Functional Convergence of Berry's Nodal Lengths

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- **Berry's planar random wave**, written

$$B_E = \left\{ B_E(x) : x \in \mathbb{R}^2 \right\}, \qquad E > 0$$

is the unique planar centred, isotropic Gaussian field such that

$$\Delta B_E + 4\pi^2 E \cdot B_E = 0 \quad \text{a.s.} \quad \text{where} \quad \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\,.$$

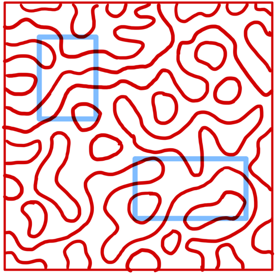
- Equivalently,

$$\mathbb{E}\left[B_E(x) B_E(y)\right] := J_0(2\pi\sqrt{E}\|x-y\|)$$

- Write $b = \left\{ b(x) : x \in \mathbb{R}^2 \right\}$ for $B_{(4\pi^2)^{-1}}$

- Think of b as a "canonical" Gaussian Laplace eigenfunction on $\mathbb{R}^2$, emerging e.g. as a universal local scaling limit for arithmetic and monochromatic RWs, random spherical harmonics ...
  See: *Marinucci & Rossi (2016), Canzani and Hanin (2021), Dierickx, Nourdin, Peccati & Rossi (2023)*.

- $B_E^{-1}(0) := \{x \in \mathbb{R}^2 : B_E(x) = 0\}$ smooth curves
- $\mathscr{L}_E(Q) :=$ length $\left(B_E^{-1}(0) \cap Q\right)$ rectangle $Q \subset \mathbb{R}^2$



To fix ideas, let $\mathcal{Q}$ be the collection of all rectangles $Q \subset [0,1]^2$.

- For every $E \geq 1$ and $Q \in \mathcal{Q}$, consider $\mathscr{L}_E(Q)$.
- **Task 1**: describe the joint fluctuations, as $E \to \infty$, of the random variables $\mathscr{L}_E(Q)$, $Q \in \mathcal{Q}$.
- **Task 2**: describe the functional fluctuations, as $E \to \infty$, of the random function

$$(s_1, s_2) \mapsto \boxed{\mathscr{L}_E(s_1, s_2) := \mathscr{L}_E([0, s_1] \times [0, s_2])} \qquad s_1, s_2 \in [0,1]$$

- *Berry (J. Phys. A, 2002)* – as $E \to \infty$:

$$\mathbb{E}\left[\mathscr{L}_E(Q)\right] = \frac{\pi \ \text{area}\, Q}{\sqrt{2}} \sqrt{E} \qquad \text{Var}\left(\mathscr{L}_E(Q)\right) \sim \frac{\text{area}\, Q}{512\pi} \log E$$

- Such an estimate follows from an analytical cancellation in Kac-Rice formulae: the natural guess for the order of the variance is $\sqrt{E}$.

- *Nourdin, Peccati & Rossi (CMP, 2019)*:

$$\sqrt{\frac{512\pi}{\log E}} \big(\mathscr{L}_E(Q) - \mathbb{E}\mathscr{L}_E(Q)\big) \xrightarrow{d} \mathcal{N}(0, \text{area}(Q)),$$

Define:

$$\widetilde{\mathscr{L}}_E(Q) := \sqrt{\frac{512\pi}{\log E}}\,\{\mathscr{L}_E(Q) - \mathbb{E}\,[\mathscr{L}_E(Q)]\}\,, \quad t \geq 1\,,$$

and similarly $\left\{\widetilde{\mathscr{L}}_E(s_1, s_2) := \widetilde{\mathscr{L}}_E([0, s_1] \times [0, s_2]) : (s_1, s_2) \in [0, 1]^2\right\}$.

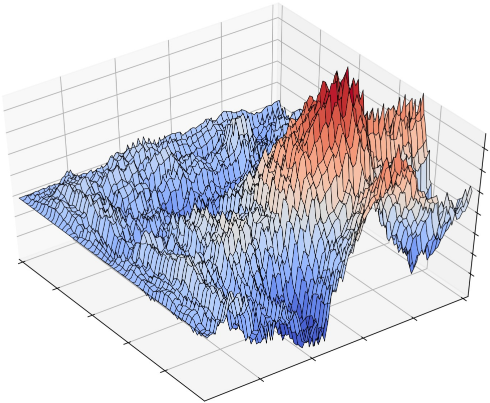
1. For all $Q_1, \ldots, Q_d \in \mathcal{Q}$, as $E \to \infty$, $\left(\widetilde{\mathscr{L}}_E(Q_1), \ldots, \widetilde{\mathscr{L}}_E(Q_d)\right)$ converges to a centered Gaussian vector with covariance function

$$\Sigma(i, j) = \text{area}(Q_i \cap Q_j).$$

2. As $E \to \infty$, the random field $\left\{\widetilde{\mathscr{L}}_E(s_1, s_2) : (s_1, s_2) \in [0, 1]^2\right\}$ converges in the f.d.d.-sense to a standard Wiener sheet.

A standard Wiener sheet $\left\{\mathbf{W}(\mathbf{s}), \mathbf{s} \in [0,1]^2\right\}$ is a centred Gaussian process with covariance

$$\mathbb{E}\left[\mathbf{W}(s_1, s_2)\mathbf{W}(t_1, t_2)\right] = (s_1 \wedge t_1)(s_2 \wedge t_2)$$



A realization of a Wiener sheet, pic by George Lowther

Question: Does $\left\{\widetilde{\mathscr{L}}_E(s_1, s_2)\right\}$ converge to a Wiener sheet *as a random function* (i.e. in $\mathcal{D}_2$, the Skorohod space of cadlag mappings on $[0,1]^2$)?

Lemma. $\{X, X_n : n \geq 1\} \subset \mathcal{D}_2, \quad X_n = U_n + V_n + W_n$

(a) as $n \to \infty$, U_n converges weakly to X in $\mathcal{D}_2$,

(b) as $n \to \infty$, V_n converges weakly to zero in $\mathcal{D}_2$,

(c) for every $\varepsilon > 0$, $\lim_{n \longrightarrow \infty} \mathbb{P}\left\{\sup_{\mathbf{t} \in [0,1]^2} |W_n(\mathbf{t})| > \varepsilon\right\} = 0$,

$\Longrightarrow$ X_n converges weakly to X in $\mathcal{D}_2$.

- $\widetilde{\mathscr{L}}_E[q] := \mathrm{proj}(\widetilde{\mathscr{L}}_E | C_q)$, C_q :=the qth Wiener chaos associated with b.
- Then,

$$\widetilde{\mathscr{L}}_E = \widetilde{\mathscr{L}}_E[2] + \widetilde{\mathscr{L}}_E[4] + R_E, \quad \text{where} \quad R_E := \sum_{q \geq 3} \widetilde{\mathscr{L}}_E[2q]\,,$$

Strategy: applying the previous lemma to

$$(X_n, U_n, V_n, W_n) = (\widetilde{\mathscr{L}}_E, \widetilde{\mathscr{L}}_E[4], \widetilde{\mathscr{L}}_E[2], R_E)$$

(I) $\widetilde{\mathscr{L}}_E[4]$ converges weakly to a standard Wiener sheet (OK – easy);

(II) $\widetilde{\mathscr{L}}_E[2]$ converges weakly to zero – *total disorder* of $E^{1/4}\mathscr{L}_E[2]$;

(III) the residual term R_E converges uniformly to zero in probability.

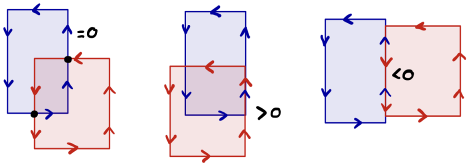
As $E \to \infty$,

$$\mathrm{Cov}\left(\mathscr{L}_E[2](Q_1), \mathscr{L}_E[2](Q_2)\right) = \frac{\lambda(\partial Q_1, \partial Q_2)}{16\pi^2\sqrt{E}} + o\left(\frac{1}{\sqrt{E}}\right),$$

where

$$\lambda(\partial Q_1, \partial Q_2) = \int_{\partial Q_1 \cap \partial Q_2} \langle n_1(x), n_2(x)\rangle \, d\mathcal{H}^1(x).$$

indicates the **signed length** of $\partial Q_1 \cap \partial Q_2$.



STEP (II) – NOTARNICOLA, PECCATI & V. (2023)

- As $E \to \infty$,

$$4\pi\, E^{1/4}\Big(\mathscr{L}_E[2](Q_1), ..., \mathscr{L}_E[2](Q_d)\Big) \xrightarrow{d} \mathcal{N}_d(0, \Sigma)$$

$$\Sigma(i,j) = \lambda(\partial Q_i, \partial Q_j)$$

- The centered Gaussian family

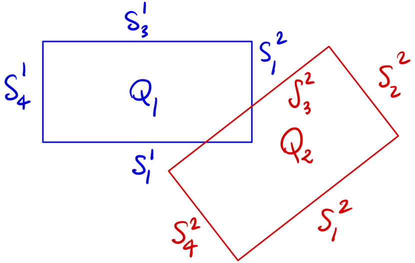
$$G = \{G(Q), Q \in \mathcal{Q}\} \quad \text{s.t.} \quad \mathbb{E}\left[G(Q_1)G(Q_2)\right] = \lambda(\partial Q_1, \partial Q_2)$$

is a **total disorder** random field: the linear span of G contains an uncountable collection of i.i.d. centered Gaussian random variables with unit variance.

STEP (II) – PROOF OF $\mathrm{Cov}\left(\mathscr{L}_E[2](Q_1), \mathscr{L}_E[2](Q_2)\right)$

- $\mathscr{L}_E(Q)[2] = \phi_E(\partial Q) = \frac{1}{8\pi\sqrt{2E}} \int_{\partial Q} B_E(x)\langle \nabla B_E(x), \mathrm{n}_{\partial Q}(x)\rangle d\mathcal{H}^1(x)$

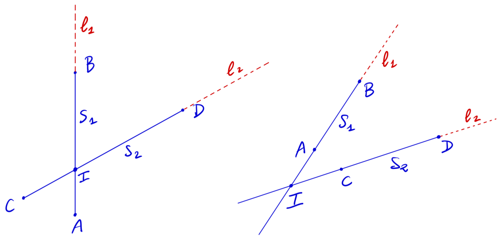
- $\mathrm{Cov}\left(\mathscr{L}_E(Q_1)[2], \mathscr{L}_E(Q_2)[2]\right) = \mathrm{Cov}\left(\sum_{k=1}^{4} \phi_E(S_k^1), \sum_{k=1}^{4} \phi_E(S_k^2)\right)$



- $\Longrightarrow$ we can reduce our study to line segments.

STEP (II) – PROOF

- $S_1 = [A, B] \quad S_2 = [C, D] \quad I = \ell_1 \cap \ell_2$



- $S_1 \cap S_2 = \{I\} \implies \mathrm{Cov}\left(\phi_E(S_1), \phi_E(S_2)\right) = \mathrm{Cov}\left(\phi_E([A, I]) + \phi_E([I, B]), \phi_E([C, I]) + \phi_E([I, D])\right)$

- $S_1 \cap S_2 = \emptyset \implies \mathrm{Cov}\left(\phi_E(S_1), \phi_E(S_2)\right) = \mathrm{Cov}\left(\phi_E([I, B]) - \phi_E([I, A]), \phi_E([I, D]) - \phi_E([I, C])\right)$

- distributional invariance under rigid motion of $B_E \implies$ we consider line segments of the type $[(0,0), P]$ (except when parallel and disjoint).

- $\lambda_i :=$ length of S_i $\quad e_i$ canonical basis vector of $\mathbb{R}^2$
  $\rho(\theta) = (\cos\theta, \sin\theta) \quad \theta \in [0, 2\pi)$

$$\begin{aligned} \gamma_1 : [0, \lambda_1] &\longrightarrow S_1 \\ t &\longmapsto te_1 \\ \gamma_2 : [0, \lambda_2] &\longrightarrow S_2 \\ t &\longmapsto t\rho(\theta) \end{aligned}$$

- $\mathrm{n}_{S_1}(x) = e_2 \quad \forall x \in S_1$

  $\mathrm{n}_{S_2}(x) = \rho(\theta)^{\perp} = (-\sin\theta, \cos\theta) \quad \forall x \in S_2$

- $\mathrm{Cov}\left(\phi_E(S_1), \phi_E(S_2)\right) = \dfrac{1}{64}\displaystyle\int_0^{\lambda_1} dt \int_0^{\lambda_2} ds$

$$\left[\psi^E_{2,2}(\gamma_1(t), \gamma_2(s))\cos\theta - \psi^E_{2,1}(\gamma_1(t), \gamma_2(s))\sin\theta\right]$$

- $\psi^E_{i,j}(x,y) := \mathbb{E}\left[B_E(x)B_E(y)\tilde{\partial}_i B_E(x)\tilde{\partial}_j B_E(y)\right], \quad i,j = 1,2$

- $\tau^E(t,s) := 2\pi\sqrt{E}\,\|\gamma_1(t) - \gamma_2(s)\| = 2\pi\sqrt{E}\sqrt{t^2 + s^2 - 2ts\cos\theta}$

- Feynmann's formula:

$$\begin{aligned}
&\mathrm{Cov}\left(\phi_E(S_1), \phi_E(S_2)\right) \\
&= \frac{2\cos\theta}{64}\int_0^{\lambda_1} dt \int_0^{\lambda_2} ds \frac{J_0(\tau^E(t,s))J_1(\tau^E(t,s))}{\tau^E(t,s)} \\
&\quad - \frac{2\sin^2\theta}{64}\int_0^{\lambda_1} dt \int_0^{\lambda_2} ds \frac{ts\left(J_0(\tau^E(t,s))J_2(\tau^E(t,s)) + J_1(\tau^E(t,s))\right)}{t^2 + s^2 - 2ts\cos\theta} \\
&= A_E(\lambda_1, \lambda_2, \theta) + B_E(\lambda_1, \lambda_2, \theta)
\end{aligned}$$

S_1 and S_2 are parallel:

- $\theta \in \{0, \pi\} \implies B_E(\lambda_1, \lambda_2, \theta) = 0$
- $0 \leq a < b, 0 \leq c < d$ and $L \geq 0$

$$\gamma_1 : t \in [a, b] \longmapsto te_1$$
$$\gamma_2 : t \in [c, d] \longmapsto te_1 + Le_2$$

- $L > 0 \iff S_1$ and S_2 are supported by parallel distinct lines.
- $L = 0 \iff S_1$ and S_2 are supported by the same line
- $\|\gamma_1(t) - \gamma_2(s)\|^2 = (t - s)^2 + L^2$

- change of variable:

$$\mathrm{Cov}\left(\phi_E(S_1), \phi_E(S_2)\right) = \frac{1}{32}\int_a^b du \frac{1}{2\pi\sqrt{E}}\int_{2\pi\sqrt{E}(u-d)}^{2\pi\sqrt{E}(u-c)} dv \frac{J_0\left(\sqrt{v^2+(2\pi^2\sqrt{E}L)^2}\right) J_1\left(\sqrt{v^2+(2\pi\sqrt{E}L)^2}\right)}{\sqrt{v^2+(2\pi\sqrt{E}L)^2}}$$

$$K^E(u; L, c, d) := \frac{1}{2\pi\sqrt{E}}\int_{2\pi\sqrt{E}(u-d)}^{2\pi\sqrt{E}(u-c)} dv \frac{J_0\left(\sqrt{v^2+(2\pi^2\sqrt{E}L)^2}\right) J_1\left(\sqrt{v^2+(2\pi\sqrt{E}L)^2}\right)}{\sqrt{v^2+(2\pi\sqrt{E}L)^2}}$$

$L > 0$:

- $|J_\nu(x)| = O(x^{-1/2})$ for $x > 0$ and $\nu = 0, 1, 2$
- $\implies$ uniformly on u

$$\sqrt{E}\left|K^E(u; L, c, d)\right| \leq \frac{O(1)}{2\pi} \int_{2\pi\sqrt{E}(u-d)}^{2\pi\sqrt{E}(u-c)} dv \frac{1}{v^2 + (2\pi\sqrt{E}L)^2} \frac{1}{E}$$
$$= O(E^{-1/2})$$

- $\sqrt{E}K^E(u; L, c, d) \to 0$ uniformly on (a, b) and

$$\mathrm{Cov}\left(\phi_E(S_1), \phi_E(S_2)\right) = O\left(\frac{1}{E}\right) \qquad E \to \infty$$

$L = 0$:

- $K^E(u;0,c,d) = \frac{1}{2\pi\sqrt{E}} \int_{2\pi\sqrt{E}(u-d)}^{2\pi\sqrt{E}(u-c)} dv \frac{J_0(|v|)J_1(|v|)}{|v|}$
- $[a,b] \cap [c,d] = \emptyset \implies \text{Cov}\left(\phi_E(S_1), \phi_E(S_2)\right) = O\left(E^{-1}\right)$ (as before)
- $[a,b] \cap [c,d] \neq \emptyset$: we start with the case $(a,b) = (c,d)$.
- $\sqrt{E}\,\text{Cov}\left(\phi(S_1), \phi(S_2)\right) = \frac{1}{32} \int_a^b du \sqrt{E} K^E(u;0,a,b)$ with

$$\sqrt{E} K^E(u;0,a,b) = \frac{1}{2\pi} \int_{2\pi\sqrt{E}(u-b)}^{2\pi\sqrt{E}(u-a)} dv \frac{J_0(v)J_1(v)}{v}$$

since J_0 is even and J_1 is odd.

- $\dfrac{d}{dv}[v(J_0(v)^2 + J_1(v)^2) - J_0(v)J_1(v)] = J_0(v)J_1(v)/v$

- $$\sqrt{E}\,\mathrm{Cov}\,(\phi(S_1), \phi(S_2)) = \underbrace{\frac{1}{64\pi}\int_a^b du\,\left[v(J_0(v)^2 + J_1(v)^2)\right]_{2\pi\sqrt{E}(u-b)}^{2\pi\sqrt{E}(u-a)}}_{(\star)} - \underbrace{\frac{1}{64\pi}\int_a^b du\,[J_0(v)J_1(v)]_{2\pi\sqrt{E}(u-b)}^{2\pi\sqrt{E}(u-a)}}_{o(1)}.$$

- $\mathfrak{f}(v) := v(J_0(v)^2 + J_1(v)^2) \implies \mathfrak{f}(-v) = -\mathfrak{f}(v)$

- $\lim\limits_{E\to\infty} [\mathfrak{f}(v)]_{2\pi\sqrt{E}(u-b)}^{2\pi\sqrt{E}(u-a)} = \lim\limits_{y\to\infty} 2\mathfrak{f}(y) = \dfrac{4}{\pi}$

- dominated convergence: $(\star) \longrightarrow \dfrac{1}{64\pi}\displaystyle\int_a^b \frac{4}{\pi}du = \frac{1}{16\pi^2}(b-a)$

- we just proved that: $\mathrm{Cov}\left(\phi(S_1), \phi(S_2)\right) = \frac{1}{16\pi^2}\frac{b-a}{\sqrt{E}} + o\left(\frac{1}{\sqrt{E}}\right)$
- $S_1 \neq S_2$ but $S_1 \cap S_2 \neq \emptyset$:
- w.l.o.g. $0 < a < c \leq b < d \implies S_1 \cap S_2 = [c, b] \times \{0\}$
- $\phi_E(S_1) = \phi_E([a, c] \times \{0\}) + \phi_E([c, b] \times \{0\})$
  $\phi_E(S_2) = \phi_E([c, b] \times \{0\}) + \phi_E([b, d] \times \{0\})$
- $$\begin{aligned} &\mathrm{Cov}\left(\phi_E(S_1), \phi_E(S_2)\right) \\ &= \mathrm{Cov}\left(\phi_E([c, b] \times \{0\}), \phi_E([c, b] \times \{0\})\right) + o\left(\frac{1}{\sqrt{E}}\right) \\ &= \frac{\lambda(S_1, S_2)}{16\pi^2\sqrt{E}} + o\left(\frac{1}{\sqrt{E}}\right) \end{aligned}$$

S_1 and S_2 are not parallel:

- $\mathrm{Cov}(\phi_E(S_1), \phi_E(S_2)) = A_E(\lambda_1, \lambda_2, \theta) + B_E(\lambda_1, \lambda_2, \theta)$
- $A_E(\lambda_1, \lambda_2, \theta) = O\left(\frac{1}{E}\right)$
- $B_E(\lambda_1, \lambda_2, \theta) = O\left(\frac{\log E}{E}\right)$
- $\implies \mathrm{Cov}(\phi_E(S_1), \phi_E(S_2)) = o\left(\frac{1}{\sqrt{E}}\right)$

Step (II) – A connection with Buckley & Sodin (2017)

- **Gaussian entire function**: $\{\zeta_n\}$ i.i.d. complex std Gaussian

$$z \mapsto f(z) = \sum_{n=0}^{\infty} \zeta_n \frac{z^n}{n!} \qquad z \in \mathbb{C}$$

- $f_R(z) := f(Rz) \qquad n_R(Q) := \#\left\{f_R^{-1}(0) \cap Q\right\}$

- $\mathrm{Var}\,(n_R(Q)) = c_0\, R \cdot \mathrm{length}\,(\partial Q) + o(R) \qquad R \to \infty$

- $\frac{1}{\sqrt{c_0\, R}}\left((n_R(Q_1) - \mathbb{E}\left[n_R(Q_1)\right]), \ldots, (n_R(Q_d) - \mathbb{E}\left[n_R(Q_d)\right])\right) \xrightarrow{d} \mathcal{N}_d(0, \Sigma)$

- *hyperuniformity*: a variance that scales as the length of the boundary of $R \cdot Q$, rather than as $\mathrm{area}(R \cdot Q) \asymp R^2$

$$\mathrm{Var}\left(\mathscr{L}(b; R \cdot Q)[2]\right) = \frac{R \cdot \mathrm{length}(\partial Q)}{8\pi} + o(R) \qquad R = 2\pi\sqrt{E}$$

$$\mathrm{Var}\left(n_R(Q)\right) = c_0\, R \cdot \mathrm{length}\left(\partial Q\right) + o(R) \qquad R \to \infty$$

- *total disorder process*:
  - $\mathscr{L}_E(Q)[2]$ and $n_R(Q)$ exhibit the same limit in the f.d.d.-sense, which is a total disorder field.
  - physics, random matrix theory.

- As $E \to \infty$, the field $\left\{\widetilde{\mathscr{L}_E}(\mathbf{s})[2] : \mathbf{s} \in [0,1]^2\right\}$ weakly converges to zero in $\mathcal{D}_2$ (tightness + estimates for sup of stationary Gaussian fields).

Fix $K \geq 1$, consider the partition Π_K of $[0,1]^2$ formed by the collection of squares of side length 2^{-K}:

- For every $i = (i_1, i_2) \in \{0, \dots, 2^K\}^2$, we define the *partition points* $\mathbf{p}_i(K,K) := (p_{i_1}(K), p_{i_2}(K)) \in [0,1]^2$ by

$$p_{i_1}(K) := \frac{i_1}{2^K}, \quad p_{i_2}(K) := \frac{i_2}{2^K}, \quad i_1, i_2 = 0, 1, \dots, 2^K.$$

- For $\mathbf{s} = (s_1, s_2) \in [0,1]^2$, we write $i_{K,K}(\mathbf{s}) = (i_{1,K}(s_1), i_{2,K}(s_2))$ for the vector verifying

$$p_{i_{1,K}(s_1)} \leq s_1 < p_{i_{1,K}(s_1)+1} \qquad p_{i_{2,K}(s_2)} \leq s_2 < p_{i_{2,K}(s_2)+1}$$

  that is, the vector $i_{K,K}(\mathbf{s})$ is such that $\mathbf{p}_{i_{K,K}(\mathbf{s})}(K,K)$ is the closest partition point to s on the left.

Discretized nodal length:

$$\mathscr{L}_E^K(s_1, s_2) := \mathscr{L}_E\left([0, p_{i_{1,K}(s_1)}(K)] \times [0, p_{i_{2,K}(s_2)}(K)]\right)$$

Take $\{K(E) : E > 0\}$ numerical sequence s.t. $K(E) \to \infty$ and $K(E) = o((\log E)^{1/10})$ as $E \to \infty$. Then,

1. for every $\varepsilon > 0$, $\quad \mathbb{P}\left\{\sup_{\mathbf{s}\in[0,1]^2} \left|R_E^{K(E)}(\mathbf{s})\right| > \varepsilon\right\} \longrightarrow 0$
2. the normalized process $\left\{\widetilde{\mathscr{L}}_E^{K(E)}(\mathbf{s})\right\}$ converges weakly to a standard Wiener sheet $\mathbf{W}$ on $[0,1]^2$ in the Skorohod space $\mathcal{D}_2$

Main difficulty for directly dealing with the residual term R_E:

- The expectation of $\mathscr{L}_E(\mathbf{s})$ (order $\sqrt{E}$) grows much faster than the normalizing factor $\log E$.

- Planar chaining argument with R_E requires

$$\left|\mathbb{E}\left[\mathscr{L}_E(\mathbf{t})\right] - \mathbb{E}\left[\mathscr{L}_E(\mathbf{p}_{i_{K,K}(\mathbf{t})}(K,K))\right]\right| \approx \frac{\sqrt{E}}{\sqrt{\log E}} \frac{1}{2^K}$$

  to be bounded.

- This requirement is incompatible with the constrained choice $K(E) = o((\log E)^{1/10})$, as is needed in the above statements.

Thank you!