Last week: – manifold
– vector bundle
– differential forms

$$\Omega^{\bullet}(X) = \bigoplus_{k=1}^{m} \Omega^k(X)$$

d : exterior diff. on $\Omega^{\bullet}$

Integrations on manifolds

$\dim X = m$

<u>Def</u>: $E \to X$ real vector bundle of rank r defined by $(G_{\alpha\beta} \in C^\infty(U_\alpha \cap U_\beta, GL_r(\mathbb{R}))$

E is called orientable iff $G_{\alpha\beta} \in GL_r^+(\mathbb{R})$

(that is $\det G_{\alpha\beta}(x) > 0$)

<u>Def</u>: The manifold X is called orientable iff TX is an orientable real vector bundle.

<u>Prop</u>: X is a connected manifold, then X is orientable iff $\Lambda^m T^*X$ is isomorphic to the trivial bundle $\underline{\mathbb{R}}$

In this case, an orientation $o(X)$ is defined as one choice of two connected components of $\Lambda^m T^*X \setminus \{0\}$.

Def: we call X an oriented manifold when it is orientable and an orientation $o(X)$ is given.

Given $(X, o(X))$, for $s \in \Omega^m(X)$

on a local chart $U_\alpha \simeq V_\alpha \subseteq \mathbb{R}^m$

$$s_\alpha = f_\alpha(x) \underbrace{dx^1 \wedge dx^2 \wedge \cdots \wedge dx^m}_{\text{oriented as } o(X)}$$

we define
$$\int_{U_\alpha} s_\alpha := \underbrace{\int_{V_\alpha} f_\alpha(x)\, dx^1 dx^2 \cdots dx^m}_{\text{Lebesgue integral on } V_\alpha \subseteq \mathbb{R}^m}$$

Prop: $X = \bigcup_\alpha U_\alpha$ open cover by local charts, locally finite

$\{\varphi_\alpha\}$ partition of 1

$\forall\ s \in \Omega^m_c(X)$

$$\int_X s := \sum_\alpha \int_{U_\alpha} \varphi_\alpha s \quad \in \mathbb{R}$$

is independent of the choice of $\{U_\alpha\}$ or $\{\varphi_\alpha\}$.

pf: $V_\alpha \xrightarrow[\psi_{\beta\alpha}]{\sim} V_\beta$ oriented $\rightsquigarrow J\psi_{\beta\alpha} = \det\left(\frac{\partial \psi_{\beta\alpha}}{\partial x}\right) > 0$

$$J\psi_{\beta\alpha} f_\beta = f_\alpha$$

$$\int_{V_\alpha} f_\alpha\, dx^\alpha = \int_{V_\alpha} J\psi_{\beta\alpha} f_\beta\, dx^\alpha = \int_{V_\beta} f_\beta\, dx^\beta \quad \#$$

Def: $\int_X : \Omega^\cdot_c(X) \longrightarrow \mathbb{R}$

$$s \longmapsto \int_X s^{[m]}$$

where $s^{[m]}$ is the degree-m component of s.

If we switch $o(X)$ to $-o(X)$ we get a minus sign for this integral.

Thm : If X is an oriented compact manifold, then $\forall\ s \in \Omega^{\cdot}(X)$

$$\int_X ds = 0 .$$

Then we can define

$$\int_X : H^{\cdot}(X) \longrightarrow \mathbb{R} .$$

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I.3 | Differential operator

E, F two vector bundle on X of rank r_1, r_2, resp.

Def : $P : C^\infty(X, E) \to C^\infty(X, F)$ is a differential operator of order k

If • $\exists\ X = \bigsqcup_{\alpha \in A} U_\alpha$ local charts s.t.

$$E|_{U_\alpha} \simeq V_\alpha \times \mathbb{C}^{r_1} \quad F|_{U_\alpha} \simeq V_\alpha \times \mathbb{C}^{r_2}$$

• $\exists\ P_\alpha := \sum_{|I| \le k} a_\alpha^I(x) \left(\frac{\partial}{\partial x}\right)^I$

$$: C^\infty(V_\alpha, \mathbb{C}^{r_1}) \to C^\infty(V_\alpha, \mathbb{C}^{r_2})$$

$I = (i_1, \cdots, i_m) \in \mathbb{N}_0^m$

$|I| = \sum_j i_j$

$\left(\frac{\partial}{\partial x}\right)^I = \left(\frac{\partial}{\partial x_1}\right)^{i_1} \cdots \left(\frac{\partial}{\partial x_m}\right)^{i_m}$

with $a_\alpha^I \in C^\infty(V_\alpha, \mathrm{Hom}(\mathbb{C}^{r_1}, \mathbb{C}^{r_2}))$

s.t. If $s = (s_\alpha)_\alpha \in C^\infty(X, E)$, then $(P_\alpha s_\alpha)_{\alpha \in A}$ defines a section in $C^\infty(X, F)$.

Example : • $C^\infty(X, \mathrm{Hom}(E, F)) \ni P$ defines a diff. op. of order 0.

• $d \circlearrowright \Omega^{\cdot}(X)$ is a diff. op. of order 1.

Prop: If $\forall f \in C^\infty(X)$, $[P, f]$ $= Pf - fP$ is a diff op of order $k-1$, then P is a diff. op. of order k.

HW 1.7

A KEY STEP: With the above hypothesis:

P is locally defined, i.e. $\forall x \in X$

if $s_1 = s_2$ on a nbd U_x of x, then $\exists$

$x \in V_x \subseteq U_x$ open, $Ps_1 = Ps_2$.

Take $\begin{cases} f \in C_c^\infty(U_x, \mathbb{R}) \subset C^\infty(X) \\ f|_{\overline{V_x}} \equiv 1 \end{cases}$ $\quad fs_1 = fs_2$ on X

$$\begin{aligned} fPs_1 &= Pfs_1 - \underbrace{[P,f]}_{\text{order} \le k-1} s_1 \quad \text{induction for the order} \\ &= Pfs_2 - [P,f]s_2 \\ &= fPs_2 \end{aligned}$$ #

$\rightsquigarrow$ reduced to diff. op. on a local chart (hence on $\mathbb{R}^m$)

Symbol: on local chart U_α, $\frac{\partial}{\partial x_j} \longleftrightarrow i\xi_j$ $\quad i = \sqrt{-1}$

$$\sigma_{total}(P_\alpha) := \sum_{|I| \le k} a_\alpha^I(x)\,(i\xi)^I$$

local total symbol

$$\in C^\infty(U_\alpha, \underbrace{\mathrm{Poly}^{\le k}(T^*X)}_{\text{polynomials along } T^*X \text{ of degree} \le k} \otimes \mathrm{Hom}(E, F))$$

$$\mathrm{Poly}^{\le k}(T^*X) \simeq \bigoplus_{j=0}^{k} S^k(TX)$$ via duality between TX and T^*X

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However

$\{\sigma_{total}(P_\alpha)\}_{\alpha\in A}$ does not define a global section of $Poly^{\le k}(T^*X)\otimes Hom(E,F)$

Prop / Def : Principal symbol

$$\sigma(P_\alpha) = \sum_{|I|=k} a_\alpha^I(x)\,(i\xi)^I \in C^\infty(U_\alpha, \underbrace{Poly^k(T^*X)\otimes Hom(E,F)}_{\simeq S^k(TX)\otimes Hom(E,F)})$$

$\{\sigma(P_\alpha)\}_{\alpha\in A}$ defines a global section $\sigma(P)$ in $C^\infty(X, S^k(TX)\otimes Hom(E,F))$

Pf : Take $f\in C^\infty(X)$, $t>0$

$e^{-itf}\,P\,e^{itf}$ is a diff. op.

On local chart U_α

$$e^{-itf}P_\alpha e^{itf} = e^{-itf}\sum_{|I|\le k} a_\alpha^I(x)\left(\frac{\partial}{\partial x}\right)^I e^{itf}$$

$$= t^k \sum_{|I|=k} a_\alpha^I(x)(i\,df)^I + O(t^{k-1})$$

$$\lim_{t\to+\infty}\frac{1}{t^k}e^{-itf}P_\alpha e^{itf} = \sum_{|I|=k} a_\alpha^I(x)\,(i\,df)^I$$

$$= \sigma(P_\alpha)(df)$$

So $\sigma(P)$ is globally well-defined by

$$\sigma(P)(x,\xi) := \lim_{t\to+\infty}\frac{1}{t^k}e^{-itf}\,P\,e^{itf} \in Hom(E_x, F_x)$$

where $df(x) = \xi \in T_x^*X$ #

Def $\mathrm{Diff}^{\le k}(E,F)$ = diff. op. of order $\le k$ ⑥

Prop: We have exact sequence

principal symbol

$$0 \longrightarrow \mathrm{Diff}^{\le k-1}(E,F) \overset{i}{\hookrightarrow} \mathrm{Diff}^{\le k}(E,F) \overset{\sigma}{\longrightarrow} C^\infty(X, S^k(TX) \otimes \mathrm{Hom}(E,F)) \longrightarrow 0$$

which means: $\begin{cases} i \text{ is injective} \\ \sigma \text{ is surjective} \\ \ker \sigma = \mathrm{Im}\, i \end{cases}$

pf: HW 1.8

Def: A diff op P is called elliptic if

$\sigma(x,\xi) \in \mathrm{Hom}(E_x, F_x)$ is invertible

for $x \in X$, and $0 \neq \xi \in T_x^* X$.

$(\Rightarrow \ \mathrm{rk}\, E = \mathrm{rk}\, F)$

Examples: — Dirac op. D

— Laplacians

on $\mathbb{R}^n$ $\quad \Delta = \sum_j \left(\frac{\partial}{\partial x_j}\right)^2$

$$\sigma(\Delta) = -\sum_j (\xi_j)^2$$

Δ is elliptic.

Statements of Atiyah - Singer index theorem

Thm : X compact manifold of dim m

E, F complex vector bdles on X

$P : C^\infty(X, E) \to C^\infty(X, F)$ elliptic diff. op.

Then

① $\operatorname{Ker} P$ and $\operatorname{Coker} P$ are finite - dimensional

$\operatorname{Ind}(P) := \dim \operatorname{Ker} P - \dim \operatorname{Coker} P$

is well-defined

② If $m = $ odd , $\operatorname{Ind}(P) = 0$

③ If $m = $ even , we have

$$\operatorname{Ind}(P) = \int_{T^*X} ch(\sigma(P))\, \pi^*(\hat{A}(T^*X)^2) \qquad (*)$$

$\leftarrow$ oriented

where $\begin{cases} ch(\sigma(P)) \in \Omega^\cdot(T^*X) & \text{integrable} \\ \hat{A}(T^*X) = 1 + \cdots\cdots \in \Omega^\cdot(X) \\ \pi : T^*X \longrightarrow X . \end{cases}$

"Approach" : ① Construct a Dirac op. D on a new manifold s.t. $\operatorname{Ind}(P) = \operatorname{Ind}(D)$

② Prove $(*)$ for D .

In this Lecture:

1) Define characteristic classes and forms $\hat{A}(T^*X)$, $ch(G(F))$
2) Dirac operator D (and spin structure)
3) A-S index theory via heat kernel
"local index theorem"
4) Geometric applications:

Thm (Gauss-Bonnet-Chern)

X cpt, even-dim, orientable

Euler class or form

$$\sum_j (-1)^j \dim H^j_{dR}(X) = \int_X e(TX)$$

Thm (Hirzebruch) X cpt, orientable, $\dim = 4k$

we have a symmetric bilinear form

$$\eta : H^{2k}_{dR}(X, \mathbb{R}) \times H^{2k}_{dR}(X, \mathbb{R}) \to \mathbb{R}$$

$$([\alpha], [\beta]) \mapsto \int_X \alpha \wedge \beta$$

HW 2.1

well-defined and non-degenerate (Poincaré duality), and

$$\text{signature}(\eta) = \int_X L(TX)$$

L-class

Thm (Riemann-Roch-Hirzebruch) X cpt complex mfd and Kähler

$$\sum_{j=0}^{\dim_{\mathbb{C}} X} (-1)^j \dim \underbrace{H^j(X, \mathcal{O}_X)}_{\text{sheaf coh.}} = \int_X Td(TX)$$

Todd class

$\simeq H^{0,j}(X)$ Dolbeault coh.

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