Introduction to the Atiyah-Singer index theory - Homework 3

Exercise 1.
[Chern–Weil theorem for superconnections] Let X be a manifold of dimension m. Let $E = E^+ \oplus E^-$ be a complex superbundle on X, and let A be a superconnection on E. Take $f(x) = \sum a_j x^j$ a formal power series with $a_j \in \mathbb{C}$. Give a detailed proof to the following statements (where φ denotes the normalization map $1/(2\pi\sqrt{-1})^{\mathrm{degree}/2}$): (1). $\varphi \operatorname{Tr_s}[f(-A^2)] \in \Omega^{\mathrm{even}}(X)$ is closed form; (2). Its cohomology class is independent of A, and

$$\left[\varphi \operatorname{Tr_s}\left[f(-A^2)\right]\right]_{\mathrm{dR}} = f_{\mathrm{a}}(E^+) - f_{\mathrm{a}}(E^-) \in H^\bullet(X).$$

Exercise 2.
[Chern character for a linear map] Let $D : E \to F$ be a linear map between two finite-dimensional vector spaces. Let h^E, h^F be Hermitian inner products on E, F respectively. Show the following results:

- There exists a unique linear operator $D^* : F \to E$ such that for $a \in E$, $b \in F$, we have
$$h^E(D^*b, a) = h^F(b, Da).$$
The operator D^* is called the adjoint of D with respect to h^E and h^F.
- Give a natural isomorphism between $\operatorname{Ker} D^*$ and $\operatorname{Coker} D := F/\mathrm{Image}(D)$.
- Prove that
$$\lim_{t\to+\infty} \operatorname{Tr}^E[e^{-tD^*D}] = \dim \operatorname{Ker} D, \quad \lim_{t\to+\infty} \operatorname{Tr}^F[e^{-tDD^*}] = \dim \operatorname{Coker} D.$$

Exercise 3.
[Clifford algebra and quaternion number field] Recall that the quaternion number field

$$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$$

with $i^2 = j^2 = k^2 = -1$ and $k = ij = -ji$. Verify that the Clifford algebra $C(\mathbb{R}^2)$ is canonically isomorphic to the quaternion number field.

Exercise 4.
[Lie algebras of linear groups] For $n \geq 2$, consider the special orthogonal group

$$\mathrm{SO}(n) := \{A \in \mathrm{M}(n, \mathbb{R}) \ : \ A^T = A^{-1}, \ \det A = 1\}.$$

Show that

- $\mathrm{SO}(n)$ is a compact Lie group.
- Set $B \in \mathrm{M}(n, \mathbb{R})$ with $B^T = -B$, then for any $t \in \mathbb{R}$,
$$\exp(tB) := \sum_{\ell=0}^{\infty} \frac{t^\ell B^\ell}{\ell!} \in \mathrm{SO}(n).$$
- For any $A \in \mathrm{SO}(n)$, there exists $B \in \mathrm{M}(n, \mathbb{R})$ with $B^T = -B$ such that $A = \exp(B)$. As a consequence, $\mathrm{SO}(n)$ is path-connected.

- The Lie algebra $\mathfrak{so}(n)$ of $\mathrm{SO}(n)$ is defined as the vector space of all left-invariant vector field on $\mathrm{SO}(n)$ together with the Lie bracket of vector fields. Then the Lie algebra $\mathfrak{so}(n)$ is isomorphic to
$$\mathfrak{so}(n) = \{B \in \mathrm{M}(n, \mathbb{R}) \ : \ B^T = -B\}$$
with Lie bracket
$$[B_1, B_2] = B_1 B_2 - B_2 B_1 \in \mathfrak{so}(n),$$
where $B_1 B_2$ is the matrix multiplication of B_1 and B_2.
- Show that $\dim_{\mathbb{R}} \mathrm{SO}(n) = \frac{n(n-1)}{2}$.

Exercise 5.
[Spin group Spin(3)] Let SU(2) denote the group of 2×2 complex unitary matrices with determinant 1. Let $\mathfrak{su}(2)$ be the Lie algebra of SU(2) which is the space of traceless skew-Hermitian matrices.

(a) Prove that the Lie algebra $\mathfrak{su}(2)$ has real dimension 3. For $u, v \in \mathfrak{su}(2)$, consider the bilinear form $\langle \cdot, \cdot \rangle$ on V
$$V \times V \ni (u, v) \mapsto \langle u, v \rangle := -\frac{1}{2} \Re(\mathrm{Tr}^{\mathbb{C}^2}[uv]) \in \mathbb{R},$$
where $\Re(\cdot)$ denotes the real part. Then $\langle \cdot, \cdot \rangle$ defines a Euclidean metric on $\mathfrak{su}(2)$.

(b) Show that SU(2) acts on $\mathbb{R}^3$ via the adjoint representation:
$$\phi(g)(X) = gXg^{-1}, \quad \text{for } X \in \mathfrak{su}(2) \cong \mathbb{R}^3,$$
and this action preserves the Euclidean metric $\langle \cdot, \cdot \rangle$.

(c) Prove that this action induces a homomorphism $\phi : \mathrm{SU}(2) \to \mathrm{SO}(3)$. Then show that
$$\mathrm{Spin}(3) \simeq \mathrm{SU}(2).$$