Almost sure GOE fluctuations of energy levels for hyperbolic surfaces of high genus

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Geometric and topological properties of random algebraic varieties

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1. Motivation: Number variance of Laplace spectrum of compact surfaces

Spectrum of surfaces

- Goal: Understand **eigenvalue statistics** of Laplacian on (random) surfaces.
- Today: **Number variance** (smooth statistics)
- (Ω, g) – Smooth closed surface (compact, no boundary)
- $\Delta = div \circ grad$ Laplacian (Laplace-Beltrami) on Ω
- **Spectrum** of $-\Delta$ ("spectrum of Ω") – purely discrete

  $\{\lambda_j\}_{j\geq 1}$: $\quad \Delta\varphi_j + \lambda_j \cdot \varphi_j = 0$
- Weyl's law: $N(\lambda) := \#\{j: \lambda_j \leq \lambda\} \sim \frac{Area(\Omega)}{4\pi} \cdot \lambda$

Number variance

- <u>Weyl's law:</u> $N(\lambda) = N_\Omega(\lambda) := \#\{j\colon \lambda_j \le \lambda\} \sim \frac{Area(\Omega)}{4\pi} \cdot \lambda$
- $\Longrightarrow$ For $L > 0$, "random" interval

  length $\hat{L} := \frac{4\pi}{Area(\Omega)} \cdot L$ expect L eigenvalues
- Take intervals $\left[x, x + \hat{L}\right]$, $x \in [E, 2E]$ random uniform
- $n(L; x) := N\left(x + \hat{L}\right) - N(x)$ ("expectation" L)
- **Number variance**

$$\Sigma^2(L; E) := \frac{1}{E} \int_E^{2E} (n(L; x) - L)^2 dx$$

Number variance (smooth)

- **Number variance**

$$\Sigma^2(L;E) := \frac{1}{E}\int_E^{2E}(n(L;x)-L)^2dx$$

- **Smooth statistics:** $f:\mathbb{R}\to\mathbb{R}$ smooth & rapidly decaying (e.g. compact support), unit mass

$$n_f(L;x) := \sum_j f\left(\frac{\lambda_j - x}{\hat{L}}\right)$$

- (Smooth) number variance

$$\Sigma_f^2(L;E) := \frac{1}{E}\int_E^{2E}\left(n_f(L;x)-L\right)^2dx$$

II. Random matrix theory as a model for number variance

Ensembles of random matrices

- Recall $\Sigma_f^2(L;E) := \frac{1}{E}\int_E^{2E}\left(n_f(L;x) - L\right)^2 dx$ surface
- Let $M = M_N$ matrix belonging to a **random ensemble** of $N \times N$ matrices. $N \leftrightsquigarrow E$
- Ensembles: GOE, GUE, Poisson
- GOE (Gaussian Orthogonal Ensemble) – symmetric matrix with i.i.d. standard Gaussian entries (save to diagonal & relations).
- GUE (Gaussian Unitary Ensemble) – Hermitian matrix with Gaussian entries.
- Poisson – diagonal matrix with i.i.d. entries.

Number variance RMT

- $M = M_N$ random $N \times N$ matrix (GOE, GUE, Poisson).
- Take **deterministic** interval $\mathrm{I} = I_N = [a, a + \hat{L}]$.
- $n(I)$ number of eigenvalues in I
- $L = N \cdot \hat{L} = \mathbb{E}_N[n(I)]$, allowed to grow with N.
- Number variance $\Sigma^2(L; N) := \mathbb{E}_N[(n(I) - L)^2]$ "**ensemble average**".
- Fact: $\Sigma^2(L; N) \sim \begin{cases} \frac{2}{\pi^2} \log(L) & GOE \\ \frac{1}{\pi^2} \log(L) & GUE \end{cases}$

(Dyson-Mehta, 1963)

Number variance RMT (cont.)

- Fact (Dyson-Mehta): $\Sigma^2(L;N) \sim \begin{cases} \frac{2}{\pi^2}\log(L) & GOE \\ \frac{1}{\pi^2}\log(L) & GUE \end{cases}$

- Can define $\sum_f^2(L;N)$ analogously

- $$\sum_f^2(L;N) \sim \begin{cases} \boxed{\sum_{GOE}^2(f) \quad GOE} \\ \frac{1}{2}\sum_{GOE}^2(f) \quad GUE \end{cases}$$

$$\sum_{GOE}^2(f) := 2\int_{\mathbb{R}} |x| \cdot \hat{f}(x)^2 dx \quad \textbf{constant}$$

- Variance small f $\leftrightsquigarrow$ very rigid structure
- $\Sigma_{Poisson}^2 \sim L$ – "easy" exercise

RMT predictions (Berry)

- Conjecture(Berry 1985):Generic chaotic Ω:

$$\Sigma^2_\Omega(L;E) \sim \begin{cases} \Sigma^2_{GOE}(L;N) \;\; \textit{time reversal symmetry} \\ \Sigma^2_{GUE}(L;N) \; \textit{time reversal violated} \end{cases}$$

$$1 \ll \hat{L} \ll \sqrt{E} \quad ; \quad \mathrm{N} = \mathrm{N(E)} = \mathrm{E} \;\; \text{Weyl's law}$$

Completely integrable systems – Poisson.

- Time reversal violated e.g. by a magnetic field.
- Goal: study the number variance for **hyperbolic surfaces** – ensemble average & window average.
- Not a single positive example to date (numerics support).

Negative examples

- Recall Berry $\Sigma^2_\Omega(L;E) \sim \frac{2}{\pi^2}\log(L)$ $\qquad \hat{L} \ll \sqrt{E}, \hat{L} \to \infty$
- Selberg 1975: Found a special "arithmetic" surface s.t. $\Sigma^2_\Omega(L;E) \gg \frac{\sqrt{E}}{\log(E)^2}$ large
- Families of arithmetic (hyperbolic) surfaces violating GOE by physicists 1985-1990 Bohigas-Giannoni-Schmit, Bogomolny-Georgeot-Giannoni-Schmit, Aurich-Steiner.
- Luo-Sarnak (1994): For all arithmetic surfaces $\Omega = \Gamma \backslash \mathbb{H}^2$ $\Sigma^2_\Omega(L;E) \gg \frac{L}{\log(L)^2}$ inconsistent to GOE.
- Results consistent with Poisson completely integrable. E.g. Bleher-Lebowitz (1995) flat Diophantine torus.

III. Weil-Petterson model random genus g hyperbolic surfaces

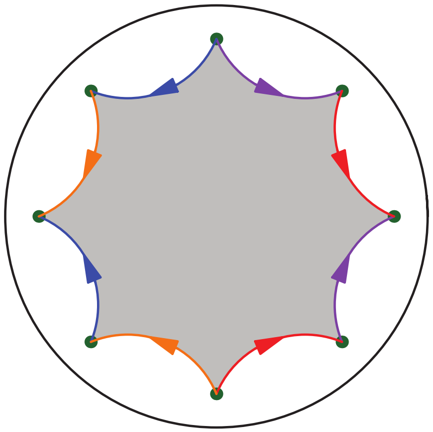
Take-home message
Random hyperbolic surfaces

- Definition: Ω is hyperbolic, if it is a **smooth surface of constant negative curvature** $(\equiv -1)$.
- For $g \geq 2$ a **moduli space** $\mathcal{M}_g$ of surfaces.

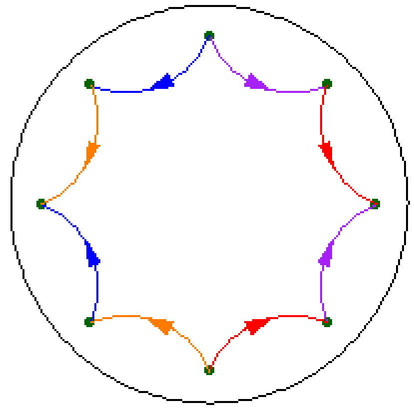
  X ∈ $\mathcal{M}_g$ is smooth closed hyperbolic, genus g
- Two equivalent ways:
  1. Different hyperbolic surfaces
  2. Endow fixed surface different hyperbolic metrics
- **Finite measure** on $\mathcal{M}_g$ "Weil-Petterson" (WP).
- ⇒ **random WP hyperbolic surfaces** genus g

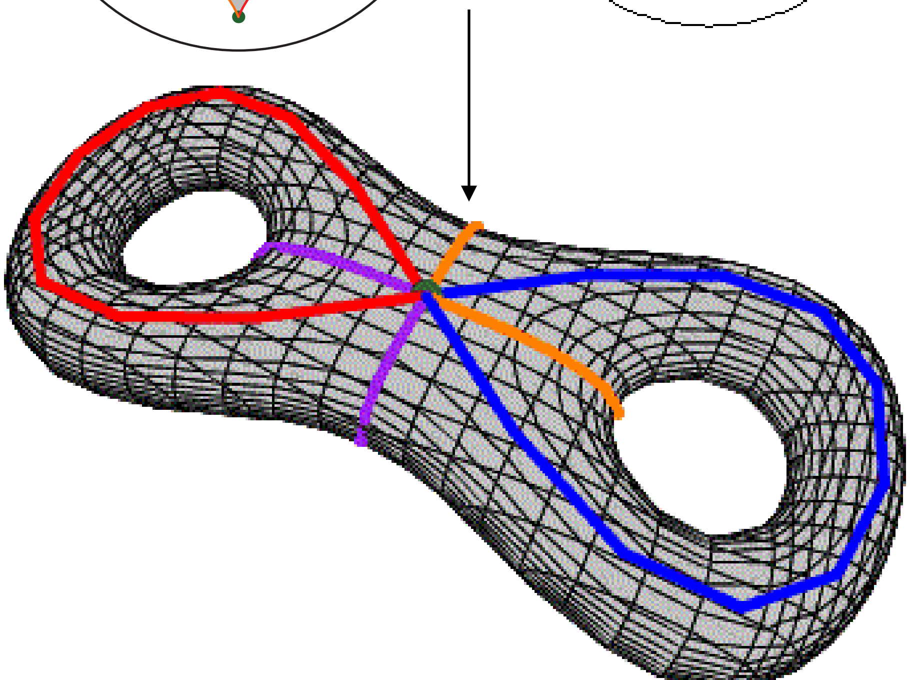
Illustration of hyperbolic surfaces

Hyperbolic plane
Poincaré disk model

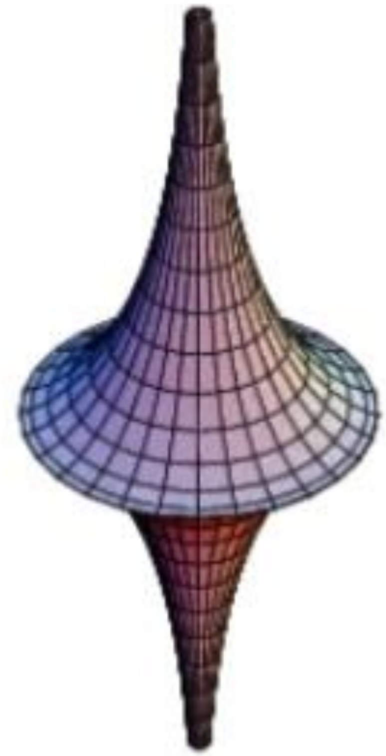


Quotient by discrete group
of isometries



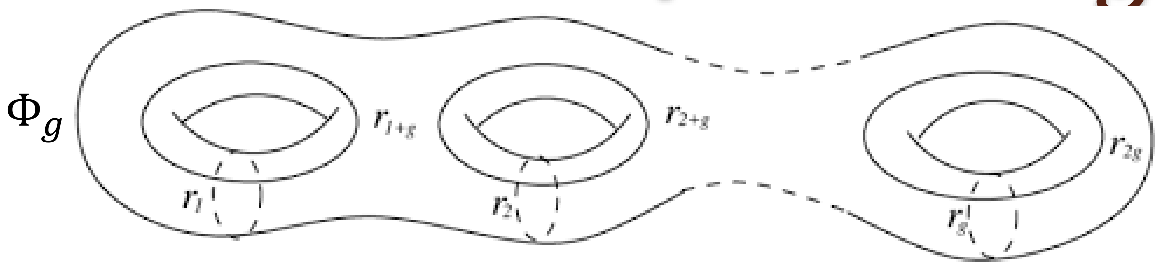


Genus 2



Pseudosphere
Not compact

Moduli space $\mathcal{M}_g$



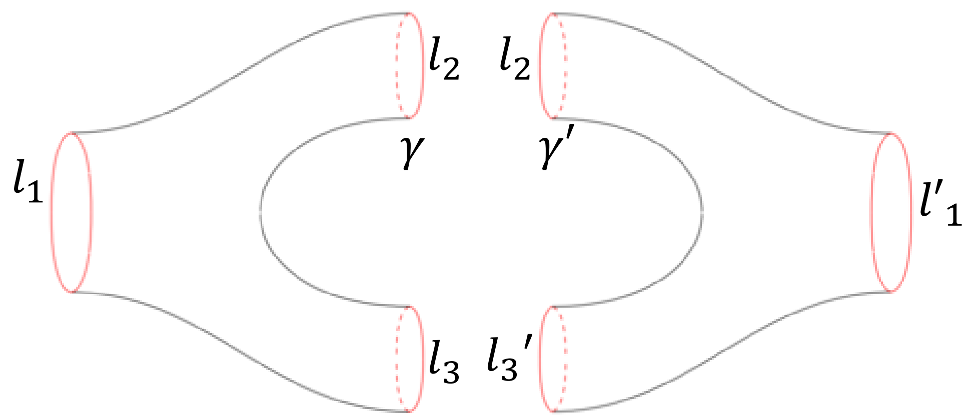
- For $g \geq 0$ let Φ_g be the (unique) genus g closed surface (compact, no boundary, topology). E.g. $g = 1 \rightsquigarrow$ torus.
- <u>Fact</u>: For $g \geq 2$ (assumed), Φ_g could be endowed with (many) hyperbolic metrics: i.e. Φ_g hyperbolic surface.
- By Gauss-Bonnet $Area(\Omega) = 4\pi(g-1)$, any Ω.

$$g \to \infty \Leftrightarrow Area(\Omega) \to \infty$$

- $\Rightarrow$Weyl's law $\#\{j\colon \lambda_j \leq \lambda\} \sim (g-1) \cdot \lambda, \quad g$ fixed
- Take into account for number variance

WP measure on $\mathcal{M}_g$

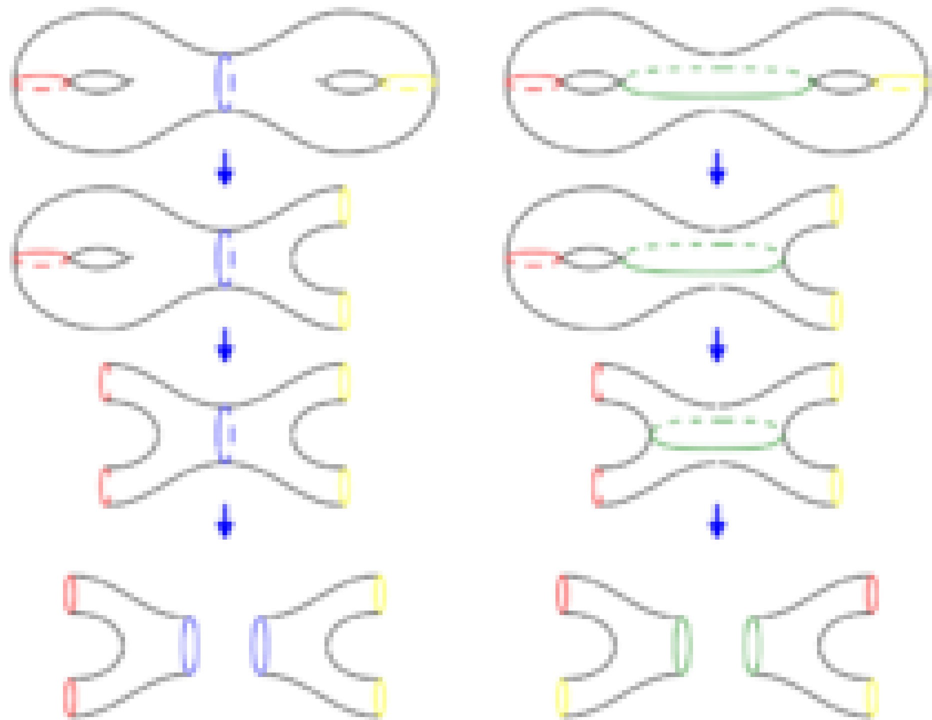
- Definition: a pair of pants is a hyperbolic surface of signature $(0,3)$ – sphere with 3 punctures.
- Fact: Given $(l_1, l_2, l_3) \in \mathbb{R}^3_{\geq 0}$ exists unique (up to isometry) pair of pants with these boundary lengths.



- Can **glue** along equal lengths. Can **twist** equal pair by α
- E.g. glue along (γ, γ') get surface signature $(0,4)$, every α

Gluing pairs of pants

- When (l_1, l_2, l_3) and $(l_1{}', l_2{}', l_3{}')$ are pairwise equal can glue 2 pants into a genus-2 closed surface.



$(l_1, l_2, l_3 = l_2), (l_1, l_2', l_3' = l_2{}')$ $(l_1, l_2, l_3) = (l_1{}', l_2{}', l_3{}')$

- 6 parameters: **lengths** (l_1, l_2, l_3) **twists** $(\alpha_1, \alpha_2, \alpha_3)$

WP measure

- More generally $g \geq 2$ take

$2g-2$ pants $\Rightarrow 6g-6$ boundary curves$\Rightarrow 3g-3$ pairs

- Can glue (combinatorial) to closed Φ_g genus g surface
- Φ_g serves as "marking"
- Fenchel-Nielsen coordinates$(l_1, \dots, l_{3g-3}\,; \alpha_1, \dots, \alpha_{3g-3})$
- $\mathcal{T}_g$ – Teichmuller, **not** Moduli
- Euclidean manifold of dimension $6g-6$.
- Admits Natural **infinite** measure WP (Wolpert)

$$dl_1 \cdot \cdots \cdot dl_{3g-3} \cdot d\alpha_1 \cdot \cdots \cdot d\alpha_{3g-3} \text{ on } \mathcal{T}_g$$

Moduli space $\mathcal{M}_g$ and WP measure

- Φ_g is (unique) genus g smooth "marking" surface.
- Teichmuller space $\mathcal{T}_g = \mathcal{T}(\Phi_g)$ - hyperbolic metrics Φ_g.
- Let $\varphi: \Phi_g \to (\Phi_g, \rho)$ be a self-homeomorphism of Φ_g
- Induces (another) metric on Φ_g. Identified within $\mathcal{M}_g$.
- Mapping class group $\mathrm{MCG}(\Phi_g)$, WP measure invariant.
- $\mathcal{M}_g = \mathrm{MCG}(\Phi_g) \backslash \mathcal{T}_g$ orbifold dimension $6g - 6$
- Induce WP **probability measure** $\mathcal{P}_g$ on $\mathcal{M}_g$.
- (WP) **random hyperbolic surface** genus g
- Study initiated by Maryam Mirzakhani (2010s).

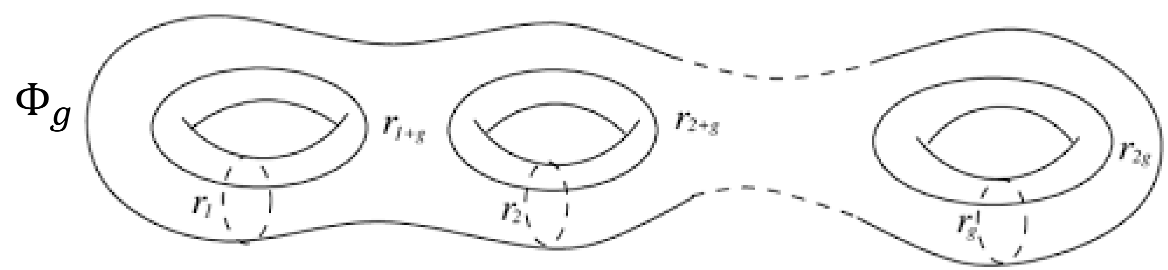
Moduli space $\mathcal{M}_g$ - summary

- Φ_g genus g surface, serves as "marking".
- Abstract definition: For $g \geq 2$:

1. Teichmuller space $\mathcal{T}_g = \mathcal{T}(\Phi_g)$ - hyperbolic metrics Φ_g
2. Mapping class group $\mathrm{MCG}(\Phi_g)$ acts on $\mathcal{T}_g$: $\varphi \in \mathcal{T}_g$

$\varphi: \Phi_g \rightarrow (\Phi_g, \rho)$ by pullback the metric ρ.

3. Moduli space $\mathcal{M}_g = \mathrm{MCG}(\Phi_g) \backslash \mathcal{T}_g$ is the genuinely different hyperbolic metrics on Φ_g.
4. Inherits WP probability measure from $\mathcal{T}_g$



IV. Statement of main results

Number variance random WP surfaces

- Recall $n_f(X; L, y) := \sum_j f\left(\frac{\lambda_j - y}{\hat{L}}\right)$ $\qquad X \in \mathcal{M}_g$
- Recall $\Sigma^2_{GOE}(f) = 2\int_{\mathbb{R}} |x| \cdot \hat{f}(x)^2 dx$ number
- Rudnick `22: GOE statistics high genus: f even smooth, s.t. $\hat{f}$ is compactly supported.

$$\lim_{\substack{\hat{L}, y \to \infty \\ \hat{L} = o(\sqrt{y})}} \lim_{g \to \infty} Var_g^{WP}\left(n_f(X; L, y)\right) = \Sigma^2_{GOE}(f)$$

- Explicated result.
- Ensemble average. Valid for individual ("typical") $X \in \mathcal{M}_g$?
- CLT (Rudnick-W `23) same regime; another work

GOE fluctuations for typical surfaces

- Rudnick-W 23': GOE variance individual $X \in \mathcal{M}_g$, high probability $g \to \infty$.
- Stronger assumptions on L, y: Assume

$$\frac{\sqrt{E}}{\log(E)} \ll \hat{L} = o\left(\sqrt{E}\right)$$

- Consider $\mathcal{V}_{E;L}(X) := \frac{1}{E}\int_E^{2E}\left(n_f(X; L, y) - L\right)^2 dy$ r.v.

$X \in \mathcal{M}_g$ random w.r.t. WP measure.

- $L = L\left(\hat{L}\right) = L(E)$- parameter. As $y \in [E, 2\mathrm{E}]$ expectation $\mathbb{E}_g^{WP}\left[n_f(X; L, y)\right]$ grows. Dominates fluctuations.
- $\tilde{V}_{E;L}(X)$ unbiased version.

GOE fluctuations for typical surfaces

- Recall $\mathcal{V}_{E;L}(X) := \frac{1}{E}\int_E^{2E}\left(n_f(X;L,y) - L\right)^2 dy$ r.v.
- $\tilde{V}_{E;L}(X)$ unbiased version.
- $\Sigma^2_{GOE}(f) = 2\int_{\mathbb{R}} |x| \cdot \hat{f}(x)^2 dx$
- Assume $\frac{\sqrt{E}}{\log(E)} \ll \hat{L} = o\left(\sqrt{E}\right)$ (LE)
- Statement Rudnick-W (23'): For every $\varepsilon > 0$

  For all $\hat{L}, E$ sufficiently large (depend on ε) subject to (LE)

  For all g sufficiently large (depending on $\hat{L}, E$):

  One has $\left|\tilde{V}_{E;L}(X) - \Sigma^2_{GOE}(f)\right| < \varepsilon$ almost full probability.

GOE fluctuations for typical surfaces

- $\tilde{V}_{E;L}(X)$ unbiased version of smooth number variance.
- $\Sigma^2_{GOE}(f) = 2\int_{\mathbb{R}} |x| \cdot \hat{f}(x)^2 dx$
- Assume $\frac{\sqrt{E}}{\log(E)} \ll \hat{L} = o\left(\sqrt{E}\right)$ (LE)
- Then $\forall \varepsilon > 0$

  For all $\hat{L}, E$ sufficiently large (depend on ε) subject to (LE)

  For all g sufficiently large (depending on $\hat{L}, E$):

  One has $\left|\tilde{V}_{E;L}(X) - \Sigma^2_{GOE}(f)\right| < \varepsilon$ almost full probability.
- Restatement, limit subject to (LE)

$$\lim_{\substack{E\to\infty \\ \hat{L}\to\infty}} \limsup_{g\to\infty} \mathcal{P}_g^{WP}\left(\left|\tilde{V}_{E;L}(X) - \Sigma^2_{GOE}(f)\right| > \varepsilon\right) = 0$$

V. On the proofs

Length spectra of hyperbolic surfaces

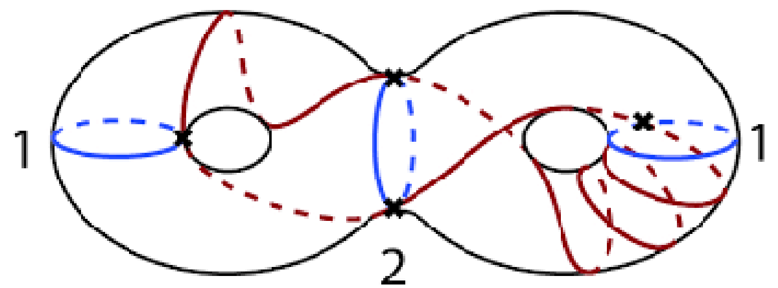
- Length spectrum (primitive) $\mathcal{L}_X^g$ of $X \in \mathcal{M}_g$: (discrete) set of lengths of distinct (primitive) closed geodesics in X.
- Fact (Huber): Spectrum of $X \leftrightsquigarrow \mathcal{L}_X^g$ (determine).
- Careful: Neither determines X (isometry class). Almost.
- **Selberg's trace formula** express $n_f(X; L, y) = \varphi\left(\mathcal{L}_X^g\right)$, $\varphi = \varphi_{f;L,y}: \mathcal{N} \to \mathbb{R}$ functional, $\mathcal{N}$ discrete measures $\mathbb{R}_{\geq 0}$.
- $\tilde{V}_{E;L}(X) = \psi\left(\mathcal{L}_X^g\right)$. φ, ψ continuous $\mathcal{N}$ (vague topology)
- $\mathcal{L}_X^g$ is random. Think as point-process. Limit $g \to \infty$?
- Geodesics are **simple** or **non-simple** (self-intersecting).

(Random) Length spectra

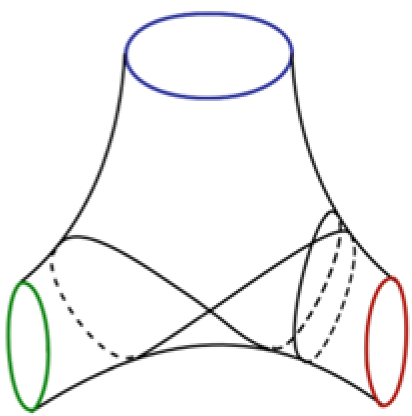
- $\mathcal{L}_X^g$ is random. Think as point-process. Limit $g \to \infty$?
- Geodesics are **simple** or **non-simple** (self-intersecting).
- For X fixed, total number of geodesics of length $\leq M$ exponential (Geodesic Prime Number Theorem).
- Number of **simple** geodesics of length $\leq M$ – polynomial (degree $6g - 6$) $M \to \infty$, X fixed (g fixed).
- $\Rightarrow$ Most geodesics are self-intersecting.
- "Mirzakhani's integration formula" – a way to average functionals on $\mathcal{M}_g$ of **simple length spectrum**.
- Too bad.

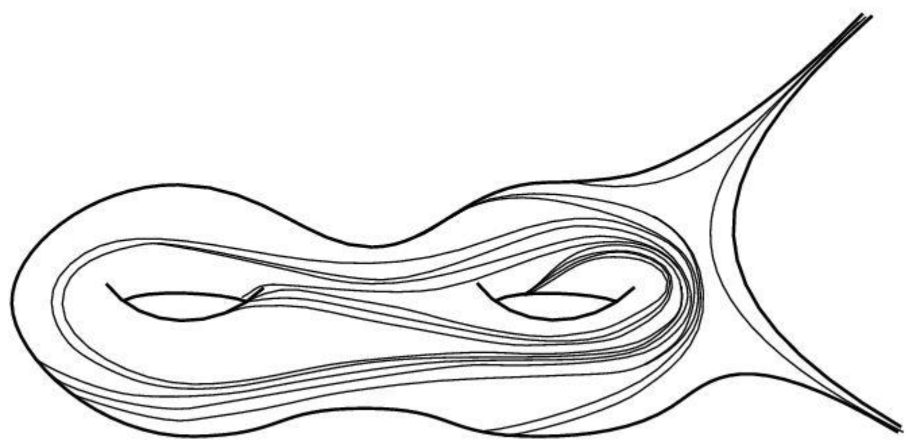
Simple vs. self-intersecting

Simple



Self-intersecting





Length spectra as point process

- Situation changes drastically regime as $g \to \infty$ (M fixed).
- Here most geodesics are **simple**.
- Number of (simple primitive) geodesics of length $\leq M$ converges to Poisson r.v., with certain parameter.
- Mirzakhani-Petri (`17) proved (simple/total) $\mathcal{L}_X^g \Rightarrow PPP$ intensity $\frac{2\sinh(x)^2}{x}$, denote $\mathcal{L}_\infty$.
- Mirzakhani's integration formula $\Leftrightarrow$ Poisson correlations.
- Recall $\tilde{V}_{E;L}(X) = \psi(\mathcal{L}_X^g), \psi = \psi_{E;L}$ continuous.
- Then $\tilde{V}_{E;L}(X) \Rightarrow \psi(\mathcal{L}_\infty)$. Can perform computations within PPP. Evaluate expectation $\Sigma_{GOE}^2(f)$, variance vanish.

Thanks!