Introduction to the Atiyah-Singer Index Theorem 2025

Bingxiao Liu, Universität zu Köln

Abstract

These notes are for a master's lecture titled *Introduction to the Atiyah-Singer Index Theorem* given in the summer semester of 2025 at the University of Cologne.

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1 Introduction: Atiyah-Singer Index Theorem

The main goal of this lecture is to explain and prove the following theorem:

Theorem 1.1 (Atiyah-Singer, 1963)**.**

$$\mathrm{Ind}(D) = \langle \widehat{A}(TX)\mathrm{ch}(E),\ [X]\rangle = \int_X \widehat{A}(TX)\mathrm{ch}(E) \in \mathbb{Z}. \tag{1.1}$$

In the left-hand side, D is a (twisted) Dirac operator, and its index is defined via the kernel space and cokernel space of D, which is mainly concerned with the analysis of the manifold X. The right-hand side is a characteristic number given as the integral of characteristic classes on X, which has a topological nature. Basically, the above theorem gives a bridge between the analysis and the topology of a given manifold.

Let us precise the **geometric setting** for the index theorem as in Theorem 1.1

- X is a compact smooth manifold.
- E, F are two (complex or real) vector bundles over X.
- $P : C^\infty(X, E) \to C^\infty(X, F)$ is an elliptic operator.

Theorem 1.1 is only for the case of some Dirac operator $P = D$ associated with certain vector bundles E and F.

The ellipticity of P implies that P is Fredholm, meaning $\ker(P)$ and $\mathrm{coker}(P)$ are finite-dimensional vector space.

Definition 1.2 (Analytical index of P)**.**

$$\mathrm{Ind}(P) \ = \ \dim\ker(P) \ - \ \dim\mathrm{coker}(P) \in \mathbb{Z}.$$

The general index theorems state that $\mathrm{Ind}(P)$ can be expressed as a topological (cohomological) formula involving the characteristic classes of X and the symbol of P.

Proposition 1.3 (Stability of index)**.** *If $\{P_t\}_{t\in\mathbb{R}}$ is a continuous family of elliptic operators of a given order, then* $\mathrm{Ind}(P_t)$ *is independent of* $t \in \mathbb{R}$.

Historical remarks:

(1) **Gel'fand's question (late 1950s–around 1960).**
Gel'fand noticed the homotopy invariance of the index and asked if there was a purely topological formula for the index of certain linear partial differential operators. This question became: *Find a topological formula for* $\mathrm{Ind}(P)$.

(2) **$\widehat{A}$-genus by Atiyah–Hirzebruch (1961).**
In the early 1960s, Borel–Hirzebruch and Atiyah–Hirzebruch had proved the integrality of the $\widehat{A}$-genus (that is, the number given by $\langle \widehat{A}(TX), [X]\rangle$ is an integer) of a spin manifold, and Atiyah suggested that this integrality could be explained if it were the index of the Dirac operator. By 1962, Atiyah and Singer attempted to provide a conceptual proof for writing the $\widehat{A}$-genus as the index of a certain elliptic operator.

(3) **Dirac operator on spin manifolds by Atiyah–Singer (1963).**
Atiyah and Singer constructed a first order geometric elliptic differential operator D on a

spin manifold X, called Dirac operator, which is a Riemannian version of Dirac's operator in physics, and established that

$$\mathrm{Ind}(D) = \langle \widehat{A}(TX),\ [X]\rangle = \int_X \widehat{A}(TX).$$

Atiyah and Singer announced their results and the sketched proof in 1963.

(4) **The general statement and extensions by Atiyah–Singer (1963, 1968–1971).** Atiyah and Singer (1963) actually obtained the index theorem for general elliptic operators, if

$$P : C^\infty(X, E) \longrightarrow C^\infty(X, F)$$

is an elliptic operator on a compact manifold X, then

$$\mathrm{Ind}(P) = \int_{T^*X} \widehat{A}(TX)^2 \ \mathrm{ch}\big(\sigma(P)\big)\,,$$

where $\sigma(P)$ is the principal symbol of P (seen as an element of K-theory on T^*X), ch is the Chern character. This became known as the *Atiyah–Singer Index Theorem*, uniting ideas from algebraic topology, differential geometry, and functional analysis. In a series of influential papers, Atiyah and Singer (1968–1971) gave a study on the index of elliptic operators in terms of a K-theoretical formulation and its extensions to manifolds with boundary, the equivariant setting, and the family version, etc.

2 Preliminaries on smooth manifolds, vector bundles, and differential operators

2.1 Manifold and partition of unity

Definition 2.1 (Smooth manifold)**.** A topological space X is called a *smooth manifold* of dimension m if:

(1) X is Hausdorff.

(2) X has a countable base (i.e. it is second-countable).

(3) X is locally Euclidean of dimension m: there is an open cover $\{U_\alpha\}_{\alpha\in A}$ of X and for each α, a homeomorphism

$$\psi_\alpha : U_\alpha \longrightarrow V_\alpha \subset \mathbb{R}^m$$

so that on overlaps $U_\alpha \cap U_\beta$, the transition maps $\psi_\beta \circ \psi_\alpha^{-1} : \psi_\alpha(U_\alpha\cap U_\beta) \to \psi_\beta(U_\alpha \cap U_\beta)$ are smooth diffeomorphisms between open sets of $\mathbb{R}^m$.

The triplet $(U_\alpha, V_\alpha, \psi_\alpha)$ is called a local chart or a local coordinate system for X.

Definition 2.2 (Smooth Functions)**.** If X is a smooth manifold, a function $f : X \to \mathbb{R}$ (or $\mathbb{C}$) is *smooth* if for every local chart $(U_\alpha, V_\alpha, \psi_\alpha)$, the composition

$$f \circ \psi_\alpha^{-1} : V_\alpha \to \mathbb{R} \text{ (or } \mathbb{C})$$

is a smooth function on the open set $V_\alpha \subset \mathbb{R}^m$ in the usual sense. We denote $\mathscr{C}^\infty(X, \mathbb{R} \text{ or } \mathbb{C})$ the set of all smooth functions on X, and let $\mathscr{C}^\infty_c(X, \mathbb{R} \text{ or } \mathbb{C}) \subset \mathscr{C}^\infty(X, \mathbb{R} \text{ or } \mathbb{C})$ be the subset consisting of smooth functions with compact support in X. In a similar way, we define the smooth maps between any two smooth manifolds.

Example 2.3.
- $\mathbb{R}^m$ is a smooth manifold of dimension m.
- The m-sphere $\mathbb{S}^m \subset \mathbb{R}^{m+1}$ is a smooth manifold.
- Any open subset $U \subset \mathbb{R}^m$ is also a smooth manifold.

Exercise 2.1 (Local charts for m-sphere)**.** We consider the m-sphere

$$\mathbb{S}^m := \{(x_1, x_2, \dots, x_{m+1}) \in \mathbb{R}^{m+1} \ : \ \sum_{j=1}^{m+1} x_j^2 = 1\} \subset \mathbb{R}^{m+1}.$$

Define the open sets

$$U_1 := \mathbb{S}^n \setminus \{(0, 0, \dots, 0, -1)\}, \quad U_2 := \mathbb{S}^n \setminus \{(0, 0, \dots, 0, 1)\}.$$

These sets cover $\mathbb{S}^n$ since every point on the sphere has $x_{n+1} \neq 1$ or $x_{n+1} \neq -1$. We consider the standard stereographic projections:

$$\psi_1 : U_1 \to \mathbb{R}^m, \quad \psi_1(x_1, x_2, \dots, x_{m+1}) = \left(\frac{x_1}{1 + x_{m+1}}, \frac{x_2}{1 + x_{m+1}}, \dots, \frac{x_n}{1 + x_{m+1}}\right),$$

$$\psi_2 : U_2 \to \mathbb{R}^m, \quad \psi_2(x_1, x_2, \dots, x_{m+1}) = \left(\frac{x_1}{1 - x_{m+1}}, \frac{x_2}{1 - x_{m+1}}, \dots, \frac{x_m}{1 - x_{m+1}}\right).$$

Prove that ψ_1 and ψ_2 are homeomorphisms, and write down explicit local charts for $\mathbb{S}^m$ using U_1 and U_2, and determine the corresponding transition function.

Remark 2.4. In these notes, manifolds are assumed to be smooth and have *no boundary.*

Definition 2.5.
- An open cover $\{U_\alpha\}_{\alpha \in A}$ of X is called locally finite if for each $x \in X$, there exists open subset $V_x \subset X$ such that

$$x \in V_x, \qquad \#\{\alpha \in A \ : \ U_\alpha \cap V_x \neq \varnothing\} < \infty.$$

- Given an open cover $\{U_\alpha\}_{\alpha \in A}$ of X, another open cover $\{U'_\beta\}_{\beta \in B}$ of X is called a refinement of $\{U_\alpha\}_{\alpha \in A}$ if for each $\beta \in B$, there exists $\alpha \in A$ such that $U'_\beta \subset U_\alpha$.

Definition 2.6 (Paracompactness)**.** A topological space X is *paracompact* if every open cover admits a locally finite refinement.

Exercise 2.2 (Paracompactness)**.** Prove that if X is a topological space which is Hausdorff, second-countable and locally compact, then X is paracompact. Therefore, every manifold is paracompact.

The above exercise shows that every manifold is paracompact.

Proposition 2.7 (Partition of Unity). *Let X be a smooth manifold and let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of X.*

(a) If the open cover $\{U_\alpha\}$ is locally finite, then there exists a smooth partition of unity $\{\rho_\alpha\}_{\alpha\in A}$ subordinate to $\{U_\alpha\}$, i.e.,

$$\operatorname{supp}(\rho_\alpha) \subset U_\alpha, \quad \textit{and} \quad \sum_{\alpha\in A} \rho_\alpha(x) = 1 \quad \textit{for all } x \in X.$$

(b) In general (i.e. without assuming the cover is locally finite) there exists a collection of smooth functions

$$\{\rho_\beta\}_{\beta\in B} \subset \mathscr{C}^\infty(X, [0,1])$$

with the property that

- *For each $\beta \in B$, there exists $\alpha \in A$ such that $\operatorname{supp}(\rho_\beta) \subset U_\alpha$;*
- *The sum $\sum_{\beta\in B} \rho_\beta(x)$ is locally finite and equals 1 for all $x \in X$, that is*

$$\sum_{\beta\in B} \rho_\beta(x) = 1.$$

Exercise 2.3. Prove the partition of unity in Proposition 2.7.

Remark 2.8. The following result is useful: let X be a smooth manifold and let $U \subset X$ be an open subset, then

- for any $x \in U$, there exists a smooth function $f \in \mathscr{C}_c^\infty(X, \mathbb{R})$ (real smooth function with compact support) such that $\operatorname{supp}(f) \subset U$, and $f \geq 0, f(x) > 0$.
- (bump function) for any open subset $W \subset U$ such that $\overline{W} \subset U$ is compact, there exists a function $f \in \mathscr{C}_c^\infty(X, \mathbb{R})$ such that $\operatorname{supp}(f) \subset U$, and $0 \leq f \leq 1, f|_W = 1$.

For a proof, we refer to [1, Lemma 1.2.3 and Theorem 1.4.1].

Remark 2.9. Partition of unity is crucial to *globalize* local constructions on a manifold. We will see such techniques in the sequel, to define objects or prove properties in the local charts (hence on an open subset of $\mathbb{R}^m$), and then apply the partition of unity to obtain the global ones.

Definition 2.10 (Submanifold). Let X be a smooth manifold of dimension m, let Y be a subset of X and let $i : Y \hookrightarrow X$ denote the inclusion map.

- We call Y an **immersed** submanifold of X if Y itself is a smooth manifold of dimension k ($k \leq m$), and for each point $y \in Y$, there exists a local chart $(U_y, V_y \subset \mathbb{R}^k, \psi_y)$ of Y and a local chart $(U, V \subset \mathbb{R}^m, \psi)$ of X near y such that the inclusion i is represented by $i : V_y \ni (x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0) \in V \subset \mathbb{R}^m$.
- We call Y an **embedded** submanifold of X if Y is a smooth manifold of dimension k ($k \leq m$) such that for each point $y \in Y$, there exists a local chart $(U, V \subset \mathbb{R}^m, \psi)$ of X near y such that $(U \cap Y, V \cap \mathbb{R}^k, \psi|_{U\cap Y})$ is a local chart for Y near y, where we

identify $\mathbb{R}^k$ with a k-subspace of $\mathbb{R}^m$.

Remark 2.11. We can have a general definition for an immersed submanifold Y of X: if $Y = f(N)$ for a smooth immersion $f : N \to X$. Here we do not require f to be injective, this means we allow the self-intersection for Y. So the narrow definition in Definition 2.10 corresponds to injective immersions.

When Y is an (injective) immersed submanifold of X and the submanifold topology of Y agrees with the induced topology of Y as a subspace of X, then Y becomes an embedded submanifold.

For an example of an immersed submanifold which is not embedded, one can consider the irrational lines inside a 2-torus.

Exercise 2.4 (Submanifolds)**.** Let $f : N \to M$ be a smooth map between smooth manifolds. We say that f is an *immersion* at a point $p \in N$ if the differential

$$df_p : T_pN \to T_{f(p)}M$$

is injective. The map f is called an *immersion* if it is an immersion at every point $p \in N$.

Now let $f : N \to M$ be a smooth injective immersion:

(a) Prove that the image $f(N)$ is an immersed submanifold of M.
Hint: Use the local immersion property of f and the constant rank theorem to show that around each point of N, there exist coordinate charts in which f is given by an inclusion.

(b) Suppose that f is a topological embedding (i.e. f is a homeomorphism onto its image). Prove that in this case, $f(N)$ is an *embedded* submanifold of M.
Hint: Show that the original topology on N and the subspace topology induced from M agree via f, so that the local charts provided by part (a) actually define a smooth structure on $f(N)$ as a subset of M.

(c) Provide an example of a smooth immersion $f : N \to M$ that is not an embedding. Explain why it is an immersion and point out which property fails for it to be an embedding.

2.2 Vector bundle and Hermitian metric

Let X be a (smooth) manifold. Recall the general linear groups $\mathrm{GL}(r, \mathbb{C})$ and $\mathrm{GL}(r, \mathbb{R})$ are canonically smooth manifolds, with the group structure, they are actually Lie groups.

Definition 2.12 (Complex vector bundle)**.** Let $\pi : E \to X$ be a map between smooth manifolds E and X. We say E is a *complex vector bundle of rank $r \in \mathbb{N}_{\geq 1}$* over X if:

(1) $E \to X$ is a smooth surjection.

(2) There is an open cover $\{U_\alpha\}_{\alpha \in A}$ of X such that for each $\alpha \in A$ we have a *local trivialization*: a diffeomorphism $G_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{C}^r$ such that the following

diagram holds

$$\begin{tikzcd} E \supset \pi^{-1}(U_\alpha) \arrow[r, "G_\alpha"] \arrow[dr, "\pi"'] & U_\alpha \times \mathbb{C}^r \arrow[d, "\mathrm{pr}_1"] \\ & X \supset U_\alpha \end{tikzcd},$$

where pr_1 denotes the projection of the first factor.

(3) Moreover, if $U_\alpha \cap U_\beta \neq \varnothing$, we have smooth map, also called a transition function, $G_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}(r, \mathbb{C})$ (invertible matrices of size $r \times r$) such that

$$\begin{tikzcd} & \pi^{-1}(U_\alpha \cap U_\beta) \arrow[dl, "G_\alpha"'] \arrow[dr, "G_\beta"] & \\ U_\alpha \cap U_\beta \times \mathbb{C}^r \arrow[rr, "G_\beta \circ G_\alpha^{-1}"'] & & U_\alpha \cap U_\beta \times \mathbb{C}^r \end{tikzcd},$$

where $G_\beta \circ G_\alpha^{-1}$ is given by $U_\alpha \cap U_\beta \times \mathbb{C}^r \ni (x, v) \mapsto (x, G_{\beta\alpha}(x) \cdot v) \in U_\alpha \cap U_\beta \times \mathbb{C}^r$, and $G_{\beta\alpha}(x)$ acts on $\mathbb{C}^r$ by matrix multiplication.

In this case, for each $x \in X$, the fiber at x is a vector space

$$E_x \;:=\; \pi^{-1}(x) \;\cong\; \mathbb{C}^r.$$

Remark 2.13 (Real vector bundle)**.** We can define *real* vector bundles in a similar fashion by using $\mathbb{R}^r$ and transition functions valued in $\mathrm{GL}(r, \mathbb{R})$.

Remark 2.14. (1) Roughly speaking, a vector bundle $E = \bigsqcup_{x \in X} E_x$ is *a smooth family of vector spaces.*
(2) Note that $G_{\alpha\alpha}$ is the constant map given by the identity matrix Id_r on U_α, and on $U_\alpha \cap U_\beta$, we have

$$G_{\alpha\beta} = G_{\beta\alpha}^{-1} \quad (\text{matrix inverse}). \tag{2.1}$$

If $U_\alpha \cap U_\beta \cap U_\gamma \neq \varnothing$, then on this intersection.

$$G_{\alpha\beta}(x)\, G_{\beta\gamma}(x) = G_{\alpha\gamma}(x). \tag{2.2}$$

For a system of transition functions $\{G_{\alpha\beta} \in \mathscr{C}^\infty(U_\alpha \cap U_\beta, \mathrm{GL}(r, \mathbb{C}))\}_{\alpha,\beta \in A}$, if (2.1) and (2.2) hold, then we call that it verify the **cocycle condition** (cf. Čech cohomology theory).

Proposition 2.15 (Constructing E from cocycle)**.** *Given a system of transition functions $\{G_{\alpha\beta}\}_{\alpha,\beta \in A}$ verifying the cocycle condition, define*

$$E \;=\; \bigsqcup_\alpha (U_\alpha \times \mathbb{C}^r) \Big/ \sim$$

where (x, v) in $U_\alpha \times \mathbb{C}^r$ is identified with $(x, G_{\beta\alpha}(x) \cdot v)$ in $U_\beta \times \mathbb{C}^r$ whenever $x \in U_\alpha \cap U_\beta$. Then E is itself a smooth manifold of real dimension $\dim X + 2r$, and the projection $\pi : E \to X$ is induced by $(x, v) \mapsto x$ on each local piece gives a smooth vector bundle on X of rank r.

Example 2.16.
- (Trivial vector bundle) A *trivial vector bundle* of rank r is given by:

$$\pi : X \times \mathbb{R}^r \to X, \quad \pi(x, v) = x.$$

or

$$\pi : X \times \mathbb{C}^r \to X, \quad \pi(x, v) = x.$$

The transition functions are simply the identity: $G_{\alpha\beta} = \mathrm{Id}_r$.

- (Tangent and cotangent bundles) Let X be a smooth manifold of dimension m. The tangent bundle TX assigns to each point $x \in X$ its tangent space T_xX, and similarly T^*X is the cotangent bundle.

  Given a local chart $U_\alpha \subset X$, with coordinates $(x_1, \dots, x_m) \in V_\alpha \subset \mathbb{R}^m$, we obtain a local trivialization:
  $$TX|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^m, \quad T^*X|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^m,$$
  where

  - The **tangent frame** associated to the chart is given by:
    $$\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\right),$$
    which forms a basis of tangent space at each point of U_α .
  - The **cotangent frame** is the dual frame:
    $$(dx^1, \dots, dx^m),$$
    which spans T^*X over U_α and satisfies:
    $$dx^\ell \left(\frac{\partial}{\partial x_j}\right) = \delta^\ell_j.$$

  To see the transition functions for TX and T^*X, let U_β be a local chart with local coordinates $(y_1, \dots, y_m) \in V_\beta \subset \mathbb{R}^m$ that intersects U_α. The transition function $\psi_{\beta\alpha}$ maps $(x_1, \dots, x_m)$ to $(y_1, \dots, y_m)$, that is, $\psi_{\beta\alpha,\ell}(x) = y_\ell$, for $\ell = 1, \dots, m$. Then we have
  $$\frac{\partial}{\partial x_j} = \sum_{\ell=1}^{m} \frac{\partial \psi_{\beta\alpha,\ell}}{\partial x_j} \frac{\partial}{\partial y_\ell},$$
  and the transition function for TX is given as
  $$G^{\tan}_{\beta\alpha}(x) = \left(\frac{\partial \psi_{\beta\alpha,\ell}}{\partial x_j}\right)_{\ell j} \in \mathrm{GL}(r, \mathbb{R}).$$
  It is easy to verify that $\{G^{\tan}_{\beta\alpha}\}_{\alpha,\beta}$ satisfies the cocycle condition so that it defines a real vector bundle.

  Analogously, we have
  $$dy^\ell = \sum_{\ell=1}^{m} \frac{\partial \psi_{\beta\alpha,\ell}}{\partial x_j} dx^j,$$
  we obtain the transition functions for T^*X given by
  $$G^{\mathrm{cotan}}_{\beta\alpha}(x) = {}^T(G^{\tan}_{\beta\alpha}(x)^{-1}),$$
  where ${}^T\cdot$ denote the matrix transpose.

Definition 2.17 (Constructing new vector bundles out of old)**.** Let $\pi : E \to X$ be a complex vector bundle with local trivializations and transition functions $\{G_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{C})\}$.

We can define new bundles via:

- **Dual bundle:** $E^* = \bigsqcup_{x\in X} E_x^*$, where $E_x^* := \mathrm{Hom}_{\mathbb{C}}(E_x, \mathbb{C})$, has transition functions $G_{\alpha\beta}^{(E^*)}(x) := {}^T(G_{\alpha\beta}(x)^{-1})$.
- **Conjugate bundle:** $\overline{E} = \bigsqcup_{x\in X} \overline{E}_x$, where $\overline{E}_x$ is the same as E_x as real vector spaces but the scalar multiplication by $\lambda \in \mathbb{C}$ is given by $\overline{\lambda}$, has transition functions $G_{\alpha\beta}^{(\overline{E})}(x) := \overline{G_{\alpha\beta}(x)}$.

If F is another complex vector bundle on X:

- **Direct sum:** $E \oplus F = \bigsqcup_{x\in X} E_x \oplus F_x$ has transition functions $G_{\alpha\beta}^{(E)} \oplus G_{\alpha\beta}^{(F)}$.
- **Tensor product:** $E \otimes F = \bigsqcup_{x\in X} E_x \otimes F_x$ has transition functions $G_{\alpha\beta}^{(E)} \otimes G_{\alpha\beta}^{(F)}$.
- **Homomorphism:** $\mathrm{Hom}(E, F) = E^* \otimes F$ has transition functions $G_{\alpha\beta}^{(E^*)} \otimes G_{\alpha\beta}^{(F)}$.
- **Tensor powers:** $E^{\otimes k}$ has transition functions $\underbrace{G_{\alpha\beta} \otimes \ldots \otimes G_{\alpha\beta}}_{k \text{ times}}$.
- **Symmetric powers:** $\mathrm{Sym}^k(E)$ (or S^kE) has transition functions $\mathrm{Sym}^k(G_{\alpha\beta})$ induced from the ones of $E^{\otimes k}$.
- **Exterior powers:** $\Lambda^k(E)$ (or Λ^kE) has transition functions $\Lambda^k(G_{\alpha\beta})$ induced from the ones of $E^{\otimes k}$.
- **Pullback bundle:** If $f : Y \to X$ is smooth, the pull-back bundle f^*E on Y is given as
$$f^*E = \bigsqcup_{y\in Y} E_{f(y)}$$
with the transition functions $G_{\alpha\beta} \circ f$ over $f^{-1}(U_\alpha \cap U_\beta)$.

We also have the definition of subbundle of a given vector bundle E.

Definition 2.18 (Subbundle)**.** Let $\pi : E \to X$ be (complex) vector bundle of rank r. A subbundle $F \subset E$ is a subset satisfying:

1. F is an embedded submanifold of E
2. For each $x \in X$, the fiber $F_x := F \cap E_x$ is a vector subspace of E_x
3. The restriction $\pi|_F : F \to X$ forms a vector bundle with the induced smooth structure

We call F a subbundle of rank k if $\dim F_x = k$ for all $x \in X$.

An equivalent characterizations for a subbundle is as follows: for each $x \in X$, there exists a local neighbourhood U_x where both E and F can be trivialized such that

$$\begin{tikzcd}
(\pi|_F)^{-1}(U_x) \arrow[r, "\mathrm{incl}"] \arrow[d, "\simeq"] & \pi^{-1}(U_x) \arrow[d, "\simeq"] \\
U_x \times \mathbb{C}^k \arrow[r, "I"] & U_x \times \mathbb{C}^r
\end{tikzcd}$$

where $I = (\mathrm{Id}_{U_x}, i)$ with an injective linear map $i : \mathbb{C}^k \hookrightarrow \mathbb{C}^r$.

Moreover, when we properly take an open cover of X, we can write the transition functions $\{G^{(E)}_{\alpha\beta}\}$ of E as :

$$G^{(E)}_{\alpha\beta} = \begin{pmatrix} G^{(F)}_{\alpha\beta} & * \\ 0 & K_{\alpha\beta} \end{pmatrix}$$

where $G^{(F)}_{\alpha\beta}$ are $k \times k$ transition functions for F.

Exercise 2.5 (Quotient vector bundles)**.** Let $\pi : E \to X$ be a vector bundle over a smooth manifold X, and let $F \subset E$ be a subbundle. Work out a system of transition functions for the quotient vector bundle E/F over X.

Definition 2.19. Given two vector bundles E, F over X, a **morphism of vector bundles** $\varphi : E \to F$ is a smooth map satisfying:

- The diagram commutes:

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d, "\pi_E"] & F \arrow[d, "\pi_F"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

- For each $x \in X$, the restriction $\varphi_x : E_x \to F_x$ is linear

Example 2.20.
- For any smooth map $f : Y \to X$ between manifolds, the **tangent map** $df_y : T_yY \to T_{f(y)}X$ gives a morphism of tangent bundles.
- The inclusion $i : F \hookrightarrow E$ of a subbundle and the quotient map $q : E \to E/F$ are canonical bundle morphisms.

Remark 2.21. For a vector bundle homomorphism $\varphi : E \to F$,

- $\ker \varphi := \bigsqcup_x \ker \varphi_x$ is a subbundle of E if $\operatorname{rank} \varphi_x$ is constant on X.
- $\operatorname{Im} \varphi := \bigsqcup_x \operatorname{Im} \varphi_x$ is a subbundle of F under the same condition.

Definition 2.22. The space of (global) smooth sections is:

$$\mathscr{C}^\infty(X, E) := \{s : X \to E \text{ smooth } | \pi \circ s = \mathrm{id}_X\}$$

Sometimes, we also use the notation $\Gamma(E)$ for the $\mathscr{C}^\infty(X, E)$ when the manifold X is clear. This forms a vector space under pointwise operations:

- $(s + s')(x) = s(x) + s'(x) \in E_x$;
- $(fs)(x) = f(x)s(x)$ for $f \in \mathscr{C}^\infty(X)$.

The support of a section $s \in \Gamma(E)$ is:

$$\operatorname{supp}(s) = \overline{\{x \in X : s(x) \neq 0 \text{ in } E_x\}} \subset X.$$

Let $\mathscr{C}^\infty_c(X, E) \subset \mathscr{C}^\infty(X, E)$ be the subspace consisting of smooth sections with compact support.

Remark 2.23. Locally, sections correspond to tuples of smooth functions:

$$s \in \mathscr{C}^\infty(X, E) \iff s_\alpha \in \mathscr{C}^\infty(U_\alpha, \mathbb{C}^r) \text{ on trivializing charts } U_\alpha \text{ of } E,$$

such that $G_{\beta\alpha} s_\alpha = s_\beta$ on $U_\alpha \cap U_\beta$.

Definition 2.24 (Hermitian metrics). A **Hermitian metric** h^E on E is a smooth section of $E^* \otimes \overline{E}^*$ such that each h^E_x is a Hermitian inner product on E_x. The pair (E, h^E) is called a Hermitian vector bundle.

Theorem 2.25. *Every vector bundle admits a Hermitian metric.*

Proof. Let $\{U_\alpha\}$ be a locally finite open cover of X such that E is trivial over each U_α, with trivializations

$$G_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^r.$$

Define a Hermitian metric h_α on $E|_{U_\alpha}$ by pulling back the standard Hermitian inner product on $\mathbb{C}^r$ via G_α.

Choose a smooth partition of unity $\{\rho_\alpha\}$ subordinate to the cover $\{U_\alpha\}$. Then define a global Hermitian metric h^E on E by:

$$h^E(x)(v, w) := \sum_\alpha \rho_\alpha(x) h_\alpha(x)(v, w), \quad \text{for all } v, w \in E_x.$$

This sum is finite at each $x \in X$ due to local finiteness. The function h^E is smooth and defines a Hermitian inner product on each fiber, as required. □

Remark 2.26. When E is a real vector bundle, then we have the Euclidean metric g^E on E. In the case $E = TX$, then $\Gamma(TX)$ is the space of all smooth vector fields on X, and a Euclidean metric g^{TX} is called a *Riemannian metric* on X.

Proposition 2.27. *Let E, F be vector bundles over X. Let $D : \mathscr{C}^\infty(X, E) \to \mathscr{C}^\infty(X, F)$ be a $\mathbb{C}$-linear operator commuting with multiplication by functions, that is, for any $s \in \mathscr{C}^\infty(X, E)$ and $f \in \mathscr{C}^\infty(X)$,*

$$[D, f]s := D(fs) - f(Ds) = 0.$$

Then D is given by a smooth section $A \in \mathscr{C}^\infty(X, \mathrm{Hom}(E, F))$, that is, for $s \in \mathscr{C}^\infty(X, E)$, we have

$$(Ds)(x) = A(s)s(x) \in F_x.$$

Proof. The proof proceeds in three main steps:
Step 1: Locality of D or D is locally defined.

We prove that for any $x \in X$, if two sections $s_1, s_2 \in \mathscr{C}^\infty(X, E)$ agree on a neighborhood U_x of x, then Ds_1 and Ds_2 agree on a possibly smaller neighborhood $V_x \subset U_x$. In fact,

- Take a bump function $f \in \mathscr{C}^\infty_c(U_x, \mathbb{R})$ with:

$$f|_{V_x} \equiv 1 \quad \text{and} \quad \mathrm{supp}(f) \subset U_x.$$

- Since $s_1 \equiv s_2$ on U_x, we have $fs_1 = fs_2$ globally on X.

- By the commutation property:

$$fDs_1 = D(fs_1) = D(fs_2) = fDs_2.$$

- On V_x where $f \equiv 1$, this implies $Ds_1|_{V_x} = Ds_2|_{V_x}$.

Step 2: Prove the conclusion on local charts.

On a trivializing chart U_α where:

$$E|_{U_\alpha} \cong U_\alpha \times \mathbb{C}^{r_1} \quad \text{and} \quad F|_{U_\alpha} \cong U_\alpha \times \mathbb{C}^{r_2};$$

the operator restricts to:

$$D|_{U_\alpha} : C^\infty(U_\alpha, \mathbb{C}^{r_1}) \to C^\infty(U_\alpha, \mathbb{C}^{r_2}).$$

- For constant sections $\{e_i\}$ (standard basis of $\mathbb{C}^{r_1}$) and $\{v_j\}$ (dual basis for $\mathbb{C}^{r_2}$), define:

$$A_\alpha(x) = \big(\langle D|_{U_\alpha} e_i, v_j\rangle\big)_{i,j} \in \mathrm{Hom}(\mathbb{C}^{r_1}, \mathbb{C}^{r_2})$$

- The commutation relation implies

$$(D|_{U_\alpha} s)(x) = A_\alpha(x) \cdot s(x).$$

Step 3: Global patching.

Let $\{\rho_\alpha\}$ be a partition of unity subordinate to a trivializing cover $\{U_\alpha\}_\alpha$ that is locally finite.

- For any section $s \in \mathscr{C}^\infty(X, E)$, write:

$$s = \sum_\alpha \rho_\alpha s.$$

- Apply D and use locality:

$$Ds = \sum_\alpha D(\rho_\alpha s) = \sum_\alpha D|_{U_\alpha}(\rho_\alpha s) = \sum_\alpha A_\alpha \cdot (\rho_\alpha s).$$

- This defines a global section $A \in \mathscr{C}^\infty(X, \mathrm{Hom}(E, F))$ by:

$$A(x) := \sum_\alpha A_\alpha(x)\rho_\alpha(x),$$

  since each $A_\alpha(x)\rho_\alpha(x) \in \mathscr{C}_c^\infty(U_\alpha, \mathrm{Hom}(E, F)) \subset \mathscr{C}^\infty(X, \mathrm{Hom}(E, F))$ with $(Ds)(x) = A(x)s(x)$ as required.

□

2.3 Differential forms and de Rham cohomology groups

Definition 2.28 (Differential forms)**.** Let X be a smooth manifold of dimension m.

- The **exterior algebra bundle**:

$$\Lambda^\bullet T^*X = \bigoplus_{k=0}^{m} \Lambda^k T^*X \quad \text{with rank } \binom{m}{k}$$

- Spaces of differential forms:

$$\Omega^\bullet(X) = \mathscr{C}^\infty(X, \Lambda^\bullet T^*X),$$

$$\Omega^k(X) = \mathscr{C}^\infty(X, \Lambda^k T^*X).$$

In particular, we have $\Omega^0(X) = \mathscr{C}^\infty(X, \mathbb{R})$. We will call elements in $\Omega^k(X)$ differential k-forms or simply k-forms.

- The **wedge product** $\wedge : \Omega^k(X) \times \Omega^\ell(X) \to \Omega^{k+\ell}(X)$ is given as $(\alpha, \beta) \mapsto \alpha \wedge \beta$. Note that it satisfies:
$$\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha.$$

Definition 2.29. A differential k-form $\omega \in \Omega^k(X)$ has compact support if $\operatorname{supp}(\omega) \subset X$ is compact. We denote this space by $\Omega_c^k(X)$.

Remark 2.30. When X is compact, $\Omega^\bullet(X) = \Omega_c^\bullet(X)$.

Proposition 2.31. *There exists a unique $\mathbb{R}$-linear operator $d : \Omega^k(X) \to \Omega^{k+1}(X)$ such that:*

1. *For all $f \in \mathscr{C}^\infty(X, \mathbb{R}) = \Omega^0(X)$, $df \in \Omega^1(X)$ is the classical differential of f.*
2. *$d \circ d \equiv 0$ (i.e., $d^2 = 0$).*
3. *For all $\alpha \in \Omega^k(X)$ and $\beta \in \Omega^\ell(X)$, the Leibniz rule holds:*
$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge d\beta.$$

Recall that the classical differential of a function f is defined as follows: for $f \in \mathscr{C}^\infty(X, \mathbb{R})$ and a local chart $(U_\alpha, V_\alpha, \psi_\alpha)$, the cotangent bundle T^*X is spanned by $dx^1, \ldots, dx^n$. The local representation of f is:
$$f_\alpha = f|_{U_\alpha} \circ \psi_\alpha^{-1} : V_\alpha \to \mathbb{R} \quad \text{(smooth)}.$$
The differential df on U_α is given by:
$$df_\alpha = \sum_j \frac{\partial f_\alpha}{\partial x_j} dx^j.$$

It is easy to verify that $\{df_\alpha\}$ patches together to a global section $df \in \Omega^1(X)$.

For a general k-form $s \in \Omega^k(X)$, its restriction to U_α can be written as:
$$s|_{U_\alpha} = \sum_{|I|=k} f_I^\alpha dx^I,$$
where $f_I^\alpha \in \mathscr{C}^\infty(U_\alpha, \mathbb{R})$ and $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ and $I = (i_1 < \cdots < i_k)$.

Proof of Proposition 2.31. Let $\{U_\alpha\}$ be a locally finite open cover by coordinate charts, with $\{\rho_\alpha\}$ a subordinate partition of unity. For any differential form $s \in \Omega^k(X)$, we write locally on U_α:
$$s|_{U_\alpha} = \sum_{|I|=k} f_I^\alpha dx^I \quad \text{(multi-index notation)}$$

Uniqueness:

The key diagram shows how properties 1–3 in the propostion determine d:

$$\begin{aligned} ds &= d\left(\sum_{\alpha} \rho_\alpha s\right) \\ &= \sum_{\alpha} d\left(\rho_\alpha \sum_I f_I^\alpha dx^I\right) \\ &= \sum_{\alpha,I} \underbrace{d(\rho_\alpha f_I^\alpha)}_{\text{Property 1}} \wedge \underbrace{dx^I}_{\text{Property 3}} + \underbrace{\rho_\alpha f_I^\alpha}\, \underbrace{d(dx^I)}_{\text{Property 2}} \\ &= \sum_{\alpha,I} \left(\sum_j \frac{\partial(\rho_\alpha f_I^\alpha)}{\partial x_j} dx^j\right) \wedge dx^I. \end{aligned}$$

This computation shows d must have this form if it satisfies the properties 1–3 in the proposition.

Existence: Define d by the formula:

$$(ds)|_{U_\alpha} := \sum_{|I|=k} \left(\sum_{j=1}^n \frac{\partial f_I^\alpha}{\partial x_j} dx^j\right) \wedge dx^I$$

On overlaps $U_\alpha \cap U_\beta$, the transition formulas for f_I^α and dx^I guarantee consistency

Axiom verification:

1. For $f \in C^\infty(X)$, reduces to classical differential.
2. $d^2 = 0$ follows from equality of mixed partials.
3. Leibniz rule holds via the construction.

□

Remark 2.32. The differential d is determined by $d|_{\Omega^0}$ and $d|_{\Omega^1}$ through the Leibniz rule.

Exercise 2.6 (Exterior differential on manifold)**.** Let X be a smooth manifold. Recall that on a local chart $(U_\alpha, V_\alpha \subset \mathbb{R}^m, \psi_\alpha)$ of X, for a 1-form

$$\beta(x) = \sum_{j=1}^m \beta_j(x) dx^j, \text{ with } \beta_j \in \mathscr{C}^\infty(V_\alpha)$$

the action of differential d on β is defined as

$$d\beta = \sum_{j=1}^m d\beta_j \wedge dx^j.$$

Recall that for two (tangent) vector fields $U, V \in \Gamma(TX)$, the Lie bracket $[U, V] \in \Gamma(TX)$ is the vector field such that for any $f \in \mathscr{C}^\infty(X)$,

$$[U, V]f := U(V(f)) - V(U(f)).$$

Prove that:

(a) For $\beta \in \Omega^1(X)$, the 2-form $d\beta \in \Omega^2(X)$ satisfies that for $U, V \in TX$,

$$(d\beta)(U, V) = U(\beta(V)) - V(\beta(U)) - \beta([U, V]).$$

(b) In general, for $\beta \in \Omega^k(X)$, and $V_0, V_1, \ldots, V_k \in TX$, we have

$$\begin{aligned}(d\beta)(V_0, V_1, \ldots, V_k) = &\sum_{j=0}^{m}(-1)^j V_j(\beta(V_0, \ldots, \widehat{V}_j, \ldots, V_k)) \\ &+ \sum_{0\le j<\ell\le m}(-1)^{j+\ell}\beta([V_j, V_\ell], V_0, \ldots, \widehat{V}_j, \ldots, \widehat{V}_\ell, \ldots, V_k)),\end{aligned}$$

where the notation $\widehat{V}_j$ means that the vector V_j is removed.

(c) Verify the Jacobi identity: for $V_1, V_2, V_3 \in TX$,

$$[V_1, [V_2, V_3]] + [V_2, [V_3, V_1]] + [V_3, [V_1, V_2]] = 0.$$

(d) Prove that $d^2 = 0$ using the formula in (b).

Proposition 2.33. *Let $f : X \to Y$ be a smooth map between two manifolds, we define the pull-back map $f^* : \Omega^k(Y) \to \Omega^k(X)$ as follows: for $\alpha \in \Omega^1(Y)$, $x \in X$, $v \in T_xX$,*

$$(f^*\alpha)_x(v_x) := \alpha_{f(x)}(df_x v).$$

Then we have

$$d^X \circ f^* = f^* \circ d^Y.$$

The **de Rham complex** is the sequence:

$$0 \to \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^m(X) \to 0.$$

Since $d^2 = 0$, the above sequence forms a complex, in particular, we have

$$\operatorname{Im} d|_{\Omega^{k-1}} \subset \ker d|_{\Omega^k}.$$

Similarly, the **de Rham complex with compact support** is defined as

$$0 \to \Omega^0_c(X) \xrightarrow{d} \Omega^1_c(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^m_c(X) \to 0.$$

Definition 2.34. The k-th **de Rham cohomology** is defined as the quotient vector space

$$H^k_{\mathrm{dR}}(X) := \frac{\ker d|_{\Omega^k}}{\operatorname{Im} d|_{\Omega^{k-1}}}.$$

The **de Rham cohomology with compact support** is defined as

$$H^k_{\mathrm{dR},c}(X) := \frac{\ker d|_{\Omega^k_c}}{\operatorname{Im} d|_{\Omega^{k-1}_c}}.$$

We will denote

$$H^\bullet_{\mathrm{dR}}(X) := \oplus_k H^k_{\mathrm{dR}}(X), H^\bullet_{\mathrm{dR},c}(X) := \oplus_k H^k_{\mathrm{dR},c}(X).$$

If we want to emphasize the number fields $\mathbb{R}$ or $\mathbb{C}$, we will put the corresponding notation inside.

A differential form $\alpha \in \Omega^\bullet(X)$ is called **closed** or d**-closed** if $d\alpha = 0$. Any closed form α defines a cohomological class $[\alpha] \in H^\bullet_{\mathrm{dR}}(X)$. A differential form $\alpha \in \Omega^\bullet(X)$ is called **exact** or

d-**exact** if there exists a form $\beta \in \Omega^\bullet(X)$ such that $\alpha = d\beta$. Two closed forms α and α' are called **cohomologous** if $\alpha - \alpha'$ is exact.

Remark 2.35. When X is compact, we always have

$$\dim H^\bullet_{\mathrm{dR}}(X) < \infty.$$

Definition 2.36 (Orientability). (1) Let $E \to X$ be a real vector bundle of rank r, then E is called orientable if there is a collection of transition functions $\{G_{\alpha\beta} \in \mathscr{C}^\infty(U_\alpha \cap U_\beta, \mathrm{GL}(r,\mathbb{R}))\}_{\alpha,\beta}$ which defines E such that $G_{\alpha\beta} \in \mathrm{GL}^+(r,\mathbb{R})$ (that is, $\det G_{\alpha\beta}(x) > 0$).

(2) A manifold X is **orientable** if its tangent bundle TX is an orientable vector bundle.

Proposition 2.37. *If X is a connected manifold of dimension m, then X is orientable if and only if $\Lambda^m T^*X$ is isomorphic to the trivial bundle $\mathbb{R}$. An orientation $o(X)$ is a choice of a nonvanishing section of $\Lambda^m T^*X$ up to multiplication of a function $f > 0$.*

We call X an oriented manifold when X is orientable and an orientation $o(X)$ is given (implicitly). If we want to emphasise the orientation, we write $(X, o(X))$.

Let X be an oriented manifold. An oriented local charts (U_α, ψ_α) with $\psi_\alpha : U_\alpha \to V_\alpha \subseteq \mathbb{R}^m$ means that we have

$$\psi_\alpha^*(dx_1 \wedge \cdots \wedge dx_m)/\omega_o > 0,$$

where ω_o is the section of $\Lambda^m T^*X$ that represents the orientation $o(X)$.

For $s \in \Omega^m(X)$ and oriented chart (U_α, ψ_α):

$$s|_{U_\alpha} = f_\alpha \, dx_1 \wedge \cdots \wedge dx_m$$

where $dx_1 \wedge \cdots \wedge dx_m$ is oriented according to $o(X)$. Define:

$$\int_{U_\alpha} s := \int_{V_\alpha} f_\alpha \circ \psi_\alpha^{-1} \, dx_1 \cdots dx_m$$

as a Lebesgue integral on $\mathbb{R}^m$.

Theorem 2.38 (Integration of m-forms). *Let $\{U_\alpha\}$ be a locally finite oriented atlas and $\{\rho_\alpha\}$ a subordinate partition of unity. For $s \in \Omega_c^m(X)$:*

$$\int_X s := \sum_\alpha \int_{U_\alpha} \rho_\alpha s$$

is independent of the choices of $\{U_\alpha\}$ and $\{\rho_\alpha\}$.

Proof. We just explain the key steps:

(a) **Chart transition**: On $U_\alpha \cap U_\beta$, the Jacobian $J\psi_{\beta\alpha}$ satisfies:

$$dx^\beta = J\psi_{\beta\alpha} dx^\alpha \quad \text{with } J\psi_{\beta\alpha} > 0,$$

where dx^β, dx^α denote the Lebesgue measure on V_β, V_α respectively. Thus:

$$\int_{U_\alpha \cap U_\beta} f_\alpha dx^\alpha = \int_{U_\alpha \cap U_\beta} f_\beta dx^\beta \quad \text{since } J\psi_{\beta\alpha} f_\beta = f_\alpha$$

(b) **Partition independence**: For another partition $\{\tilde{\rho}_\beta\}$:

$$\sum_\alpha \int \rho_\alpha s = \sum_{\alpha,\beta} \int \rho_\alpha \tilde{\rho}_\beta s = \sum_\beta \int \tilde{\rho}_\beta s,$$

where the sum is locally finite, so that we can exchange the order of summations.

$\square$

Remark 2.39 (Orientation Reversal)**.** For $-X = (X, -o(X))$ denoting X with opposite orientation:

$$\int_{-X} s = -\int_X s$$

Remark 2.40. If $f : Y \to X$ is an orientation-preserving diffeomorphism:

$$\int_Y f^* s = \int_X s.$$

Definition 2.41. We define the linear functional:

$$\int_X : \Omega_c^\bullet(X) \to \mathbb{R}$$

by $\int_X s := \int_X s^{[m]}$, where $s^{[m]}$ denote the degree-m component of s.

Theorem 2.42. *If X is an oriented compact manifold, then for all $s \in \Omega^\bullet(X)$,*

$$\int_X ds = 0.$$

Then it induces a linear functional

$$\int_X : H^\bullet(X) \to \mathbb{R}.$$

2.4 Differential operator and principal symbol

Definition 2.43 (Differential Operator)**.** Let E, F be vector bundles over X of ranks r_1, r_2 respectively. A linear operator:

$$P : \mathscr{C}^\infty(X, E) \to \mathscr{C}^\infty(X, F)$$

is a **differential operator of order** k if locally on trivializing charts U_α where

$$E|_{U_\alpha} \simeq U_\alpha \times \mathbb{C}^{r_1}, \qquad F|_{U_\alpha} \simeq U_\alpha \times \mathbb{C}^{r_2}$$

we have the local expression of P as

$$P_\alpha = \sum_{|I| \le k} a_\alpha^I(x) \left(\frac{\partial}{\partial x}\right)^I \tag{2.3}$$

where $a_\alpha^I \in C^\infty(U_\alpha, \mathrm{Hom}(\mathbb{C}^{r_1}, \mathbb{C}^{r_2}))$, and our multi-index notation in (2.3) is given as: for $I = (i_1, \ldots, i_m) \in \mathbb{N}_0^m$, we put $\left(\frac{\partial}{\partial x}\right)^I = \Pi_{\ell=1}^m \left(\frac{\partial}{\partial x_\ell}\right)^{i_\ell}$, and $|I| := \sum_{\ell=1}^m i_\ell$.

In particular, if $s = \{s_\alpha\}_\alpha \in \mathscr{C}^\infty(X, E)$, then $\{P_\alpha s_\alpha\}_\alpha$ patches together as a global section of F.

Example 2.44.

- Any $P \in \mathscr{C}^\infty(X, \mathrm{Hom}(E, F))$ is a differential operator of order zero.
- A vector field $V \in \Gamma(TX)$ acting on $\mathscr{C}^\infty(X)$ is a first-order differential operator.
- The exterior differential $d : \Omega^\bullet(X) \to \Omega^{\bullet+1}(X)$ is a first-order differential operator.

The following proposition is an extension of Proposition 2.27 to differential operators.

Proposition 2.45. *Let P be a differential operator on X. If for $f \in \mathscr{C}^\infty(X)$, $[P, f] := Pf - fP$ is a differential operator of order $k-1$, then P is a differential operator of order k.*

One key step to prove the above proposition is show that P is locally defined: for any $x \in X$, if $s_1, s_2 \in \Gamma(E)$ coincide in an open neighbourhood U_x of x, then Ps_1 and Ps_2 also coincide in a possibly smaller open neighbourhood V_x of x. To see this, we may take any $V_x \Subset U_x$, and a bump function $\rho \in \mathscr{C}_c^\infty(U_x, [0,1])$ such that $\rho|_{V_x} \equiv 1$, then $\rho s_1 = \rho s_2$ on whole X by our assumption, we get

$$\rho P s_1 = P(\rho s_1) - [P, \rho] s_1 = P(\rho s_2) - [P, \rho] s_1 = \rho P s_2 + [P, \rho] s_2 - [P, \rho] s_1.$$

Now we need to compare $[P, \rho]s_2$ and $[P, \rho]s_1$, while $[P, \rho]$ is a differential operator of order $k-1$, so we can apply the induction on the orders.

Exercise 2.7 (Differential operators)**.** Let X be a smooth manifold and let E, F be two vector bundles over X. Consider a differential operator

$$P : \mathscr{C}^\infty(X, E) \to \mathscr{C}^\infty(X, F).$$

Prove the following assertions:

(a) If P is a differential operator of order k, then for any smooth function $f \in \mathscr{C}^\infty(X)$, the commutator
$$[P, f] : s \mapsto P(fs) - f\,P(s)$$
is a differential operator of order $k-1$.

(b) Prove that every differential operator is locally defined.

(c) In the case where X is an open subset U of $\mathbb{R}^n$ and E, F are trivial vector bundles, prove the following assertion: if P is a differential operator on U such that for any $f \in \mathscr{C}^\infty(U)$, $[P, f]$ is a differential operator of order $k-1$ on U, then P is a differential operator of order k.

(d) Use a partition of unity to extend the above assertion to the general case.

Definition 2.46 (Local total symbol)**.** On local chart U_α, we can define the **total symbol** of P as

$$\sigma_{\mathrm{total}}(P_\alpha)(x, \xi) = \sum_{|I| \le k} a_\alpha^I(x)(\sqrt{-1}\xi)^I \in \mathrm{Hom}(E_x, F_x).$$

So that $\sigma_{\text{total}}(P_\alpha) \in \mathscr{C}^\infty(U_\alpha, \text{Poly}^{\leq k}(T^*X) \otimes \text{Hom}(E,F))$, where $\text{Poly}^{\leq k}(T^*X)$ denotes the bundle of polynomials functions along the fiber of T^*X of degree $\leq k$. In general, $\{\sigma_{\text{total}}(P_\alpha)\}_\alpha$ **does not** give a global section on X.

Note that, by the (fibrewise) duality between TX and T^*X, as a vector bundle,

$$\text{Poly}^{\leq k}(T^*X) \simeq S^{\leq k}(TX) := \oplus_{j=0}^k S^j(TX).$$

Remark 2.47. Note if we write $\xi = \sum_j \xi_j dx^j$, then the notation $(\sqrt{-1}\xi)^I$ with $I = (i_1, \ldots, i_)$means From the differential operator to its symbol, it is roughly replacing $\frac{\partial}{\partial x_j}$ by $\sqrt{-1}\xi_j$, this can be explained by the Fourier transform on $\mathbb{R}^m$.

Theorem 2.48 (Definition of principal symbol)**.** *If P is differential operator of order k, the local principal symbol $\sigma_k(P_\alpha)$ on chart U_α is given by the highest-order terms:*

$$\sigma_k(P_\alpha)(x,\xi) = \sum_{|I|=k} a_\alpha^I(x)(\sqrt{-1}\xi)^I.$$

*Then $\{\sigma_k(P_\alpha)\}$ defines a global section $\sigma_k(P) \in \mathscr{C}^\infty(X, S^k(TX) \otimes \text{Hom}(E,F))$, which is called the **principal symbol** of P. Moreover, it is determined by the following limit:*

$$\sigma_k(P)(x,\xi) = \lim_{t\to+\infty} \frac{1}{t^k} e^{-\sqrt{-1}tf} P e^{\sqrt{-1}tf} \in \text{Hom}(E_x, F_x),$$

where $f \in \mathscr{C}^\infty(X,\mathbb{R})$ and $df(x) = \xi \in T_x^*X$. Sometimes, we denote the principal symbol simply by $\sigma(P)$.

Proof. On the local chart U_α, by (2.3) , we compute

$$\begin{aligned} e^{-\sqrt{-1}tf} P_\alpha e^{\sqrt{-1}tf} &= e^{-\sqrt{-1}tf} \sum_{|I|\leq k} a_\alpha^I(x) \left(\frac{\partial}{\partial x}\right)^I e^{\sqrt{-1}tf} \\ &= t^k \sum_{|I|\leq k} a_\alpha^I(x) \left(\sqrt{-1}df\right)^I + \text{ terms in lower power of } t. \end{aligned}$$

$\square$

Let $\text{Diff}^{\leq k}(E,F)$ (or $\text{Diff}_X^{\leq k}(E,F)$) denote the space of differential operators on X from E to F of order $\leq k$. By definition, we have

$$\text{Diff}^0(E,F) = \mathscr{C}^\infty(X, \text{Hom}(E,F)).$$

Proposition 2.49 (Symbol sequence)**.** *There is an exact sequence for differential operators:*

$$0 \to \text{Diff}^{\leq k-1}(E,F) \xhookrightarrow{i} \text{Diff}^{\leq k}(E,F) \xrightarrow{\sigma_k} \mathscr{C}^\infty(X, S^k(TX) \otimes \text{Hom}(E,F)) \to 0$$

where:

- *i denotes the natural inclusion, and σ_k extracts the principal symbol.*
- $\text{Im}(i) = \ker \sigma_k$.
- *σ_k is surjective.*

Exercise 2.8. Prove Proposition 2.49. To prove that σ_k is surjective, construct it on local charts and then patch them together using a partition of unity.

Definition 2.50 (Elliptic operator)**.** A differential operator P of order k is **elliptic** if for all $x \in X$ and $\xi \in T_x^*X \setminus \{0\}$:

$$\sigma_k(P)(x,\xi) : E_x \to F_x \quad \text{is invertible}$$

This definition implies that $\operatorname{rank}(E) = \operatorname{rank}(F)$.

Example 2.51. The Laplacians are elliptic. On $\mathbb{R}^m$, we consider

$$\Delta = \sum_{j=1}^{m} \frac{\partial^2}{\partial x_j^2}.$$

Its principal symbol is: $\sigma_2(\Delta)(x,\xi) = -\|\xi\|^2$.

Remark 2.52. Here the principal symbol of a differential operator is always homogeneous polynomial functions along the cotangent space T^*X, that is why we regard it as elements in $S^k(TX)$. In general, we can define the principal symbol $\sigma_k(P) \in \mathscr{C}^\infty(T^*X, \pi^* \operatorname{Hom}(E,F))$ where $\pi : T^*X \to X$. This way, we can consider the non-polynomial symbols, that correspond to the pseudodifferential operators.

2.5 Atiyah–Singer index theorem and its applications

With the above preliminaries, we can give more explanations for the Atiyah–Singer index theorem.

Theorem 2.53 (Atiyah–Singer, 1963)**.** *Let X be a compact manifold of dimension m, and let E, F be two complex vector bundles on X. Let $P : \Gamma(E) \to \Gamma(F)$ be an elliptic differential operator. Then*

- $\ker(P)$ *and* $\operatorname{coker}(P)$ *are finite dimensional, then*

  $$\operatorname{Ind}(P) := \dim\ker(P) - \dim\operatorname{coker}(P) \in \mathbb{Z}$$

  is well-defined.

- *If m is odd, then we have* $\operatorname{Ind}(P) = 0$.

- *If m is even, then*

  $$\operatorname{Ind}(P) = \int_{T^*X} \pi^*(\widehat{A}(TX)^2)\operatorname{ch}(\sigma(P)), \tag{2.4}$$

  where $\pi : T^*X \to X$, the manifold T^*X is canonically oriented, and $\widehat{A}(TX) = 1 + \cdots \in \Omega^\bullet(X)$ is a closed form, and $\operatorname{ch}(\sigma(P)) \in \Omega^\bullet(T^*X)$ is a closed form that is integrable (in fact, every component is integrable on the non-compact manifold T^*X).

Roughly speaking, an approach to prove the above general theorem is: at first, construct a Dirac operator D, a certain first-order differential operator, on a new manifold, such that $\operatorname{Ind}(P) = \operatorname{Ind}(D)$; then prove (2.4) for D, that is (1.1) . In this **Lecture**, we focus on (1.1) and we will cover the following topics:

(a) Define the characteristic classes and characteristic forms, in particular, define $\widehat{A}(TX)$ and $\mathrm{ch}(\sigma(P))$.

(b) Dirac operator D and spin manifold.

(c) Heat kernel approach for Atiyah–Singer index theorem: local index theorem.

(d) Geometric applications of Atiyah–Singer index theorem.

Let's mention three important geometric applications of the Atiyah–Singer index theorem, specially the formula (1.1) for Dirac operator. They were proven by different methods before the Atiyah–Singer index theorem (1963) was established, but now we can treat them uniformly.

(i) (Gauss-Bonnet-Chern theorem) For an even-dimensional oriented compact manifold, we have

$$\mathrm{Eul}(X) = \sum_{j=0}^{\dim X} (-1)^j \dim H^j_{\mathrm{dR}}(X) = \int_X e(TX),$$

where $\mathrm{Eul}(X) = \sum_{j=0}^{\dim X} (-1)^j \dim H^j_{\mathrm{dR}}(X)$ is the Euler number of X, and $e(TX) \in H^{\dim X}_{\mathrm{dR}}(X)$ is the Euler class of X.

(ii) (Hirzebruch signature theorem) If X is an oriented compact manifold of dimension $m = 4k$. We can define a symmetric bilinear form

$$\eta : H^{2k}_{\mathrm{dR}}(X) \times H^{2k}_{\mathrm{dR}}(X) \to \mathbb{R}$$

by

$$\eta(\alpha, \beta) := \int_X \alpha \wedge \beta.$$

By Poincaré duality, we know that η is a non-degenerate bilinear form. The signature of η equals the number of positive eigenvalues of η minus the number of negative eigenvalues. Then Hirzebruch signature theorem says that

$$\mathrm{sign}(\eta) = \int_X L(TX),$$

where $L(TX) \in H^{4\bullet}_{\mathrm{dR}}(X)$ is the L-class of X.

(iii) (Riemann-Roch-Hirzebruch theorem) Let X be a compact complex manifold (that is, local charts are given by open subsets of $\mathbb{C}^m$ and the transition functions are holomorphic), we assume that X is Kähler, then we have

$$\sum_{j=0}^{\dim_{\mathbb{C}} X} (-1)^j \dim H^j(X, \mathscr{O}_X) = \int_X \mathrm{Td}(T_h X),$$

where $\mathrm{Td}(T_h X) \in H^{2\bullet}_{\mathrm{dR}}(X)$ is the Todd class for the holomorphic tangent bundle $T_h X$, and $H^j(X, \mathscr{O}_X)$ is the j-th sheaf chomology group of the structure sheaf $\mathscr{O}_X$ (of holomorphic functions). Here we can replace $H^j(X, \mathscr{O}_X)$ by the Dolbeault cohomology group $H^{0,j}(X)$.

Exercise 2.9 (Wedge products of cohomological classes)**.** Let X be a manifold of dimension m. Prove the following results:

(a) If α and β are closed forms, then $\alpha \wedge \beta$ is also closed.

(b) If α is closed and β is exact, then $\alpha \wedge \beta$ is exact.

(c) Assume X to be compact and oriented: for any $k \in \{0, \ldots, m\}$, the bilinear form

$$\eta_k : H^k_{\mathrm{dR}}(X) \times H^{m-k}_{\mathrm{dR}}(X) \to \mathbb{R}$$

by

$$\eta_k(\alpha, \beta) := \int_X \alpha \wedge \beta$$

is well-defined.

(d) For any smooth map $f : Y \to X$ between manifolds, the pull-back map f^* on differential forms induces a linear map

$$f^* : H^\bullet_{\mathrm{dR}}(X) \to H^\bullet_{\mathrm{dR}}(Y)$$

which preserves the degrees.

3 Connection, characteristic classes, and Chern–Weil theory

3.1 Connection and curvature

We recall the definitions of connection and curvature, which play the central role in the construction of characteristic classes and Chern–Weil theory. We will often use the notation

$$\Omega^\bullet(X, E) = \mathscr{C}^\infty(X, \Lambda^\bullet(T^*X) \otimes E)$$

for a vector bundle $E \to X$.

3.1.1 Connection

Definition 3.1 (Connection)**.** Let $E \to X$ be a vector bundle. A **connection** ∇^E is a linear operator:

$$\nabla^E : \mathscr{C}^\infty(X, E) \to \Omega^1(X, E) = \mathscr{C}^\infty(X, T^*X \otimes E)$$

satisfying the Leibniz rule:

$$\nabla^E(fs) = df \otimes s + f\nabla^E s \quad \forall f \in \mathscr{C}^\infty(X), s \in \mathscr{C}^\infty(X, E).$$

Remark 3.2. The definition of ∇^E is equivalent to say that ∇^E is a first order differential operator from $\mathscr{C}^\infty(X, E)$ to $\Omega^1(X, E)$ with principal symbol $\sigma(\nabla^E)(x, \xi) = \sqrt{-1}\xi\wedge$.

Proposition 3.3 (Existence and affine structure)**.** *The space of connections $\mathcal{A}(E)$ of E is:*

- *non-empty, and*
- *affine: For $\nabla_1, \nabla_2 \in \mathcal{A}(E)$, their difference $\nabla_1 - \nabla_2 \in \Omega^1(X, \mathrm{End}(E))$, and we can write $\mathcal{A}(E) = \nabla + \Omega^1(X, \mathrm{End}(E))$ for any connection ∇.*

Proof. (1) By Proposition 2.49, for the section $\sqrt{-1}\xi\wedge \in \mathscr{C}^\infty(X, TX \otimes \mathrm{Hom}(E, T^*X \otimes E))$ (to make it clear, we take a local frame $\{e_j\}$ of TX and the dual frame $\{e^j\}$ of T^*X, then we write $\sqrt{-1}\xi\wedge = \sqrt{-1}\sum_j e_j \otimes (e^j \otimes \mathrm{Id}_E)$), there always exists $\nabla \in \mathrm{Diff}^{\leq 1}(E, T^*X \otimes E)$ such that $\sigma(\nabla) = \sqrt{-1}\xi\wedge$.

(2) The difference $A := \nabla_1 - \nabla_2$ satisfies $A(f\,s) = f\,A(s)$ for all f, s, so by Proposition 2.27, $A \in \Omega^1(X, \mathrm{End}(E))$. Moreover, adding $A \in \Omega^1(X, \mathrm{End}(E))$ to any connection ∇ preserves the Leibniz rule. ∎

On a trivializing chart U of X for E where $E|_U \simeq U \times \mathbb{C}^r$:

$$\nabla^E|_U = d + \Gamma^E, \quad \Gamma^E \in \Omega^1(U, \mathrm{End}(\mathbb{C}^r)),$$

where d is the classical differential on functions. A local section $s \in \mathscr{C}^\infty(U, E)$ now becomes $(f_1, \dots, f_r)^T$ with each $f_j \in \mathscr{C}^\infty$, then

$$\nabla^E|_U \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix} = \begin{pmatrix} df_1 \\ \vdots \\ df_r \end{pmatrix} + \Gamma^E \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix}.$$

Remark 3.4. Given a connection ∇^E on E. For $V \in \Gamma(TX)$ a smooth vector field on X, and for $s \in \mathscr{C}^\infty(X, E)$, the covariant derivative of s along V is defined as $\nabla^E_V s := \iota_V \nabla^E s \in \mathscr{C}^\infty(X, E)$, where ι_V means that we pair V pointwisely with the (T^*X)-factor of $\nabla^E s$.

Definition 3.5 (Induced connections)**.** Given (E, ∇^E) and (F, ∇^F), we have induced connections on:

- **Dual Bundle** E^*: For $s^* \in \mathscr{C}^\infty(X, E^*)$, $s \in \mathscr{C}^\infty(X, E)$:

  $$\langle \nabla^{E^*} s^*, s \rangle = d\langle s^*, s \rangle - \langle s^*, \nabla^E s \rangle.$$

  Locally: $\nabla^{E^*}|_U = d - {}^T(\Gamma^E)$

- **Conjugate Bundle** $\overline{E}$: For $s \in \mathscr{C}^\infty(X, E)$,

  $$\nabla^{\overline{E}} \overline{s} = \overline{\nabla^E s},$$

  Locally: $\nabla^{\overline{E}}|_U = d + \overline{\Gamma}^E$.

- **Tensor Product** $E \otimes F$:

  $$\nabla^{E \otimes F}(s_1 \otimes s_2) = (\nabla^E s_1) \otimes s_2 + s_1 \otimes (\nabla^F s_2).$$

  Locally: $\nabla^{E \otimes F}|_U = d + \Gamma^E \otimes \mathrm{Id}_F + \mathrm{Id}_E \otimes \Gamma^F$.

- Similarly for $E^{\otimes k}$, $S^k E$, $\Lambda^k E$, and $\mathrm{Home}(E, F)$.

Proposition 3.6. *Given a vector bundle (E, ∇^E) on X and a smooth map $f : Y \to X$: for the **pullback bundle** f^*E, we have an induced connection ∇^{f^*E} which is defined locally by*

$$\nabla^{f^*E} = d + f^*\Gamma^E.$$

It is the unique connection on $f^*E \to Y$ such that for $s \in \mathscr{C}^\infty(X, E)$, $v \in T_yY$,

$$(\nabla^{f^*E}_v (s \circ f))(y) = (\nabla^E_{df_y(v)} s)(f(y)) \in E_{f(y)}.$$

Definition 3.7 (Adjoint connection)**.** For a Hermitian bundle (E, h^E), for a connection ∇^E, the adjoint connection $(\nabla^E)^*$ of ∇^E with respect to h^E is the unique connection satisfying

$$d\langle s_1, s_2\rangle_{h^E} = \langle(\nabla^E)^* s_1, s_2\rangle_{h^E} + \langle s_1, \nabla^E s_2\rangle_{h^E},\ \forall\, s_1, s_2 \in \mathscr{C}^\infty(X, E).$$

Definition 3.8 (Metric connection or Hermitian connection)**.** For a Hermitian bundle (E, h^E), ∇^E is **metric** (or **Hermitian**) if:

$$d\langle s_1, s_2\rangle_{h^E} = \langle\nabla^E s_1, s_2\rangle_{h^E} + \langle s_1, \nabla^E s_2\rangle_{h^E}.$$

Equivalently, $\nabla^E = (\nabla^E)^*$ (self-adjoint w.r.t. h^E). In this case, we also say ∇^E preserves the metric h^E.

Proposition 3.9. *Every Hermitian bundle admits a metric connection.*

Proof. Given any Hermitian vector bundle (E, h^E), let ∇^E be any connection on E. At first, we take the adjoint connection $(\nabla^E)^*$ of ∇^E with respect to h^E, then we put

$$\nabla^{\mathrm{u}} = \frac{1}{2}(\nabla^E + (\nabla^E)^*),$$

it is clear that $\nabla^{\mathrm{u}} \in \mathcal{A}(E)$ preserves the metric h^E. $\square$

3.1.2 Curvature

Any connection ∇^E extends uniquely to a first-order differential operator $\nabla^E : \Omega^k(X, E) \to \Omega^{k+1}(X, E)$ by the rule

$$\nabla^E(\alpha \otimes s) = d\alpha \otimes s + (-1)^k\, \alpha \wedge \nabla^E s,$$

for $\alpha \in \Omega^k(X), s \in \mathscr{C}^\infty(X, E)$.

Definition 3.10 (Curvature)**.** Given a connection ∇^E, we define its **curvature** as:

$$R^E := (\nabla^E)^2 : \mathscr{C}^\infty(X, E) \to \Omega^2(X, E).$$

Note that since ∇^E is first-order, a priori, $R^E \in \mathrm{Diff}^{\leq 2}(E, \Lambda^2(T^*X) \otimes E)$, a direct calculation shows that $\sigma_2(R^E) = -\xi \wedge \xi \equiv 0$, hence $R^E \in \mathrm{Diff}^{\leq 1}(E, \Lambda^2(T^*X) \otimes E)$.

Proposition 3.11. *R^E is a zeroth-order differential operator, that is $R^E \in \Omega^2(X, \mathrm{End}(E))$.*

Proof. For $f \in C^\infty(X)$, $s \in C^\infty(X, E)$:

$$R^E(fs) = \nabla^E(df \wedge s + f\nabla^E s) = ddf \wedge s - df \wedge \nabla^E s + df \wedge \nabla^E s + f(\nabla^E)^2 s = fR^E s$$

Thus $[R^E, f] = 0$, then we apply Proposition 2.27. $\square$

Proposition 3.12. *For any two vector fields $U, V \in \Gamma(TX)$, we have*

$$R^E(U, V) = \nabla^E_U \nabla^E_V - \nabla^E_V \nabla^E_U - \nabla^E_{[U,V]}. \tag{3.1}$$

Proof. Let $\{e_j\}_{j=1}^m$ be a local frame for TX with dual basis $\{e^j\}$ of T^*X. Then we can write

$$\nabla^E = \sum_j e^j \wedge \nabla^E_{e_j}.$$

Therefore

$$(\nabla^E)^2 = \sum_j de^j \wedge \nabla^E_{e_j} + \sum_{j,k} e^k \wedge e^j \nabla^E_{e_k}\nabla^E_{e_j}.$$

For vector fields U, V, we can write

$$U = \sum_j e^j(U)e_j,\ V = \sum_j e^j(V)e_j.$$

We also have

$$de^j(U,V) = Ue^j(V) - Ve^j(U) - e^j([U,V]).$$

Then

$$\begin{aligned}(\nabla^E)^2(U,V) =& \sum_j de^j(U,V)\nabla^E_{e_j} + \sum_{j,k}\left(e^k(U)e^j(V) - e^k(V)e^j(U)\right)\nabla^E_{e_k}\nabla^E_{e_j}\\ =& \sum_j \left(Ue^j(V)\nabla^E_{e_j} + \sum_{j,k} e^k(U)e^j(V)\nabla^E_{e_k}\nabla^E_{e_j}\right)\\ & - \sum_j \left(Ve^j(U)\nabla^E_{e_j} + \sum_{j,k} e^k(V)e^j(U)\nabla^E_{e_k}\nabla^E_{e_j}\right) - \nabla^E_{[U,V]}\\ =& \nabla^E_U\nabla^E_V - \nabla^E_V\nabla^E_U - \nabla^E_{[U,V]}.\end{aligned}$$

□

The following result is important and clear by the definition of R^E.

Proposition 3.13 (Bianchi identity)**.** *We have*

$$[\nabla^E, R^E] = 0$$

Proposition 3.14. *For metric connection ∇^E on a Hermitian vector bundle (E, h^E), the curvature is skew-adjoint, that is*

$$R^E \in \Omega^2(X, \mathrm{End}^{\mathrm{anti}}(E)),$$

where the fibers of $\mathrm{End}^{\mathrm{anti}}(E)$ are the anti-hermitian endomorphism of E with respect to h^E.

Proof. For $s_1, s_2 \in \mathscr{C}^\infty(X, E)$, then we have

$$\begin{aligned}0 = d^2\langle s_1, s_2\rangle_{h^E} &= d\left(d\langle s_1, s_2\rangle_{h^E}\right)\\ &= d\left(\langle \nabla^E s_1, s_2\rangle_{h^E} + \langle s_1, \nabla^E s_2\rangle_{h^E}\right)\\ &= d\left(\langle \nabla^E s_1, s_2\rangle_{h^E}\right) + d\left(\langle s_1, \nabla^E s_2\rangle_{h^E}\right)\\ &= \left(\langle R^E s_1, s_2\rangle_{h^E} - \langle \nabla^E s_1, \nabla^E s_2\rangle_{h^E}\right) + \left(\langle \nabla^E s_1, \nabla^E s_2\rangle_{h^E} + \langle s_1, R^E s_2\rangle_{h^E}\right)\\ &= \langle R^E s_1, s_2\rangle_{h^E} + \langle s_1, R^E s_2\rangle_{h^E}.\end{aligned}$$

□

3.1.3 Parallel transport

Definition 3.15. Given a smooth curve $\gamma : [0,1] \to X$ and vector bundle $(E, \nabla^E) \to X$, a section $s \in \mathscr{C}^\infty([0,1], \gamma^* E)$ is called **parallel** along γ with respect to ∇^E if:

$$\nabla^{\gamma^* E}_{\frac{\partial}{\partial t}} s \equiv 0,$$

which is also denoted simply by

$$\nabla^E_{\dot\gamma} s \equiv 0,$$

where $\dot\gamma$ is the speed vector of the curve.

Proposition 3.16 (Existence & uniqueness of parallel transport)**.** *For any $v \in E_{\gamma(0)}$, there exists a unique parallel section s with $s(0) = v$. This defines the* ***parallel transport****:*

$$\tau^0_t : E_{\gamma(0)} \to E_{\gamma(t)}, \quad v \mapsto s(t)$$

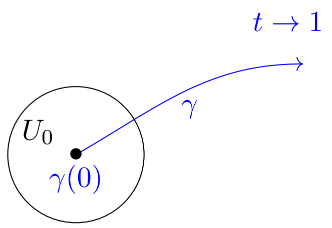
which is a linear isomorphism and satisfies $\tau^{t_1}_{t_2} \circ \tau^0_{t_1} = \tau^0_{t_2}$ for $t_1 \le t_2$.

Proof. Let U_0 be a local chart of X where $\gamma(0) \in U_0$ and $E|_{U_0} \simeq U_0 \times \mathbb{C}^r$, moreover, we write

$$\nabla^E|_{U_0} = d + \Gamma^E_0.$$

The the equation for a parallel section becomes a first-order ordinary differential equation:

$$\begin{cases} \frac{\partial s(t)}{\partial t} + \Gamma^E_{0,\gamma(t)}(\dot\gamma(t)) s(t) = 0; \\ s(0) = v \in E_{\gamma(0)} \simeq \mathbb{C}^r. \end{cases}$$

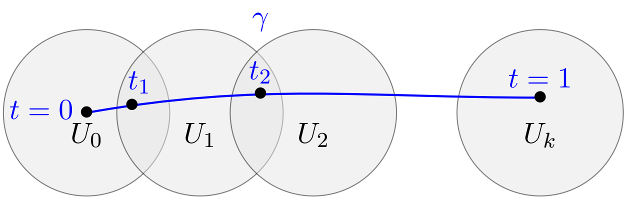


By the existence and uniqueness of the solution of the ODE, we get a parallel section s for $t \in [0, t_1]$ whenever $\gamma([0,t_1]) \subset U_0$.

Now, as showed in the picture below, we take a sequence of local charts U_j to cover γ, since $\gamma([0,1])$ is compact, we only need finite number of them, say $(k+1)$ local charts:

$$\gamma([0,1]) \subset \bigcup_{j=0}^{k} U_j$$

We then repeat the above arguments of ODEs on each chart consecutively, obtaining a sequence of parallel sections: $s|_{[0,t_1]}$, $s|_{[t_1,t_2]}$, ….



Finally, by considering the parallel sections on the intersections of these local charts, we conclude that they patch together smoothly as a parallel section s along whole γ, which is uniquely determined by $s(0) \in E_{\gamma(0)}$. The rest part of this proposition is clear. $\square$

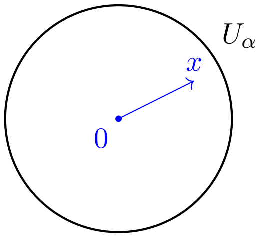
Note that we can use the parallel transport to get a **canonical local trivialization** of a vector bundle. Let

$$X = \bigcup_{\alpha} U_\alpha, \quad U_\alpha \simeq B(0,1) \subset \mathbb{R}^m,$$

where $B(0,1)$ denote the unit open ball. Given a vector bundle $\pi : E \to X$ with a connection ∇^E, we can define the following identification of vector bundles

$$\pi^{-1}(U_\alpha) \simeq B(0,1) \times E_0,$$
$$(x, \tau_x^0 v) \longleftrightarrow (x, v),$$

where $E_0 \simeq \mathbb{C}^r$ is the fiber of E at the center point of U_α, and $\tau_x^0 v$ denotes the parallel transport of v to the point x along the path $t \mapsto tx$ in $B(0,1)$ with respect to the connection ∇^E.



Example 3.17. Let $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, with coordinate t, and define the bundle $E = \mathbb{S}^1 \times \mathbb{C}$ to be the trivial bundle.

Let ∇^E be the connection defined by

$$\nabla^E = dt \wedge \frac{\partial}{\partial t} + \alpha(t)\, dt,$$

where $\alpha \in \mathscr{C}^\infty(\mathbb{S}^1, \mathbb{C})$ is a smooth function. A parallel section S along the path $\mathbb{R} \ni t \mapsto \gamma(t) = t \in S^1$ is a map

$$S : \mathbb{R} \to \mathbb{C}, \quad \text{such that} \quad \frac{\partial S(t)}{\partial t} + \alpha(t) S(t) = 0.$$

The solution is:

$$S(t) = S(0) \cdot \exp\left(-\int_0^t \alpha(s)\, ds\right).$$

Going from $t = 0$ to $t = 1$, the path is exactly one round of the circle - a primitive loop, we see that the **holonomy** for this loop is given by

$$\exp\left(-\int_0^1 \alpha(s)\, ds\right) \in \mathbb{C}^*.$$

If α is purely imaginary, then $\exp\left(-\int_0^1 \alpha(s)\, ds\right) \in \mathbb{U}(1)$. If $\int_0^1 \alpha(s)\, ds \in 2\pi\sqrt{-1}\mathbb{Z}$, then the holonomy is trivial.

3.2 First Chern form and first Chern class of complex line bundles

Let (L, ∇^L) be a complex line bundle on X with connection ($\operatorname{rank} L = 1$). The curvature is

$$R^L = (\nabla^L)^2 \in \Omega^2(X, \operatorname{End}(L)) \cong \Omega^2(X, \mathbb{C}),$$

where we have the canonical identification $\operatorname{End}(L) \xrightarrow[\mathrm{Tr}]{\sim} \mathbb{C}$.

Definition 3.18. The **first Chern form** of (L, ∇^L) is defined as

$$c_1(L, \nabla^L) := \frac{\sqrt{-1}}{2\pi} R^L \in \Omega^2(X, \mathbb{C})$$

Proposition 3.19. *We have the following properties:*
1) $c_1(L, \nabla^L)$ is closed, that is $dc_1(L, \nabla^L) = 0$.
2) For two connections $\nabla_1^L - \nabla_2^L = A \in \Omega^1(X, \mathbb{C})$:

$$c_1(L, \nabla_1^L) = c_1(L, \nabla_2^L) + \frac{\sqrt{-1}}{2\pi} dA$$

Thus the cohomology class $[c_1(L, \nabla^L)] \in H^2_{\mathrm{dR}}(X, \mathbb{C})$ is well-defined and independent of the connection ∇^L.

Proof. 1) Locally on a trivializing chart U, we have $\nabla^L = d + \Gamma$. Since L is a line bundle, so $\Gamma \in \Omega^1(U, \mathbb{C})$ and $\Gamma \wedge \Gamma = 0$. Then locally, $R^L = d\Gamma$, so $dR^L = dd\Gamma = 0$.

2) Locally, we write $\nabla_j^L = d + \Gamma_j$:

$$R_1^L - R_2^L = d\Gamma_1 - d\Gamma_2 = d(\Gamma_1 - \Gamma_2) = dA$$

$\square$

Remark 3.20. If we have two complex line bundles L_1 and L_2 on X that are isomorphic as vector bundles, then we can verify

$$[c_1(L_1, \nabla^{L_1})] = [c_1(L_2, \nabla^{L_2})] \in H^2_{\mathrm{dR}}(X, \mathbb{C}).$$

Definition 3.21. For any complex line bundle $L \to X$, the **first Chern class** $c_1(L) \in H^2_{\mathrm{dR}}(X, \mathbb{C})$ is defined as the cohomological class $[c_1(L, \nabla^L)]$ for any connection ∇^L. This way, we get a map

$$\begin{aligned} c_1 : \{\text{Complex line bundles on } X\}/\sim \quad &\to H^2_{\mathrm{dR}}(X, \mathbb{C}) \\ [L] \quad &\mapsto c_1(L)\,, \end{aligned}$$

where $\sim$ means the equivalence relation given by isomorphisms of complex line bundles.

Exercise 3.1 (First Chern class)**.** Let $L \to X$ be a complex line bundle on a smooth manifold:

- For any $k \in \mathbb{N}$, prove that $L^{\otimes k}$ is a complex line bundle on X.
- Prove the identity of first Chern class

$$c_1(L^{\otimes k}) = kc_1(L) \in H^2_{\mathrm{dR}}(X, \mathbb{C}).$$

- Show that $L^* \otimes L$ is a trivial line bundle on X, and first Chern class of a trivial line bundle is zero.

- Let L' be another complex line bundle on X, we have

$$c_1(L \otimes L') = c_1(L) + c_1(L') \in H^2_{\mathrm{dR}}(X, \mathbb{C}).$$

- Show that we always have the isomorphism $\overline{L} \cong L^*$, and

$$c_1(\overline{L}) = c_1(L^*) = -c_1(L) \in H^2_{\mathrm{dR}}(X, \mathbb{C}).$$

- For any smooth map $f : Y \to X$, we have

$$f^* c_1(L) = c_1(f^* L) \in H^2_{\mathrm{dR}}(Y, \mathbb{C}).$$

Proposition 3.22. *For a metric connection ∇^L on (L, h^L), we have*

$$c_1(L, \nabla^L) \in \Omega^2(X, \mathbb{R}),$$

therefore, we always have $c_1(L) \in H^2_{\mathrm{dR}}(X, \mathbb{R})$.

Proof. Locally on a trivializing chart, we have $\nabla^L = d + \sqrt{-1}\Gamma$, where $\Gamma \in \Omega^1(U, \mathbb{R})$ is real since ∇^L is a metric connection. Then locally

$$c_1(L, \nabla^L)|_U = -\frac{1}{2\pi} d\Gamma,$$

which is clearly a real-valued differential form. □

Remark 3.23. If ∇^L is any connection on L, we can write

$$c_1(L, \nabla^L) = c_1^{\mathrm{Re}}(L, \nabla^L) + \sqrt{-1} c_1^{\mathrm{Im}}(L, \nabla^L),$$

where $c_1^{\mathrm{Re}}(L, \nabla^L)$ and $c_1^{\mathrm{Im}}(L, \nabla^L)$ are real forms. Then by Proposition 3.22, we can conclude $c_1(L) = [c_1^{\mathrm{Re}}(L, \nabla^L)]$ and $c_1^{\mathrm{Im}}(L, \nabla^L)$ is an exact form.

Exercise 3.2 (Complex line bundles on Riemann sphere)**.** Let $\mathbb{CP}^1 \cong S^2$ be the Riemann sphere, or called 1-dimensional complex projective space, with two standard charts:

- The north pole chart $U_N \cong \mathbb{C}$ with coordinate $z = x + \sqrt{-1}y \in \mathbb{C}$
- The south pole chart $U_S \cong \mathbb{C}$ with coordinate $w = 1/z$

Let $\mathcal{O}(-1) \to \mathbb{CP}^1$ denote the tautological line bundle, i.e., $\mathcal{O}(-1) = \{([z], \lambda z) \in \mathbb{CP}^1 \times \mathbb{C}^2, \lambda \in \mathbb{C}\}$.

(a) Prove that $\mathcal{O}(-1) \to \mathbb{CP}^1$ is a well-defined complex line bundle.

(b) Prove that $\mathbb{CP}^1$ is orientable, and we can take the orientation on $\mathbb{CP}^1$ induced by $\mathbb{C}$ through the chart U_N, U_S.

(c) On U_N, we define a 1-form

$$A = \frac{\bar{z}\, dz}{1 + |z|^2},$$

where $dz = dx + \sqrt{-1}dy$. Define a Hermitian connection $\nabla = d + A$ on $\mathcal{O}(-1)|_{U_N}$ using the local frame $e_N(z) = (1, z)$ of $\mathcal{O}(-1)$. Show that ∇ can extend to a global connection ∇ on $\mathcal{O}(-1) \to \mathbb{CP}^1$.

(d) Compute on local charts U_N and U_S the curvature form $R = \nabla^2$, and then give a formula for the first Chern form of $c_1(\mathcal{O}(-1), \nabla)$.

(e) Prove that for any connection ∇ on $\mathcal{O}(-1)$, we have

$$\int_{\mathbb{CP}^1} c_1(\mathcal{O}(-1), \nabla) = -1.$$

Set the line bundle $\mathcal{O}(k) = \begin{cases} \mathcal{O}(-1)^{\otimes |k|} & \text{for } k \in \mathbb{Z} \text{ and } k < 0 \\ (\mathcal{O}(-1)^*)^{\otimes k} & \text{for } k \in \mathbb{Z} \text{ and } k \geq 0 \end{cases}$, show that for any $k \in \mathbb{Z}$,

$$\int_{\mathbb{CP}^1} c_1(\mathcal{O}(k), \nabla^k) = k.$$

To conclude this section, we give a different point-view of the first Chern classes for the complex line bundles. Recall that for a general topological space X, we have several cohomology theories:

- by considering the injective resolution of an abelian sheaf $\mathcal{F}$ on X, we can define the sheaf cohomology groups $H^\bullet(X, \mathcal{F})$.
- by considering the open covers of X together with a sheaf $\mathcal{F}$, we have the Čech cohomology groups $\check{H}^\bullet(X, \mathcal{F})$.

If X is a manifold (which is always Hausdorff and paracompact), then we always have

$$H^\bullet(X, \mathcal{F}) \simeq \check{H}^\bullet(X, \mathcal{F}).$$

Hence here it is not necessary to distinguish them.

Proposition 3.24. *A complex line bundle L over X is completely determined by its transition functions $\{g_{\alpha\beta}\}$ with:*

- $g_{\alpha\beta} \in \mathscr{C}^\infty(U_\alpha \cap U_\beta, \mathbb{C}^*)$
- *Cocycle condition:* $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$ *on* $U_\alpha \cap U_\beta \cap U_\gamma$

This data defines a cohomology class $[L] \in \check{H}^1(X, \mathscr{C}_X^{\infty,*}) = H^1(X, \mathscr{C}_X^{\infty,*})$ *where* $\mathscr{C}_X^{\infty,*}$ *is the sheaf of non-vanishing smooth complex-valued functions.*

Now we consider the **exponential sheaf sequence** on X,

$$0 \to \mathbb{Z} \to \mathscr{C}_X^\infty \xrightarrow{\exp(2\pi\sqrt{-1}\,\bullet\,)} \mathscr{C}_X^{\infty,*} \to 0$$

where:

- $\mathbb{Z}$ is the constant sheaf of integers on X;
- $\mathscr{C}_X^\infty$ is the sheaf of smooth $\mathbb{C}$-valued functions.

The above sheaf sequence is exact on X, therefore it induces a long exact sequence in sheaf cohomology:

$$\begin{tikzcd} \cdots \arrow[r] & H^1(X,\mathbb{Z}) \arrow[r] & H^1(X,\mathscr{C}_X^\infty) \arrow[r] & H^1(X,\mathscr{C}_X^{\infty,*}) \arrow[dll] \\ & H^2(X,\mathbb{Z}) \arrow[r] & H^2(X,\mathscr{C}_X^\infty) \arrow[r] & \cdots \end{tikzcd}$$

Since $\mathscr{C}_X^\infty$ is a soft sheaf, then $H^p(X, \mathscr{C}_X^\infty) = 0$ for $p \geq 1$. Therefore, the connecting map $\delta : H^1(X, \mathscr{C}_X^{\infty,*}) \xrightarrow{\sim} H^2(X, \mathbb{Z})$ is an isomorphism, so that we define a **topological version of first Chern class**

$$c_1^{\mathrm{topo}}(L) = \delta([L]) \in H^2(X, \mathbb{Z}).$$

For smooth manifolds, we have isomorphisms:

$$H^\bullet(X, \mathbb{Z}) \otimes \mathbb{C} \simeq H^\bullet(X, \mathbb{C}) \simeq H^\bullet_{\mathrm{dR}}(X, \mathbb{C})$$

Then we have the following correspondence, which says that the topological version of first Chern class agrees with the one defined by the first Chern forms:

$$\begin{tikzcd}
{[L] \in H^1(X, \mathscr{C}_X^{\infty,*})} \arrow[r, "\delta"] \arrow[dr] & c_1^{\mathrm{topo}}(L) \in H^2(X, \mathbb{Z}) \arrow[d, "\otimes\mathbb{C}"] \\
& {[c_1(L, \nabla^L)] \in H^2_{\mathrm{dR}}(X, \mathbb{C})}
\end{tikzcd},$$

where we can replace $\mathbb{C}$ by $\mathbb{R}$ for the de Rham cohomology.

Exercise 3.3 (Poincaré Lemma and injective resolution)**.** Let X be a smooth m-dimensional manifold. We study the relationship between closed differential forms and sheaf cohomology via the de Rham complex.

Prove that

(a) (Poincaré lemma for closed forms) Let $U \subseteq X$ be a contractible open set (e.g., diffeomorphic to $\mathbb{R}^m$). For any closed k-form $\omega \in \Omega^k(U)$ (i.e., $d\omega = 0$), show that there exists $\eta \in \Omega^{k-1}(U)$ such that $\omega = d\eta$.

(b) Find a closed 1-form ω on $X = \mathbb{R}^2 \setminus \{0\}$ that is not exact.

Assume X to be connected. Let $\underline{\mathbb{R}}$ denote the constant sheaf of $\mathbb{R}$ on X, that means, for each open subset $U \subset X$,

$$\underline{\mathbb{R}}(U) := \{\text{locally constant real functions on } U\}.$$

For $k \geq 0$, define the sheaf Ω^k as

$$\Omega^k(U) := \{\text{real-valued smooth } k \text{ forms on } U\}.$$

For each $x \in X$, let $\underline{\mathbb{R}}_x$, Ω^k_x denote the stalks at x, which are the germs of functions or forms.

Consider the **de Rham complex** as a resolution:

$$0 \to \underline{\mathbb{R}} \xrightarrow{\iota} \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^m \to 0,$$

where ι is given by the inclusion $\underline{\mathbb{R}}(U) \subset \Omega^0(U)$, and d is given by the exterior differential.

(c) (Exactness of sequence) For each $x \in X$, we have the sequence of spaces of germs:

$$0 \to \underline{\mathbb{R}}_x \xrightarrow{\iota} \Omega^0_x \xrightarrow{d} \Omega^1_x \xrightarrow{d} \cdots \xrightarrow{d} \Omega^m_x \to 0,$$

Verify exactness at each Ω^k_x for $k \geq 0$, and show ι is injective. This means that the **de Rham complex** gives an injective resolution for the constant sheaf $\underline{\mathbb{R}}$.

This way, we identify the sheaf cohomology of $\underline{\mathbb{R}}$ on X with the de Rham cohomology of X.

In subsequent sections, we will use the abbreviated notation:

$$H^k(X) := H^k_{\mathrm{dR}}(X,\mathbb{C}) \simeq H^k(X,\mathbb{C}) \simeq \check{H}^k(X,\mathbb{C}).$$

When working with smooth real-valued forms, we use $H^k(X,\mathbb{R})$. We are not going to distinguish different cohomology theories, since they are canonically isomorphic to each other.

3.3 Characteristic classes of vector bundles (topological version)

Based on the first Chern class class of complex line bundles, we will define several typical characteristic classes for vector bundles, more precisely, we are going to define

- for complex vector bundles: $\begin{cases} \text{Chern class} \\ \text{Chern character} \\ \text{Todd class} \end{cases}$
- for real vector bundles: $\begin{cases} \text{Pontrjagin class} \\ \widehat{A}\text{-class} \\ L\text{-class} \end{cases}$
- for oriented real vector bundles: Euler class.

Constructing the characteristic classes of vector bundles means to associate the cohomological classes in $H^\bullet(X)$ to those vector bundles that satisfy the functoriality property.

To get the constructions of these characteristic classes, we admit the **Splitting Principle** for the vector bundles.

Theorem 3.25 (Splitting principle). *For complex vector bundle $E \to X$ of rank r, there exists a manifold M and a smooth proper submersion $\pi : M \to X$ such that:*

$$\pi^* E \cong L_1 \oplus \cdots \oplus L_r$$

with complex line bundles $L_j \to M$, and the induced map $\pi^* : H^\bullet(X) \hookrightarrow H^\bullet(M)$ is injective. In this case, we will call M a ***split manifold*** *for E.*

Remark 3.26. The smooth map $\pi : M \to X$ is called a proper submersion if

- (submersion) for all $p \in M$, the tangent map $d\pi_p : T_pM \to T_{\pi(p)}X$ is surjective;
- (properness) for each $x \in X$, the preimage $\pi^{-1}(x)$ is compact subset of M.

Exercise 3.4 (Projectivization, universal line bundle, and splitting principle). Given a complex vector bundle $E \to X$ of rank $r \geq 2$ over a smooth manifold X, let $\mathbb{P}(E)$ denote its projectivisation and $\pi : \mathbb{P}(E) \to X$ the natural projection. Specifically, for each $x \in X$, $\pi^{-1}(x) = \mathbb{P}(E_x) \simeq \mathbb{CP}^{r-1}$ via $E_x \simeq \mathbb{C}^r$.

(a) Prove that $\mathbb{P}(E)$ is a smooth manifold, in particular, to describe the local charts and transition functions for $\mathbb{P}(E)$ based on the local charts of E and X.

(b) Prove that $\pi : \mathbb{P}(E) \to X$ is a smooth proper submersion.

(c) Show that the pull-back map: $\pi^* : \Omega^\bullet(X) \to \Omega^\bullet(\mathbb{P}(E))$ is injective.

(d) Define the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(-1)$ on $\mathbb{P}(E)$ whose fibre over $[v] \in \mathbb{P}(E_x)$ is the line $\mathbb{C}v \subset E_x$. Show that $\mathcal{O}_{\mathbb{P}(E)}(-1) \subset \pi^*E$ is a subbundle of rank one.

(e) Based on the above results, show that there exists a proper submersion $\pi : M \to X$ such that $\pi^*E \simeq L_1 \oplus \dots L_r$ with each L_j being a complex line bundle on M.

Theorem 3.27 (Chern class)**.** *There exists a unique map*

$$\begin{aligned} c : \{\textit{complex vector bundles on } X\} &\to H^{2\bullet} = \oplus_k H^{2k}(X) \\ E\ (\textit{up to isomorphisms}) &\mapsto c(E) \end{aligned}$$

for all smooth manifold X, such that

a) $c(E) = 1 + c_1(E) + \ldots + c_k(E) + \cdots$ *with* $c_k(E) \in H^{2k}(X)$.

b) (Whitney sum) $c(E \oplus F) = c(E)c(F)$ *(*$= c(E) \wedge c(F) = c(F) \wedge c(E)$ *since the degrees are even).*

c) (Functoriality) For any smooth map $f : Y \to X$, we have

$$c(f^*E) = f^*c(E) \in H^{2\bullet}(Y).$$

d) For a complex line bundle $L \to X$, $c(L) = 1 + c_1(L)$, where $c_1(L)$ is the first Chern class given by first Chern form $c_1(L, \nabla^L)$.

This map c is called Chern class, and $c_j(E)$ is called j-th Chern class of E.

Proof. **Uniqueness**:

Suppose such map c exists. For any $E \to X$, we can take a split manifold M of E such that for $\pi : M \to X$, we get $\pi^*E \simeq L_1 \oplus \ldots \oplus L_r$. Then by properties a) – d), we have

$$\pi^*c(E) := \prod_{j=1}^{r}(1 + c_1(L_j)) \in H^{2\bullet}(M). \tag{3.2}$$

This determines $c(E)$ uniquely since π^* is injective.

Existence: We sketch the idea. We can define $c(E)$ by considering a split manifold M for E and then take the definition (3.2). Consider the curvature

$$R^{\pi^*E}_{\text{split}} = \text{diag}(R^{L_1}, \ldots, R^{L_r}),$$

which corresponds to the direct sum of the connections on each L_j. So that we can rewrite (3.2) as

$$\pi^*c(E) := \det\left(\text{Id}_{\pi^*E} + \frac{\sqrt{-1}}{2\pi}R^{\pi^*E}_{\text{split}}\right) \in H^{2\bullet}(M), \tag{3.3}$$

where we can use the properties of the **elementary symmetric polynomials** to understand each $c_j(E)$.

If we have two split manifolds M_1 and M_2 for E (with the submersions π_1 and π_2 respectively), then we can define $M := M_1 \times_X M_2$, which is a again a split manifold of E, this way, we can identify $\pi_1^*c(E)$ and $\pi_2^*c(E)$ on M. So that (3.2) is independent of the choice of split manifolds.

Recall that $\pi^* : H^\bullet(X) \hookrightarrow H^\bullet(M)$ is injective, then to get a well-defined $c(E) \in H^\bullet(X)$ from (3.2), it is enough to verify that

$$\prod_{j=1}^{r}(1 + c_1(L_j)) \in \text{Im}\ \pi^*.$$

Let us explain it in a conceptual way. We will see from the Chern-Weil theory that this class is independent of the choices of the connections or curvatures on π^*E, therefore, if we take any connection ∇^E on E with curvature R^E, then

$$\pi^*c(E) = \det\left(\mathrm{Id}_{\pi^*E} + \frac{\sqrt{-1}}{2\pi}\pi^*R^E\right) = \pi^*\det\left(\mathrm{Id}_{\pi^*E} + \frac{\sqrt{-1}}{2\pi}R^E\right),$$

for the class $\det\left(\mathrm{Id}_{\pi^*E} + \frac{\sqrt{-1}}{2\pi}R^E\right) \in H^{2\bullet}(X)$. $\square$

Remark 3.28. The collection $\{c_1(L_j)\}_{j=1}^r$ is called the Chern roots of E. So each Chern class $c_j(E)$ is a symmetric polynomials in terms of these Chern roots.

(1). For complex vector bundles

Definition 3.29 (Multiplicative class)**.** For $f : \mathbb{R} \to \mathbb{R}$ an analytic function with $f(0) = 1$. The multiplicative class $f_{\mathrm{m}}(E) \in H^{2\bullet}(X)$ for the complex vector bundle $E \to X$ is defined by

$$\pi^* f_{\mathrm{m}}(E) = \prod_{j=1}^r f(c_1(L_j)) = \prod_{j=1}^r \left(\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(0)(c_1(L_j))^k\right),$$

for any split manifold $\pi : M \to X$, and due to the degree's reason, each sum in the above is a finite sum.

Example 3.30.

- $f(x) = 1 + x$, we obtain Chern class $c(E)$
- $f(x) = \frac{x}{1-e^{-x}}$, then it is called Todd class $\mathrm{Td}(E)$

Definition 3.31 (Additive class)**.** For $f : \mathbb{R} \to \mathbb{R}$ real analytic, the additive class $f_{\mathrm{a}}(E) \in H^{2\bullet}(X)$ is defined by

$$\pi^* f_{\mathrm{a}}(E) = \sum_{j=1}^r f(c_1(L_j)).$$

Example 3.32. If $f(x) = e^x$, $f_{\mathrm{a}}(E)$ is called Chern character of E.

(2). For real vector bundles

At first, we need the following real version of the splitting principle. For real bundle $F \to X$ of rank r, then there exists a smooth submersion $\pi : M \to X$ such that

$$\pi^*(F \otimes_{\mathbb{R}} \mathbb{C}) \cong \begin{cases} L_1 \oplus \overline{L}_1 \oplus \cdots \oplus L_k \oplus \overline{L}_k & r = 2k, \\ L_1 \oplus \overline{L}_1 \oplus \cdots \oplus L_k \oplus \overline{L}_k \oplus \underline{\mathbb{C}} & r = 2k+1, \end{cases} \tag{3.4}$$

Note that in the above splitting, we **can not** distinguish L_j with its conjugate $\overline{L}_j$.

Lemma 3.33. *For any complex line bundle $L \to X$, then*

$$c_1(\overline{L}) = -c_1(L) \in H^2(X).$$

Proof. See also Exercise 3.1. We give a local proof. Fix a Hermitian metric h^L and a metric connection ∇^L, in a local trivializing chart of L, we can write

$$\nabla = d + \sqrt{-1}\Gamma,$$

where Γ is a real-valued 1-form.

Then $\overline{\nabla} = d - \sqrt{-1}\Gamma$ defines a connection on $\overline{L}$. Locally, we have $c_1(\overline{L}) = -[\frac{1}{\pi}d\Gamma] = -c_1(L)$. $\square$

Definition 3.34 (Multiplicative class)**.** For $f : \mathbb{R} \to \mathbb{R}$ a real even analytic function ($f(x) = f(-x)$) with $f(0) = 1$. The multiplicative class $f_{\mathrm{m}}(F) \in H^{4\bullet}(X)$ for a real vector bundle $F \to X$ is defined by

$$\pi^* f_{\mathrm{m}}(F) = \prod_{j=1}^{r} f(c_1(L_j)) = \prod_{j=1}^{r} \left(\sum_{k=0}^{\infty} \frac{1}{(2k)!} f^{(2k)}(0)(c_1(L_j))^{2k} \right),$$

for any split manifold $\pi : M \to X$, and due to the degree's reason, each sum in the above is a finite sum.

Example 3.35.

- $f(x) = 1 + x^2$, we obtain Pontrjagin class $p(F)$.
- $f(x) = \frac{x/2}{\sinh x/2}$, then it is called $\widehat{A}$-class $\widehat{A}(F)$.
- $f(x) = \frac{x/2}{\tanh x/2}$, then it is called L-class $L(F)$.

(3). Euler Class for oriented real vector bundle

When F is an oriented real vector bundle of rank $r = 2k$, the given orientation of F allows us to distinguish L_j and $\overline{L}_j$ in (3.4). In fact, we have a split manifold $\pi : M \to X$ such that

$$\pi^* F \cong F_1 \oplus \ldots \oplus F_k$$

with F_j a real vector bundle on M of rank 2. Moreover, we have

$$F_j \otimes_{\mathbb{R}} \mathbb{C} \cong L_j \oplus \overline{L}_j.$$

Definition 3.36. For an oriented real bundle F of rank $2k$, the Euler class

$$e(F) = \prod_{j=1}^{k} c_1(L_j) \in H^{2k}(X, \mathbb{R}).$$

Exercise 3.5 (Complex structure on real vector space)**.** Denote $V = \mathbb{R}^{2n}$ a real vector space of real dimension $2n$. Let e_1, e_2, $\ldots$, e_{2n-1}, e_{2n} denote the canonical basis of V such that the vector $v = (x_1, x_2, \ldots, x_{2n-1}, x_{2n}) = \sum_{j=1}^{2n} x_j e_j$. Define an endomorphism J of V as follows, for $j = 1, 2, \ldots, n$,

$$\begin{aligned} Je_{2j-1} &= e_{2j}, \\ Je_{2j} &= -e_{2j-1}. \end{aligned}$$

Let $g^{T\mathbb{R}^{2n}}$ denote the standard Euclidean inner product on V, equivalently, we can write

$$g^{T\mathbb{R}^{2n}} = \sum_{j=1}^{2n} dx_j \otimes dx_j.$$

a) We have the following identity:

$$J^2 = -\mathrm{Id}_V, \;\; g^{T\mathbb{R}^{2n}}(J\cdot, J\cdot) = g^{T\mathbb{R}^{2n}}(\cdot, \cdot).$$

b) Consider the action of complex number $a + b\sqrt{-1} \in \mathbb{C}(\, a, b \in \mathbb{R}\,)$ on $v \in V$ via

$$(a + b\sqrt{-1})v := av + bJv \in V.$$

This way, we make (V, J) a complex vector space of dimension n with a $\mathbb{C}$-basis given by $\{e_1, e_3, \ldots, e_{2n-1}\}$.

For $j = 1, \ldots, n$, set $z_j = x_{2j-1} + \sqrt{-1}x_{2j} \in \mathbb{C}$, then $(z_1, \ldots, z_n) \in C^n$ denotes the standard complex coordinate system on (V, J). More precisely, we have the following identification

$$\mathbb{C}^n \ni (z_1, \ldots, z_n) \mapsto \sum_{j=1}^{n} z_j e_{2j-1} \in V.$$

c) Set $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ and $J_{\mathbb{C}} := J \otimes_{\mathbb{R}} \mathrm{Id}_{\mathbb{C}} \in \mathrm{End}(V_{\mathbb{C}})$. Here $\mathbb{C}$ acts on $V_{\mathbb{C}}$ via the second tensor factor $\mathbb{C}$. Then $J_{\mathbb{C}}$ has exactly two eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$. The corresponding eigenspaces are given as follows:

$$V^{1,0} := \mathrm{Span}_{\mathbb{C}}\{e_{2j-1} - \sqrt{-1}e_{2j}\,;\, j = 1, \ldots, n\},$$
$$V^{0,1} := \mathrm{Span}_{\mathbb{C}}\{e_{2j-1} + \sqrt{-1}e_{2j}\,;\, j = 1, \ldots, n\}.$$

In particular, we have $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$.

d) Using the complex coordinates $(z_1, \ldots, z_n)$ for (V, J), set

$$\omega = \frac{\sqrt{-1}}{2} \sum_{j=1}^{n} dz_j \wedge d\overline{z}_j.$$

Then ω is a $(1,1)$-form on V. Prove that $\omega = \overline{\omega}$ (that is ω is a real differential form), moreover, we have

$$\omega = \sum_{j=1}^{n} dx_{2j-1} \wedge dx_{2j} \in \Omega^2(V).$$

e) We have the following relation between $g^{T\mathbb{R}^{2n}}$ and ω: for $v, v' \in V$, we have

$$\omega(v, v') = g^{T\mathbb{R}^{2n}}(Jv, v').$$

In particular, for any $0 \neq v \in V$, $\omega(v, Jv) > 0$ (that is, ω is positive).

f) $g^{T\mathbb{R}^{2n}}$ extends $\mathbb{C}$-linearly on as an bilinear form on $V_{\mathbb{C}}$, for $W, W' \in V^{1,0}$, set $h^{V^{1,0}}(W, W') := g^{T\mathbb{R}^{2n}}(W, \overline{W'})$, then $h^{V^{1,0}}$ defines a hermitian metric on $V^{1,0}$, an orthonormal basis is given as follows:

$$\boldsymbol{f}_j := \frac{1}{\sqrt{2}}(e_{2j-1} - \sqrt{-1}e_{2j}), j = 1, \ldots, n.$$

A similar result holds for $V^{0,1}$.

3.4 Characteristic forms by Chern–Weil theory

At first, we introduce the supersymmetric convention for the Chern–Weil theory. In fact, it is not necessary to use the supersymmetric formulation from beginning, but it is more convenient when we deal with the superconnection, Dirac operators and index theorem in later sections.

3.4.1 Supersymmetric formulation

Definition 3.37. A **superspace** is a $\mathbb{Z}_2$-graded vector space $E = E^+ \oplus E^-$, where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{\pm\}$.

A **superalgebra** is an associative $\mathbb{Z}_2$-graded algebra with unit such that $\mathscr{A} = \mathscr{A}^+ \oplus \mathscr{A}^-$ as a superspace with $1 \in \mathscr{A}^+$ and

$$\mathscr{A}^+\mathscr{A}^-, \mathscr{A}^-\mathscr{A}^+ \subset \mathscr{A}^-, \quad \mathscr{A}^-\mathscr{A}^-, \mathscr{A}^+\mathscr{A}^+ \subset \mathscr{A}^+.$$

A usual vector space E is a superspace as $E = E \oplus 0$, that is $E = E^+$. A usual associative algebra with unit is a superalgebra as $\mathscr{A} = \mathscr{A}^+ \oplus 0$.

Example 3.38.

- A vector E has a $\mathbb{Z}$-grading if we can write it as a direct sum

$$E = \oplus_{j \in \mathbb{Z}} E^j,$$

then we have the induced superspace

$$E = E^+ \oplus E^-$$

with $E^+ = E^{\text{even}} = \oplus_{j\in\mathbb{Z}} E^{2j}$, $E^- = E^{\text{odd}} = \oplus_{j\in\mathbb{Z}} E^{2j-1}$.

- The differential forms $\Omega^\bullet(X) = \Omega^{\text{even}}(X) \oplus \Omega^{\text{odd}}(X)$ form a superalgebra with wedge product.
- Let $E = E^+ \oplus E^-$ be a superspace, and then $\mathrm{End}(E)$ is naturally a superalgebra with

$$\mathrm{End}^+(E) = \{f \in \mathrm{End}(E) \,:\, f \text{ preserves the splitting } E^+ \oplus E^-\} = \mathrm{End}(E^+) \oplus \mathrm{End}(E^-),$$

and

$$\mathrm{End}^-(E) = \mathrm{Hom}(E^+, E^-) \oplus \mathrm{Hom}(E^-, E^+).$$

In the block matrix form, we can write

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \in E^+ \oplus E^-.$$

Definition 3.39. Let $\mathscr{A} = \mathscr{A}^+ \oplus \mathscr{A}^-$ be a superalgebra. The supercommutator or super-bracket is defined as:

$$[a, b]_{\mathrm{s}} := ab - (-1)^{|a|\cdot|b|} ba,$$

where $|a|$ denotes the parity of a, that is 1 for $a \in \mathscr{A}^-$ and is 0 for $\mathscr{A}^+$. If $[a, b]_{\mathrm{s}} = 0$ for all $a, b \in \mathscr{A}$, then the superalgebra is said to be supercommutative, or simply, commutative. In the most case, we will omit the subscript s and denote it as $[a, b]$. Then we uniform the notation for the usual algebra and superalgebra.

Remark 3.40. (i) For $a, b, c \in A$, the super-Jacobi identity holds:

$$[a, [b, c]] = [[a, b], c] + (-1)^{|a|\cdot|b|}[b, [a, c]].$$

(ii) $\Omega^\bullet(X)$ is commutative.

Exercise 3.6. Prove the results in Remark 3.40.

Definition 3.41 (Supertrace)**.** Let $\mathscr{A} = \mathscr{A}^+ \oplus \mathscr{A}^-$ be a superalgebra. A $\mathbb{C}$-linear map $\alpha : \mathscr{A} \to \mathbb{C}$ is called **supertrace** if for all $a, b \in \mathscr{A}$:

$$\alpha([a, b]) = 0.$$

Proposition 3.42. *Let $E = E^+ \oplus E^-$ be a superspace (of finite dimension). The* ***super-trace*** $\mathrm{Tr_s} : \mathrm{End}(E) \to \mathbb{C}$ *is defined by:*

$$\mathrm{Tr_s}\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right] := \mathrm{Tr}^{E^+}[A] - \mathrm{Tr}^{E^-}[D].$$

Then we have

$$\mathrm{Tr_s}\,[[M,N]] = 0 \quad \forall M,N \in \mathrm{End}(E),$$

which makes Tr_s *a supertrace on the superalgebra* $\mathrm{End}(E)$.

Proof. Note that:

$$\mathrm{Tr_s}\left[\left[\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \begin{pmatrix} A' & 0 \\ 0 & D' \end{pmatrix}\right]\right] = \mathrm{Tr_s}\left[\begin{pmatrix} AA' - A'A & 0 \\ 0 & DD' - D'D \end{pmatrix}\right] = 0,$$
$$\mathrm{Tr_s}\left[\left[\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \begin{pmatrix} 0 & B' \\ C' & 0 \end{pmatrix}\right]\right] = \mathrm{Tr_s}\left[\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}\right] = 0.$$

The last case is that

$$\begin{aligned}\mathrm{Tr_s}\left[\left[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \begin{pmatrix} 0 & B' \\ C' & 0 \end{pmatrix}\right]\right] &= \mathrm{Tr_s}\left[\begin{pmatrix} BC' + B'C & 0 \\ 0 & CB' + C'B \end{pmatrix}\right] \\ &= \mathrm{Tr}^{E^+}[BC' + B'C] - \mathrm{Tr}^{E^-}[CB' + C'B] = 0.\end{aligned}$$

□

Remark 3.43. Take $\tau = \begin{pmatrix} \mathrm{Id}_{E^+} & 0 \\ 0 & -\mathrm{Id}_{E^-} \end{pmatrix}$, then for any $M \in \mathrm{End}(E)$, we have

$$\mathrm{Tr_s}[M] = \mathrm{Tr}^E[\tau M],$$

where Tr^E denote the usual trace on E without $\mathbb{Z}_2$-grading.

Definition 3.44. Let $\mathscr{A}$, $\mathscr{B}$ be two superalgebras, then the super tensor product $\mathscr{A}\widehat{\otimes}\mathscr{B}$ is a superalgebra defined as the space $\mathscr{A} \otimes \mathscr{B}$ with the $\mathbb{Z}_2$-grading

$$[\mathscr{A}\widehat{\otimes}\mathscr{B}]^+ = \mathscr{A}^+ \otimes \mathscr{B}^+ \oplus \mathscr{A}^- \otimes \mathscr{B}^-\,, \quad [\mathscr{A}\widehat{\otimes}\mathscr{B}]^- = \mathscr{A}^+ \otimes \mathscr{B}^- \oplus \mathscr{A}^- \otimes \mathscr{B}^+.$$

The product is given as follows: for $a, a' \in \mathscr{A}$, $b, b' \in \mathscr{B}$,

$$(a\widehat{\otimes}b)(a'\widehat{\otimes}b') = (-1)^{|a'|\cdot|b|}(aa')\widehat{\otimes}(bb').$$

We also write $a \otimes b$ if there is no confusion. If $\mathscr{A}^- = 0$ or $\mathscr{B}^- = 0$, then it is the same as the usual tensor product.

Definition 3.45. Let $\mathscr{A}$ be a superalgebra over $\mathbb{C}$, and let $E = E^+ \oplus E^-$ be a superspace, then we define a supertrace

$$\mathrm{Tr_s} : \mathscr{A}\widehat{\otimes}\,\mathrm{End}(E) \to \mathscr{A}$$

by

$$\mathrm{Tr_s}[a\otimes M] = a\,\mathrm{Tr_s}[M], \text{ for } a\in\mathscr{A},\ M\in\mathrm{End}(E).$$

Proposition 3.46. *Let $\mathscr{A}$ be a* ***commutative*** *superalgebra and E be a superspace. For $A, B \in \mathcal{A}\widehat{\otimes}\,\mathrm{End}(E)$,*

$$\mathrm{Tr_s}\left[[A,B]\right] = 0$$

Proof. Write $A = a\otimes M_1$, $B = b\otimes M_2$ where $a, b\in\mathcal{A}$ and $M_1, M_2\in\mathrm{End}(E)$. Then:

$$[A,B] = (-1)^{|b|\cdot|M_1|} ab\otimes[M_1, M_2],$$

since $\mathcal{A}$ is supercommutative.

Note that $\mathrm{Tr_s}$ vanishes on supercommutators in $\mathrm{End}(E)$, the result follows. □

3.4.2 Chern–Weil theory for characteristic forms

In this subsection, we consider a usual vector bundle $E = E\oplus 0$ on a smooth manifold X, we always use our supersymmetric convention on the differential forms on X. The analogous constructions and proofs presented in this subsection always hold for a superbundle $E = E^+\oplus E^-$ (that means, fiberwisely a superspace, see next section).

Let X be a manifold and $E\to X$ a vector bundle. For each $x\in X$, $\Lambda^\bullet T_x^*X$ be a commutative superalgebra. By taking the (usual) trace on $\mathrm{End}(E_x) = \mathrm{End}(E)_x$, we have

$$\mathrm{Tr} : \Lambda^\bullet T_x^*X\otimes\mathrm{End}(E_x)\to\Lambda^\bullet T_x^*X,$$

which vanishes on the commutators. Based on this pointwise version, we have the global versions: we have commutative superalgebra $\Omega^\bullet(X)$, a superspace $\Omega^\bullet(X,E) := \mathscr{C}^\infty(X,\Lambda^\bullet(T^*X)\otimes E)$, and an associated superalgebra $\Omega^\bullet(X,\mathrm{End}(E))$, where the $\mathbb{Z}_2$-gradings are induced from the one of $\Lambda^\bullet(T^*X)$.

We also have the trace map

$$\mathrm{Tr} : \Omega^\bullet(X,\mathrm{End}(E))\to\Omega^\bullet(X)$$

given by the pointwise trace map on X. Moreover,

- (connection) $\nabla^E : \Omega^\bullet(X,E)\to\Omega^{\bullet+1}(X,E)$ is an odd operator (which exchanges the $\pm$-components of a superspace);
- (curvature) $R^E = (\nabla^E)^2 = \frac{1}{2}[\nabla^E,\nabla^E]$ is an even operator (preserving the $\pm$-components of a superspace).

Proposition 3.47. *For any $A\in\Omega^\bullet(X,\mathrm{End}(E))$, we have*

$$d\,\mathrm{Tr}[A] = \mathrm{Tr}[[\nabla^E, A]]\in\Omega^\bullet(X).$$

Proof. Consider a local trivializing chart U of E, the connection $\nabla^E = d+\Gamma^E$ is an odd operator. Then on U,

$$[\nabla^E, A] = [d, A] + [\Gamma^E, A],$$

and thus:

$$\mathrm{Tr}[[\nabla^E, A]] = \mathrm{Tr}[[d,A]] + \mathrm{Tr}[[\Gamma^E, A]],$$

where we always have $\mathrm{Tr}[[\Gamma^E, A]] = 0$.

For $A \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, and on U, $\mathrm{End}(E)$ is identified with the square matrices, then $[d, A] = dA$, and:

$$\mathrm{Tr}[dA] = d(\mathrm{Tr}[A]).$$

If $A = \alpha \wedge B$ with $\alpha \in \Omega^\bullet(X), B \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, then:

$$[d, A] = dA = d\alpha \wedge B + (-1)^{|\alpha|}\alpha \wedge dB,$$

so

$$\mathrm{Tr}[[d, A]] = d\alpha \wedge \mathrm{Tr}[B] + f d\,\mathrm{Tr}[B] = d(\mathrm{Tr}[A]).$$

The proof is complete. □

Definition 3.48. Let $(E, \nabla^E) \to X$ be a complex vector bundle with connection. Let $f : \mathbb{R} \to \mathbb{R}$ be an analytic function. Then define:

$$f_{\mathrm{a}}(E, \nabla^E) := \mathrm{Tr}[f(\frac{\sqrt{-1}}{2\pi}R^E)] \in \Omega^\bullet(X),$$

where $f(\frac{\sqrt{-1}}{2\pi}R^E)$ is defined by Taylor series of f. Such forms are called additive characteristic forms associated to f and E, the word *additive* is used in the sense

$$f_{\mathrm{a}}(E \oplus F, \nabla^E \oplus \nabla^F) = f_{\mathrm{a}}(E, \nabla^E) + f_{\mathrm{a}}(F, \nabla^F).$$

Theorem 3.49 (Chern–Weil theory)**.** *Let f be as in the above definition. Then*

a) $f_{\mathrm{a}}(E, \nabla^E)$ is a closed form on X.

b) If ∇_0, ∇_1 are two connections on E, then:

$$f_{\mathrm{a}}(E, \nabla_0) - f_{\mathrm{a}}(E, \nabla_1) = d\eta$$

for some differential form $\eta \in \Omega^\bullet(X)$. That is, the difference is exact.

c) The cohomological class $[f_{\mathrm{a}}(E, \nabla^E)] \in H^\bullet(X)$ is independent of the choice of connection ∇^E.

d) We have $[f_{\mathrm{a}}(E, \nabla^E)] = f_{\mathrm{a}}(E)$ defined by the splitting principle (the topological version).

Proof. a). This is a consequence of the Bianchi identity and Proposition 3.47.

$$\begin{aligned} df_{\mathrm{a}}(E, \nabla^E) &= d\,\mathrm{Tr}[f(\frac{\sqrt{-1}}{2\pi}R^E)] \\ &= \mathrm{Tr}[[\nabla^E, f(\frac{\sqrt{-1}}{2\pi}R^E)]] = 0 \end{aligned}$$

b). Let $\nabla_t = (1-t)\nabla_0 + t\nabla_1$ be a smooth one-parameter family of connections for $t \in \mathbb{R}$. Denote the corresponding curvature forms by $R_t \in \Omega^2(X, , \mathrm{End}(E))$.

Now we set $X' = X \times \mathbb{R}$ and the projection $p_1 : X' \to X$. We consider the vector bundle p_1^*E on X', and set a connection on X'

$$\nabla^{p_1^*E} = dt \wedge \frac{\partial}{\partial t} + \nabla_t.$$

Then the curvature

$$R^{p_1^*E} \in \Omega^2(X, p_1^* \operatorname{End}(E)),$$

and we write

$$R^{p_1^*E} = dt \wedge \alpha_t + p_1^* R_t,$$

where $\alpha_t := \frac{\partial}{\partial t}\nabla_t \in \Omega^1(X, \operatorname{End}(E))$.

Take the differential form on X',

$$\operatorname{Tr}^{p_1^*E}[f(\frac{\sqrt{-1}}{2\pi}R^{p_1^*E})] = dt \wedge \beta_t + p_1^* \operatorname{Tr}[f(\frac{\sqrt{-1}}{2\pi}R_t)],$$

where $\beta_t \in \Omega^\bullet(X)$ that depends smoothly on $t \in \mathbb{R}$. By a), we know $\operatorname{Tr}^{p_1^*E}[f(\frac{\sqrt{-1}}{2\pi}R^{p_1^*E})]$ is a closed form on X'.

Using the fact $d^{X'} = dt \wedge \frac{\partial}{\partial t} + d^X$ Hence:

$$(dt \wedge \frac{\partial}{\partial t} + d^X)(dt \wedge \beta_t + p_1^* \operatorname{Tr}[f(\frac{\sqrt{-1}}{2\pi}R_t)]) = 0,$$

Then

$$\frac{\partial}{\partial t} \operatorname{Tr}[f(\frac{\sqrt{-1}}{2\pi}R_t)]) = d^X \beta_t.$$

As a consequence, we obtain

$$\operatorname{Tr}[f(\frac{\sqrt{-1}}{2\pi}R_1)]) - \operatorname{Tr}[f(\frac{\sqrt{-1}}{2\pi}R_0)]) = d^X \int_0^1 \beta_t \in \Omega^\bullet(X),$$

so we just take $\eta = \int_0^1 \beta_t$ to complete the proof of b).

c). Result c) is a consequence of a) and b).

d). Let $\pi : M \to X$ be a split manifold for E so that

$$\pi^* E \simeq L_1 \oplus \ldots \oplus L_r.$$

Then

$$\begin{aligned}
\pi^* f_{\mathrm{a}}(E, \nabla^E) &= f_{\mathrm{a}}(\pi^* E, \pi^* \nabla^E) \\
&= f_{\mathrm{a}}(L_1 \oplus \ldots \oplus L_r, \pi^* \nabla^E) \\
&\sim f_{\mathrm{a}}(L_1 \oplus \ldots \oplus L_r, \nabla^{L_1} \oplus \ldots \oplus \nabla^{L_r}) \\
&= \operatorname{Tr}\left[\begin{pmatrix} f(\frac{\sqrt{-1}}{2\pi}R^{L_1}) & 0 & \ldots & 0 \\ 0 & f(\frac{\sqrt{-1}}{2\pi}R^{L_2}) & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & f(\frac{\sqrt{-1}}{2\pi}R^{L_r}) \end{pmatrix}\right] \\
&= \sum_{j=1}^r f(c_1(L_j, \nabla^{L_j})) = \pi^* f_{\mathrm{a}}(E),
\end{aligned}$$

where $\sim$ means cohomologous, and the last equation is in the cohomology group $H^\bullet(X)$. This way, we obtain $\pi^*[f_{\mathrm{a}}(E, \nabla^E)] = \pi^* f_{\mathrm{a}}(E)$, and the result d) follows from that $\pi^* : H^\bullet(X) \to H^\bullet(M)$ is injective.

□

References

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