Introduction to the Atiyah-Singer index theory - Homework 1

Exercise 1.
[Local charts for m-sphere] We consider the m-sphere

$$\mathbb{S}^m := \{(x_1, x_2, \ldots, x_{m+1}) \in \mathbb{R}^{m+1} \ : \ \sum_{j=1}^{m+1} x_j^2 = 1\} \subset \mathbb{R}^{m+1}.$$

Define the open sets

$$U_1 := \mathbb{S}^n \setminus \{(0, 0, \ldots, 0, -1)\}, \quad U_2 := \mathbb{S}^n \setminus \{(0, 0, \ldots, 0, 1)\}.$$

These sets cover $\mathbb{S}^n$ since every point on the sphere has $x_{n+1} \neq 1$ or $x_{n+1} \neq -1$. We consider the standard stereographic projections:

$$\psi_1 : U_1 \to \mathbb{R}^m, \quad \psi_1(x_1, x_2, \ldots, x_{m+1}) = \left(\frac{x_1}{1 + x_{m+1}}, \frac{x_2}{1 + x_{m+1}}, \ldots, \frac{x_n}{1 + x_{m+1}} \right),$$

$$\psi_2 : U_2 \to \mathbb{R}^m, \quad \psi_2(x_1, x_2, \ldots, x_{m+1}) = \left(\frac{x_1}{1 - x_{m+1}}, \frac{x_2}{1 - x_{m+1}}, \ldots, \frac{x_m}{1 - x_{m+1}} \right).$$

Prove that ψ_1 and ψ_2 are homeomorphisms, and write down explicit local charts for $\mathbb{S}^m$ using U_1 and U_2, and determine the corresponding transition function.

Exercise 2.
[Paracompactness] Prove that if X is a topological space which is Hausdorff, second-countable and locally compact, then X is paracompact.

Exercise 3.
[Partition of Unity] Let X be a smooth manifold and let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of X.

(a) Show that if the open cover $\{U_\alpha\}$ is locally finite, then there exists a smooth partition of unity $\{\varphi_\alpha\}_{\alpha \in A}$ subordinate to $\{U_\alpha\}$, i.e.,

$$\operatorname{supp}(\varphi_\alpha) \subset U_\alpha, \quad \text{and} \quad \sum_{\alpha \in A} \varphi_\alpha(x) = 1 \quad \text{for all } x \in X.$$

(b) Prove that in general (i.e. without assuming the cover is locally finite) there exists a collection of smooth functions

$$\{\varphi_\alpha\}_{\beta \in B} \subset \mathscr{C}^\infty(X, [0, 1])$$

with the property that
- For each $\beta \in B$, there exists $\alpha \in A$ such that $\operatorname{supp} \varphi_\beta \subset U_\alpha$;
- The sum $\sum_{\beta \in B} \varphi_\beta(x)$ is locally finite and equals 1 for all $x \in X$:

$$\sum_{\beta \in B} \varphi_\beta(x) = 1.$$

Exercise 4.
[Submanifolds] Let $f : N \to M$ be a smooth map between smooth manifolds. We say that f is an *immersion* at a point $p \in N$ if the differential

$$df_p : T_pN \to T_{f(p)}M$$

is injective. The map f is called an *immersion* if it is an immersion at every point $p \in N$.
Now let $f : N \to M$ be an injective smooth immersion:

(a) Prove that the image $f(N)$ is an immersed submanifold of M.
Hint: Use the local immersion property of f and the constant rank theorem to show that around each point of N, there exist coordinate charts in which f is given by an inclusion.

(b) Suppose that f is a topological embedding (i.e. f is a homeomorphism onto its image). Prove that in this case, $f(N)$ is an *embedded* submanifold of M.
Hint: Show that the smooth structure on N and the subspace topology induced from M agree via f, so that the local charts provided by part (a) actually define a smooth structure on $f(N)$ as a subset of M.

(c) Provide an example of a smooth immersion $f : N \to M$ that is not an embedding. Explain why it is an immersion and point out which property fails for it to be an embedding.

Exercise 5.
[Quotient Vector Bundles] Let $\pi : E \to X$ be a vector bundle over a smooth manifold X, and let $F \subset E$ be a subbundle. Work out a system of transition functions for the quotient vector bundle E/F over X.

Exercise 6.
[Exterior differential on manifold] Let X be a smooth manifold. Recall that on a local chart $(U_\alpha, V_\alpha \subset \mathbb{R}^m, \psi_\alpha)$ of X, for a 1-form

$$\beta(x) = \sum_{j=1}^{m} \beta_j(x) dx^j, \text{ with } \beta_j \in \mathscr{C}^\infty(V_\alpha)$$

the action of differential d on β is defined as

$$d\beta = \sum_{j=1}^{m} d\beta_j \wedge dx^j.$$

Prove that:

(a) For $\beta \in \Omega^1(X)$, the 2-form $d\beta \in \Omega^2(X)$ satisfies that for $U, V \in TX$,
$$(d\beta)(U, V) = U(\beta(V)) - V(\beta(U)) - \beta([U, V]).$$

(b) In general, for $\beta \in \Omega^k(X)$, and $V_0, V_1, \dots, V_k \in TX$, we have
$$(d\beta)(V_0, V_1, \dots, V_k) = \sum_{j=0}^{m} (-1)^j V_j(\beta(V_0, \dots, \widehat{V_j}, \dots, V_k)) \\ + \sum_{0 \le j < \ell \le m} (-1)^{j+\ell} \beta([V_j, V_\ell], V_0, \dots, \widehat{V_j}, \dots, \widehat{V_\ell}, \dots, V_k)),$$
where the notation $\widehat{V_j}$ means that the vector V_j is removed.

(c) Verify the Jacobi identity: for $V_1, V_2, V_3 \in TX$,
$$[V_1, [V_2, V_3]] + [V_2, [V_3, V_1]] + [V_3, [V_1, V_2]] = 0.$$

(d) Prove that $d^2 = 0$ using the formula in (b).

Exercise 7.
[Differential Operators] Let X be a smooth manifold and let E, F be two vector bundles over X. Consider a differential operator

$$P : \mathscr{C}^\infty(X, E) \to \mathscr{C}^\infty(X, F).$$

Prove the following assertions:

(a) If P is a differential operator of order k, then for any smooth function $f \in \mathscr{C}^\infty(X)$, the commutator

$$[P, f] : s \mapsto P(fs) - f\,P(s)$$

is a differential operator of order $k-1$.

(b) Prove that every differential operator is locally defined.

(c) In the case where X is an open subset U of $\mathbb{R}^n$ and E, F are trivial vector bundles, prove the following assertion: if P is a differential operator on U such that for any $f \in \mathscr{C}^\infty(U)$, $[P, f]$ is a differential operator of order $k-1$ on U, then P is a differential operator of order k.

(d) Use a partition of unity to extend the above assertion to the general case.

Exercise 8.
[Exact Sequence of Differential Operators] Consider the sequence

$$0 \longrightarrow \mathrm{Diff}_X^{\leq k-1}(E, F) \xrightarrow{i} \mathrm{Diff}_X^{\leq k}(E, F) \xrightarrow{\sigma} \mathscr{C}^\infty\Big(X, \mathrm{Sym}^k\, TX \otimes \mathrm{Hom}(E, F)\Big) \longrightarrow 0,$$

where i is the natural inclusion of differential operators (hence it is injective), and σ denotes the principal symbol map. Prove that this sequence is exact, i.e.,

$$\mathrm{Ker}\,\sigma = \mathrm{Im}(i) \quad \text{and} \quad \sigma \text{ is surjective.}$$