}})} \phi(x)\phi(y)\mathcal{D}_d(x,y)\,|\mathrm{d}V_M|^2$$

$$\simeq d^{-\frac{n}{2}}\int_{x\in M}\left(\int_{\|z\|<K\log d}\phi(x)\phi\left(x+\tfrac{z}{\sqrt{d}}\right)\mathcal{D}_d\left(x,x+\tfrac{z}{\sqrt{d}}\right)\mathrm{d}z\right)|\mathrm{d}V_M|$$

$$\simeq d^{r-\frac{n}{2}}\left(\int_{x\in M}\phi(x)^2\,|\mathrm{d}V_M|\right)\left(\int_{\mathbb{R}^n}\mathcal{D}(\|z\|)\,\mathrm{d}z\right).$$

The end

Thank you for your attention!