Lecture 5 30/04/2025 ①

Last time: connection & curvature

Today: Applications to characteristic classes & characteristic forms (Chern-Weil Theory)

II.3] First chern forms and chern classes of complex line bdles

(L, ∇^L) is a complex line bdle on manifold X
↖ connection $rk = 1$, $L_x \cong \mathbb{C}$

Then $R^L = (\nabla^L)^2 \in \Omega^2(X, \mathrm{End}(L)) \cong \Omega^2(X, \mathbb{C})$

$$\mathrm{End}(L) \xrightarrow[\text{isomorphism}]{tr} \mathbb{C}$$

Def: $c_1(L, \nabla^L) := \frac{\sqrt{-1}}{2\pi} R^L \in \Omega^2(X, \mathbb{C})$
is called the first chern form of (L, ∇^L)

Prop: 1) $c_1(L, \nabla^L)$ is a closed form, i.e., $d c_1(L, \nabla^L) = 0$

2) If $\nabla^L_1 - \nabla^L_2 = A \in \Omega^1(X, \mathbb{C} = \mathrm{End}(L))$, then
$$c_1(L, \nabla^L_1) = c_1(L, \nabla^L_2) + \frac{\sqrt{-1}}{2\pi} dA$$
so the coh. class $[c_1(L, \nabla^L)] \in H^2_{dR}(X, \mathbb{C})$ is independent of the connection ∇^L.

Pf: Locally, $\nabla^L = d + \Gamma$ $\quad \Gamma \in \Omega^1(U, \mathbb{C})$ local chart

$$R^L = (\nabla^L)^2 = (d+\Gamma)(d+\Gamma)$$
$$= \underbrace{d^2}_{=0} + d\Gamma + \underbrace{\Gamma\wedge\Gamma}_{=0} = d\Gamma$$

$\Rightarrow \quad dR^L = dd\Gamma = 0 \quad \Rightarrow$ 1)

For 2): we write $\nabla^L_j = d + \Gamma_j$ on U

$$\Gamma_1 - \Gamma_2 = A$$
$$R^L_1 - R^L_2 = d\Gamma_1 - d\Gamma_2 = d(\Gamma_1 - \Gamma_2) = dA \quad \#$$

Def: For a complex line bdle $L \to X$, the first chern class $c_1(L) \in H^2_{dR}(X, \mathbb{C})$ is defined as $[c_1(L, \nabla^L)]$ for a connection ∇^L on L.

HW 2.3) More properties

Prop: If ∇^L is a metric connection for (L, h^L), then
$$c_1(L, \nabla^L) \in \Omega^2(X, \mathbb{R})$$
Hence, for any L, $\quad c_1(L) \in H^2_{dR}(X, \mathbb{R})$.

Pf: In this case, locally $\nabla^L = d + \sqrt{-1}\,\Gamma$, $\quad \Gamma \in \Omega^1(X, \mathbb{R})$

$$R^L = \sqrt{-1}\, d\Gamma$$
$$c_1(L, \nabla^L) = -\frac{1}{2\pi} d\Gamma \quad \text{real 2-form} \qquad \#$$

Rmk: In general, $c_1(L, \nabla^L) = c_1^{Re}(L, \nabla^L) + \sqrt{-1}\, c_1^{Im}(L, \nabla^L)$

then $c_1^{Im}(L, \nabla^L) = d\beta$ exact.

HW 2.3 : Concrete example on S^2.

<u>Rmk</u>: A complex line bdle L is determined by $\{ G_{\alpha\beta} \in C^\infty(U_\alpha \cap U_\beta, \mathbb{C}^*) \}$ with cocycle condition

$\rightsquigarrow$ defines an element $[L]$ in $\check{H}^1(X, (C_X^\infty)^*) = H^1(X, (C_X^\infty)^*)$

first Čech cohomology = first sheaf cohomology via injective resolution

Consider the exact sequence of sheaves on X

$$0 \to \mathbb{Z} \to C_X^\infty \xrightarrow{\exp(2\pi\sqrt{-1}\,\cdot)} (C_X^\infty)^* \to 0$$

constant sheaf; germs of C^∞-fcts; germs of nonvanishing C^∞ fcts

Then we get the long exact sequence for sheaf cohomology

$$\cdots \to \underset{\substack{\| \\ 0}}{H^1(X, C_X^\infty)} \to H^1(X, (C_X^\infty)^*) \xrightarrow{\sim} H^2(X, \mathbb{Z}) \to \underset{\substack{\| \\ 0}}{H^2(X, C_X^\infty)} \to \cdots$$

$$[L] \xmapsto{} c_1^{top}(L)$$

topological first Chern class

Note that since C_X^∞ is soft

$$H^p(X, C_X^\infty) = 0 \quad \text{for} \quad p \geq 1$$

Consider $\quad H^2(X, \mathbb{Z}) \otimes \mathbb{C} \simeq H^2(X, \mathbb{C}) \simeq H^2_{dR}(X, \mathbb{C})$

$$c_1^{top}(L) \longleftrightarrow c_1(L) \text{ given by } c_1(L, \nabla^L)$$

Rmk: For a smooth manifold X,

$$\check{H}^k(X, \underset{\text{or } \mathbb{R}}{\mathbb{C}}) \simeq H^k(X, \underset{\text{or } \mathbb{R}}{\mathbb{C}}) \simeq H^k_{dR}(X, \underset{\text{or } \mathbb{R}}{\mathbb{C}})$$

HW 2.4

constant sheaf $\mathbb{C}$ or $\mathbb{R}$ over X. In the sequel, we denote $H^k(X)$.

II. 4) Characteristic classes (topological)

We will define the characteristic classes for

- complex vector bundles | Chern class, chern character, Todd class
- real vector bundles | Pontrjagin class, $\hat{A}$ - class, L - class
- oriented real vector bundles : Euler class.

Today : topological version

Next time : Chern-Weil theory

We admit the following Splitting Principle

tangent map is surjective

Complex version: Given $E \to X$ complex vector bundle of rank r, then $\exists$ $\pi : M \to X$ smooth submersion s.t.

- $\pi^* : H^\bullet(X) \hookrightarrow H^\bullet(M)$ injective
- $\pi^* E \simeq L_1 \oplus L_2 \oplus \cdots \oplus L_r$ as vector bdls on M, $\mathrm{rk}(L_j) = 1$.

In this, M is called a split manifold for E.

HW 2.5

(Actually π is a proper fibration)

Ref: Lawson, Spin Geometry, Chap III §11.

Thm: $\exists\,!$ map, called Chern class

$$\forall X \quad \left\{\begin{array}{c}\text{complex vector bundles}\\ \text{on } X\end{array}\right\} \longrightarrow H^{2\cdot}(X) = \bigoplus_{k=1}^{[n/2]} H^{2k}(X)$$

$$E/\text{isomorphism} \longmapsto c(E)$$

s.t.: ① $c(E) = 1 + c_1(E) + c_2(E) + \cdots \in H^{2\cdot}(X)$

$c_j(E) \in H^{2j}(X)$

② $c(E\oplus F) = c(E)\wedge c(F)\ (= c(E)\,c(F))$.

③ For any ~~smooth~~ $f: Y \to X$

$f^* c(E) = c(f^*E)$

④ $c(L) = 1 + c_1(L)$ if rank $L = 1$

$c_1(L)$ is the first Chern class of L.

Pf: Uniqueness: If the map c exists for all X with ① - ④

Then for any $E \to X$, $\exists\ \pi: M \to X$

$\pi^* E \simeq L_1 \oplus \cdots \oplus L_r$

Then $\pi^* c(E) = c(L_1 \oplus \cdots \oplus L_r) = \prod_{j=1}^{r} (1 + c_1(L_j))$

This determines $c(E)$ uniquely!

Existence: we define $c(E)$ by

$$\pi^* c(E) = \prod_j (1 + c_1(L_j)) \in H^{2\cdot}(M)$$

① If $L_1' \oplus \cdots \oplus L_r' \simeq L_1 \oplus \cdots \oplus L_r$ on M

then $\prod_j (1 + c_1(L_j)) = \prod_j (1 + c_1(L_j'))$

2. If $\pi_j : M_j \to X$ two split manifolds

$$\pi : M_1 \times_X M_2 =: M \longrightarrow X$$

M is also split

$$\Rightarrow \quad \pi_1^* c(E) = \pi_2^* c(E) \cong \pi^* c(E)$$

$$\pi^* : H^{2\cdot}(X) \hookrightarrow H^{2\cdot}(M)$$

So it is enough to prove that

$$\prod_j (1 + c_1(L_j)) \in \text{Image of } \pi^*_M \; H^{2\cdot}(M)$$

Using symmetric polynomials

$$c(E) = \left[\det^E \left(\mathrm{Id} + \tfrac{\sqrt{-1}}{2\pi} R^E \right) \right] \in H^{2\cdot}(X)$$

since $\pi^*\left(\frac{\sqrt{-1}}{2\pi} R^E\right) = \frac{\sqrt{-1}}{2\pi} R^{\pi^* E} \underset{\text{coh.}}{\simeq} \begin{bmatrix} c_1(L_1) & & \\ & \ddots & \\ & & c_1(L_r) \end{bmatrix}$ #

Chern-Weil

Def: $f : \mathbb{R} \to \mathbb{R}$ real analytic fct with $f(0) = 1$

The multiplicative class $f_m(E)$ for cplx v.b. $E \to X$ is defined by

$$\pi^* f_m(E) = \prod_{j=1}^r f(c_1(L_j))$$

$$= \prod_{j=1}^r \underbrace{\left(\sum_{k=0}^\infty \frac{1}{k!} f^{(k)}(0)\, c_1(L_j)^k \right)}_{\text{finite sum}}$$

Example:
- If $f(x) = 1 + x \rightsquigarrow$ chern class $c(E)$
- If $f(x) = \dfrac{x}{1 - e^{-x}} \rightsquigarrow$ Todd class $Td(E)$

Def: The additive class $f_a(E)$ is defined by

$$\pi^* f_a(E) = \sum_{j=1}^{r} f(c_1(L_j))$$

Example: if $f(x) = e^x$, $f_a(E) \rightsquigarrow$ chern character $ch(E)$

Real version: Splitting principle

$E \to X$ real vector bdle of $\overset{\text{real}}{\wedge}$ rank r. Then $\exists\ \pi : M \to X$ smooth proper fibration s.t.

- $\pi^* : H^\cdot(X) \hookrightarrow H^\cdot(M)$ injective
- $$\pi^*(E \otimes_{\mathbb{R}} \mathbb{C}) = \begin{cases} L_1 \oplus \bar{L}_1 \oplus \cdots \oplus L_k \oplus \bar{L}_k & \text{if } r = 2k \\ L_1 \oplus \bar{L}_1 \cdots \oplus L_k \oplus \bar{L}_k \oplus \underline{\mathbb{C}} & \text{if } r = 2k+1 \end{cases}$$

$\mathrm{rk}\, L_j = 1$

Rmk: We can not distinguish L_j and $\bar{L}_j$.

Lemma: $c_1(L) = -c_1(\bar{L}) \in H^2(X)$ $\quad L$ cplx line bdle

pf: Fix (L, h^L) and a metric connection ∇^L

locally $\nabla^L = d + \sqrt{-1}\,\Gamma$ $\quad \Gamma$ real 1-form

So the induced connection $\nabla^{\bar{L}}$ on $\bar{L}$

$$\nabla^{\bar{L}} = d - \sqrt{-1}\,\Gamma$$

$$\rightsquigarrow \quad c_1(\bar{L}) = \left[\tfrac{\sqrt{-1}}{2\pi} R^{\bar{L}}\right] = \left[\tfrac{1}{2\pi} d\Gamma\right] = -c_1(L). \quad \#$$

Def: $f : \mathbb{R} \to \mathbb{R}$ real analytic, $\overset{f(0)=1}{\vee}$ even fct $(f(x) = f(-x))$

The multiplicative class $f_M(E)$ is defined by

$$\pi^* f_M(E) = \prod_{j=1}^{k} f(c_1(L_j)) \in H^{4\cdot}$$

we can also use $c_1(\bar{L}_j) = -c_1(L_j)$

Examples :

① $f(x) = 1+x^2$ Pontrjagin class $p(E)$

② $f(x) = \dfrac{x/2}{\sinh(x/2)}$ $\hat{A}$ - class $\hat{A}(E)$

③ $f(x) = \dfrac{x/2}{\tanh(x/2)}$ L - class $L(E)$.

Rmk : In the above construction, we can replace real analytic f by a formal power series $f(x) = \sum_j a_j x^j$ $a_j \in \mathbb{R}$