# Lecture 6



Recall: last time, we define the characteristic classes for complex/real vector bundles by the splitting principles based on the first Chern classes/forms for complex line bundles

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When $E$ is real and oriented, of rank $r=2k$, then we can distinguish $\underbrace{L_j \text{ and } \bar{L}_j}$

In fact: $\pi: M \to X$

$$\pi^* E \cong E_1 \oplus \cdots \oplus E_k$$

HW 2.6 each $E_j$ is a real bundle of rank 2.

s.t. $E_j \otimes \mathbb{C} \simeq L_j \oplus \bar{L}_j$

Euler class

Def: $e(E) = \prod_{j=1}^{k} c_1(L_j) \in H^{2k}(X)$

## II.5 Characteristic forms by Chern-Weil theory

We need to use $\begin{cases} \text{super symmetric formulation} \\ \text{Connection theory} \end{cases}$

$\mathbb{Z}_2$-graded str. $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$

### Superspace and supertrace

Def: ① A superspace is a $\mathbb{Z}_2$-graded vector space

$$E = E^+ \oplus E^-$$

② A superalgebra is an associative $\mathbb{Z}_2$-graded algebra with identity (unital associative algebra)

$$\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^-$$

s.t. $1 \in \mathcal{A}^+$ & $\begin{cases} \mathcal{A}^+\mathcal{A}^-, \ \mathcal{A}^-\mathcal{A}^+ \subset \mathcal{A}^- \\ \mathcal{A}^+\mathcal{A}^+, \ \mathcal{A}^-\mathcal{A}^- \subset \mathcal{A}^+ \end{cases}$

Rmk: A usual vector space $E$ is a superspace

$$E = \underset{+}{E} \oplus \underset{-}{0}$$

A usual algebra $\mathcal{A}$ is a superalg. $\mathcal{A} = \underset{+}{\mathcal{A}} \oplus \underset{-}{0}$

Example: ① $\Omega^{\cdot}(X) = \underset{\text{even degree}}{\Omega^+(X)} \oplus \underset{\text{odd degree}}{\Omega^-(X)}$

superalgebra

② For $E = E^+ \oplus E^-$ superspace

$\mathrm{End}(E)$ is a superalgebra

s.t.

$$\mathrm{End}^+(E) = \{ f \in \mathrm{End}(E) : f \text{ preserves the splitting } E = E^+ \oplus E^- \}$$ 3

$$= \mathrm{End}(E^+) \oplus \mathrm{End}(E^-)$$

$$\mathrm{End}^-(E) = \mathrm{Hom}(E^+, E^-) \oplus \mathrm{Hom}(E^-, E^+)$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}}_{\mathrm{End}^+(E)} + \underbrace{\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}}_{\mathrm{End}^-(E)}$$

**Def**: • If $\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^-$ is a superalgebra: $a, b \in \mathcal{A}$

$$|a| = \begin{cases} 0 & a \in \mathcal{A}^+ \\ 1 & a \in \mathcal{A}^- \end{cases}$$

supercommutator / super-bracket

$$[a, b]_s := ab - (-1)^{|a|\cdot|b|} ba$$

or simply $[a, b]$

• $\mathcal{A}$ is (super) commutative if $[a, b] = 0 \quad \forall a, b \in \mathcal{A}$.

Rmk: ① $\forall a, b, c \in \mathcal{A}$

$$[a, [b, c]] = [[a, b], c] + (-1)^{|a|\cdot|b|} [b, [a, c]]$$

HW 2.7.

② $\Omega^\cdot(X)$ is commutative

**Def**: $\alpha : \mathcal{A} \to \mathbb{C}$ $\mathbb{C}$-linear map is called a supertrace if $\forall a, b \in \mathcal{A}$

$$\alpha([a, b]) = 0.$$

**Prop**: $E = E^+ \oplus E^-$ superspace. Define

$$\mathrm{Tr}_s : \mathrm{End}(E) \longrightarrow \mathbb{C} \quad (\text{or } \mathbb{R} \text{ if } E \text{ is } \mathbb{R}\text{-space})$$

$$\rotatebox{90}{$\in$}$$

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \longmapsto \mathrm{Tr}^{E^+}[A] - \mathrm{Tr}^{E^-}[A]$$

Then $\mathrm{Tr}_s$ is a supertrace on superalgebra $\mathrm{End}(E)$.

(Rmk: here we consider the finite dimensional $E$ )

Pf: 1) $\mathrm{Tr}_s\left[\left[\begin{pmatrix} A & 0 \\ 0 & D\end{pmatrix}, \begin{pmatrix} A' & 0 \\ 0 & D'\end{pmatrix}\right]\right] = \mathrm{Tr}_s\left[\begin{bmatrix} [A, A'] & 0 \\ 0 & [D, D']\end{bmatrix}\right] = 0$

2) $\mathrm{Tr}_s\left[\left[\begin{pmatrix} A & 0 \\ 0 & D\end{pmatrix}, \begin{pmatrix} 0 & B \\ C & 0\end{pmatrix}\right]\right] = \mathrm{Tr}_s\left[\begin{bmatrix} 0 & * \\ * & 0\end{bmatrix}\right] = 0$

3) $\mathrm{Tr}_s\left[\left[\begin{pmatrix} 0 & B \\ C & 0\end{pmatrix}, \begin{pmatrix} 0 & B' \\ C' & 0\end{pmatrix}\right]\right] = \mathrm{Tr}_s\left[\begin{pmatrix} BC' + B'C & 0 \\ 0 & CB' + C'B\end{pmatrix}\right]$

$= \mathrm{Tr}^{E^+}[BC' + B'C] - \mathrm{Tr}^{E^-}[CB' + C'B]$

$= 0$ cancel each other

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Rmk: Take $\tau = \begin{bmatrix} \mathrm{Id}_{E^+} & 0 \\ 0 & -\mathrm{Id}_{E^-}\end{bmatrix} \in \mathrm{End}^+(E)$, then for $M \in \mathrm{End}(E)$

$$\mathrm{Tr}_s[E] = \mathrm{Tr}^E[\tau M]$$

usual trace of endomorphism.

Def: $\mathcal{A}$, $\mathcal{B}$ two superalgebras, $\mathcal{A} \hat{\otimes} \mathcal{B}$ superalgebra is defined as

$$\left.\begin{aligned} \mathcal{A} \hat{\otimes} \mathcal{B} &= \mathcal{A} \otimes \mathcal{B} \\ [\mathcal{A} \hat{\otimes} \mathcal{B}]^+ &= \mathcal{A}^+ \otimes \mathcal{B}^+ \oplus \mathcal{A}^- \otimes \mathcal{B}^- \\ [\mathcal{A} \hat{\otimes} \mathcal{B}]^- &= \mathcal{A}^+ \otimes \mathcal{B}^- \oplus \mathcal{A}^- \otimes \mathcal{B}^+ \end{aligned}\right\} \text{ as sets}$$

we put

$$(a \hat{\otimes} b)(a' \hat{\otimes} b') = (-1)^{|a'| \cdot |b|} aa' \hat{\otimes} bb'$$

when there is no confusion, we also write $a \otimes b$.

If $B^- = 0$, then $\mathcal{A} \hat{\otimes} \mathcal{B}$ is just the normal $\mathcal{A} \otimes \mathcal{B}$

Def: $\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^-$ superalgebra, commutative

$E = E^+ \oplus E^-$ superspace

$$\begin{aligned} \mathrm{Tr}_s : \mathcal{A} \hat{\otimes} \mathrm{End}(E) &\longrightarrow \mathcal{A} \\ a \hat{\otimes} M &\longmapsto a\, \mathrm{Tr}_s[M] \end{aligned}$$

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Prop: $\forall A, B \in \mathcal{A} \hat{\otimes} \mathrm{End}(E)$, where $\mathcal{A}$ is commutative

$\mathrm{Tr}_s[[A, B]] = 0$ in $\mathcal{A}$.

pf: $A = a \hat{\otimes} M_1, \quad B = b \hat{\otimes} M_2$

$$[A, B] = (a \hat{\otimes} M_1)(b \hat{\otimes} M_2) - (-1)^{(|a|+|M_1|)(|b|+|M_2|)} (b \hat{\otimes} M_2)(a \hat{\otimes} M_1)$$

$$= ab \hat{\otimes} (-1)^{|b|\cdot|M_1|} M_1 M_2 - (-1)^{|a|\cdot|b| + |b|\cdot|M_1| + |M_1|\cdot|M_2|} ba \hat{\otimes} M_2 M_1$$

$$= (-1)^{|b|\cdot|M_1|} ab \hat{\otimes} [M_1, M_2]$$

since $ab = (-1)^{|a|\cdot|b|} ba$

$\mathcal{A}$ is commutative

$$\Rightarrow \mathrm{Tr}_s[[A, B]] = (-1)^{|b|\cdot|M_1|} ab\, \mathrm{Tr}_s[[M_1, M_2]] = 0.$$

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## Chern-Weil theory

$X$ manifold, $\quad E \to X$ vector bundle

Pointwise version

- $\forall x \in X, \quad \Lambda^\cdot T_x^* X$ commutative superalgebra ($\alpha \wedge \beta$)
- $E_x = \underset{+}{E_x} \oplus \underset{-}{0}$ super-trace now is usual trace
- $\mathrm{Tr}: \Lambda^\cdot T_x^* X \otimes \mathrm{End}(E_x) \to \Lambda^\cdot T_x^* X$ vanishes on the $[\ ,\ ]$.

global version

- $\Omega^\cdot(X)$ superalgebra, commutative
- $\Omega^\cdot(X, \mathrm{End}(E))$ superalgebra
- $\nabla^E : \Omega^\cdot(X, E) \to \Omega^{\cdot+1}(X, E)$ odd operator
- $R^E = (\nabla^E)^2 = \frac{1}{2}[\nabla^E, \nabla^E]$ even operator

we consider

$$\mathrm{Tr} : \Omega^{\bullet}(X, \mathrm{End}(E)) \longrightarrow \Omega^{\bullet}(X)$$

taking trace of $\mathrm{End}(E_x)$ pointwisely

Prop: For $A \in \Omega^{\bullet}(X, \mathrm{End}(E))$, we have

$$d\,\mathrm{Tr}[A] = \mathrm{Tr}[[\nabla^E, A]]$$

Pf: Consider a local chart

$$\nabla^E|_{U_\alpha} = d + \Gamma^E_\alpha \qquad \text{odd operator}$$

$$[\nabla^E|_{U_\alpha}, A] = [d, A] + [\Gamma^E_\alpha, A]$$

$$\mathrm{Tr}[[\nabla^E|_{U_\alpha}, A]] = \mathrm{Tr}[[d, A]] + 0$$

If $A \in C^\infty(X, \mathrm{End}(E))$

$$[d, A] = dA$$

$$\mathrm{Tr}[dA] = d\,\mathrm{Tr}[A]$$

If $A = \alpha \wedge B \quad \alpha \in \Omega^{\bullet}(X), \ B \in C^\infty(X, \mathrm{End}(E))$

$$dA = d\alpha \wedge B + (-1)^{|\alpha|} \alpha \wedge dB$$

$$\mathrm{Tr}[dA] = d\alpha\, \mathrm{Tr}[B] + (-1)^{|\alpha|} \alpha \wedge d\,\mathrm{Tr}[B]$$

$$= d(\alpha\, \mathrm{Tr}[B]) = d\,\mathrm{Tr}[A] \qquad \#$$

Def: $(E, \nabla^E) \to X$ complex vector bundle with connection $\nabla^E$

$f : \mathbb{R} \to \mathbb{R}$ analytic fct

$$f_a(E, \nabla^E) := \mathrm{Tr}\left[f\left(\frac{\sqrt{-1}}{2\pi} R^E\right)\right] \in \Omega^{2\bullet}(X, \mathbb{C})$$

$\uparrow$

additive

**Thm** (Chern – Weil)

1) $f_a(E, \nabla^E)$ is a closed form.
2) $f_a(E, \nabla_1^E) - f_a(E, \nabla_0^E)$ is an exact form
3) $[f_a(E, \nabla^E)] \in H^\bullet(X, \mathbb{C})$ is independent of choices of $\nabla^E$
4) $[f_a(E, \nabla^E)] = f_a(E) \leftarrow$ the topological version of additive $f$-class.

Pf: 1) $d f_a(E, \nabla^E)$

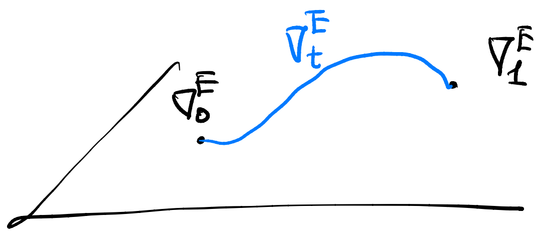
$$= d\, \mathrm{Tr}\left[ f_a\left(\tfrac{\sqrt{-1}}{2\pi} R^E\right)\right] = \mathrm{Tr}\Big[ \underbrace{\left[\nabla^E, f_a\left(\tfrac{\sqrt{-1}}{2\pi} R^E\right)\right]}_{\substack{\| \\ 0}} \Big]$$

since Bianchi identity

$$[\nabla^E, R^E] = 0$$

$$\rightsquigarrow [\nabla^E, (R^E)^k] = 0$$

2) $(\nabla_t^E)_{t \in \mathbb{R}}$ a smooth family of smooth connections of $E$




e.g. $\nabla_t^E = (1-t)\nabla_0^E + t\nabla_1^E$

$$\begin{array}{ccc} \pi^* E & & E \\ \downarrow & & \downarrow \\ X \times \mathbb{R} & \xrightarrow{\pi} & X \end{array}$$

$C^\infty(X \times \mathbb{R}, \pi^* E) = C^\infty(X, E) \otimes C^\infty(\mathbb{R})$

Define the connection

$$\nabla^{\pi^* E} = dt \wedge \frac{\partial}{\partial t} + \nabla_t^E$$

Then $R^{\pi^*E} \in \Omega^2(X\times\mathbb{R})$ 8

$$R^{\pi^*E} = dt\wedge a_t + R^E_t$$

where $\begin{cases} a_t = \frac{\partial}{\partial t}\nabla^E_t \in \Omega^1(X, \mathrm{End}(E)) \\ R^E_t = (\nabla^E_t)^2 \end{cases}$

Take $\mathrm{Tr}^{\pi^*E}\left[f\left(\frac{\sqrt{-1}}{2\pi}R^{\pi^*E}\right)\right] = dt\wedge\beta_t + \pi^*\mathrm{Tr}^E\left[f\left(\frac{\sqrt{-1}}{2\pi}R^E_t\right)\right]$

By 1), it is $d^{X\times\mathbb{R}}$-closed, i.e.

$$\left(dt\wedge\frac{\partial}{\partial t} + d^X\right)\left(dt\wedge\beta_t + f_a(E,\nabla^E_t)\right) = 0$$

$$dt\wedge\left(-d^X\beta_t + \frac{\partial}{\partial t}f_a(E,\nabla^E_t)\right) = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial t}f_a(E,\nabla^E_t) = d^X\beta_t$$

Therefore $f_a(E,\nabla^E_1) - f_a(E,\nabla^E_0) = \underline{d^X\left(\int_0^1 \beta_t\, dt\right)}$

$\in \Omega^{\cdot}(X)$. $\underline{d\text{-exact}}$

1) + 2) $\Rightarrow$ 3)

4)
$$\begin{array}{ccc} & & E \\ & & \downarrow \\ M & \xrightarrow{\pi} & X \end{array} \qquad \pi^*E = L_1\oplus L_2\cdots\oplus L_r$$

$\pi^*f_a(E,\nabla^E) = f_a(\pi^*E, \nabla^{\pi^*E})$ since $R^{\pi^*E} = \pi^*R^E$

$= f_a(L_1\oplus\cdots\oplus L_r, \nabla^{\pi^*E})$

$\underset{\text{cohomologous}}{\sim} f_a(L_1\oplus\cdots\oplus L_r, \nabla^{L_1}\oplus\cdots\oplus\nabla^{L_r})$

$$= \mathrm{Tr}\left[f\begin{bmatrix}\frac{\sqrt{-1}}{2\pi}R^{L_1} & & \\ & \ddots & \\ & & \frac{\sqrt{-1}}{2\pi}R^{L_r}\end{bmatrix}\right]$$

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$$= \operatorname{Tr}\left[\begin{pmatrix} f(c_1(L_1, \nabla^{L_1})) & & \\ & \ddots & \\ & & f(c_1(L_r, \nabla^{L_r})) \end{pmatrix}\right]$$

$$= \sum_j f(c_1(L_j, \nabla^{L_j})) \longrightarrow \pi^* f_a(E) \quad \#$$