Introduction to the Atiyah-Singer index theory - Homework 4

Exercise 1.
[Irreducible representation of matrix algebra] Let $M_k(\mathbb{C})$ be the algebra of all $k \times k$ complex matrices. Let (V, ρ) be a representation of $M_k(\mathbb{C})$ over $\mathbb{C}$, that is, a morphism of algebras

$$\rho : M_k(\mathbb{C}) \to \mathrm{End}(V).$$

We say that (V, ρ) is irreducible if it does not contain any proper nontrivial subspace invariant by the action of $\rho(M_k(\mathbb{C}))$.
Show that

(1) Consider the matrix action of $M_k(\mathbb{C})$ on $\mathbb{C}^k$, then it defines an irreducible representation.
(2) The only nonzero irreducible (complex) representation of $M_k(\mathbb{C})$ is above $V = \mathbb{C}^k$ with the matrix action.

Exercise 2.
[Supertrace of Clifford actions on spinors] Let $(V, \langle\cdot,\cdot\rangle)$ be a Euclidean space of dimension $m = 2k$, and let $J \in O(V)$ be a complex structure of V (that is, $J^2 = -\mathrm{Id}_V$). Associated with J, we have the splitting $V \otimes_{\mathbb{R}} \mathbb{C} = W \oplus \overline{W}$ and the spinor space

$$S = \Lambda^\bullet \overline{W}^*.$$

Recall that $\mathrm{End}(S) = \mathrm{Cl}(V) \otimes_{\mathbb{R}} \mathbb{C}$, where $\mathrm{Cl}(V)$ is Clifford algebra of V.
Show the following results:

(1) If $g \in \mathrm{O}(V)$, then g acts on $\Lambda^\bullet V$, prove that
$$\det{}_V(1-g) = \mathrm{Tr_s}^{\Lambda^\bullet V}[g].$$
(2) If $g \in \mathrm{SO}(V)$, let g act on $\Lambda^\bullet V^*$ induced by $\alpha \in V^*$, $v \in V$, $(g\alpha, v) = (\alpha, g^{-1}v)$. Then we have
$$\det{}_V(1-g) = \mathrm{Tr_s}^{\Lambda^\bullet V^*}[g].$$
(3) If $\{e^j\}_{j=1}^m$ is an ordered orthonormal basis of V, then we have
$$\mathrm{Tr_s}^S[\sqrt{-1}^k c(e_1)c(e_2)\cdots c(e_{m-1})c(e_m)] = 2^k.$$
(4) If $a \in \mathrm{Cl}^{m-1}(V) := \mathrm{Im}(\oplus_{j\le m-1} V^{\otimes j} \to \mathrm{Cl}(V))$, then we have
$$\mathrm{Tr_s}^S[c(a)] = 0.$$
(5) Let $\rho : \mathrm{Spin}(V) \to \mathrm{SO}(V)$ be the canonical Lie group morphism, and recall the symbol map $\sigma : \mathrm{Cl}(V) \to \Lambda^\bullet V^*$. For any $\alpha \in \Lambda^\bullet V^*$, let $[\alpha]^{\max} \in \mathbb{R}$ denote the coefficient of $e^1 \wedge \cdots \wedge e^m$ in α. Show that for $g \in \mathrm{Spin}(V)$, we have
$$([\sigma(g)]^{\max})^2 = 2^{-k}\det{}_V(1-\rho(g)).$$
(6) Show that for any $g \in \mathrm{SO}(V)$, the function $\det{}_V(1-g\exp(A))$ of $A \in \mathfrak{so}(V)$ has an analytic square root.

Exercise 3.
[Exterior differential and Cartan formula] Let (X, g^{TX}) be an oriented Riemannian manifold of dimension m, let $d : \Omega^\bullet(X) \to \Omega^{\bullet+1}(X)$ be the exterior differential acting on the forms. Show that

(1) The Levi-Civita connection ∇^{TX} associated with g^{TX} is determined by the following formula: for $U, V, W \in \mathscr{C}^\infty(X, TX)$,

$$\begin{aligned}&\langle \nabla^{TX}_U V, W\rangle_{g^{TX}}\\ =&\frac{1}{2}\Big\{U\langle V, W\rangle_{g^{TX}} + V\langle W, U\rangle_{g^{TX}} - W\langle U, V\rangle_{g^{TX}}\\ &\quad + \langle [U,V], W\rangle_{g^{TX}} - \langle [V,W], U\rangle_{g^{TX}} - \langle [U,W], V\rangle_{g^{TX}}\Big\}.\end{aligned}$$

(2) Let $\nabla^{\Lambda^\bullet T^*X}$ denote connection on $\Lambda^\bullet T^*X$ induced by the Levi-Civita connection ∇^{TX}. If $\{e_j\}_{j=1}^m$ is a local orthonormal basis of (TX, g^{TX}) with the dual frame $\{e^j\}_{j=1}^m$ of T^*X, then we have

$$d = \sum_{j=1}^m e^j \wedge \nabla^{\Lambda^\bullet T^*X}_{e_j}.$$

(3) For $V \in \mathscr{C}^\infty(X, TX)$, let L_V denote the Lie derivative of V on $\Omega^\bullet(X)$ (by taking the derivative of the pull-back by the diffeomorphism $\exp(tV)$ on X). Then we have

$$L_V = [d, \iota_V] = d\iota_V + \iota_V d.$$

Exercise 4.
[Formal adjoint of exterior differential] Let (X, g^{TX}) be an oriented Riemannian manifold of dimension m, let $\langle\cdot,\cdot\rangle_{L^2}$ denote the L^2-inner product on $\Omega^\bullet(X, \mathbb{R})$ induced by g^{TX} and the corresponding Riemannian volume form. Let $\nabla^{\Lambda^\bullet T^*X}$ denote connection on $\Lambda^\bullet T^*X$ induced by the Levi-Civita connection on X. Let $d : \Omega^\bullet(X) \to \Omega^{\bullet+1}(X)$ be the exterior differential, and let $d^* : \Omega^\bullet(X) \to \Omega^{\bullet-1}(X)$ be its formal adjoint of d with respect to the L^2-inner product, that is, for any $s_1, s_2 \in \Omega^\bullet_c(X)$,

$$\langle d^* s_1, s_2\rangle_{L^2} = \langle s_1, ds_2\rangle_{L^2}.$$

Prove that, if $\{e_j\}_{j=1}^m$ is a local orthonormal basis of (TX, g^{TX}), then we have

$$d^* := -\sum_{j=1}^m \iota_{e_j} \nabla^{\Lambda^\bullet T^*X}_{e_j}.$$

Exercise 5.
[Principal symbol of Dirac operator] Let (X, g^{TX}) be an oriented Riemannian manifold of dimension m, and let (E, h^E) be a $\mathbb{Z}_2$-graded Clifford module on X with a Hermitian Clifford connection ∇^E. Let $D^E := \sum_{j=1}^m c(e_j)\nabla^E_{e_j}$ be the associated Dirac operator. Show that for $f \in \mathscr{C}^\infty(X, \mathbb{R})$, we have

$$[D^E, f] = c(df^*),$$

where $df^* \in TX$ denote the metric dual of $df \in T^*X$ with respect to g^{TX}.