16th April 2025

0. Introduction :

Atiyah - Singer index theorem

Thm (Atiyah - Singer, 1963)

Goal of this lecture : $\mathrm{Ind}(D) = \int_X \underline{\hat{A}(TX)\,\mathrm{ch}(E)}$

Dirac operator

cohomological classes
characteristic classes

Analysis of the manifold $\xleftrightarrow{\text{A-S.}}$ Topology of the manifold

Geometric setting :

X compact manifold

E, F two vector bundles over X

$P : C^\infty(X, E) \longrightarrow C^\infty(X, F)$

is an elliptic operator

$\Longrightarrow$ P is Fredholm operator

that is $\mathrm{Ker}(P)$, $\mathrm{CoKer}(P)$ are finite dimensional vector spaces

Def : $\mathrm{Ind}(P) := \dim \mathrm{Ker}(P) - \dim \mathrm{Coker}(P) \in \mathbb{Z}$

Prop : If $P_t : C^\infty(X, E) \to C^\infty(X, F)$ is a continuous family of elliptic operators of order k. Then ②

continuously depending on $t \in \mathbb{R}$.

$\underbrace{\mathrm{Ind}(P_t)}_{\in \mathbb{Z}}$ is independent of $t \in \mathbb{R}$.

Historical remarks:

① Around 1960, Gel'fand's question: find a topological formula for $\mathrm{Ind}(P)$.

② 1962, Atiyah and Singer found a conceptual argument to show: if X is spin

$$\underbrace{\int_X \hat{A}(TX)}_{\hat{A}\text{-genus of } X} \in \mathbb{Z}$$

(A result by Atiyah–Hirzebruch)

③ They conjectured:

$$\int_X \hat{A}(TX) = \mathrm{Ind}(P) \text{ for some elliptic operator } P$$

④ 1963: $P = D$ Dirac op. when X is spin

⑤ Atiyah–Singer (1963, 1968): P elliptic

$$\mathrm{Ind}(P) = \int_{T^*X} \hat{A}(T^*X)^2 \, ch(\sigma(P))$$

$\sigma(P)$ principal symbol of P, as an element in K-group of T^*X.

I. Preliminary on smooth manifold, vector bundle and differential operator

I.1 Manifold and vector bundle

Def: A topological space X is a smooth manifold of dim m if:

1) X is Hausdorff, that is, $\forall x, y \in X$ if $x \neq y$, $\exists$ U, V open subsets s.t. $x \in U$, $y \in V$, $U \cap V = \emptyset$

2) X has a countable base: (X is second-countable $\Leftrightarrow$) $\exists \{U_i\}_{i=1}^{\infty}$, $U_i \underset{\text{open}}{\subset} X$ s.t. any open subset of X is a union of subfamily of $\{U_i\}_{i=1}^{\infty}$

3) X is locally Euclidean:

① $\exists$ $X = \bigcup_{\alpha \in A} U_\alpha$ open cover, A an index set.

② $\exists$ $V_\alpha \subset \mathbb{R}^m$ open subset and $\varphi_\alpha : U_\alpha \xrightarrow{\sim} V_\alpha$ homeomorphism ($\triangleq$ φ_α is continuous and bijective, φ_α^{-1} is also continuous)

③ If $U_\alpha \cap U_\beta \neq \emptyset$

$$\begin{array}{ccc} \varphi_\alpha(U_\alpha \cap U_\beta) & \xrightarrow[\sim]{\varphi_\beta \circ \varphi_\alpha^{-1}} & \varphi_\beta(U_\alpha \cap U_\beta) \\ \cap\!| & & \cap\!| \\ \mathbb{R}^m & & \mathbb{R}^m \end{array}$$

is a diffeomorphism (it is smooth and it has a smooth inverse)

<u>Def</u>: Triplet $(U_\alpha, V_\alpha, \varphi_\alpha)$ is called a local chart of X, $V_\alpha \subset \mathbb{R}^n$ is called local coordinate system.

<u>Def</u>: X manifold, $f: X \to \mathbb{R}$ or $\mathbb{C}$ is a called a smooth function if $\forall$ any local chart $(U_\alpha, V_\alpha, \varphi_\alpha)$

$$f|_{U_\alpha} \circ \varphi_\alpha^{-1} : \underset{\mathbb{R}^m}{\underset{\cap}{V_\alpha}} \to \mathbb{R} \text{ or } \mathbb{C}$$

is smooth.

<u>Examples</u>: ① $\mathbb{R}^m$, $S^1 = \mathbb{R}/\mathbb{Z}$

$T^2 = \mathbb{R}^2/\mathbb{Z}^2$

m-sphere $S^m = \{(x_0, x_1, \cdots, x_m) \in \mathbb{R}^{m+1} : \sum_j x_j^2 = 1\}$

② $\mathbb{D} \subset \mathbb{R}^2$, or any <u>open subset</u> $V \subset \mathbb{R}^m$

(HW 1.1) $\mathbb{D} = \{(x,y) : x^2+y^2 < 1\}$ here our manifolds always have no boundary.

<u>Rmk</u> ① A topological space X is paracompact if for $\forall$ $X = \bigcup_{\alpha \in A} U_\alpha$ open cover

$\exists$ a refined open cover $X = \bigcup_{\beta \in B} U'_\beta$ that is <u>locally finite</u>

$\bigcup_{\beta \in B} U'_\beta$ is called a refinement of $\bigcup_{\alpha \in A} U_\alpha$

(that is $\forall \beta \in B$, $\exists \alpha \in A$
$U'_\beta \subset U_\alpha$)

$\forall x \in X$, $\exists V \underset{\text{open}}{\subset} X$, $x \in V$
s.t.
$\# \{\beta \in B \mid U'_\beta \cap V \neq \phi\} < \infty$.

⑤

$\rightsquigarrow$ such space admit
partition of unity
subordinate to any open cover.

② Hausdorff
second countable
locally compact
$\Bigg\} \underset{\text{HW1.2}}{\Rightarrow}$ paracompact

therefore, manifolds are always paracompact

Prop (Partition of unity) X manifold
$X = \bigcup_{\alpha \in A} U_\alpha$ open cover. Then

1) If $\{U_\alpha\}_{\alpha \in A}$ is locally finite.
Then $\exists\ \varphi_\alpha \in C^\infty(X, [0, 1])$ such that
- $\operatorname{supp} \varphi_\alpha \subset U_\alpha$
- $\sum_{\alpha \in A} \varphi_\alpha \equiv 1$

2) In general, $\exists \{\varphi_\beta \in C^\infty(X, [0,1])\}_{\beta \in B}$ s.t.
- $\forall \beta \in B$, $\exists \alpha \in A$ s.t.
$\operatorname{supp} \varphi_\beta \subset U_\alpha$

- $\sum_{\beta\in B} \varphi_\beta \equiv 1$, where the sum is locally finite.

Pf: HW 1.3

Use the fact that X is paracompact!

Rmk: local objects / constructions (or local charts) $\rightsquigarrow$ global objects on manifold X

↑ partition of unity.

Def (vector bundle) $\pi: E \to X$ is a complex vector bundle of rank r

if: ① E, X both are smooth manifolds, π is a smooth surjection

② $\exists$ $X = \bigcup_{\alpha\in A} U_\alpha$ open cover s.t. by local charts

$$\forall \alpha \ \exists \ G_\alpha : \pi^{-1}(U_\alpha) \xrightarrow[\text{diffeo}]{\sim} U_\alpha \times \mathbb{C}^r$$

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[r, "\sim"] \arrow[dr, "\pi"'] & U_\alpha \times \mathbb{C}^r \arrow[d, "\pi_1"] \\ & U_\alpha \end{tikzcd}$$

this is called a local trivialization of E over U_α.

③ If $U_\alpha \cap U_\beta = \phi$, then

$$\exists\ G_{\beta\alpha} \in C^\infty(U_\alpha \cap U_\beta, GL_r(\mathbb{C}))$$

↑ transition function

↑ invertible matrices of size $r \times r$

s.t.

$$\begin{tikzcd} & \pi^{-1}(U_\alpha \cap U_\beta) \arrow[dl, "G_\alpha"'] \arrow[dr, "G_\beta"] & \\ (U_\alpha \cap U_\beta) \times \mathbb{C}^r \arrow[rr, "G_\beta \circ G_\alpha^{-1}", "\sim"'] & & (U_\alpha \cap U_\beta) \times \mathbb{C}^r \end{tikzcd}$$

$$(x, v) \longmapsto (x, G_{\beta\alpha}(x) \cdot v)$$

↑ matrix acts on $\mathbb{C}^r$

In this case, $x \in X$, $E_x := \pi^{-1}(x) \simeq \mathbb{C}^r$
called the fiber of E at x.

Rmk: 1) Formally $E = \bigsqcup_{x \in X} E_x$

"vector bundle is a smooth family of vector spaces"

We can define real vector bundle in similar ($\mathbb{C}^r$) way
$E_x \simeq \mathbb{R}^r$ and $G_{\alpha\beta} \in GL_r(\mathbb{R})$

2) We have $G_{\alpha\beta} = G_{\beta\alpha}^{-1}$ (matrix inverse)

(*) $G_{\alpha\alpha} = Id_{r \times r}$

Moreover

If $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$, then

(**) $G_{\gamma\beta} \cdot G_{\beta\alpha} = G_{\gamma\alpha}$ on $U_\alpha \cap U_\beta \cap U_\gamma$

(*) (**) are called cocycle condition

3) For $X = \bigsqcup_{\alpha \in A} U_\alpha$ manifold

if $G_{\alpha\beta} \in C^\infty(U_\alpha \cap U_\beta, GL_r(\mathbb{C}))$

$\{G_{\alpha\beta}\}_{\alpha,\beta \in A}$ satisfies the cocycle condition

Then
$$E = \bigsqcup_\alpha U_\alpha \times \mathbb{C}^r \Big/ \sim$$

$$(x, v) \sim (y, \omega) \iff \begin{cases} x \in U_\alpha, \ y \in U_\beta \\ x = y \in U_\alpha \cap U_\beta \\ \omega = G_{\beta\alpha}(x) \cdot v \in \mathbb{C}^r \end{cases}$$

Prop : E is a smooth manifold and

$\pi : E \longrightarrow X$ induced by $U_\alpha \times \mathbb{C}^r \to U_\alpha \subset X$

is a smooth vector bdle on X of rank r.

$(\dim_{\mathbb{R}} E = \dim X + 2r)$.

Ex : ① Trivial vector bundle of rank r

$\underline{\mathbb{C}}^r := X \times \mathbb{C}^r \xrightarrow{\pi = Pr_1} X$

or $\underline{\mathbb{R}}^r := X \times \mathbb{R}^r \xrightarrow{\pi} X$

Here we can take the transition functions
$$G_{\alpha\beta} = Id_{r\times r}$$

② Tangent space or tangent vector bundle

X smooth manifold of $\dim = m$

For any local chart
$$U_\alpha \underset{\sim}{\xrightarrow{\varphi_\alpha}} V_\alpha \subset \mathbb{R}^m \ni (x_1^\alpha, \dots, x_m^\alpha)$$

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Consider $\sqcup_\alpha \times \mathbb{R}^m$

$$\left(x, \sum_{j=1}^{m} a_j \frac{\partial}{\partial x_j^\alpha}\right) \qquad a_j \in \mathbb{R}$$

Transition function, on $U_\alpha \cap U_\beta$

$$\mathbb{R}^m \supseteq \varphi_\alpha(U_\alpha \cap U_\beta) \xrightarrow{\varphi_{\beta\alpha} = \varphi_\beta \circ \varphi_\alpha^{-1}} \varphi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^m$$

$$(x_1^\alpha, \cdots, x_m^\alpha) \longmapsto (x_1^\beta, \cdots, x_m^\beta)$$

$$\frac{\partial}{\partial x_j^\alpha} = \sum_{k=1}^{m} \frac{\partial \varphi_{\beta\alpha, k}}{\partial x_j^\alpha} \frac{\partial}{\partial x_k^\beta}$$

$$G_{\beta\alpha}^{tan}(x) = \left(\frac{\partial \varphi_{\beta\alpha, k}(x)}{\partial x_j^\alpha}\right)_{kj} \in GL_r(\mathbb{R})$$

$\{G_{\beta\alpha}^{tan}\}_{\beta,\alpha}$ satisfies cocycle condition

This defines tangent vector bundle TX

3 $$dx_k^\beta = \frac{\partial \varphi_{\beta\alpha, k}}{\partial x_j^\alpha} dx_j^\alpha$$

$\Rightarrow$ we get cotangent vector bundle T^*X spanned by dx_j^α on local chart $(U_\alpha, V_\alpha, \varphi_\alpha)$

t = transpose
no conjugation

$$G_{\alpha\beta}^{cotan}(x) = {}^t G_{\alpha\beta}^{tan}(x)^{-1}.$$

Constructing new vector bundles out of old

Def: If $\pi : E \to X$ is a complex vector bdle of rank r given by transition fcts $\{G_{\alpha\beta}\}_{\alpha,\beta}$

- $E^* = \bigsqcup_{x \in X} E_x^*$ $\quad E_x^* := \mathrm{Hom}_{\mathbb{C}}(E_x, \mathbb{C})$
  is a complex v.b. defined by $\{ {}^t G_{\alpha\beta}^{-1} \}$
- $\overline{E} = \bigsqcup_{x \in X} \overline{E}_x$ defined by $\{\overline{G_{\alpha\beta}}\}$

  $\lambda \in \mathbb{C}$, $v \in \overline{E}$, $\lambda \cdot v = \bar{\lambda} v \in E$
- $\forall k \in \mathbb{N}$ $\quad E^{\otimes k}$ defined by $\{ \underbrace{G_{\alpha\beta} \otimes \cdots \otimes G_{\alpha\beta}}_{k \text{ times}} \}$
  $E^{\otimes 0} := \underline{\mathbb{C}}$
- $S^k(E)$ or $S^k E$ symmetric tensor product of E
  $= \bigsqcup_{x \in X} S^k E_x$

  $\Lambda^k(E)$ or $\Lambda^k E$ anti-symmetric tensor product of E
  $= \bigsqcup_{x \in X} \Lambda^k E_x$

E, F are vector bdles given by $\{G^E_{\alpha\beta}\}$, $\{G^F_{\alpha\beta}\}$

- E, F two vector bdles
  $E \otimes F = \bigsqcup_{x \in X} E_x \otimes F_x$ defined by $G^E_{\alpha\beta} \otimes G^F_{\alpha\beta}$
- $\mathrm{Hom}(E, F) := E^* \otimes F$
- $f : Y \to X$ smooth map between two manifolds

$f^* E = \bigsqcup_{y \in Y} E_{f(y)}$ vector bundle on Y defined by $\{G_{\alpha\beta} \circ f\}$
called pull-back bundle of E over Y by f.
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