Proposition: If ∇^E is a metric connection of (E, h^E), then for any $f : \mathbb{R} \to \mathbb{R}$ real analytic, the form

$$f_a(E, \nabla^E) := \mathrm{Tr}\left[f\left(\frac{\sqrt{-1}}{2\pi} R^E\right)\right] \in \Omega^{\text{even}}(X, \mathbb{R})$$

proof: Note that $R^E \in \Omega^2(X, \mathrm{End}^{\text{anti}}(E))$

We define an endomorphism

$$* : \Omega^{\bullet}(X, \mathrm{End}(E)) \longrightarrow \Omega^{\bullet}(X, \mathrm{End}(E))$$

$$A = \alpha_1 \wedge \cdots \wedge \alpha_k \otimes u \longmapsto A^* = (-\alpha_k) \wedge (-\alpha_{k-1}) \wedge \cdots \wedge (-\alpha_2) \wedge (-\alpha_1) \otimes u^*$$

α_j 1-form

$u \in \Gamma(\mathrm{End}(E))$

u^* pointwise adj. of u w.r.t h^E

Then ① $(A^*)^* = A$, $(A \wedge B)^* = B^* \wedge A^*$

Moreover $(R^E)^* = R^E$

② For $A \in \Omega^k(X, \mathrm{End}(E))$, we have

$$\mathrm{Tr}[A^*] = (-1)^{\frac{k(k+1)}{2}} \overline{\mathrm{Tr}[A]} \leftarrow \text{complex conj.}$$

So $(R^E)^* = R^E$, then for any $k \in \mathbb{N}$

(2)

$$\mathrm{Tr}[(R^E)^k] = \mathrm{Tr}[((R^E)^k)^*]$$
$$= (-1)^{\frac{2k(2k+1)}{2}} \overline{\mathrm{Tr}[(R^E)^k]}$$
$$= (-1)^k \overline{\mathrm{Tr}[(R^E)^k]}$$

Then $\mathrm{Tr}[(\sqrt{-1}R^E)^k] = \overline{\mathrm{Tr}[(\sqrt{-1}R^E)^k]}$

$$\Rightarrow \quad \overline{f_a(E,\nabla^E)} = f_a(E,\nabla^E) \qquad \#$$

Lemma : Let $f(x) = \sum_{j=1}^{+\infty} a_j x^j$, $a_j \in \mathbb{R}$ be a formal power series

Then
$$f : \Omega^{\mathrm{even}}(X) \longrightarrow \Omega^{\mathrm{even}}(X)$$
$$\alpha \longmapsto f(\alpha) := \sum a_j \alpha^j$$
is well-defined. Moreover, if $d\alpha = 0$, then $df(\alpha) = 0$.

<u>Def</u> (characteristic forms)

(E, ∇^E) a complex vector bundle equipped with connection ∇^E.

- $ch(E,\nabla^E) = \mathrm{Tr}\left[\exp\left(\frac{\sqrt{-1}}{2\pi}R^E\right)\right] \in \Omega^{2\bullet}(X)$ Chern character form
- $c(E,\nabla^E) = \det\left(1 + \frac{\sqrt{-1}}{2\pi}R^E\right)$ total Chern form
$$= \exp\left(\mathrm{Tr}\left[\log\left(1+\frac{\sqrt{-1}}{2\pi}R^E\right)\right]\right)$$

"defined via the Taylor series of $\log(1+x)$"

Todd form

- $$\mathrm{Td}(E,\nabla^E) = \det\left[\frac{\frac{\sqrt{-1}}{2\pi}R^E}{1-e^{-\frac{\sqrt{-1}}{2\pi}R^E}}\right] \in \Omega^{2\cdot}(X)$$
$$= \exp(\mathrm{Tr}[\log[\ \ \ ]])$$

They are closed forms on X, and they are real if ∇^E is metric.

Def: For a real vector bundle (E,∇^E), and for $f:\mathbb{R}\to\mathbb{R}$ real analytic, even function and $f(0)=1$. Then define

$$f_m(E,\nabla^E) = \det_E\left[f\left(\frac{\sqrt{-1}}{2\pi}R^E\right)\right]^{1/2}$$
$$= \exp\left(\frac{1}{2}\mathrm{Tr}\left[\log f\left(\frac{\sqrt{-1}}{2\pi}R^E\right)\right]\right) \in \Omega^{4\cdot}(X,\mathbb{R})$$

since $\log f(x)$ has a well-defined Taylor series at $x=0$

Prop: (E,∇^E) real, $f(0)=1$.

1) $f_m(E,\nabla^E)$ is a closed form

2) The cohomological class $[f_m(E,\nabla^E)]$ is independent of ∇^E.

3) $[f_m(E,\nabla^E)] = f_m(E) \in H^{4\cdot}(X,\mathbb{C})$

The topological version

Proof: 1) & 2) follow from the same arguments as for $f_a(E,\nabla^E)$.

For 3): Recall that $f_m(E)$ is defined by

$\pi : M \longrightarrow X$ a C^∞-proper fibration

s.t. $\pi^* E \otimes \mathbb{C} \simeq L_1 \oplus \bar{L}_1 \oplus \cdots \oplus L_k \oplus \bar{L}_k$ if $\mathrm{rk} E = 2k$

$\simeq L_1 \oplus \bar{L}_1 \oplus \cdots \oplus L_k \oplus \bar{L}_k \oplus \mathbb{C}$ if $\mathrm{rk} E = 2k+1$

Then we compute

$$\pi^* f_m(E, \nabla^E) = \det_{E\otimes\mathbb{C}}\left(f\left(\tfrac{\sqrt{-1}}{2\pi} R^{\pi^* E}\right)\right)^{1/2}$$

$$\overset{\text{cohomo}}{\simeq} \det{}^{1/2}_{E\otimes\mathbb{C}} \left(f \begin{bmatrix} \frac{\sqrt{-1}}{2\pi} R^{L_1} & & \\ & \frac{\sqrt{-1}}{2\pi} R^{\bar{L}_1} & \\ & & \ddots \end{bmatrix} \right)$$

$R^{\mathbb{C}} = 0$

$f(0) = 1$

suppose ∇^L is metric

$$\simeq \det{}^{1/2}_{E\otimes\mathbb{C}} \; f \begin{bmatrix} c_1(L_1, \nabla^{L_1}) & & \\ & -c_1(L_1, \nabla^{L_1}) & \\ & & \ddots \end{bmatrix}$$

f is even

$$= \det{}^{1/2}_{E\otimes\mathbb{C}} \begin{bmatrix} f(c_1(L_1)) & & & & \\ & f(c_1(L_1)) & & & \\ & & f(c_1(L_2)) & & \\ & & & f(c_1(L_2)) & \\ & & & & \ddots \end{bmatrix}$$

$$= \prod_j f(c_1(L_j)) = \pi^* f(E). \qquad \#$$

Def :

$$\hat{A}(E, \nabla^E) = \det\left[\frac{\sqrt{-1}R^E/4\pi}{\sinh(\sqrt{-1}R^E/4\pi)}\right]^{1/2} \in \Omega^{4\cdot}(X)$$

$$L(E, \nabla^E) = \det\left(\frac{\sqrt{-1}R^E/4\pi}{\tanh(\sqrt{-1}R^E/4\pi)}\right)^{1/2}$$

Euler form:

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Let $(V, \langle , \rangle)$ be an oriented Euclidean space (over $\mathbb{R}$)

Def: Let $A: V \to V$ be an anti-symmetric linear map.

We define $\eta(A) \in \Lambda^2 V^*$ by

$$\eta(A) = \frac{1}{2} \sum_{j,\ell} \langle e_j, A e_\ell \rangle e^j \wedge e^\ell$$

where $\{e_j\}$ is ONB of $(V, \langle , \rangle)$

$\{e^j\}$ dual basis of V^*

Rmk: $-\eta(A)$ is the usual correspondence of A in $\Lambda^2 V^*$.

Def [Pfaffian] $n = \dim_{\mathbb{R}} V$, let $\{e_j\}$ be an ONB (orthonormal basis) and let $e^1 \wedge \cdots \wedge e^n$ be the unit element in $\Lambda^n V^*$ representing the orientation of V. For $A \in \mathrm{End}^{\text{anti}}(V)$, we define $\mathrm{Pf}[A] \in \mathbb{R}$ by

$$\exp(\eta(A)) = \underbrace{\mathrm{Pf}[A]\, e^1 \wedge \cdots \wedge e^n}_{\text{top degree}} + \text{terms of lower degree}$$

Note $\mathrm{Pf}[A]$ depends on the orientation of V and $\langle , \rangle$.

Rmk: $V = \mathbb{R}^2$ $\quad A = \begin{bmatrix} 0 & \theta \\ -\theta & 0 \end{bmatrix}$ $\quad \theta \in \mathbb{R}$

$$\eta(A) = \langle e_1, A e_2 \rangle e^1 \wedge e^2 = \theta\, e^1 \wedge e^2$$

$$\exp(\eta(A)) = 1 + \theta\, e^1 \wedge e^2$$

so $\mathrm{Pf}[A] = \theta$, note that $\det A = \theta^2$

Rmk: If $n = \dim_{\mathbb{R}} V = $ odd, we have $\mathrm{Pf}[A] = 0$

Prop: $\mathrm{Pf}[A]^2 = \det A$

Proof: If $n = $ odd, $A \in \mathrm{End}^{\mathrm{anti}}(V)$, $\mathrm{Pf}[A] = \det A = 0$

If $\underset{= 2k}{n = \text{even}}$, $\exists$ $e_1, e_2, e_3, e_4, \cdots, e_{2k-1}, e_{2k}$

oriented ONB of V s.t.

orthonormal basis

we can wrote

$$A = \begin{bmatrix} \begin{bmatrix} 0 & \theta_1 \\ -\theta_1 & 0 \end{bmatrix} & & & \\ & \begin{bmatrix} 0 & \theta_2 \\ -\theta_2 & 0 \end{bmatrix} & & \\ & & \ddots & \\ & & & \begin{bmatrix} 0 & \theta_k \\ -\theta_k & 0 \end{bmatrix} \end{bmatrix}$$

with $\theta_j \in \mathbb{R}$

$$\eta(A) = \sum_{j=1}^{k} \theta_j e^{2j-1} \wedge e^{2j}$$

$$\Rightarrow \quad \mathrm{Pf}[A] = \prod_j \theta_j \qquad \det A = \prod_j \theta_j^2 \qquad \#$$

Def (Euler form) $E \to X$ oriented real vector bundle

g^E Euclidean metric on E

∇^E metric connection

Define $e(E, g^E, \nabla^E) := \mathrm{Pf}\left[\frac{R^E}{2\pi}\right] \in \Omega^{r}(X, \mathbb{R})$ $\quad r = \mathrm{rk}\, E$

defined pointwisely on (E_x, g^E_x)

Rmk: If $\operatorname{rk} E = \text{odd}$, $e(E, g^E, \nabla^E) = 0$.

Prop: 1) $e(E, g^E, \nabla^E)$ is closed

2) Its class $[e(E, g^E, \nabla^E)]$ is independent of (g^E, ∇^E)

3) $[e(E, g^E, \nabla^E)] = e(E) \in H^{\bullet}(X, \mathbb{R})$.

Proof: Pf is not a $\mathrm{Tr}[f(\cdot)]$, so the previous arguments do not apply.

1) e_i ONB of (E, g^E)

e^i dual basis of E^*

Then $R^E \rightsquigarrow \eta(R^E) = \sum_{i,j} \frac{1}{2} \langle e_i, R^E e_j \rangle_{g^E} e^i \wedge e^j \in \Omega^2(X, \Lambda^2 E^*)$

Claim 1: $g^E : E \to E^*$ (since it is real)

$$a \longmapsto (g^E a)(b) = g^E(a, b)$$

Then $[\nabla^E, g^E] = 0$ since ∇^E is metric connection

$$\Big(\Leftrightarrow \underset{\substack{\uparrow \\ \text{induced connection}}}{\nabla^{E^*}} g^E - g^E \nabla^E = 0 \text{ as } \mathrm{Hom}(E, E^*) \Big)$$

$\forall\ a, b \in E$

$$\begin{aligned} \langle \nabla^{E^*} g^E a, b \rangle &= d\, g^E(a, b) - (g^E a)(\nabla^E b) \\ &= g^E(\nabla^E a, b) + \cancel{g^E(a, \nabla^E b)} \\ &\quad - \cancel{(g^E a)(\nabla^E b)} \\ &= \langle g^E \nabla^E a, b \rangle \end{aligned}$$

#

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claim 2 : $\nabla^{\Lambda^\cdot E^*}(\eta(R^E)) = 0$

HW 2.8

Note that $\eta(R^E) = \frac{1}{2}\sum (g^E R^E e_j)(e_i)\, e^i \wedge e^j$

it follows from Claim 1 + Bianchi $[\nabla^E, R^E] = 0$

Claim 3 : $\nabla^{\Lambda^\cdot E^*}(e^1 \wedge \cdots \wedge e^r) = 0$

↑ since $\forall\, u \in TX$

$$\nabla_u^{\Lambda^\cdot E^*}(e^1\wedge\cdots\wedge e^r) = \sum_j \langle \nabla_u^{E^*} e^j, e_j \rangle\, e^1\wedge\cdots\wedge e^r$$

$$= -\sum_j \underbrace{\langle \nabla_u^E e_j, e_j \rangle_{g^E}}_{\overset{\shortparallel}{0}\ \text{since } \nabla^E \text{ is metric}}\, e^1\wedge\cdots\wedge e^r$$

Therefore

Claim 2 → $\nabla^{\Lambda^\cdot E^*}(\exp(\eta(R^E))) = 0$

$$\nabla^{\Lambda^\cdot E^*}(\mathrm{Pf}[R^E]\, e^1\wedge\cdots\wedge e^r + \text{lower terms}) = 0$$

$$\underset{\overset{\shortparallel}{0}}{d\,\mathrm{Pf}[R^E]}\, e^1\wedge\cdots\wedge e^r + \mathrm{Pf}[R^E]\big(\underbrace{\nabla^{\Lambda^\cdot E^*}(e^1\wedge\cdots\wedge e^r)}_{=0 \text{ by Claim 3}}\big) + \text{lower terms} = 0$$

$\Longrightarrow \quad d\,\mathrm{Pf}[R^E] = 0$.

2) Now we consider two pairs

$$\begin{cases} \nabla_0^E & \text{metric connection w.r.t. } g_0^E \\ \nabla_1^E & \text{metric connection w.r.t. } g_1^E \end{cases}$$

Claim : $\exists$ a smooth (piecewise smooth) family of pairs (g_t^E, ∇_t^E) connecting (g_0^E, ∇_0^E) for $t=0$ and (g_1^E, ∇_1^E) for $t=1$

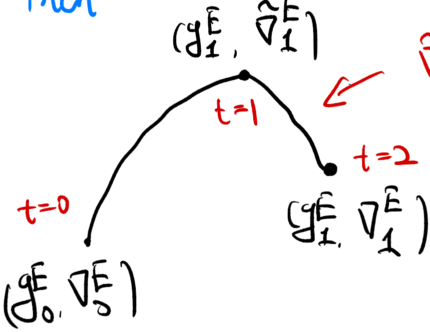
↖ metric connection

At first : Take g_t^E $\quad g_0^E \xrightarrow{\;g_t^E\;} g_1^E \quad t \in [0,1]$

Define $\tilde{\nabla}^E_t := \frac{1}{2}\left(\nabla^E_0 + (\nabla^E_0)^*_t\right)$

← adj connection w.r.t g^E_t of ∇^E_0

Then



$\tilde{\nabla}^E_t = (t-1)\nabla^E_1 + (2-t)\tilde{\nabla}^E_1$

$g^E_t = g^E_1$ for $t \in [1,2]$

$\underset{\text{rescaling } t}{\rightsquigarrow} \quad (g^E_t, \nabla^E_t)_{t\in[0,1]}$

$$\begin{array}{ccc} \pi_1^*E & & E \\ \downarrow & & \downarrow \\ X\times[0,1] & \xrightarrow{\pi_1} & X \end{array}$$

$$g^{\pi^*E}_{(x,t)} := g^E_{t,x}$$

$$\nabla^{\pi^*E} := \nabla^E_t + \frac{1}{2}\Big(dt\wedge\frac{\partial}{\partial t} + \underbrace{\left(dt\wedge\frac{\partial}{\partial t}\right)^*_{g^E_t}}_{=\, dt\wedge\frac{\partial}{\partial t} + dt\wedge A_t(x)}\Big)$$

$$g^E_t\left(\left(dt\wedge\frac{\partial}{\partial t}\right)^* a\right)(b)$$

$$= dt\,\frac{\partial}{\partial t} g^E_t(a,b)$$

$$= dt\,\frac{\partial g^E_t}{\partial t}(a,b)$$

$$= \nabla^E_t + dt\left(\frac{\partial}{\partial t} + \frac{1}{2}A_t(x)\right)$$

metric connection

Then $R^{\pi^*E} = R^E_t + dt\wedge B_t \qquad B_t \in \Omega^1(X)$

$$e(\pi^*E, g^{\pi^*E}, \nabla^{\pi^*E}) = e(E, g^E_t, \nabla^E_t) + dt\wedge\beta_t \quad \leftarrow \Omega^\bullet(X)$$

↑ d-closed on $X\times[0,1]$ $\Rightarrow \frac{\partial}{\partial t} e(E, g^E_t, \nabla^E_t) = d^X\beta_t$

$$\Rightarrow e(E, g^E_1, \nabla^E_1) - e(E, g^E_0, \nabla^E_0) = d^X\int_0^1 \beta_t\, dt$$

3) Using the split manifold $\pi : M \to X \quad r = 2k$ (10)

$$\pi^* E \simeq E_1 \oplus \cdots \oplus E_k$$

each E_j real vector bdle of rk 2

Since E is oriented $\Rightarrow$ each E_j is oriented.

$$\pi^* e(E, g^E, \nabla^E) = e(\pi^* E, g^{\pi^* E}, \nabla^{\pi^* E})$$

$$\overset{\text{coh.}}{\simeq} e(\pi^* E, g^{E_1 \oplus \cdots \oplus E_k}, \nabla^{E_1} \oplus \cdots \oplus \nabla^{E_k})$$

$$= \prod_j e(E_j, g^{E_j}, \nabla^{E_j}) \simeq \prod_j c_1(L_j) = \pi^* e(E).$$

Since E_j is oriented

$$E_j \simeq \underset{(e_1, e_2)}{\mathbb{R}^2}$$

Define the complex structure $J : e_1 \to e_2$, $e_2 \to -e_1$

$$(\mathbb{R}^2, J) \simeq \mathbb{C}$$

$$J \mapsto \sqrt{-1} \quad \text{since } J^2 = -Id_{\mathbb{R}^2}$$

$J(e_1 - \sqrt{-1} e_2) = e_2 + \sqrt{-1} e_1$

$$\mathbb{R}^2 \otimes \mathbb{C} \simeq \mathbb{C} \oplus \overline{\mathbb{C}}$$

$J \otimes \mathbb{C} \qquad \sqrt{-1} \quad -\sqrt{-1}$ eigenspaces of $J \otimes \mathbb{C}$

$$A = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \circlearrowright \mathbb{R}^2 \longmapsto (-\sqrt{-1}\theta, \sqrt{-1}\theta)$$

$$\Rightarrow \quad E_j \otimes \mathbb{C} \simeq L_j \oplus \overline{L_j}$$

$$\eta(R^{E_j}) = \theta e^1 \wedge e^2 \qquad R^{L_j} = -\sqrt{-1}\theta e^1 \wedge e^2$$

$$e\left(\frac{R^{E_j}}{2\pi}\right) = \frac{\theta}{2\pi} e^1 \wedge e^2 = c_1(L_j) = \frac{\theta}{2\pi} e^1 \wedge e^2$$

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