II. Theory of connections, characteristic classes & Chern-Weil theory

II.1 Connection and curvature

X manifold, $E \to X$ vector bundle

Def: $\nabla^E : C^\infty(X, E) \to \Omega^1(X, E) := C^\infty(X, T^*X \otimes E)$ is a connection if

1) ∇^E is $\mathbb{C}$-linear

2) $\forall f \in C^\infty(X)$, $s \in C^\infty(X, E)$

(Leibniz rule) $\nabla^E(fs) = df \wedge s + f \nabla^E s$

$\Leftrightarrow [\nabla^E, f] = df \wedge$

Rmk: 1) & 2) $\Leftrightarrow$ ∇^E is a first order diff. op. with principal symbol $\sigma(\nabla^E)(x, \xi) = i\xi \wedge$

Prop: Space of connections is non-empty and affine, more precisely: ① There always exists a connection ∇^E

② If we have two connections ∇^E_1, ∇^E_2 then $\nabla^E_1 - \nabla^E_2 \in \Omega^1(X, \mathrm{End}(E))$

Pf: ① Recall $\sigma : \mathrm{Diff}^{\leq 1}_X(E, T^*X \otimes E) \to C^\infty(X, T X \otimes \mathrm{Hom}(E, T^*X \otimes E))$ is surjective

$\sigma(\nabla^E)(x, \xi) = i\xi \wedge$

② $\sigma(\nabla^E_1 - \nabla^E_2) = 0 \Rightarrow \nabla^E_1 - \nabla^E_2 \in \mathrm{Diff}^{\leq 0}_X(E, T^*X \otimes E)$

$\quad = C^\infty(X, T^*X \otimes E^* \otimes E)$

$\quad = \Omega^1(X, \mathrm{End}(E))$ #

Rmk : On local chart U_α

$$E|_{U_\alpha} \simeq U_\alpha \times \mathbb{C}^r$$

$$\nabla^E|_{U_\alpha} = d + \Gamma^E_\alpha \qquad \Gamma^E_\alpha \in \Omega^1(U_\alpha, \mathrm{End}(\mathbb{C}^r))$$

(d: exterior differential)

$$\nabla^E|_{U_\alpha}\begin{bmatrix} f_1 \\ \vdots \\ f_r \end{bmatrix} = \begin{bmatrix} df_1 \\ \vdots \\ df_r \end{bmatrix} + \Gamma^E_\alpha \begin{bmatrix} f_1 \\ \vdots \\ f_r \end{bmatrix}$$

$f_j \in C^\infty(U_\alpha)$

Rmk : For $v \in C^\infty(X, TX)$ vector field, $s \in C^\infty(X, E)$

$\nabla^E_v s \in C^\infty(X, E)$ "covariant derivative by v" (or along v)

Def (induced connections) (E, ∇^E)

- For E^* : $s^* \in C^\infty(X, E^*)$, $s \in C^\infty(X, E)$

$$\langle \nabla^{E^*} s^*, s \rangle + \langle s^*, \nabla^E s \rangle = d \underbrace{\langle s^*, s \rangle}_{C^\infty(X)}$$

locally, $\nabla^{E^*}|_{U_\alpha} = d - {}^t\Gamma^E_\alpha$

- For $\overline{E}$: $\nabla^{\overline{E}} \overline{s} = \overline{\nabla^E s}$ locally $\nabla^{\overline{E}}|_{U_\alpha} = d + \overline{\Gamma^E_\alpha}$

- For $E \otimes F$, ∇^E & ∇^F induce a connection

$$\nabla^{E\otimes F}(\underset{\substack{\cap \\ C^\infty(X,E)}}{s_1} \otimes \underset{\substack{\cap \\ C^\infty(X,F)}}{s_2}) = (\nabla^E s_1) \otimes s_2 + s_1 \otimes (\nabla^F s_2)$$

locally $\nabla^{E\otimes F}|_{U_\alpha} = d + \Gamma^E_\alpha \otimes \mathrm{Id}_F + \mathrm{Id}_E \otimes \Gamma^F_\alpha$

- Similarly for $E^{\otimes k}$, $\Lambda^k E$, $S^k E$, $\mathrm{Hom}(E, F)$.

Prop: $f: Y \longrightarrow X$, $E \to X$ vector bdle with ∇^E

Then ∇^E induces a connection ∇^{f^*E} for $f^*E \to Y$

locally defined by $\nabla^{f^*E}|_{U_\alpha} = d + f^*\Gamma^E_\alpha$

$U_\alpha \subset Y$ local chart

It is the unique connection on f^*E s.t.

$\forall\ s \in C^\infty(X, E)$, $v \in T_y Y$

$$\nabla^{f^*E}_v (s \circ f)(y) = (\nabla^E_{f_* v} s)(f(y)) \qquad \#$$

$f_* v = df_y v$

<u>Now</u> Let h^E be a smooth Hermitian metric on E, ∇^E

<u>Def</u>: (1) The adjoint connection $\nabla^{E'}$ of ∇^E w.r.t. h^E is defined by

$$\underbrace{d\langle s_1, s_2\rangle_{h^E}}_{C^\infty(X)} = \langle \nabla^{E'} s_1, s_2\rangle_{h^E} + \langle s_1, \nabla^E s_2\rangle_{h^E} \qquad \forall\ s_1, s_2 \in C^\infty(X, E)$$

(2) The connection ∇^E is called metric or hermitian on (E, h^E) iff $\nabla^E = \nabla^{E'}$.

$$(\Leftrightarrow\ d\langle s_1, s_2\rangle_{h^E} = \langle \nabla^E s_1, s_2\rangle_{h^E} + \langle s_1, \nabla^E s_2\rangle_{h^E})$$

Prop: Given (E, h^E), always $\exists$ metric connection.

Pf: Take any $\nabla^E \rightsquigarrow \nabla^{E'}$

Define $\nabla^u := \frac{1}{2}(\nabla^E + \nabla^{E'})$ a connection on E

$$(\nabla^u)' = \frac{1}{2}(\nabla^{E'} + (\nabla^{E'})') = \frac{1}{2}(\nabla^{E'} + \nabla^E) = \nabla^u \ \#$$

Curvature:

Given $\nabla^E : C^\infty(X, E) \to \Omega^1(X, E)$

we define $\nabla^E : \underset{\substack{\| \\ C^\infty(X,\, \Lambda^\bullet T^*X \otimes E)}}{\Omega^\bullet(X, E)} \longrightarrow \Omega^{\bullet+1}(X, E)$

$$\alpha \wedge s \longmapsto d\alpha \wedge s + (-1)^{|\alpha|} \alpha \wedge \nabla^E s$$

<u>Def</u>: $R^E := (\nabla^E)^2 : C^\infty(X, E) \to \Omega^2(X, E)$ is called curvature of ∇^E (on E).

Note that ∇^E is of first order, so a priori,

$$R^E \in \mathrm{Diff}_X^{\leq 2}(E, \Lambda^2 T^*X \otimes E)$$

But $\sigma(R^E) = -\xi \wedge \xi = 0 \Rightarrow R^E \in \mathrm{Diff}_X^{\leq 1}$

<u>Prop</u>: R^E is diff. op. of order 0, that is $R^E \in \underset{\substack{\| \\ C^\infty(X,\, \Lambda^2 T^*X \otimes \mathrm{End}(E))}}{\Omega^2(X, \mathrm{End}(E))}$.

pf: Take $f \in C^\infty(X)$, $s \in C^\infty(X, E)$

$$\begin{aligned} R^E(fs) &= \nabla^E(\nabla^E(fs)) = \nabla^E(df \wedge s + f\nabla^E s) \\ &= \underbrace{ddf}_{\substack{\| \\ 0}} \wedge s - df \wedge \nabla^E s + df \wedge \nabla^E s + f(\nabla^E)^2 s \\ &= f(R^E s) \end{aligned}$$

$\Rightarrow [R^E, f] = 0$ by Prop in Lecture 1

$\Rightarrow R^E \in C^\infty(X, \mathrm{Hom}(E, \Lambda^2 T^*X \otimes E)) = \Omega^2(X, \mathrm{End}(E))$ #

<u>Prop</u>: For $U, V \in TX$,

$$R^E(U, V) = \nabla^E_U \nabla^E_V - \nabla^E_V \nabla^E_U - \nabla^E_{[U,V]}$$

$[U, V] \in TX$ is the Lie bracket of U & V $= UV - VU$ as diff. operator on $C^\infty(X)$.

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Pf: Let $e_1, \cdots, e_m$ be a local basis of TX

$e^1, \cdots, e^m$ dual basis of T^*X

Then $\nabla^E \underset{\text{locally}}{=} \sum_{j=1}^{m} e^j \wedge \nabla^E_{e_j}$

For $u \in TX$ $\quad u = \sum_j e^j(u) e_j \in TX$

$\Omega^2(X) \ni$ also $de^j(u, v) = u\, e^j(v) - v\, e^j(u) - e^j([u, v])$

Then $(\nabla^E)^2 = \nabla^E \left(\sum_j e^j \wedge \nabla^E_{e_j} \right)$

$$= \sum_j de^j \wedge \nabla^E_{e_j} + \sum_{j,k} e^k \wedge e^j \nabla^E_{e_k} \nabla^E_{e_j}$$

Then

$$(\nabla^E)^2(u, v) = u(e^j(v)) \cdot \nabla^E_{e_j} - v(e^j(u)) \nabla^E_{e_j}$$
$$+ \sum_{j,k} \left(e^k(u) e^j(v) - e^k(v) e^j(u) \right) \nabla^E_{e_k} \nabla^E_{e_j}$$
$$- e^j([u, v]) \nabla^E_{e_j}$$
$$= \underbrace{\left(u(e^j(v)) \nabla^E_{e_j} + e^j(v) \nabla^E_u \nabla^E_{e_j} \right)}_{= \nabla^E_u \nabla^E_v} - \underbrace{\left(v(e^j(u)) \nabla^E_{e_j} + e^j(u) \nabla^E_v \nabla^E_{e_j} \right)}_{= \nabla^E_v \nabla^E_u} - \nabla^E_{[u, v]}$$

#

Prop (Bianchi identity) $[R^E, \nabla^E] \equiv 0$

$= R^E \circ \nabla^E - \nabla^E \circ R^E$

Prop: Let ∇^E be a metric connection of (E, h^E)

Then $R^E \in \Omega^2(X, \mathrm{End}^{\text{anti}}(E))$

anti-hermitian endomorphism w.r.t h^E

Pf: For $s_1, s_2 \in C^\infty(X, E) = \Gamma(E)$ 6

$$0 = d^2 \langle s_1, s_2 \rangle_{h^E} = d\left(d \langle s_1, s_2 \rangle_{h^E} \right)$$
$$= d\left(\langle \nabla^E s_1, s_2 \rangle_{h^E} + \langle s_1, \nabla^E s_2 \rangle_{h^E} \right)$$
$$= \langle R^E s_1, s_2 \rangle_{h^E} \cancel{- \langle \nabla^E s_1, \nabla^E s_2 \rangle_{h^E}}$$
$$\cancel{+ \langle \nabla^E s_1, s_2 \rangle_{h^E}}$$
$$+ \langle s_1, R^E s_2 \rangle_{h^E} \qquad \#$$

II.2 Parallel transport

$\gamma : [0, 1] \longrightarrow X$ a smooth curve

$(E, \nabla^E) \longrightarrow X$

Def: A section $s_\cdot \in C^\infty([0,1], \gamma^* E)$ ($\Leftrightarrow s(t) \in E_{\gamma(t)}$) is parallel along γ w.r.t. ∇^E if it satisfies

$$\nabla^{\gamma^* E}_{\frac{\partial}{\partial t}} s \equiv 0 \quad \left(\text{simply denoted as } \nabla^E_{\dot\gamma} s \equiv 0 \right)$$

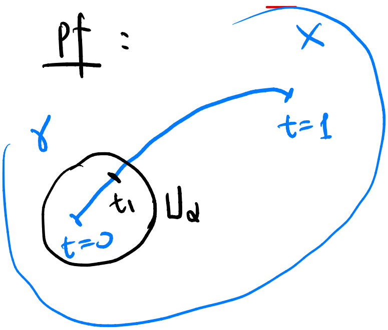
$\dot\gamma$ speed vector of γ

Prop: Given any vector $v \in E_{\gamma(0)}$, $\exists !$ $s_\cdot \in C^\infty([0,1], \gamma^* E)$ s.t. (unique)

$$(*) \begin{cases} \nabla^{\gamma^* E}_{\frac{\partial}{\partial t}} s \equiv 0 \\ s_0 = v \end{cases}$$

"initial problem for first order linear differential equation"

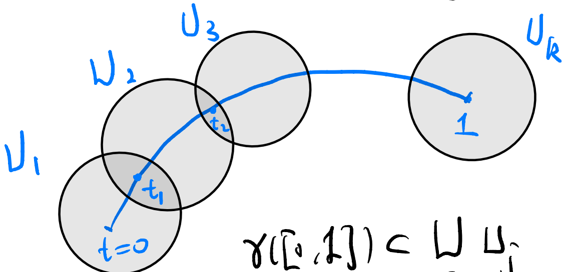
Pf :



$$\begin{cases} U_\alpha \text{ a small local chart where} \\ E|_{U_\alpha} \simeq U_\alpha \times \mathbb{C}^r \\ \nabla^E = d + \Gamma^E_\alpha \end{cases}$$

$$(*) \text{ on } U_\alpha \iff \begin{cases} \frac{\partial S(t)}{\partial t} + \Gamma^E_{\alpha,\gamma(t)}(\dot{\gamma}(t)) S(t) = 0 \\ S(0) = v \end{cases}$$

By the existence and uniqueness of the solution of ODE, we get a parallel section S for $t \in [0, t_1]$



$\gamma([0,1]) \subset \bigcup_j U_j$

U_j a sequence of local charts

$\gamma([0,1])$ is cpt

finite # of U_j's.

$S|_{[0,t_1]}$

$S|_{[t_1,t_2]}$ ......

Finally, they path ~~together~~ smoothly as a parallel section S along γ, which is uniquely determined by $S(0) \in E_{\gamma(0)}$.

#

Def Given $\gamma : [0,1] \xrightarrow{C^\infty} X$, for $t \in [0,1]$

$\tau^0_t : E_{\gamma(0)} \to E_{\gamma(t)}$ is the linear map s.t.

$\forall s_0 \in E_{\gamma(t)}$, $t \mapsto \tau^0_t s_0$ is the unique parallel section along γ with initial value s_0 at $t = 0$

Rmk: $\tau_{t_2}^{t_1} \circ \tau_{t_1}^{t_0} = \tau_{t_1}^{t_0}$ #

Parallel transport $\rightsquigarrow$ "canonical" local trivialization of vector bundle

$X = \bigcup_\alpha U_\alpha \qquad U_\alpha \simeq B(0,1) \subset \mathbb{R}^n$

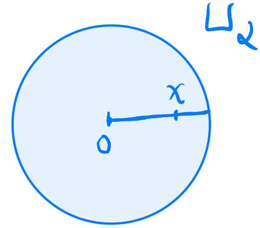
$\pi : E \to X$

Then

$$\pi^{-1}(U_\alpha) \xrightarrow{\sim} B(0,1) \times E_0$$

$$(x, \tau_x^0 v) \longleftrightarrow (x, v)$$

$\uparrow$ parallel transport along $t \mapsto tx$ in $B(0,1)$ w.r.t. ∇^E



Ex: $S^1 = \mathbb{R}/\mathbb{Z} \ni t \qquad E = S^1 \times \mathbb{C}$

$$\nabla^E = dt \wedge \frac{\partial}{\partial t} + \sqrt{-1}\, \alpha(t)\, dt$$

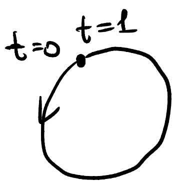
$\alpha \in C^\infty(S^1, \mathbb{R})$ real fct. $\alpha(t) = \alpha(1+t)$

A parallel section s along $\gamma(t) = t \in S^1$ is given as $s : \mathbb{R} \to \mathbb{C}$ such that

$$\frac{\partial s(t)}{\partial t} + \sqrt{-1}\, \alpha(t) s(t) = 0$$

$$\Rightarrow \quad s(t) = s(0)\, e^{-\sqrt{-1}\int_0^t \alpha(s)\, ds}$$

#



from $t=0$ back $t=1 \sim t=0$

the holonomy is given by $e^{-\sqrt{-1}\int_0^1 \alpha(s)\, ds} \in U(1)$.