# Introduction to the Atiyah-Singer index theory - Homework 5

**Exercise 1.**
[SO($m$)-principal bundle] Let $(X, g^{TX})$ be an oriented Riemannian manifold of dimension $m$. Define

$$\mathrm{SO}(X) := \{(x, f) \ : \ x \in X \ , \ f : (T_x X, g_x^{TX}) \to (\mathbb{R}^m, g_{\mathrm{Eucl}}) \text{ oriented linear isometrie } \}.$$

Then

- Prove that $\mathrm{SO}(X)$ has a smooth structure such that the map
$$p : \mathrm{SO}(X) \to X \, , \quad (x, f) \mapsto x$$
is a smooth submersion.
- Verify that $p : \mathrm{SO}(X) \to X$ is a $\mathrm{SO}(m)$-principal bundle on $X$.
- Prove that $E := \mathrm{SO}(X) \times_{\mathrm{SO}(m)} \mathbb{R}^m$ defines an orientable vector bundle over $X$, and $g_{\mathrm{Eucl}}$ on $\mathbb{R}^m$ induces a Euclidean metric $g^E$ on $E$.
- Prove that there is a canonical isomorphism of vector bundles on $X$
$$E = \mathrm{SO}(X) \times_{\mathrm{SO}(m)} \mathbb{R}^m \simeq TX$$
- Prove that the above isomorphism identify $g^E$ with $g^{TX}$.

**Exercise 2.**
[Connection on $G$-pincipal bundle] Let $(X, g^{TX})$ be an oriented Riemannian manifold of dimension $m$. Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$, and let $p : P_G \to X$ be a $G$-principal bundle on $X$.

- If $(E, \rho^E)$ is a representation of $G$, set
$$F := P_G \times_G E$$
where $(p, v) \sim (pg, \rho^E(g^{-1})v)$ for any $p \times P_G, v \in E, g \in G$. Show that $F$ is a vector bundle on $X$.
- Let $\mathscr{C}^\infty(P_G, E)^G$ denote the set of all smooth map $s : P_G \to E$ such that $s(pg) = \rho^E(g^{-1})s(p)$. Show that there exists a natural identification between
$$\mathscr{C}^\infty(X, F) \simeq \mathscr{C}^\infty(P_G, E)^G.$$

For a compact Lie group, we have an important property: for each nonzero vector $A \in \mathfrak{g}$, there exists a unique 1-dimensional subgroup $\{\gamma(t) := \exp(tA), t \in \mathbb{R}\} \subset G$ with the property $\gamma(t)\gamma(s) = \gamma(t+s)$, $\gamma(0) = 1 \in G$ and $\gamma'(0) = A$. We need a bit recap on the adjoint action:

- For $g \in G$, $A \in \mathfrak{g}$, define
$$\mathrm{Ad}(g)A := \frac{\partial}{\partial t}|_{t=0} g \exp(tA) g^{-1}.$$
Show that $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ gives a representation of $G$. This is called adjoint representation of $G$.
- Show that for $g \in G$, $A \in \mathfrak{g}$, we have
$$g \exp(tA) g^{-1} = \exp(t\mathrm{Ad}(g)A) \, , \, t \in \mathbb{R}.$$

A connection 1-form on $P_G$ is an element $\omega \in \Omega^1(P_G, \mathfrak{g})$ satisfying:

(1) $\omega(A_P) = A$ for all $A \in \mathfrak{g}$, where $A_P$ is the smooth vector field on $P_G$ defined by
$$A_P(p) := \frac{\partial}{\partial t}|_{t=0} p \exp(tA).$$
(2) $R_g^* \omega = \mathrm{Ad}_{g^{-1}} \omega$ for all $g \in G$, where $R_g$ is the right action of $g$ on $P_G$, and $\mathrm{Ad}(g^{-1})$ is the adjoint action of $g^{-1}$ on $\mathfrak{g}$.

Then

- For $s \in \mathscr{C}^\infty(P_G, E)^G$, prove that
$$ds + \rho^E(\omega)s \in \mathscr{C}^\infty(P_G, T^*P_G \otimes E)^G,$$
where $\mathscr{C}^\infty(P_G, T^*P_G \otimes E)^G$ is the space of invariant sections $\beta \in \Omega^1(P_G) \otimes E$ such that for all $g \in G$, $(R_g^* \otimes \rho^E(g))\beta = \beta$.
- Show that if $\beta \in \mathscr{C}^\infty(P_G, T^*P_G \otimes E)^G$ is such that for any $A \in \mathfrak{g}$, $\beta(A_P) = 0$, then $\beta$ defines uniquely a smooth section of $F := P_G \times_G E$ on $X$.
- Show that from the connection 1-form $\omega$, we can define a connection $\nabla^F$ on the vector bundle $F \to X$ via the formula
$$ds + \rho^E(\omega)s.$$
- Prove that for any connection $\nabla^F$ on a vector bundle $F \to X$ of rank $r$, we can find a unique connection 1-form on frame bundle ($\mathrm{GL}(r)$-principal bundle) $\mathrm{GL}(F) \to X$.
- Explain when $X$ is spin, we can define a connection on spinor bundle from the Levi-Civita connection on $TX$.

**Exercise 3.**

[Clifford connection] Let $(X, g^{TX})$ be an even-dimensional oriented spin Riemannian manifold with spinor bundle $S^{TX}$, and let $E$ be a $\mathbb{Z}_2$-graded Clifford module on $X$ with Clifford connection $\nabla^E$. Recall that we have the identification of Clifford module
$$E \simeq S^{TX} \otimes W$$
with a vector bundle
$$W_x := \mathrm{Hom}_{C(TX)_x \otimes_{\mathbb{R}} \mathbb{C}}(S_x^{TX}, E_x).$$
Let $\nabla^{TX}$ be the Clifford connection on $S^{TX}$ induced from Levi-Civita connection on $(TX, g^{TX})$. For $s \in \mathscr{C}^\infty(X, S^{TX})$, $w \in \mathscr{C}^\infty(X, W)$, show that

- $w \otimes s$ is naturally a smooth section of $E$ on $X$.
- Define $\nabla^W : \mathscr{C}^\infty(X, W) \to \Omega^1(X, W)$ by
$$(\nabla^W w) \otimes s := \nabla^E(w \otimes s) - w \otimes (\nabla^{S^{TX}} s),$$
then $\nabla^W$ is a well-defined connection on vector bundle $W$.

**Exercise 4.**

[Dirac operator and Laplacian]

(1). Let $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ be the circle with coordinate $t \in \mathbb{R}/\mathbb{Z}$, and the standard Riemannian metric induced from $\mathbb{R}$. Consider the de Rham complex $(\Omega^\bullet(\mathbb{S}^1, \mathbb{R}), d)$ and the $L^2$-inner product on the differential forms.

- Write down the formal adjoint $d^*$ in coordinate $t$.
- Write down the Levi-Civita connection in coordinate $t$ and verify that
$$D = d + d^*$$
is a Dirac operator.

- Compute a formula for the Hodge Laplacian
$$\Box = D^2.$$
- Work out the spectrum of $\Box$ acting on $\Omega^\bullet(\mathbb{S}^1, \mathbb{R})$, in particular, finding the eigenvalues and eigensections.

(2). Let $\mathbb{H} = \{z = x + \sqrt{-1}y \in \mathbb{C} \ : \ y > 0\}$ be upper half-plane equipped with the hyperbolic metric
$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$
Consider the de Rham complex $(\Omega^\bullet(\mathbb{H}, \mathbb{R}), d)$ and the $L^2$-inner product on the differential forms.

- Write down the formal adjoint $d^*$ in coordinate $(x, y)$.
- Write down the Levi-Civita connection in coordinate $(x, y)$ and verify that
$$D = d + d^*$$
is a Dirac operator.
- Compute a formula for the Hodge Laplacian
$$\Box = D^2.$$
- Is $\mathbb{H}$ a spin manifold? If yes, work out a spin structure (that is, Spin(2)-principal bundle which gives a double cover of SO($\mathbb{H}$))