Introduction to the Atiyah-Singer index theory - Homework 6

Exercise 1.
[Vanishing of $\widehat{A}$-genus] Let X be a compact oriented even-dimensional spin manifold, and let g^{TX} be a Riemannian metric on X such that the scalar curvature $r^X \geq 0$ and there exists a point $x_0 \in X$ with $r^X(x_0) > 0$. Using the statement of Atiyah-Singer index theorem to prove that

$$\int_X \widehat{A}(TX) = 0.$$

Exercise 2.
[Heat kernel on Euclidean space] Let $p_t(x, y)$ be the heat kernel on Euclidean space given as follows, for $x, y \in \mathbb{R}^m$, $t > 0$,

$$p_t(x, y) = \frac{1}{(4\pi t)^{m/2}} e^{-\frac{\|x-y\|^2}{4t}}.$$

Show that for any smooth function with compact support $f \in \mathscr{C}_c^\infty(\mathbb{R}^m, \mathbb{R})$, we have

$$\lim_{t\to 0} \int_{\mathbb{R}^m} p_t(x, y) f(y) d\lambda(y) = f(x).$$

Exercise 3.
[Heat kernel and heat trace on circle] Let $p_t^{\mathbb{S}^1}(x, y)$ be the heat kernel on the standard circle $\mathbb{S}^1 := \mathbb{R}/\mathbb{Z}$ associated to the Laplacian $\Delta = -\frac{\partial^2}{\partial x^2}$, $x \in \mathbb{S}^1$.

- Verify the following identity

$$p_t^{\mathbb{S}^1}(x, y) = \frac{1}{\sqrt{4\pi t}} \sum_{k\in\mathbb{Z}} e^{-\frac{\|x-y-k\|^2}{4t}}.$$

- Give the full spectrum of Δ acting on $L^2(\mathbb{S}^1)$.
- Compute separately $\mathrm{Tr}[e^{-t\Delta}]$ using the spectrum and the heat kernel to conclude the Poisson summation formula for the Gaussian function.

Exercise 4.
[Trace class operator] Recall that a trace class operator on a separable Hilbert space is defined as the operator which is given by the composition of two Hilbert-Schmidt operators.
Let $\mathcal{H}$ be a separable Hilbert space, and let $A \in \mathrm{End}(\mathcal{H})$ be a bounded linear operator such that $A^* = A$ and $A \geq 0$. Fix an orthonormal basis $\{e_j\}_j$ of $\mathcal{H}$. Show that A is trace class if and only if

$$\sum_j \langle Ae_j, e_j\rangle_{\mathcal{H}} < \infty.$$

Exercise 5.
[Supertrace vanishing on trace class operators]Let X be a compact manifold with a volume form dv. Consider a $\mathbb{Z}_2$-graded vector bundle on X given by $E = E^+ \oplus E^-, h^E = h^{E^+} \oplus h^{E^-}$. Let K_1 and K_2 be two integral linear operators acting on $L^2(X, E) = L^2(X, E^+) \oplus L^2(X, E^-)$. We assume that these operators have smooth

Schwartz kernels $K_1(x,y)$ and $K_2(x,y)$ on $X\times X$ with respect to the volume form dv. Show that

- The Hilbert-Schmidt norm of K_j is
$$\|K_j\|_{\mathrm{HS}}^2=\int_{X\times X}\mathrm{Tr}[K_j(x,y)^*K_j(x,y)]dv(x)dv(y),$$
where $K_j(x,y)^*\in E_y\otimes E_x^*$ is the adjoint of $K_j(x,y)\in E_x\otimes E_y^*$.
- The supercommutator $[K_1,K_2]$ is a trace class linear operator.
- We have
$$\mathrm{Tr_s}[[K_1,K_2]]=0.$$

Exercise 6.
[Connection and curvature in geodesic coordinates]Let (X,g^{TX}) be a compact Riemannian manifold of dimension m. Let (E,h^E) be a Hermitian vector bundle on X with a Hermitian connection ∇^E. Let R^E be the curvature of (E,∇^E).
Fix $x_0\in X$ and let $\delta>0$ be sufficiently small such that we have the geodesic coordinates centred at x_0.

$$\exp_{x_0}:B^{T_{x_0}X}(0,\delta)\to B^X(x_0,\delta)$$
where $B^{T_{x_0}X}(0,\delta)\ni Z\mapsto \exp_{x_0}(Z)\in X$.
We trivialize $E|_{B^{T_{x_0}X}(0,\delta)}$ by the parallel transport with respect to ∇^E along the path $[0,1]\ni s\mapsto sZ\in B^{T_{x_0}X}(0,\delta)$. This identifies E_Z with $E_0=E_{x_0}$.
Therefore, we have

$$E|_{B^{T_{x_0}X}(0,\delta)}\simeq B^{T_{x_0}X}(0,\delta)\times E_0.$$
Under this trivialization with local coordinates $Z=(Z_1,\cdots,Z_m)$, we write
$$\nabla^E=d+\Gamma^E$$
with $\Gamma^E\in\Omega^1(B^{T_{x_0}X}(0,\delta),\mathrm{End}(E_{x_0})$.

- Show that $\Gamma^E\in\Omega^1(B^{T_{x_0}X}(0,\delta),\mathrm{End}^{\mathrm{anti}}(E_{x_0},h_{x_0}))$.
- Show that $\Gamma^E_0=0$, the value of Γ^E at $Z=0$ vanishes.
- Define the local tangent vector field
$$\mathcal{R}:=\sum_{j=1}^m Z_j\frac{\partial}{\partial Z_j}.$$
Show that $\iota_{\mathcal{R}}\Gamma^E\equiv 0$.
- Show that
$$L_{\mathcal{R}}\Gamma^E=\iota_{\mathcal{R}}R^E.$$
As a consequence, deduce a formula relating the derivatives of Γ^E to the derivatives of R^E in coordinates Z.