Lecture 2 17th April 2025 ①

Recall: X^m smooth manifold, $Y \subset X$ subset

- Y is an immersed submanifold of : Y is a smooth mfd of dim $k \le m$

  (HW 1.4) injective immersion ‖

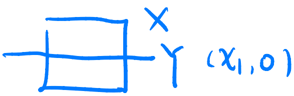
  $\forall y \in Y$, $\exists$ local chart $U_y \underset{\text{open}}{\subset} Y$ s.t. we can find a local chart of X $\in \mathbb{R}^m$

  inclusion $U_y \hookrightarrow X$ is given by $(x_1, \dots, x_k) \hookrightarrow (x_1, \dots, x_k, 0 \dots 0)$

- Y is an embedded submanifold $\Leftrightarrow$ $\forall y \in Y$, $\exists$ local chart (U, V, φ) of X, $y \in U$

  s.t. $\varphi(U \cap Y) = V \cap \mathbb{R}^k$ $\quad \mathbb{R}^k \subset \mathbb{R}^m$

  and $(U \cap Y, V \cap \mathbb{R}^k, \varphi|_{U \cap Y})$ is a local chart of Y.



Def: $\pi : E \to X$ vector bundle

$F \subset E$ embedded submanifold

F is a called a subbundle of E if

$\pi|_F : F \to X$ is a vector bundle (of rank $r' \le r$)

& $F_x = E_x \cap F \subset E_x$ a vector subspace

$\Leftrightarrow$ $\forall x \in X$, local chart U_x near x

s.t. $(\pi|_F)^{-1}(U_x) \hookrightarrow \pi^{-1}(U_x)$

$$\begin{array}{ccc} (\pi|_F)^{-1}(U_x) & \hookrightarrow & \pi^{-1}(U_x) \\ \wr| & & \wr| \\ U_x \times \mathbb{C}^{r'} & \hookrightarrow & U_x \times \mathbb{C}^r \\ (x, v) & \longmapsto & (x, I(v)) \end{array}$$

$I : \mathbb{C}^{r'} \hookrightarrow \mathbb{C}^r$ injective linear map

Prop: $F \subset E$ subbundle

E/F quotient bundle is well-defined by

$$E/F = \bigsqcup_{x \in X} E_x / F_x$$

HW 1.5: work out a system of transition functions for E/F

Def: $E, F \to X$ two vector bundle

$\varphi : E \to F$ morphism of vector bundles

is called a

if
- φ is smooth and

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr] & \arrow[d, phantom, "\circlearrowright"] & F \arrow[dl] \\ & X & \end{tikzcd}$$

- $\varphi_x : E_x \to F_x$ is linear

Ex:
- $f : Y \to X$ smooth map

  $df_y : T_yY \to T_{f(y)}X$ tangent map is a morphism of real vector bundles.

- subbundle $i : F \hookrightarrow E$ inclusion

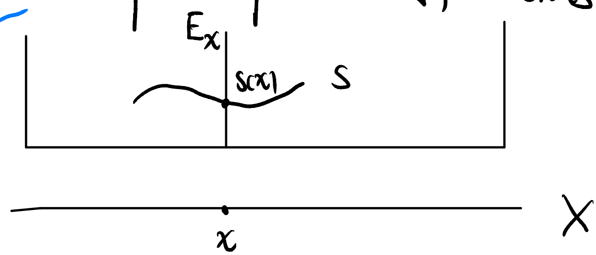
- quotient map $E \to E/F$

Def: $\pi : E \to X$ vector bundle

$$C^\infty(X, E) := \left\{ s : X \to E \;\; \begin{array}{l} \text{smooth map} \\ \text{s.t.} \\ \pi \circ s = Id_X \end{array} \right\}$$

$\Gamma(E) :=$ if no confusion

space of (smooth) sections



a section $s = \{ s(x) \in E_x \}_{x \in X}$

$s \in C^\infty(X, E) \iff \{s_\alpha\}_{\alpha \in A}$ | $s_\alpha \in C^\infty(U_\alpha, \mathbb{C}^r)$ s.t. $G_{\beta\alpha} s_\alpha = s_\beta$

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$\forall\ s, s' \in C^\infty(X, E)$

$(s+s')(x) := s(x) + s'(x) \in E_x$ $\rightsquigarrow$ $C^\infty(X, E)$ is a vector space.

$\Rightarrow\ s+s' \in C^\infty(X, E)$

$\forall\ \lambda \in \mathbb{C}\quad (\lambda s)(x) = \lambda s(x) \in E_x$

Def (Hermitian metric) $\pi : E \to X$

a Hermitian metric h^E on E is

- $h^E \in C^\infty(X, E^* \otimes \bar{E}^*)$
- $\forall\ x \in X\quad h^E_x$ is a Hermitian metric on fiber E_x

we will call (E, h^E) a Hermitian vector bundle.

Prop: Hermitian metric always exists.

Pf: $\pi : E \to X$ vector bundle

Take $X = \bigcup_{\alpha \in A} U_\alpha$ s.t. $G_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{C}^r$

We can assume $\{U_\alpha\}$ is locally finite cover.

- For each α, fix h^E_α Hermitian metric on $\pi^{-1}(U_\alpha)$ induced by standard Hermitian metric on $\mathbb{C}^r$
- Take $\{\varphi_\alpha\}$ partition of unity subordinate to $\{U_\alpha\}$

Define $h^E := \sum_{\alpha \in A} \varphi_\alpha h_\alpha \in C^\infty(X, E^* \otimes \bar{E}^*)$

which is a Hermitian metric #

Rmk: When E is a real vector bundle, we consider g^E the Euclidean metric on E.

In the case $E = TX$, g^{TX} is called a Riemannian metric of X.

Prop: E, F two complex vector bundles on X

$D : C^\infty(X, E) \longrightarrow C^\infty(X, F)$ $\mathbb{C}$-linear operator

s.t. $\forall f \in C^\infty(X)$, $s \in C^\infty(X, E)$

$$D(fs) = f D(s)$$

$$(\text{or} \quad [D, f] = 0)$$

Then $\exists$ $A \in C^\infty(X, \mathrm{Hom}(E, F))$ s.t.

$$[Ds](x) = A(x)\, s(x)$$

Pf: Step 1: D is locally defined, that is:

$\forall x \in X$, if $s_1 \equiv s_2$ in an open neighborhood U_x of x

then $Ds_1|_{V_x} \equiv Ds_2|_{V_x}$ for some open V_x nbd of x.

In fact: take $f \in C_c^\infty(U_x, \mathbb{R})$, $x \in V_x \subset U_x$

s.t. $f|_{\overline{V_x}} \equiv 1$, we have $f \in C^\infty(X)$ via extending it by 0.

$fs_1 \equiv fs_2$ on whole X

$fDs_1 = D(fs_1) = D(fs_2) = fDs_2$

on X $\Rightarrow$ $Ds_1|_{V_x} \equiv Ds_2|_{V_x}$.

Step 2: Consider local chart U_α

where E & F are locally trivialized as $\mathbb{C}^{r_1}$ & $\mathbb{C}^{r_2}$

$$D|_{U_\alpha} : C^\infty(U_\alpha, \mathbb{C}^{r_1}) \to C^\infty(U_\alpha, \mathbb{C}^{r_2})$$

⑤

Then $\forall f \in C^\infty(U_\alpha)$ $[D|_{U_\alpha}, f] = 0$

$$\Rightarrow (D|_{U_\alpha} s)(x) = A_\alpha(x)\, s(x)$$

where $A_\alpha(x) = (\langle D|_{U_\alpha} e_i, f_j \rangle)_{ij}$

$\in C^\infty(U_\alpha, \mathrm{Hom}(\mathbb{C}^{r_1}, \mathbb{C}^{r_2}))$

$\begin{cases} \{e_i\} \text{ basis of } \mathbb{C}^{r_1}, \text{ constant section} \\ \{f_j\} \quad \text{''} \quad \mathbb{C}^{r_2}, \text{ constant section} \end{cases}$

<u>Step 3</u> : Partition of unity $\{\varphi_\alpha\}$

$s \in C^\infty(X, E)$ $\qquad s = \sum_\alpha \varphi_\alpha s$

$$\begin{aligned} Ds &= D(\textstyle\sum_\alpha \varphi_\alpha s) \\ &= \sum_\alpha D|_{U_\alpha}(\varphi_\alpha s) \\ &= \underbrace{\sum_\alpha (A_\alpha \cdot \varphi_\alpha)}_{=: A \,\in\, C^\infty(X, \mathrm{Hom}(E, F))} s \end{aligned}$$

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I.2 Recall : differential and integral on manifolds

Def (differential forms) X manifold of dim $= m$

$$\Lambda^\bullet T^*X = \bigoplus_{k=0}^{m} \Lambda^k T^*X$$

rank $\Lambda^k T^*X = \binom{m}{k} = \frac{m!}{k!(m-k)!}$

Set

$\Omega^\bullet(X) := C^\infty(X, \Lambda^\bullet T^*X)$ differential forms on X (over $\mathbb{R}$)

$\cup|$ subspace

$\Omega^k(X) := C^\infty(X, \Lambda^k T^*X)$ differential forms of degree k (or k-forms)

$\Omega^0(X) = C^\infty(X, \mathbb{R})$

Similarly
$$\Omega^\bullet_c(X) := C^\infty_c(X, \Lambda^\bullet T^*X)$$
$$= \{ s \in C^\infty(X, \Lambda^\bullet T^*X) : \text{supp}\, s \text{ is compact in } X \}$$
$$\Omega^k_c(X) = \Omega^\bullet_c(X) \cap \Omega^k(X)$$
differential forms with compact support.

Recall: $s \in C^\infty(X, E)$
$$\text{supp}\, s := \overline{\{ x \in X : s(x) \neq 0 \text{ in } E_x \}} \subset X$$

If X is a compact manifold $\Omega^\bullet(X) = \Omega^\bullet_c(X)$.

$$\wedge : \Omega^k(X) \times \Omega^\ell(X) \to \Omega^{k+\ell}(X)$$
$$(\alpha \,,\, \beta) \mapsto \alpha \wedge \beta$$
wedge product or exterior product
$$\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha$$

Prop: $\exists !\ d : \Omega^\bullet(X) \to \Omega^{\bullet+1}(X)$, $\mathbb{R}$-linear operator s.t.

1) $\forall f \in C^\infty(X, \mathbb{R}) = \Omega^0(X)$
$df \in \Omega^1(X)$ is the classical differential of function f.

2) $d \circ d \equiv 0 \quad (d^2 = 0)$

3) $\forall \alpha \in \Omega^k(X)$, $\beta \in \Omega^\ell(X)$

Leibniz rule $\to$ $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge d\beta$

Recall: $f \in C^\infty(X, \mathbb{R})$ on a local chart $(U_\alpha, V_\alpha, \psi_\alpha)$

T^*X is spanned by $dx^1, \cdots, dx^m$

$(x_1, \cdots, x_m) \in V_\alpha$ local coordinates

$$f_\alpha = f|_{U_\alpha} \circ \psi_\alpha^{-1} : \underset{\mathbb{R}^m}{\overset{}{V_\alpha}} \to \mathbb{R} \quad C^\infty$$

($V_\alpha \subseteq \mathbb{R}^m$)

df on U_α is given by $df_\alpha = \sum_j \frac{\partial f_\alpha}{\partial x_j} dx^j$

$df \in \Omega^1(M)$ is defined by $\{df_\alpha\}_\alpha$

Generally, $\forall\, s \in \Omega^k(M)$

$s|_{U_\alpha}$ can be presented by $\sum_{|I|=k} f_I^\alpha\, dx^I$

where $\begin{cases} f_I^\alpha & \text{smooth function on } V_\alpha \\ dx^I = dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k} \\ I = (i_1 < i_2 \cdots < i_k) \end{cases}$

Pf: fix $X = \bigcup_{\alpha \in A} U_\alpha$ open cover by local charts locally finite

$\{\varphi_\alpha\}$ partition of unity subordinate to it

Uniqueness: $s \in \Omega^k(X)$

$$\begin{aligned} ds &= d\Big(\sum_\alpha \varphi_\alpha s\Big) \\ &= d\Big(\sum_\alpha \varphi_\alpha \sum_I f_I^\alpha dx^I\Big) \\ &= \sum_{\alpha, I} d(\underbrace{\varphi_\alpha f_I^\alpha}_{\text{fcts}}) \wedge dx^I \qquad (*) \end{aligned}$$

If d satisfies 1), 2) & 3)

Existence: Let d be defined as in $(*)$

HW 1.6

① d_S is independent of the cover $\bigsqcup_{\alpha\in A} U_\alpha$ and $\{\varphi_\alpha\}$

② $d^2 \equiv 0$, since it holds true on each local chart. #

Rk: d is determined by $d|_{\Omega^0}$ & $d|_{\Omega^1}$ through Leibniz rule.

Prop / Def: $f: X \to Y$ smooth map of two manifolds

$f^*: \Omega^k(Y) \to \Omega^k(X)$ pull-back on differential forms

via for $\alpha \in \Omega^1(Y)$ $(f^*\alpha)_x(v_x) = \alpha_{f(x)}(df\, v_x)$ for $\forall x \in X$, $v_x \in T_x X$

Moreover, we have

$$d^X \circ f^* = f^* \circ d^Y$$

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de Rham complex $(\Omega^\bullet(X), d)$

$$0 \to \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(X) \to 0$$

Since $d^2 = 0$, we get

$$\operatorname{Im} d|_{\Omega^{k-1}} \subseteq \ker d|_{\Omega^k}$$

Def: • $\alpha \in \Omega^\bullet(X)$ is called closed (or d-closed) if $d\alpha = 0$

• $\alpha \in \Omega^\bullet(X)$ is called exact (or d-exact) if $\exists \beta \in \Omega^\bullet(X)$ s.t. $\alpha = d\beta$

Exact forms are always closed!

Def: $H^k_{dR}(X) := \ker d|_{\Omega^k} \big/ \operatorname{Im} d|_{\Omega^{k-1}}$ de Rham cohomology (vector space)

Similarly

$H^k_{dR,c}(X) := \ker d|_{\Omega^k_c} \big/ \operatorname{Im} d|_{\Omega^{k-1}_c}$ de Rham coh with compact support.

For $\alpha \in \Omega^\bullet(X)$, if $d\alpha = 0$ ($\Leftrightarrow$ α is closed)
then α defines a cohomological class

$$[\alpha] \in H^\bullet_{dR}(X)$$

Two closed forms α and α' are called cohomologous
If $\alpha - \alpha'$ is exact: $\exists\, \beta \in \Omega^\bullet(X)$ s.t.

$$\alpha - \alpha' = d\beta.$$

$$H^k_{dR}(X) = \{\text{closed } k\text{-forms}\} \big/ \{\text{exact } k\text{-forms}\}$$

Rmk: When X is compact, we always have

$$\dim H^\bullet_{dR}(X) < \infty$$

In this case, $H^\bullet_{dR}(X)$ can be computed by the simplicial complex using triangulations of X.