Introduction to the Atiyah-Singer index theory - Homework 2

Exercise 1.
[Wedge products of cohomological classes] Let X be a manifold of dimension m. Prove the following results:

(a) If α and β are closed forms, then $\alpha \wedge \beta$ is also closed.
(b) If α is closed and β is exact, then $\alpha \wedge \beta$ is exact.
(c) Assume X to be compact and oriented: for any $k \in \{0, \dots, m\}$, the bilinear form

$$\eta_k : H^k_{\mathrm{dR}}(X) \times H^{m-k}_{\mathrm{dR}}(X) \to \mathbb{R}$$

by

$$\eta_k(\alpha, \beta) := \int_X \alpha \wedge \beta$$

is well-defined.
(d) For any smooth map $f : Y \to X$ between manifolds, the pull-back map f^* on differential forms induces a linear map

$$f^* : H^\bullet_{\mathrm{dR}}(X) \to H^\bullet_{\mathrm{dR}}(Y)$$

which preserves the degrees.

Exercise 2.
[First Chern class] Let $L \to X$ be a complex line bundle on a smooth manifold:

- For any $k \in \mathbb{N}$, prove that $L^{\otimes k}$ is a complex line bundle on X.
- Prove the identity of first Chern class

$$c_1(L^{\otimes k}) = kc_1(L) \in H^2_{\mathrm{dR}}(X, \mathbb{C}).$$

- Show that $L^* \otimes L$ is a trivial line bundle on X, and first Chern class of a trivial line bundle is zero.
- Let L' be another complex line bundle on X, we have

$$c_1(L \otimes L') = c_1(L) + c_1(L') \in H^2_{\mathrm{dR}}(X, \mathbb{C}).$$

- Show that we always have the isomorphism $\overline{L} \cong L^*$, and

$$c_1(\overline{L}) = c_1(L^*) = -c_1(L) \in H^2_{\mathrm{dR}}(X, \mathbb{C}).$$

- For any smooth map $f : Y \to X$, we have

$$f^* c_1(L) = c_1(f^* L) \in H^2_{\mathrm{dR}}(Y, \mathbb{C}).$$

Exercise 3.
[Complex line bundles on Riemann sphere] Let $\mathbb{CP}^1 \cong S^2$ be the Riemann sphere, or called 1-dimensional complex projective space, with two standard charts:

- The north pole chart $U_N \cong \mathbb{C}$ with coordinate $z = x + \sqrt{-1}y \in \mathbb{C}$
- The south pole chart $U_S \cong \mathbb{C}$ with coordinate $w = 1/z$

Let $\mathcal{O}(-1) \to \mathbb{CP}^1$ denote the tautological line bundle, i.e., $\mathcal{O}(-1) = \{([z], \lambda z) \in \mathbb{CP}^1 \times \mathbb{C}^2, \lambda \in \mathbb{C}\}$.

(a) Prove that $\mathcal{O}(-1) \to \mathbb{CP}^1$ is a well-defined complex line bundle.
(b) Prove that $\mathbb{CP}^1$ is orientable, and we can take the orientation on $\mathbb{CP}^1$ induced by $\mathbb{C}$ through the chart U_N, U_S.

(c) On U_N, we define a 1-form

$$A = \frac{\bar{z}\, dz}{1+|z|^2},$$

where $dz = dx + \sqrt{-1}dy$. Define a Hermitian connection $\nabla = d + A$ on $\mathcal{O}(-1)|_{U_N}$ using the local frame $e_N(z) = (1, z)$ of $\mathcal{O}(-1)$. Show that ∇ can extend to a global connection ∇ on $\mathcal{O}(-1) \to \mathbb{CP}^1$.

(d) Compute on local charts U_N and U_S the curvature form $R = \nabla^2$, and then give a formula for the first Chern form of $c_1(\mathcal{O}(-1), \nabla)$.

(e) Prove that for any connection ∇ on $\mathcal{O}(-1)$, we have

$$\int_{\mathbb{CP}^1} c_1(\mathcal{O}(-1), \nabla) = -1.$$

Set the line bundle $\mathcal{O}(k) = \begin{cases} \mathcal{O}(-1)^{\otimes |k|} & \text{for } k \in \mathbb{Z} \text{ and } k < 0 \\ (\mathcal{O}(-1)^*)^{\otimes k} & \text{for } k \in \mathbb{Z} \text{ and } k \geq 0 \end{cases}$, show that for any $k \in \mathbb{Z}$,

$$\int_{\mathbb{CP}^1} c_1(\mathcal{O}(k), \nabla^k) = k.$$

Exercise 4.

[Poincaré Lemma and injective resolution] Let X be a smooth n-dimensional manifold. We study the relationship between closed differential forms and sheaf cohomology via the de Rham complex.

Prove that

(a) (Poincaré lemma for closed forms) Let $U \subseteq X$ be a contractible open set (e.g., diffeomorphic to $\mathbb{R}^n$). For any closed k-form $\omega \in \Omega^k(U)$ (i.e., $d\omega = 0$), show that there exists $\eta \in \Omega^{k-1}(U)$ such that $\omega = d\eta$.

(b) Find a closed 1-form ω on $X = \mathbb{R}^2 \setminus \{0\}$ that is not exact.

Assume X to be connected. Let $\underline{\mathbb{R}}$ denote the constant sheaf of $\mathbb{R}$ on X, that means, for each open subset $U \subset X$,

$$\underline{\mathbb{R}}(U) := \{\text{locally constant real functions on } U\}.$$

For $k \geq 0$, define the sheaf Ω^k as

$$\Omega^k(U) := \{\text{real-valued smooth } k \text{ forms on } U\}.$$

For each $x \in X$, let $\underline{\mathbb{R}}_x$, Ω^k_x denote the stalks at x, which are the germs of functions or forms.

Consider the **de Rham complex** as a resolution:

$$0 \to \underline{\mathbb{R}} \xrightarrow{\iota} \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n \to 0,$$

where ι is given by the inclusion $\underline{\mathbb{R}}(U) \subset \Omega^0(U)$, and d is given by the exterior differential.

(c) (Exactness of sequence) For each $x \in X$, we have the sequence of spaces of germs:

$$0 \to \underline{\mathbb{R}}_x \xrightarrow{\iota} \Omega^0_x \xrightarrow{d} \Omega^1_x \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_x \to 0,$$

Verify exactness at each Ω^k_x for $k \geq 0$, and show ι is injective. This means that the **de Rham complex** gives an injective resolution for the constant sheaf $\underline{\mathbb{R}}$.

This way, we identify the sheaf cohomology of $\underline{\mathbb{R}}$ on X with the de Rham cohomology of X.

Exercise 5.
[Projectivization, Universal Line Bundle, and Splitting Principle] Given a complex vector bundle $E \to X$ of rank $r \geq 2$ over a smooth manifold X, let $\mathbb{P}(E)$ denote its projectivisation and $\pi : \mathbb{P}(E) \to X$ the natural projection. Specifically, for each $x \in X$, $\pi^{-1}(x) = \mathbb{P}(E_x) \simeq \mathbb{CP}^{r-1}$ via $E_x \simeq \mathbb{C}^r$.

(a) Prove that $\mathbb{P}(E)$ is a smooth manifold, in particular, to describe the local charts and transition functions for $\mathbb{P}(E)$ based on the local charts of E and X.
(b) Prove that $\pi : \mathbb{P}(E) \to X$ is a smooth proper submersion.
(c) Show that the pull-back map: $\pi^* : \Omega^\bullet(X) \to \Omega^\bullet(\mathbb{P}(E))$ is injective.
(d) Define the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(-1)$ on $\mathbb{P}(E)$ whose fibre over $[v] \in \mathbb{P}(E_x)$ is the line $\mathbb{C}v \subset E_x$. Show that $\mathcal{O}_{\mathbb{P}(E)}(-1) \subset \pi^* E$ is a subbundle of rank one.
(e) Based on the above results, show that there exists a proper submersion $\pi : M \to X$ such that $\pi^* E \simeq L_1 \oplus \dots L_r$ with each L_j being a complex line bundle on M.

Exercise 6.
[Complex structure on real vector space] Denote $V = \mathbb{R}^{2n}$ a real vector space of real dimension $2n$. Let e_1, e_2, $\dots$, e_{2n-1}, e_{2n} denote the canonical basis of V such that the vector $v = (x_1, x_2, \dots, x_{2n-1}, x_{2n}) = \sum_{j=1}^{2n} x_j e_j$. Define an endomorphism J of V as follows, for $j = 1, 2, \dots, n$,

$$\begin{aligned} Je_{2j-1} &= e_{2j}, \\ Je_{2j} &= -e_{2j-1}. \end{aligned}$$

Let $g^{T\mathbb{R}^{2n}}$ denote the standard Euclidean inner product on V, equivalently, we can write

$$g^{T\mathbb{R}^{2n}} = \sum_{j=1}^{2n} dx_j \otimes dx_j.$$

a) We have the following identity:

$$J^2 = -\mathrm{Id}_V, \quad g^{T\mathbb{R}^{2n}}(J\cdot, J\cdot) = g^{T\mathbb{R}^{2n}}(\cdot, \cdot).$$

b) Consider the action of complex number $a + b\sqrt{-1} \in \mathbb{C}(\ a, b \in \mathbb{R}\ )$ on $v \in V$ via

$$(a + b\sqrt{-1})v := av + bJv \in V.$$

This way, we make (V, J) a complex vector space of dimension n with a $\mathbb{C}$-basis given by $\{e_1, e_3, \dots, e_{2n-1}\}$.

For $j = 1, \dots, n$, set $z_j = x_{2j-1} + \sqrt{-1}x_{2j} \in \mathbb{C}$, then $(z_1, \dots, z_n) \in C^n$ denotes the standard complex coordinate system on (V, J). More precisely, we have the following identification

$$\mathbb{C}^n \ni (z_1, \dots, z_n) \mapsto \sum_{j=1}^{n} z_j e_{2j-1} \in V.$$

c) Set $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ and $J_{\mathbb{C}} := J \otimes_{\mathbb{R}} \mathrm{Id}_{\mathbb{C}} \in \mathrm{End}(V_{\mathbb{C}})$. Here $\mathbb{C}$ acts on $V_{\mathbb{C}}$ via the second tensor factor $\mathbb{C}$. Then $J_{\mathbb{C}}$ has exactly two eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$. The corresponding eigenspaces are given as follows:

$$V^{1,0} := \mathrm{Span}_{\mathbb{C}}\{e_{2j-1} - \sqrt{-1}e_{2j}\,;\, j = 1, \ldots, n\},$$
$$V^{0,1} := \mathrm{Span}_{\mathbb{C}}\{e_{2j-1} + \sqrt{-1}e_{2j}\,;\, j = 1, \ldots, n\}.$$

In particular, we have $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$.

d) Using the complex coordinates $(z_1, \ldots, z_n)$ for (V, J), set

$$\omega = \frac{\sqrt{-1}}{2} \sum_{j=1}^{n} dz_j \wedge d\overline{z}_j.$$

Then ω is a $(1,1)$-form on V. Prove that $\omega = \overline{\omega}$ (that is ω is a real differential form), moreover, we have

$$\omega = \sum_{j=1}^{n} dx_{2j-1} \wedge dx_{2j} \in \Omega^2(V).$$

e) We have the following relation between $g^{T\mathbb{R}^{2n}}$ and ω: for $v, v' \in V$, we have

$$\omega(v, v') = g^{T\mathbb{R}^{2n}}(Jv, v').$$

In particular, for any $0 \neq v \in V$, $\omega(v, Jv) > 0$ (that is, ω is positive).

f) $g^{T\mathbb{R}^{2n}}$ extends $\mathbb{C}$-linearly on as an bilinear form on $V_{\mathbb{C}}$, for $W, W' \in V^{1,0}$, set $h^{V^{1,0}}(W, W') := g^{T\mathbb{R}^{2n}}(W, \overline{W'})$, then $h^{V^{1,0}}$ defines a hermitian metric on $V^{1,0}$, an orthonormal basis is given as follows:

$$\boldsymbol{f}_j := \frac{1}{\sqrt{2}}(e_{2j-1} - \sqrt{-1}e_{2j}), j = 1, \ldots, n.$$

A similar result holds for $V^{0,1}$.

Exercise 7.
[Jacobi identity for superalgebra]

- Let $\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^-$ be a superalgebra, prove that for $a, b, c \in \mathcal{A}^{\pm}$, we have
$$[a, [b, c]] = [[a, b], c] + (-1)^{|a|\cdot|b|}[b, [a, c]].$$
- Verify that $\Omega^{\bullet}(X)$ with wedge product is (super)commutative superalgebra.

Exercise 8.
[Vanishing by connection action] Let (E, g^E) be a real vector bundle on X of rank r and with Euclidean metric g^E, and let ∇^E be a metric connection with curvature R^E. Recall that $\eta(R^E) \in \Omega^2(X, \Lambda^2 E^*)$ is defined as follows: let $\{e_j\}_{j=1}^r$ be a local orthonormal frame of (E, g^E), and let $\{e^j\}_{j=1}^r$ be its dual frame of E^*, then

$$\eta(R^E) = \frac{1}{2} \sum_{j,\ell=1}^{r} \langle e_j, R^E e_\ell \rangle_{g^E} e^j \wedge e^\ell.$$

Show that we have

$$\nabla^{\Lambda^{\bullet} E^*} \eta(R^E) = 0,$$

where $\nabla^{\Lambda^{\bullet} E^*}$ is induced by ∇^E.