Michigan Math. J. **(Advance Publication)**
https://doi.org/10.1307/mmj/20226253

Large Deviations for Zeros of Holomorphic Sections on Punctured Riemann Surfaces

Alexander Drewitz,
Bingxiao Liu, & George Marinescu

Abstract. In this article we obtain large deviation estimates for zeros of random holomorphic sections on punctured Riemann surfaces. These estimates are then employed to yield estimates for the respective hole probabilities. A particular case of relevance that is covered by our setting is that of cusp forms on arithmetic surfaces. Most of the results we obtain also allow for reasonably general probability distributions on holomorphic sections, which underlines the universal character of these estimates. Finally, we also extend our results to the case of certain higher dimensional complete Hermitian manifolds, which are not necessarily assumed to be compact.

1. Introduction

1.1. Zeros of Random Holomorphic Sections

One particularly important aspect in the study of random functions or stochastic processes has been the investigation of their zero sets, see [1] and [5] and the references therein. We will impose some further assumptions here, focusing on the case of geometric generalizations of random polynomials, that is, random holomorphic sections.

To motivate and introduce our setting, we begin with recalling that for analytic functions $\sum_{n=0}^{\infty} a_n z^n$ whose coefficients a_n are assumed to be independent random variables, Offord proved in his fundamental article [20] the exponential decay of the tail probabilities of an analytic function having an excess or deficiency of zeros in a given region. More recently, Sodin [26] used Offord's method to improve Offord's exponential bound on the probability that a random analytic function has no zeros in a disk of radius r (hole probability) by showing that it decays at least at the rate $\mathcal{O}(e^{-Cr^2})$. This result has since been refined and extended in various ways in a series of papers [17; 21; 27; 28; 29; 30].

Shiffman, Zelditch, and Zrebiec [25] significantly enlarged the scope of these results by generalizing the situation described above to compact Kähler manifolds

Received June 28, 2022. Revision received January 19, 2023.

The authors are partially supported by the DFG Priority Program 2265 'Random Geometric Systems' (Project-ID 422743078). G. M. is partially supported by the DFG funded project SFB TRR 191 'Symplectic Structures in Geometry, Algebra and Dynamics' (Project-ID 281071066 – TRR 191) and the ANR-DFG project 'QuasiDy – Quantization, Singularities, and Holomorphic Dynamics' (Project-ID 490843120).

and zeros of holomorphic sections of powers of a positive line bundle. A principal interest in this setting is the study of the distribution of zeros as $p \to \infty$. In this situation, the power series representation of an analytic function is not canonical anymore, and, as a consequence, one has to replace the arguments based on the power series by more analytic and geometric methods that are appropriate for the study of holomorphic sections. In particular, these include tools such as Bergman kernels and coherent states asymptotics, which are by now deeply rooted in the study of the geometry of Kähler manifolds.

In this paper, we pursue the above results in two directions: From a geometric point of view, we are now concerned with the case of noncompact (complete) complex manifolds. On the probabilistic side, we allow for probability measures that are no longer Gaussian; instead, these probability measures will be assumed to fulfill some rather general conditions, which therefore entail a certain universality of the results we obtain. In Bayraktar, Coman, and Marinescu [9] (see also [8] for a survey) it is shown that the equidistribution of zeros takes place for a large class of probability measures satisfying a certain moment condition (e.g. measures with heavy tail probability and small ball probability, or measures with support contained in totally real subsets of the complex probability space). Analogous equidistribution results for non-Gaussian ensembles are proved in [6; 7; 10; 13]. In this paper we consider probability measures satisfying very mild conditions in terms of their densities (see Section 1.3).

We primarily focus on the case of a Riemann surface with cusps and prove large deviation estimates for zeros of random holomorphic sections of high powers L^p of a holomorphic line bundle L whose curvature equals the Poincaré metric near the cusps. A special case is that of cusp forms of high degree $2p$. For such a bundle L, Auvray, Ma, and Marinescu [3] (cf. also [4]) gave a very precise description of the Bergman kernel near the cusps; in particular, they provided an optimal uniform estimate of the supremum norm of the Bergman kernel, involving the fractional growth order $p^{3/2}$ in the tensor power (the growth order is p in the compact case). Using this estimate we obtain in Theorem 1.4 asymptotic bounds for the expectation and the tail probability of the maximum modulus of a random section on an open set. What is more, we also establish extensions to the case of higher dimensional Hermitian manifolds under suitable conditions.

We now introduce the setting for our bounds on the excess or deficiency probabilities of zeros. Indeed, for a compact Kähler manifold (X, ω) endowed with a Hermitian holomorphic line bundle (L, h) with positive curvature $\omega = c_1(L, h)$, Shiffman and Zelditch [23] showed that the normalized currents of integration $\frac{1}{p}[\mathrm{Div}(s_p)]$ over zero divisors of a random sequence of sections $s_p \in H^0(X, L^p)$ converge almost surely to $c_1(L, h)$ as $p \to \infty$. This result was generalized to the noncompact setting in [14] and to the setting of singular metrics whose curvature is a Kähler current in [9; 11; 12]. It holds also in our present setting and implies that the number of zeros (counted with multiplicity) of a random section s_p in an open set U with negligible boundary is asymptotically equal to p times the area of U in the metric given by $c_1(L, h)$. In Theorem 1.5 we prove this result in our setting and show that the probability that a section s_p has an excess or deficiency of zeros in U (when centered around its typical value) decreases at rate $\exp(-Cp^2)$ at least, this being consistent with the decay obtained in Sodin [26] cited above.

1.2. Geometric Setting: Punctured Riemann Surfaces

Let $\overline{\Sigma}$ be a compact Riemann surface and let $D = \{a_1, \ldots, a_N\} \subset \overline{\Sigma}$ be a finite set. The induced punctured Riemann surface will be denoted by $\Sigma = \overline{\Sigma} \backslash D$, and ω_Σ will be a Hermitian form on Σ. We furthermore let L be a holomorphic line bundle on $\overline{\Sigma}$ and denote by h a singular Hermitian metric on L satisfying the following properties:

(α) h is smooth over Σ, and for all $j \in \{1, \ldots, N\}$ there is a trivialization of L in the complex neighborhood $\overline{V}_j$ of a_j in $\overline{\Sigma}$ with associated coordinate z_j such that $|1|_h^2(z_j) = |\log(|z_j|^2)|$.

(β) There exists $\varepsilon_0 > 0$ such that the (smooth) curvature R^L of h satisfies $iR^L \geq \varepsilon_0 \omega_\Sigma$ over Σ; moreover, $iR^L = \omega_\Sigma$ on $V_j := \overline{V}_j \backslash \{a_j\}$; in particular, $\omega_\Sigma = \omega_{\mathbb{D}^*}$ in the local coordinate z_j on V_j and (Σ, ω_Σ) is complete.

Here, $\omega_{\mathbb{D}^*}$ denotes the Poincaré metric on the punctured unit disc $\mathbb{D}^*$, normalized as

$$\omega_{\mathbb{D}^*} = \frac{i\, dz \wedge d\overline{z}}{|z|^2 \log^2(|z|^2)}. \tag{1.1}$$

Since h is assumed to be a Hermitian metric, on the local chart V_j as in assumption (α), the coordinate z_j has norm strictly less than 1, so that the area (volume) of V_j with respect to measure ω_Σ is finite.

Let $J \in \mathrm{End}(T\Sigma)$ denote the complex structure of Σ and write $g^{T\Sigma} = \omega_\Sigma(\cdot, J\cdot)$ for the complete Riemannian metric on Σ so that the corresponding Riemannian volume element is exactly ω_Σ. For $x \in \Sigma$ and $v \in T_x\Sigma$, we denote by $\|v\|$ the norm of v with the metric $g_x^{T\Sigma}$. For $x, y \in \Sigma$, we write $\mathrm{dist}(x, y)$ for their Riemannian distance. Furthermore, for $x \in \Sigma$ we set

$$a(x) = iR_x^L / \omega_{\Sigma,x} \geq \varepsilon_0 > 0. \tag{1.2}$$

For $p \geq 1$, we denote by $h^p := h^{\otimes p}$ the metric induced by h on $L^p|_\Sigma$. We write $H^0(\Sigma, L^p)$ for the space of holomorphic sections of L^p on Σ and $\mathcal{L}^2(\Sigma, L^p)$ for the space of $\mathcal{L}^2$-sections of L^p on Σ. Set

$$\begin{aligned} H^0_{(2)}(\Sigma, L^p) &= H^0(\Sigma, L^p) \cap \mathcal{L}^2(\Sigma, L^p) \\ &= \left\{ s \in H^0(\Sigma, L^p) : \|s\|^2_{\mathcal{L}^2} := \int_\Sigma |s|^2_{h^p} \omega_\Sigma < \infty \right\}, \end{aligned} \tag{1.3}$$

which we tacitly assume to be endowed with the $\mathcal{L}^2$-metric. Then the sections in $H^0_{(2)}(\Sigma, L^p)$ extend to holomorphic sections of L^p over $\overline{\Sigma}$, that is,

$$H^0_{(2)}(\Sigma, L^p) \subset H^0(\overline{\Sigma}, L^p). \tag{1.4}$$

Moreover, for $p \geq 2$, elements in $H^0_{(2)}(\Sigma, L^p)$ are exactly the sections in $H^0(\overline{\Sigma}, L^p)$ vanishing on the puncture divisor D.

In the sequel, we write $c_1(L,h)$ for the first Chern form of (L,h), that is,

$$c_1(L,h) = \frac{i}{2\pi} R^L. \tag{1.5}$$

Hence, $c_1(L,h) \geq \frac{\varepsilon_0}{2\pi}\omega_\Sigma$ due to (α). We set

$$\begin{aligned} d_p &:= \dim H^0_{(2)}(\Sigma, L^p) = \dim H^0(\overline{\Sigma}, L^p \otimes \mathscr{O}_{\overline{\Sigma}}(-D)) \\ &= \deg(L) \quad p + 1 - g - N, \end{aligned} \tag{1.6}$$

where $\deg(L)$ is the degree of L over $\overline{\Sigma}$, and g is the genus of $\overline{\Sigma}$.

Furthermore, we denote the Schwartz kernel of the orthogonal projection from $\mathcal{L}^2(\Sigma, L^p)$ onto $H^0_{(2)}(\Sigma, L^p)$, called Bergman kernel, by $B_p(x,y)$ for $x, y \in \Sigma$. If S^p_j, $j = 1, \ldots, d_p$ is an orthonormal basis of $H^0_{(2)}(\Sigma, L^p)$ with respect to the $\mathcal{L}^2$ inner product, then

$$B_p(x,y) = \sum_{j=1}^{d_p} S^p_j(x) \otimes S^{p,*}_j(y) \in L^p_x \otimes L^{p,*}_y \quad \text{for } x, y \in \Sigma, \tag{1.7}$$

where the dual $S^{p,*}_j(y) = \langle \cdot, S^p_j(y)\rangle_{h^p} \in L^{p,*}_y$ is defined by h^p. In particular, $B_p(x,x)$ is a positive function in $x \in \Sigma$.

1.3. Probabilistic Setting

For each $p \in \mathbb{N}$, we will endow $H^0_{(2)}(\Sigma, L^p)$ with a probability measure Υ_p and hence obtain a sequence of probability spaces $(H^0_{(2)}(\Sigma, L^p), \Upsilon_p)_{p\in\mathbb{N}}$. In order to construct the sequence $\{\Upsilon_p\}_{p\in\mathbb{N}}$, we proceed as follows. For each $p \in \mathbb{N}$, we fix an orthonormal basis $O_p = \{S^p_j\}^{d_p}_{j=1}$ for $H^0_{(2)}(\Sigma, L^p)$ with respect to the respective $\mathcal{L}^2$-inner products. We assume given a family of independent $\mathbb{C}$-valued random variables $\{\eta^p_j\}_{p\in\mathbb{N}, 1\leq j\leq d_p}$ such that the following are satisfied:

- uniformly bounded densities: each η^p_j admits a probability density function (PDF) f^p_j on $\mathbb{C}$ with respect to the standard Lebesgue measure on $\mathbb{C} \simeq \mathbb{R}^2$, and there exists a constant $M_0 > 0$ such that for all p and all $1 \leq j \leq d_p$,
$$\sup_{z\in\mathbb{C}} f^p_j(z) \leq M_0; \tag{1.8}$$
- uniform lower bound for variances: for each p, the random variables η^p_j, $1 \leq j \leq d_p$ are centered (i.e., $\mathbb{E}[\eta^p_j] = 0$) and have the same variance $\sigma^2_p > 0$ ($\sigma_p > 0$). Moreover, there exists $c_0 > 0$ such that for all p,
$$c_0 \leq \sigma^2_p < \infty; \tag{1.9}$$
- moment bounds: there exists $C_0 > 0$ such that for all p, $1 \leq j \leq d_p$ we have
$$\mathbb{E}[|\eta^p_j|^{d_p}] \leq C_0 (d_p)^{d_p}. \tag{1.10}$$

REMARK 1.1. All of the above conditions are rather natural to avoid degeneracies. Indeed, condition (1.8) limits the concentration of η_j^p in small areas of $\mathbb{C}$, and condition (1.10) avoids an overly fast growth of moments. The conditions are relatively mild in that it is easily seen to be verified for a wide range of distributions including for example sub-Gaussian or exponential distributions.

For each $p \in \mathbb{N}$, the orthonormal basis O_p induces an identification $H^0_{(2)}(\Sigma, L^p) \simeq \mathbb{C}^{d_p}$, where the section $s_p = \sum_{j=1}^{d_p} z_j S_j^p$ maps to the vector $(z_1, \dots, z_{d_p}) \in \mathbb{C}^{d_p}$. Denoting by dVol_p the standard Lebesgue measure on $\mathbb{C}^{d_p} \simeq \mathbb{R}^{2d_p}$, this naturally induces a probability measure Υ_p on $H^0_{(2)}(\Sigma, L^p)$ via

$$\prod_{j=1}^{d_p} f_j^p(z_j)\, \mathrm{dVol}_p. \tag{1.11}$$

For later use, we will abbreviate the respective density as

$$f^p(z_1, \dots, z_{d_p}) = \prod_{j=1}^{d_p} f_j^p(z_j). \tag{1.12}$$

Hence, using the above identification, a random section in $(H^0_{(2)}(\Sigma, L^p), \Upsilon_p)$ with distribution Υ_p can be written as

$$s_p = \sum_{j=1}^{d_p} \eta_j^p S_j^p. \tag{1.13}$$

In general, Υ_p (and f^p) depends on both the choice of orthonormal basis O_p and the sequence $\{\eta_j^p\}_{j=1}^{d_p}$.

In the sequel, we fix—once and for all—a choice of the constants M_0, c_0, C_0. Moreover, most of the constants appearing in our computations throughout this paper depend on this choice, but we do not make this dependence explicit in our notation.

Some examples of families of probability measures satisfying the above assumptions are given in the following.

EXAMPLE 1.2 (Gaussian ensembles). A natural choice for Υ_p is taking the sequence $\{\eta_j^p\}_{p\in\mathbb{N}, 1\le j\le d_p}$ to be i.i.d. centered complex Gaussian random variables with positive variance. Then conditions (1.8)–(1.10) are clearly satisfied, and in this case Υ_p is independent of the choice of basis O_p.

EXAMPLE 1.3 (Random sections with bounded coefficients). Let r_p, $p \in \mathbb{N}$, be a sequence of positive numbers uniformly bounded below by $r > 0$. Let U_p denote a complex random variable that is uniformly distributed on the disk $D(0, r_p) \subset \mathbb{C}$. For each p, we take η_j^p, $1 \le j \le d_p$, to be a sequence of i.i.d. random variables with the same distribution as U_p. Then

$$\mathbb{E}[|U_p|^{d_p}] = \frac{2}{d_p+2} r_p^{d_p}, \tag{1.14}$$

and hence, to satisfy condition (1.10), we shall choose $r_p \leq d_p$ for all p.

1.4. Main Results for Punctured Riemann Surfaces

We start with investigating the supremum norm of random holomorphic sections on open sets. For this purpose, for U a nonempty open subset of Σ and $s_p \in H^0_{(2)}(\Sigma, L^p)$, we set

$$\mathcal{M}_p^U(s_p) = \sup_{x \in U} |s_p(x)|_{h^p} < +\infty. \tag{1.15}$$

For sections s_p of unit $\mathcal{L}^2$-norm, an optimal upper bound for $\mathcal{M}_p^U(s_p)$ is given by the square root of the supremum of Bergman kernel function $B_p(x,x)$ on U. Using the results of [3] mentioned previously, one can get an upper bound for $\mathcal{M}_p^U(s_p)$, which grows as $p^{1/2}$ if U is relatively compact in Σ, and as $p^{3/4}$ otherwise. Note that for the case of cusp forms on arithmetic surfaces (see Section 4), $\mathcal{M}_p^U(s_p)$ has its own interest, and such upper bounds are also obtained by other methods; we refer to [16; 22] for more details.

Our first main result concerns the expectation and concentration properties for the random variables $\mathcal{M}_p^U(s_p)$.

THEOREM 1.4. *Let Σ and (L,h) be a punctured Riemann surface and a line bundle satisfying conditions (α) and (β), respectively, and let Υ_p be the measures considered in Section 1.3. Let U be an open subset of Σ. Then there exists a constant $C_U > 0$ such that for all $p \in \mathbb{N}$ we have*

$$\frac{1}{C_U} p^{-2} \leq \mathbb{E}[\mathcal{M}_p^U(s_p)] \leq C_U p^{9/4}. \tag{1.16}$$

For any $\delta > 0$, there exists a constant $C_{U,\delta} > 0$ such that for any $p \in \mathbb{N}$ we have

$$\Upsilon_p(\{s_p : |\log \mathcal{M}_p^U(s_p)| \geq \delta p\}) \leq e^{-C_{U,\delta} p^2}. \tag{1.17}$$

For a holomorphic line bundle $E \to \Sigma$ and a holomorphic section $s \in H^0(\Sigma, E)$, which is not identically zero, we denote by $\mathrm{Div}(s) = \sum_{s(x)=0} m_x \cdot x$ the divisor of zeros of s, where the sum runs over the zeros $x \in \Sigma$ of s and $m_x = \mathrm{ord}_x(s)$ is the multiplicity of s at x. Note that the zero set of s in any compact subset of Σ is finite due to the identity theorem for holomorphic functions.

If $s \in H^0(\Sigma, E)$, then we define the measure of zeros of s by

$$[\mathrm{Div}(s)] = \sum_{x \in \Sigma, s(x)=0} m_x \delta_x. \tag{1.18}$$

In view of the higher dimensional case we note that $[\mathrm{Div}(s)]$ can be identified with a $(1,1)$-current on Σ. We say that a sequence $(\mu_p)_{p\in\mathbb{N}}$ of $(1,1)$-currents (or measures) on Σ converges weakly to a $(1,1)$-current μ on Σ if

$$\lim_{p\to\infty}(\mu_p,\phi)=(\mu,\phi)\quad \text{for all } \phi\in\mathcal{C}_0^\infty(\Sigma), \tag{1.19}$$

where $\mathcal{C}_0^\infty(\Sigma)$ denotes the space of smooth compactly supported functions on Σ.

Now we go back to our setting where $E=L^p$, $p=1,2,\ldots$, and $s_p\in H^0_{(2)}(\Sigma,L^p)$. If $U\subset\Sigma$ is an open set, then we write

$$\mathcal{N}_p^U(s_p)=\int_U[\mathrm{Div}(s_p)] \tag{1.20}$$

to denote the number of zeros (with multiplicities) of s_p in U and $\mathrm{Area}^L(U)$ to denote the area of U defined by the volume form iR^L. As a consequence of assumption (β), we have that $\mathrm{Area}^L(U)$ is finite.

Next we will apply the results in Theorem 1.4, using essentially the well-known Lelong–Poincaré formula (cf. (1.33)), to study the zeros of random holomorphic section s_p. In particular, we can infer an upper bound for hole probabilities. Using Borel–Cantelli type arguments, we then also obtain the almost sure convergence of zeros of sequences of holomorphic sections. For this purpose let us introduce the product probability space

$$(\mathcal{H},\Upsilon)=\prod_{p=1}^{\infty}(H^0_{(2)}(\Sigma,L^p),\Upsilon_p). \tag{1.21}$$

An element in $\mathcal{H}$ is a sequence $(s_p)_{p\in\mathbb{N}}$, $s_p\in H^0_{(2)}(\Sigma,L^p)$.

The results we obtain are stated in the following.

THEOREM 1.5. *Let Σ and (L,h) be a punctured Riemann surface and a line bundle satisfying conditions* (α) *and* (β), *respectively, and let Υ_p be the measures considered in Section* 1.3.

(a) Υ*-almost surely, we have the weak convergence of measures*

$$\lim_{p\to\infty}\frac{1}{p}[\mathrm{Div}(s_p)]=c_1(L,h)\quad \text{on } \Sigma. \tag{1.22}$$

(b) *If U is an open set of Σ with ∂U having zero measure with respect to some given smooth volume measure on $\overline{\Sigma}$, then for any $\delta>0$, there exists a constant $C_{\delta,U}>0$ such that for $p\gg 0$ the following holds*:

$$\Upsilon_p\left(\left\{s_p:\left|\frac{1}{p}\mathcal{N}_p^U(s_p)-\frac{\mathrm{Area}^L(U)}{2\pi}\right|>\delta\right\}\right)\le e^{-C_{\delta,U}p^2}. \tag{1.23}$$

We will give a quick proof of item (a) by using Theorem 1.10. It follows actually from [11, Theorem 5.1] that the convergence of currents in (a) takes place on $\overline{\Sigma}$. Our emphasis here is on item (b), which is a large deviation estimate in this context. As a consequence of (1.23), choosing $\delta=\mathrm{Area}^L(U)/2\pi$, we infer the following estimates of hole probabilities.

COROLLARY 1.6. *If U is a nonempty open set of Σ with ∂U having zero measure in Σ, then there exists a constant $C_U > 0$ such that for $p \gg 0$,*

$$\Upsilon_p(\{s_p : \mathcal{N}_p^U(s_p) = 0\}) \leq e^{-C_U p^2}. \tag{1.24}$$

Note that in the above statements, we can take U to be noncompact in Σ, that is, an open neighborhood of the punctured points. In particular, for the cusped hyperbolic surfaces investigated in Section 4, our results can be used to study the zeros of cusp forms near cusps.

Moreover, in the case of Gaussian ensembles, we also have a lower bound estimate for the hole probabilities of matching exponential order for Corollary 1.6.

PROPOSITION 1.7. *Suppose that $\{\Upsilon_p\}_{p\in\mathbb{N}}$ is defined as in Example 1.2 with $\sigma_p = 1$. If U is a relatively compact open subset of Σ such that ∂U has zero measure in Σ, and if there exists a section $\tau \in H^0_{(2)}(\Sigma, L)$ such that it does not vanish in $\overline{U} \subset \Sigma$, then there exists $C'_U > 0$ such that for $p \gg 0$,*

$$\Upsilon_p(\{s_p : \mathcal{N}_p^U(s_p) = 0\}) \geq e^{-C'_U p^2}. \tag{1.25}$$

Fix an integer $k_0 \geq 2$. For each $a_j \in D$, there exist $r_j \in]0, \frac{1}{2}[$ and $\tau_j \in H^0_{(2)}(\Sigma, L^{k_0})$ such that τ_j has no zeros in $\mathbb{D}^*_{2r_j} \subset V_j$ described in assumption (β). For $0 < r < r_j$, set $\mathbb{D}(r, r_j) = \{z \in \mathbb{C} : r < |z| < r_j\} \subset \mathbb{D}^*_{r_j} \subset V_j$. Then there exists $c_j > 0$ such that for $0 < r < r_j$, we have

$$\Upsilon_{pk_0}(\{s_{pk_0} : \mathcal{N}_{pk_0}^{\mathbb{D}(r,r_j)}(s_{pk_0}) = 0\}) \geq e^{-c_j|\log r|p^2} = r^{c_j p^2}, \quad \forall p \gg 0. \tag{1.26}$$

In the next subsection we provide some intermediate results, which are of independent interest and which will play an important role on our way to proving the results given previously.

1.5. Intermediate Results: An Approach to Theorem 1.5

The normalized Bergman kernel is defined as

$$P_p(x, y) = \frac{|B_p(x, y)|_{h_x^p \otimes h_y^{p,*}}}{\sqrt{B_p(x, x)}\sqrt{B_p(y, y)}}, \quad x, y \in \Sigma. \tag{1.27}$$

Near-diagonal estimates for $P_p(x, y)$ play a central role in our computations. In the case of compact Kähler manifolds, such results were established in [24, Propositions 2.6 and 2.7] and in [25, Proposition 2.1]. In our setting, we will take advantage of the Bergman kernel expansion for complete, possibly noncompact, Hermitian manifolds obtained by Ma and Marinescu in [18, Theorems 4.2.1 and 6.1.1].

THEOREM 1.8. *Let U be a relatively compact open subset of Σ, then we have the following uniform estimate on the normalized Bergman kernel. Fix $k \geq 1$ and $b > \sqrt{16k/\varepsilon_0}$. Then:*

(a) *There exists* $C > 0$ *such that for all* $p \in \mathbb{N}_{\geq 2}$ *and all* $x, y \in U$ *with* $\operatorname{dist}(x, y) \geq b\sqrt{\frac{\log p}{p}}$ *we have* $P_p(x, y) \leq Cp^{-k}$.

(b) *For* $p \geq 2$, *there exist functions*

$$G_p : \left\{(x, y) \in U \times U : \operatorname{dist}(x, y) \leq b\sqrt{\frac{\log p}{p}}\right\} \to \mathbb{R}$$

such that $\sup |G_p| \to 0$ *as* $p \to \infty$, *and such that*

$$P_p(x, y) = (1 + G_p(x, y)) \exp\left(-\frac{a(x)p}{4} \operatorname{dist}(x, y)^2\right), \tag{1.28}$$

where $a(x)$ *is defined by* (1.2).

Note that in the higher dimensional setting considered in Section 1.6, an analog of the above results still holds true (cf. Theorem 5.1). These estimates, together with a crucial inequality for the marginal densities of Υ_p proved in Proposition 3.7, are the key ingredients of our proof of (1.17) in Theorem 1.4. As a consequence, we obtain the following estimate for the logarithm of the modulus of holomorphic sections.

PROPOSITION 1.9. *Let* U *be a relatively compact open subset in* Σ. *For any* $\delta > 0$, *there exists* $C_{U,\delta} > 0$ *such that for all* $p \gg 0$,

$$\Upsilon_p\left(\left\{s_p : \int_U |\log |s_p|_{h^p}| \omega_\Sigma \geq \delta p\right\}\right) \leq e^{-C_{U,\delta}p^2}. \tag{1.29}$$

Note that estimate (1.29) is a version of [25, Lemma 1.6]. To prove it, we use here Theorem 1.4 (cf. Section 3.4) instead of [25, Theorem 3.1]. But since ω_Σ is singular near punctures, estimate (1.29) does not hold if we take $U = \Sigma$. Indeed, as we will see in Section 3.4, $|\log |s_p|_{h^p}|$ is not integrable with respect to ω_Σ near the punctures.

Using the Lelong–Poincaré formula, Proposition 1.9 leads to a large deviation estimate stated in the next result. Theorem 1.5 will be one of its consequences.

THEOREM 1.10. *If* $\phi \in \mathcal{C}^\infty(\overline{\Sigma})$ *is such that* ϕ *is locally constant in an open neighborhood of* D, *then for* $\delta > 0$ *there exists* $C_{\phi,\delta} > 0$ *such that for* $p \gg 0$ *we have*

$$\Upsilon_p\left(\left\{s_p : \left|\left(\frac{1}{p}[\operatorname{Div}(s_p)], \phi\right) - \int_\Sigma \phi c_1(L, h)\right| > \delta\right\}\right) \leq e^{-C_{\phi,\delta}p^2}, \tag{1.30}$$

where the sum in (1.30) *takes into account the multiplicities of the zeros.*

We would like to point out the difference between (1.30) here and the one proved in [25, Theorem 1.5]. Indeed, for $p \geq 2$, the section s_p always vanishes at the punctures as specified by D. Denoting by $\operatorname{ord}_{a_j}(s_p) \geq 1$ the vanishing order of s_p at $a_j \in D$, we infer that

$$\mathcal{N}_p^{\overline{V}_j}(s_p) = \mathcal{N}_p^{V_j}(s_p) + \operatorname{ord}_{a_j}(s_p), \tag{1.31}$$

where $\overline{V}_j$, V_j are open sets as in assumptions (α) and (β). In terms of divisors on $\overline{\Sigma}$, we can then rewrite (1.31) as

$$[\mathrm{Div}_{\overline{\Sigma}}(s_p)] = [\mathrm{Div}(s_p)] + \sum_j \mathrm{ord}_{a_j}(s_p)\delta_{a_j}, \tag{1.32}$$

where we view $[\mathrm{Div}(s_p)]$ as a divisor on $\overline{\Sigma}$.

Note that h is a singular Hermitian metric of L over $\overline{\Sigma}$, but for any smooth function ϕ on $\overline{\Sigma}$, the Lelong–Poincaré formula still holds true [18, Theorem 2.3.3], that is,

$$([\mathrm{Div}_{\overline{\Sigma}}(s_p)], \phi) = \frac{i}{\pi}(\partial\overline{\partial}\log|s_p|_{h^p}, \phi) + p(c_1(L,h), \phi). \tag{1.33}$$

Comparing (1.32) and (1.33) with the event in (1.30), we see that to obtain Theorem 1.10, it is sufficient to control the vanishing orders $\mathrm{ord}_{a_j}(s_p)$ in a uniform way for $p \gg 0$ and for arbitrary s_p, except for possibly subsets of small probability. Indeed, we have the following result.

LEMMA 1.11. *There exist $p_0 > 0$, $k_0 > 0$ such that for any $p \geq p_0$ the following inequalities hold Υ_p-almost surely*:

$$\mathrm{ord}_{a_j}(s_p) \leq k_0, \quad \forall a_j \in D. \tag{1.34}$$

This lemma will be restated as Lemma 3.9 in a more concrete way. Its proof, given in Section 3.5, relies on the positivity of L on $\overline{\Sigma}$.

1.6. Higher Dimensional Hermitian Manifolds

In Section 5, we provide extensions of our results (with suitable adaptations) to higher dimensional complex manifolds. Since our method relies on the Bergman kernel expansions, we work in the geometric settings considered in [14] and [18, Chapter 6].

Let (X, J, ω) be an m-dimensional complex Hermitian (not necessarily compact) manifold where J denotes the complex structure and ω is a positive $(1,1)$ form. To ω we associate a J-invariant Riemannian metric g^{TX} defined by $g^{TX}(u,u) = \omega(u, Jv)$ for all $u, v \in T_xX$ and $x \in X$. We assume that (X, g^{TX}) is complete. If $U \subset X$ is open, then let $\Omega_0^{p,q}(U)$ denote the set of smooth differential forms on U of bi-degree (p,q) that have compact support in U. In particular, $\mathcal{C}_0^\infty(U) = \Omega_0^{0,0}(U)$.

Let (L,h) be a holomorphic line bundle over X. We still denote the Chern curvature form of L by R^L, and let $R^{\det}$ be the curvature of the holomorphic connection $\nabla^{\det}$ on $K_X^* = \det(T^{(1,0)}X)$ with the Hermitian metric induced by g^{TX}. In addition we assume that there exist $\varepsilon_1 > 0$, $C_1 > 0$ such that

$$iR^L > \varepsilon_1\omega, \qquad iR^{\det} > -C_1\omega, \qquad |\partial\omega|_{g^{TX}} < C_1. \tag{1.35}$$

Some remarks:

(1) If (X, ω) is Kähler, then $\partial\omega = 0$ and the second condition in (1.35) is trivially satisfied. Moreover, in this case, $iR^{\det} = \mathrm{Ric}_\omega$, where Ric_ω is the Ricci curvature associated with g^{TX}.
(2) The assumptions in (1.35) imply the full asymptotics of the Bergman kernel on compact sets of X (cf. [18, Theorem 6.1.1]).

Let $\mathcal{C}_0^\infty(X, L^p)$ denote the space of compactly supported smooth sections on which we define a scalar inner product by

$$\langle s_1, s_2\rangle := \int_X \langle s_1(x), s_2(x)\rangle_{h_p} \,\mathrm{dV}(x), \tag{1.36}$$

where $h^p = (h^L)^{\otimes p}$ and $\mathrm{dV} = \frac{1}{m!}\omega^m$ is the volume form induced by ω. We also let $\mathcal{L}^2(X, L^p)$ be the Hilbert space obtained by completing $\mathcal{C}_0^\infty(X, L^p)$ with respect to the norm $\|\cdot\|_p$ induced by (1.36). Here, we consider Hilbert space of holomorphic sections

$$H^0_{(2)}(X, L^p) := \mathcal{L}^2(X, L^p) \cap H^0(X, L^p). \tag{1.37}$$

In addition, we assume that for $p \in \mathbb{N}$, $d_p = \dim_{\mathbb{C}} H^0_{(2)}(X, L^p)$ is finite, and that as $p \to \infty$,

$$d_p = \mathcal{O}(p^m). \tag{1.38}$$

This hypothesis is satisfied in several geometric situations. The punctured Riemann surface discussed in previous subsections is an example of complex dimension one. We will give other examples in Section 5.

For $s_p \in H^0_{(2)}(X, L^p)$, let Z_{s_p} denote the zero set of s_p, that is,

$$Z_{s_p} = \{x \in X : s_p(x) = 0\}. \tag{1.39}$$

For a nontrivial section s_p, the zero set Z_{s_p} is a complex $(m-1)$-dimensional hypersurface. We define the divisor of s_p by $\mathrm{Div}(s_p) = \sum_V \mathrm{ord}_V(s_p) \cdot V$, where the sum runs over all irreducible analytic hypersurfaces V of Z_{s_p} and $\mathrm{ord}_V(s_p) \in \mathbb{Z}$ is the order of s_p along V. For any hypersurface V, we denote by $[V]$ the current of integration on V and by $[\mathrm{Div}(s_p)] = \sum_V \mathrm{ord}_V(s_p)[V]$ the current of integration on $\mathrm{Div}(s_p)$. This is a $(1,1)$-current.

Consider the product probability space

$$(\mathcal{H}, \Upsilon) = \prod_{p=1}^{\infty} (H^0_{(2)}(X, L^p), \Upsilon_p). \tag{1.40}$$

When Υ_p, $p \in \mathbb{N}$, are defined from Gaussian ensembles (Example 1.2), Dinh, Marinescu, and Schmidt [14, Theorem 1.2] showed that the zero divisors of generic random sequences $(s_p)_p \in \prod_{p=1}^\infty H^0_{(2)}(X, L^p)$ are equidistributed with respect to $c_1(L, h^L)$. For proving this result, they actually gave a convergence speed for the divisors as follows.

THEOREM 1.12 ([14, Theorem 1.5]). *If U is a relatively compact open subset of X, then there exist a constant $c = c(U) > 0$ and a positive integer $p(U)$*

with the following property. For any positive number sequence $(\lambda_p)_{p\in\mathbb{N}}$ *with* $\lim_{p\to\infty}\lambda_p/\log p = \infty$, *and for any* $p \geq p(U)$ *and* $\phi \in \Omega_0^{m-1,m-1}(U)$, *we have*

$$\Upsilon_p\left(\left\{s_p : \left|\left(\frac{1}{p}[\mathrm{Div}(s_p)] - c_1(L,h), \phi\right)\right| > \frac{\lambda_p}{p}\|\phi\|_{\mathcal{C}^2}\right\}\right) \leq cp^{2m}e^{-\lambda_p/c}, \quad (1.41)$$

where $\|\cdot\|_{\mathcal{C}^2}$ *denotes the* $\mathcal{C}^2$*-norm of smooth sections.*

To get a probability bound of order $e^{-c'p^{m+1}}$ in (1.41) via setting $\lambda_p = p^{m+1}$, we obtain from (1.41) that

$$\Upsilon_p\left(\left\{s_p : \left|\left(\frac{1}{p}[\mathrm{Div}(s_p)] - c_1(L,h), \phi\right)\right| > p^m\|\phi\|_{\mathcal{C}^2}\right\}\right) \leq cp^{2m}e^{-p^{m+1}/c}. \quad (1.42)$$

This is clearly a weaker version of the estimate from Theorem 1.5.

Now, for each $p \in \mathbb{N}$, we let Υ_p be the probability measure on $H^0_{(2)}(X, L^p)$, constructed in Section 1.3 (not necessarily assumed to be Gaussian). Note that in condition (1.10), we have $d_p = \mathcal{O}(p^m)$. In this higher dimensional setting we prove the following results.

THEOREM 1.13. *Let* U *be a relatively compact open subset of* X. *Then there exists a constant* $C_U > 0$ *such that for any* $p \in \mathbb{N}$,

$$\frac{1}{C_U}p^{-m-1} \leq \mathbb{E}[\mathcal{M}_p^U(s_p)] \leq C_U p^{2m}. \quad (1.43)$$

For any $\delta > 0$, *there exists a constant* $C_{U,\delta} > 0$ *such that for any* $p \in \mathbb{N}$,

$$\Upsilon_p(\{s_p : |\log \mathcal{M}_p^U(s_p)| \geq \delta p\}) \leq e^{-C_{U,\delta}p^{m+1}}. \quad (1.44)$$

Then we can get the following improvement of (1.42).

THEOREM 1.14. *If* U *is a relatively compact open subset of* X, *then for any* $\delta > 0$ *and* $\phi \in \Omega_0^{m-1,m-1}(U)$, *there exists a constant* $c = c(U,\delta,\phi) > 0$ *such that for* $p \in \mathbb{N}$ *we have*

$$\Upsilon_p\left(\left\{s_p : \left|\left(\frac{1}{p}[\mathrm{Div}(s_p)] - c_1(L,h), \phi\right)\right| > \delta\right\}\right) \leq e^{-cp^{m+1}}. \quad (1.45)$$

Moreover, Υ*-almost surely we have the weak convergence of* $(1,1)$*-currents*

$$\lim_{p\to\infty}\frac{1}{p}[\mathrm{Div}(s_p)] = c_1(L,h). \quad (1.46)$$

Since $c_1(L,h)$ is positive, $c_1(L,h)^m/m!$ defines a positive volume element on X. If $U \subset X$ is open, then set

$$\mathrm{Vol}_{2m}^L(U) = \int_U \frac{c_1(L,h)^m}{m!}. \quad (1.47)$$

We will see in (5.6) that this volume is always finite.

For $s_p \in H^0_{(2)}(X, L^p)$, we define the $(2m-2)$-dimensional volume (with respect to $c_1(L,h)$) of Z_{s_p} in an open subset $U \subset X$ as follows:

$$\mathrm{Vol}^L_{2m-2}(Z_{s_p} \cap U) = \int_{Z_{s_p}\cap U} \frac{c_1(L,h)^{m-1}}{(m-1)!}. \tag{1.48}$$

As a consequence of Theorem 1.14, we have the following theorem.

THEOREM 1.15. *If U is a relatively compact open subset of X such that ∂U has zero measure in X, then for any $\delta > 0$, there exists a constant $c_{U,\delta} > 0$ such that for p large enough we have*

$$\Upsilon_p\left(\left\{s_p : \left|\frac{1}{p}\mathrm{Vol}^L_{2m-2}(Z_{s_p} \cap U) - m\mathrm{Vol}^L_{2m}(U)\right| > \delta\right\}\right) \leq e^{-c_{U,\delta}p^{m+1}}. \tag{1.49}$$

If U is a nonempty open (possibly not relatively compact) set of X with ∂U having zero measure in X, then there exists a constant $C_U > 0$ such that

$$\Upsilon_p(\{s_p : Z_{s_p} \cap U = \emptyset\}) \leq e^{-C_U p^{m+1}}, \quad \forall p \gg 0. \tag{1.50}$$

Note that when X is compact and $\omega = iR^L$, as well as for the special choice of Υ_p, $p \in \mathbb{N}$, as a Gaussian ensemble (cf. Example 1.2), the results in Theorems 1.13, 1.14, and 1.15 recover the main results proved in [25].

1.7. Organization of the Paper

This paper is organized as follows. In Section 2, we recall the estimates of the Bergman kernels for the punctured Riemann surface Σ. In Section 2.3, we give a proof of Theorem 1.8.

In Section 3, we give the proofs of other results stated in Sections 1.4 and 1.5. In particular, Sections 3.1–3.3 are devoted to the proof of Theorem 1.4. In Section 3.5, we prove at first Lemma 1.11 and then Theorem 1.10. In Section 3.6, we prove Theorem 1.5 using Theorem 1.10. At last, in Section 3.7, we prove Proposition 1.7.

In Section 4, we give a discussion for hyperbolic surfaces with cusps and of high genus. They are important examples of punctured Riemann surfaces where our results apply.

Finally, in Section 5, we study the higher dimensional complex Hermitian manifolds and give the proofs of the results stated in Section 1.6.

2. Estimates on Bergman Kernel

In this section, we recall some results on the Bergman kernel expansions for our punctured Riemann surface Σ obtained by Ma and Marinescu [18, Chapter 6] and by Auvray, Ma, and Marinescu [3]. Note that the results in [18, Chapter 6] are applicable to general Hermitian manifolds and line bundles such as the ones in Section 1.6; we refer to Section 5 for a more detailed discussion. In this section, we focus on Σ.

2.1. On-Diagonal Estimates

Recall that the positive smooth function a on Σ is defined as follows, for $x \in \Sigma$:

$$a(x) = \frac{i R^L_x}{\omega_{\Sigma,x}} \geq \varepsilon_0. \tag{2.1}$$

In our setting (with assumptions (α) and (β)), due to [18, Theorem 6.1.1] we have the following result.

THEOREM 2.1. *For any compact set $K \subset \Sigma$, we have the uniform asymptotic expansion for $x \in K$:*

$$B_p(x,x) = \frac{p}{2\pi} a(x) + \mathcal{O}_K(1), \quad \text{as} p \to +\infty. \tag{2.2}$$

In [3], an asymptotic expansion of B_p near the punctured points is obtained by studying the Bergman kernel expansion for the punctured disk endowed with the Poincaré metric. Furthermore, they obtained a global optimal upper bound for B_p. By [3, Corollary 1.4], we have

$$\sup_{x\in\Sigma} B_p(x,x) = \left(\frac{p}{2\pi}\right)^{3/2} + \mathcal{O}(p), \quad \text{as} p \to +\infty. \tag{2.3}$$

REMARK 2.2. The uniform upper bound of $B_p(x,x)$ given in (2.3) plays an important role in the proof of Theorem 1.4. In the absence of such a uniform upper bound on the noncompact manifold, we should assume U to be relatively compact in Theorem 1.4. As we will see in Section 3.7, the upper bound of $B_p(x,x)$ in (2.3) is also necessary in the proof of (1.26).

2.2. Off- and Near-Diagonal Estimates

For the off- and near-diagonal expansion of B_p, we still apply [18, Theorem 6.1.1] to our punctured Riemann surface.

PROPOSITION 2.3 ([18, Theorem 6.1.1]). *For any $\ell \in \mathbb{N}$ and $\delta > 0$, for any compact subset $K \subset \Sigma$, there exists $C_{\ell,\delta,K} > 0$ such that for all $p \in \mathbb{N}$ and $x, y \in K$ with* $\operatorname{dist}(x,y) \geq \delta$,

$$|B_p(x,y)| \leq C_{l,\delta,K}\, p^{-\ell}. \tag{2.4}$$

Fix any compact subset K, and for any $N \in \mathbb{N}$, there exist $\varepsilon > 0$, functions $\mathcal{F}_r$, and constants $C, C' > 0$ such that for $x_0 \in K$, $v, v' \in (T_{x_0}\Sigma, g^{T\Sigma}_{x_0})$, $\|v\|, \|v'\| \leq 2\varepsilon$, we have, as $p \to \infty$,

$$\begin{aligned}
&\left| \frac{1}{p} B_p(\exp_{x_0}(v), \exp_{x_0}(v')) - \sum_{r=0}^{N} \mathcal{F}_r\big(\sqrt{p}v, \sqrt{p}v'\big)\kappa^{-1/2}(v)\kappa^{-1/2}(v')p^{-r/2} \right| \\
&\quad \leq Cp^{-(N+1)/2}\big(1 + \sqrt{p}\|v\| + \sqrt{p}\|v'\|\big)^{2N+6} \exp\big(-C'\sqrt{p}\|v - v'\|\big) \\
&\qquad + \mathcal{O}(p^{-\infty}).
\end{aligned} \tag{2.5}$$

The norm on the left-hand side of (2.5) is taken at the point x_0 after trivializing the line bundle L near x_0 along the radial geodesic path centered at x_0 with respect to the Chern connection of (L, h). The function κ, and the functions $\mathcal{F}_r$, $r \in \mathbb{N}$, all depending smoothly on x_0, will be described more explicitly in what follows (cf. (2.6), (2.9)). The term $\mathcal{O}(p^{-\infty})$ is used to denote a decay faster than $p^{-\ell}$ for any $\ell \in \mathbb{N}$.

As in the previous proposition, for $x_0 \in K$ and $\varepsilon > 0$ sufficiently small, we can identify the Euclidean ball $B^{T_{x_0}\Sigma}(0, 4\varepsilon) \subset (T_{x_0}\Sigma, g^{T\Sigma}_{x_0})$ with the geodesic ball $B^{\Sigma}(x_0, 4\varepsilon) \subset \Sigma$ via the local geodesic coordinate centered at x_0. Let g^{Σ_0} be a metric on $\Sigma_0 := T_{x_0}\Sigma \simeq \mathbb{R}^2$ that coincides with $g^{T\Sigma}$ on $B^{T_{x_0}\Sigma}(0, 2\varepsilon)$ and $g^{T\Sigma}_{x_0}$ outside $B^{T_{x_0}\Sigma}(0, 4\varepsilon)$. Let dv_{Σ_0} be the Riemannian volume form of (Σ_0, g^{Σ_0}), and let $dv_{T_{x_0}\Sigma}$ denote the Riemannian volume form of $(T_{x_0}\Sigma, g^{T\Sigma}_{x_0})$. The function κ is a positive function on $T_{x_0}\Sigma$ such that for $v \in T_{x_0}\Sigma$,

$$dv_{\Sigma_0}(v) = \kappa(v)\, dv_{T_{x_0}\Sigma}(v). \tag{2.6}$$

In particular, $\kappa(0) = 1$. Moreover, when x_0 varies in the compact set K, for $v \in T_{x_0}\Sigma$ with $\|v\| \leq 2\varepsilon$, the function $\kappa(v)$ is uniformly bounded.

To describe the function $\mathcal{F}_r$, we need to explain the complex coordinate near x_0. Let $\mathbf{f}$ denote a unit vector of $T^{(1,0)}_{x_0}\Sigma$, that is, $g^{T\Sigma}_{x_0}(\mathbf{f}, \bar{\mathbf{f}}) = 1$. Set

$$\mathbf{e}_1 = \frac{1}{\sqrt{2}}(\mathbf{f} + \bar{\mathbf{f}}), \qquad \mathbf{e}_2 = \frac{i}{\sqrt{2}}(\mathbf{f} - \bar{\mathbf{f}}). \tag{2.7}$$

Then $\{\mathbf{e}_1, \mathbf{e}_2\}$ is an oriented orthonormal basis of the (real) tangent space $(T_{x_0}\Sigma, g^{T\Sigma}_{x_0})$. If $v = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 \in T_{x_0}\Sigma$, $v_1, v_2 \in \mathbb{R}$, then

$$v = (v_1 + iv_2)\frac{1}{\sqrt{2}}\mathbf{f} + (v_1 - iv_2)\frac{1}{\sqrt{2}}\bar{\mathbf{f}}, \tag{2.8}$$

and we associate it with a complex coordinate $z = v_1 + iv_2 \in \mathbb{C}$. In this coordinate, we have $\frac{\partial}{\partial z} = \frac{1}{\sqrt{2}}\mathbf{f}$ and $\|\frac{\partial}{\partial z}\| = |\frac{\partial}{\partial z}|_{g^{T\Sigma}} = \frac{1}{2}$. Note that, for $z \in \mathbb{C}$, $|z|$ still denotes the standard norm of z as complex number.

Now, for $v, v' \in T_{x_0}\Sigma$, let z, z' denote the corresponding complex coordinates. Set

$$\mathcal{F}_r(v, v') = \mathcal{P}(v, v')\mathcal{J}_r(v, v'), \tag{2.9}$$

where

$$\mathcal{P}(v, v') = \frac{a(x_0)}{2\pi}\exp\left(-\frac{1}{4}a(x_0)(|z|^2 + |z'|^2 - 2z\bar{z}')\right), \tag{2.10}$$

and

$$\begin{gathered}\mathcal{J}_r(v, v') \text{is a polynomial in } v, v' \text{ of degree at most } 3r, \\ \text{whose coefficients are smooth in } x_0 \in \Sigma.\end{gathered} \tag{2.11}$$

In particular,

$$\mathcal{J}_0 = 1. \tag{2.12}$$

The following lemma is elementary.

LEMMA 2.4. *The norm of $\mathcal{P}$ satisfies*

$$|\mathcal{P}(v, v')| = \frac{a(x_0)}{2\pi} \exp\Big(-\frac{1}{4}a(x_0)\|v - v'\|^2\Big). \tag{2.13}$$

Proof. This follows directly from (2.10) in combination with the formula

$$\begin{aligned} |z - z'|^2 &= |z|^2 + |z'|^2 - 2\Re(z\overline{z}') \\ &= |z|^2 + |z'|^2 - 2z\overline{z}' + 2i\Im(z\overline{z}'), \end{aligned} \tag{2.14}$$

where $\Re(\cdot)$, $\Im(\cdot)$ denote, respectively, the real and imaginary parts. By definition, we have $|z - z'| = \|v - v'\|$. □

2.3. *Normalized Bergman Kernel: Proof of Theorem 1.8*

We start by proving the first estimate of the theorem. Note that U is relatively compact in Σ, so $\overline{U}$ is compact and Proposition 2.3 is applicable. Let $\varepsilon > 0$ be the sufficiently small quantity stated in the second part of Proposition 2.3. Then, by the first part of the same proposition, if $x, y \in U$ is such that $\mathrm{dist}(x, y) \geq \varepsilon$, we have

$$|B_p(x, y)| \leq C_{k+1,\varepsilon,K}\, p^{-k-1}. \tag{2.15}$$

We fix large enough $p_0 \in \mathbb{N}$ such that

$$b\sqrt{\frac{\log p_0}{p_0}} \leq \frac{\varepsilon}{2}. \tag{2.16}$$

For $p > p_0$, if $x, y \in U$ is such that $b\sqrt{\frac{\log p}{p}} \leq \mathrm{dist}(x, y) < \varepsilon$, then we take advantage of the expansion in (2.5) with $N = 2k + 1$, $x_0 = x$, $v = 0$, $y = \exp_x(v')$, and $v' \in T_x\Sigma$, to obtain

$$\begin{aligned} &\left|\frac{1}{p}B_p(x, y) - \sum_{r=0}^{2k+1} \mathcal{F}_r\big(0, \sqrt{p}v'\big)\kappa^{-1/2}(v')p^{-r/2}\right| \\ &\quad \leq Cp^{-k-1}\big(1 + \sqrt{p}\|v'\|\big)^{4k+8} \exp\big(-C'\sqrt{p}\|v'\|\big) + \mathcal{O}(p^{-k-1}). \end{aligned} \tag{2.17}$$

Now, for $k \geq 1$, there exists a constant $C_k > 0$ such that for any $r > 0$,

$$(1 + r)^{4k+8} \exp(-C'r) \leq C_k. \tag{2.18}$$

Note that $\|v'\| = \mathrm{dist}(x, y)$. By (2.9), Lemma 2.4 and the fact that $\|v'\| \geq b\sqrt{\frac{\log p}{p}}$, we get

$$|\mathcal{F}_r\big(0, \sqrt{p}v'\big)| \leq Cp^{3r/2} \exp\Big(-\frac{\varepsilon_0}{4}b^2 \log p\Big), \tag{2.19}$$

where the constant $C > 0$ does not depend on $x \in U$, and the number ε_0 from assumption (β) can be taken smaller than 1.

Since we take $b > \sqrt{16k/\varepsilon_0}$, then for $r = 0, \dots, 2k + 1$, we get

$$|\mathcal{F}_r\big(0, \sqrt{p}v'\big)\kappa^{-1/2}(v')p^{-r/2}| \leq Cp^{-(2k-1)}. \tag{2.20}$$

Finally, combining (2.15)–(2.20), we get the desired estimate for any $p > 1$.

We next prove the second part of our theorem. For this purpose, we only need to consider sufficiently large p such that $b\sqrt{\frac{\log p}{p}} \leq \frac{\varepsilon}{2}$, where ε is given in Step 1.

In expansion (2.5), we take $x_0 = x$, $y = \exp_x(v')$, $N = 1$, so $\operatorname{dist}(x, y) = \|v'\| = |z'| \leq b\sqrt{\frac{\log p}{p}}$, where $z' \in \mathbb{C}$ is the complex coordinate for v'. We infer

$$\begin{aligned} B_p(x, y) = {} & p\kappa^{-1/2}(v')\frac{a(x)}{2\pi}\exp\Big(-\frac{1}{4}a(x)p\|v'\|^2\Big) \\ & + p^{1/2}\kappa^{-1/2}(v')\frac{a(x)}{2\pi}\exp\Big(-\frac{1}{4}a(x)p\|v'\|^2\Big)\mathcal{J}_1\big(0, \sqrt{p}v'\big) \\ & + \mathcal{O}(|\log p|^4). \end{aligned} \tag{2.21}$$

Since $\|v'\| \leq b\sqrt{\frac{\log p}{p}}$, using (2.11) we infer that $|\mathcal{J}_1(0, \sqrt{p}v')| \leq C|\log p|^{3/2}$. The previous in combination with (2.2) then supplies us with

$$\begin{aligned} \frac{\exp(\frac{1}{4}a(x)p\|v'\|^2)B_p(x, y)}{\sqrt{B_p(x, x)}\sqrt{B_p(y, y)}} &= \frac{pa(x)\kappa^{-1/2}(v')}{\sqrt{B_p(x, x)}\sqrt{B_p(\exp_x(v'), \exp_x(v'))}} \\ &\quad + \mathcal{O}(p^{-1/2}|\log p|^{3/2} + p^{-1}|\log p|^4) \\ &= 1 + \mathcal{O}(\|v'\| + p^{-1/2}|\log p|^{3/2} + p^{-1}|\log p|^4) \\ &= 1 + o(1), \quad \text{as } p \to +\infty. \end{aligned} \tag{2.22}$$

Note that in the definition of P_p we have the Hermitian norm of $B_p(x, y)$. Since in the asymptotic expansion (2.21) we have trivialized the line bundle near x using the Chern connections, we have

$$|B_p(x, y)|_{h_x^p \otimes h_y^{p,*}} = |B_p(x, y)|. \tag{2.23}$$

Combining (2.22) and (2.23), we get the estimate (1.28) by taking the term $G_p(x, y)$ to be the $o(1)$-term in the last equation in (2.22). This completes the proof of Theorem 1.8.

3. Proofs of the Results for Punctured Riemann Surfaces

In the sequel, we adopt the following notation and conventions: for positive functions $f, g : \mathbb{N} \to \mathbb{R}$, we write $f(p) \lesssim g(p)$ if there exists a constant $C > 0$ (possibly depending on some given data) such that $f(p) \leq Cg(p)$ for all (sufficiently large) $p \in \mathbb{N}$. Similarly, we write $f(p) \gtrsim g(p)$ if $f(p) \geq cg(p)$ for some constant $c > 0$ and all (sufficiently large) $p \in \mathbb{N}$. Moreover, we write $f(p) \simeq g(p)$ if both $f(p) \lesssim g(p)$ and $f(p) \gtrsim g(p)$ hold.

Since the computations of this section are also applicable in the higher dimensional case and for a relatively compact open subset U as described in Section 1.6 and Section 5, we will always emphasize the quantity d_p appearing in various estimates of this section. We use punctured Riemann surfaces as an important example when $d_p \simeq p$. Another advantage of this class is that due to [3] we can study divisors on open subsets that are not relatively compact.

3.1. Supremum Norm of Random Holomorphic Sections

As introduced above, s_p will denote a random section with probability measure Υ_p. Then $\mathcal{M}_p^U(s_p)$ is a positive random variable. In this subsection, we study the expectation $\mathbb{E}[\mathcal{M}_p^U(s_p)]$ to understand the *typical* value of $\mathcal{M}_p^U(s_p)$.

For a vector $\eta^p = (\eta_1^p, \dots, \eta_{d_p}^p) \in \mathbb{C}^{d_p}$, set $\|\eta^p\|^2 = \sum_{j=1}^{d_p} |\eta_j^p|^2$. For $x \in U$ and for $s_p = \sum_{j=1}^{d_p} \eta_j^p S_j^p$, we have

$$|s_p(x)|_{h^p} \leq \|\eta^p\| B_p(x,x)^{1/2}. \tag{3.1}$$

The bounds in (1.10) also give the bounds for σ_p^2, $p \in \mathbb{N}$.

LEMMA 3.1. *There exists a constant $K_0 > 0$ such that for $p \in \mathbb{N}$,*

$$\sigma_p^2 \leq K_0 d_p^2. \tag{3.2}$$

Proof. Since $\mathbb{E}[\eta_j^p] = 0$, then $\sigma_p^2 = \mathbb{E}[|\eta_j^p|^2]$. Let p be sufficiently large such that $d_p > 2$. By Jensen's inequality, we get

$$\mathbb{E}[|\eta_j^p|^2]^{d_p/2} \leq \mathbb{E}[|\eta_j^p|^{d_p}]. \tag{3.3}$$

Then (3.2) follows from assumption (1.10) for η_j^p. □

LEMMA 3.2. *We have the following inequalities of moments of $\|\eta^p\|$ for $p \in \mathbb{N}$ sufficiently large*:

$$\begin{aligned} c_0 d_p \leq \mathbb{E}[\|\eta^p\|^2] &\leq K_0 d_p^3, \\ \mathbb{E}[\|\eta^p\|^{d_p}] &\leq C_0 (d_p)^{2d_p+1}. \end{aligned} \tag{3.4}$$

Therefore, we have the lower bound estimate for all $p \in \mathbb{N}$ sufficiently large

$$\mathbb{E}[\|\eta^p\|] \geq C_0^{-\frac{1}{d_p-2}} (d_p)^{-\frac{2d_p+1}{d_p-2}} (c_0 d_p)^{\frac{d_p-1}{d_p-2}} \gtrsim (d_p)^{-1-\frac{4}{d_p-2}}. \tag{3.5}$$

Proof. Note that by the assumption (cf. (1.9)) for η_j^p, $1 \leq j \leq d_p$, we have

$$\mathbb{E}[\|\eta^p\|^2] = d_p \sigma_p^2. \tag{3.6}$$

Then the first inequality in (3.4) follows directly from (1.9) and (3.2), we now prove the second one.

For p sufficiently large, we have $d_p > 3$, so we can set

$$q = \frac{1}{1 - \frac{2}{d_p}} > 1. \tag{3.7}$$

Then

$$q \frac{d_p}{2} \leq d_p, \tag{3.8}$$

and by Hölder's inequality, we have

$$\|\eta^p\|^{d_p} \le \left(\sum_{j=1}^{d_p} |\eta_j^p|^{d_p}\right)(d_p)^{q\frac{d_p}{2}}. \tag{3.9}$$

Then the second inequality in (3.4) follows directly from (1.10), (3.8), and (3.9).

Set $a = \frac{d_p-2}{d_p-1}$, $p_1 = 1/a = \frac{d_p-1}{d_p-2}$, $p_2 = d_p - 1$, then by Hölder's inequality for the pair (p_1, p_2), we get

$$\mathbb{E}[\|\eta^p\|^2] = \mathbb{E}[\|\eta^p\|^a \|\eta^p\|^{2-a}] \le \mathbb{E}[\|\eta^p\|]^{1/p_1} \mathbb{E}[\|\eta^p\|^{d_p}]^{1/p_2}. \tag{3.10}$$

Using (3.4), inequality (3.5) follows. This completes our proof. □

By (2.3), there is a constant $p_0 \in \mathbb{N}$ (independent of open set U) such that for all $p > p_0$,

$$\sup_{x\in U} B_p(x,x) < \frac{1}{\pi\sqrt{2\pi}} p^{3/2}. \tag{3.11}$$

Then we have

$$\mathcal{M}_p^U(s_p) \le \|\eta^p\| p^{3/4}. \tag{3.12}$$

PROPOSITION 3.3. *We have the following inequalities for sufficiently large p:*

$$\begin{aligned} \mathbb{E}[\mathcal{M}_p^U(s_p)] &\le (K_0)^{1/2} p^{3/4} d_p^{3/2} \lesssim p^{9/4}, \\ \mathbb{E}[\mathcal{M}_p^U(s_p)^{d_p}] &\le C_0(d_p)^{2d_p+1} p^{3d_p/4} \\ &\lesssim (Cp)^{Cd_p} \quad \text{forsomeconstant} C > 0. \end{aligned} \tag{3.13}$$

Moreover,

$$\mathbb{E}[\mathcal{M}_p^U(s_p)] \gtrsim p^{-2}. \tag{3.14}$$

Proof. The inequalities in (3.13) follow directly from $d_p \simeq p$, (3.4), (3.12), and

$$\mathbb{E}[\|\eta^p\|]^2 \le \mathbb{E}[\|\eta^p\|^2]. \tag{3.15}$$

Now we prove the lower bound in (3.14). We fix a point $x_0 \in U$. Then

$$|s_p|_{h^p}^2(x_0) = \sum_{j,l=1}^{d_p} \eta_j^p \overline{\eta}_l^p h^p(S_j^p(x_0), S_l^p(x_0)), \tag{3.16}$$

and using (1.9) we infer that

$$\mathbb{E}[|s_p(x_0)|_{h^p}^2] = \sum_{j=1}^{d_p} \mathbb{E}[|\eta_j^p|^2] |S_j^p(x_0)|_{h^p}^2 = \sigma_p^2 B_p(x_0,x_0) \ge c_0 B_p(x_0,x_0). \tag{3.17}$$

The second inequality of (3.13) implies

$$\mathbb{E}[|s_p(x_0)|_{h^p}^{d_p}] \le C_0(d_p)^{2d_p+1} p^{3d_p/4}. \tag{3.18}$$

Also, by Hölder's inequality for $|s_p(x_0)|_{h^p}$ as in (3.10),

$$\mathbb{E}[|s_p(x_0)|_{h^p}^2] \le \mathbb{E}[|s_p(x_0)|_{h^p}]^{\frac{d_p-2}{d_p-1}} \mathbb{E}[|s_p(x_0)|_{h^p}^{d_p}]^{\frac{1}{d_p}}. \tag{3.19}$$

Combining this with (3.17), (3.18), we get

$$\mathbb{E}[|s_p(x_0)|_{h^p}] \geq (c_0 B_p(x_0,x_0))^{\frac{d_p-1}{d_p-2}} \left(\frac{1}{C_0 (d_p)^{2d_p+1} p^{3d_p/4}} \right)^{\frac{1}{d_p-2}} \gtrsim d_p^{-\frac{7/4+2/d_p}{1-2/d_p}}. \tag{3.20}$$

Since $d_p \simeq p$ and $\mathbb{E}[|s_p(x_0)|_{h^p}] \leq \mathbb{E}[\mathcal{M}_p^U(s_p)]$, we get (3.14). This finishes the proof. □

REMARK 3.4. Note that the lower bound p^{-2} in (3.14) is clearly non-optimal. Using the asymptotic expansion of $B_p(x,x)$ given in Theorem 2.1, and by (3.20), we get the following limit:

$$\liminf_{p\to\infty} p^{7/4} \mathbb{E}[\mathcal{M}_p^U(s_p)] \geq c_0 a(x_0) > 0, \tag{3.21}$$

where c_0 is the constant in (1.9), and $a(x)$ is defined in (2.1). Note that we always have $a(x_0) \geq \varepsilon_0 > 0$. Since $x_0 \in U$ is arbitrarily chosen, we get

$$\liminf_{p\to\infty} p^{7/4} \mathbb{E}[\mathcal{M}_p^U(s_p)] \geq c_0 \sup_{x\in U} a(x). \tag{3.22}$$

REMARK 3.5. If we take a relatively compact open subset U in Σ, then the estimates in Proposition 3.3 can be improved as follows: as p sufficiently large,

$$p^{-3/2-\delta} \lesssim \mathbb{E}[\mathcal{M}_p^U(s_p)] \lesssim p^2, \tag{3.23}$$

where $\delta > 0$ is any sufficiently small number.

Applying the Chebyshev inequality to the second inequality in (3.13), we get the following result.

COROLLARY 3.6. *There exists a constant $C > 0$ such that for any sequence $\{\lambda_p\}_{p\in\mathbb{N}}$ of strictly positive numbers, we have*

$$\Upsilon_p(\{s_p : \mathcal{M}_p^U(s_p) \geq \lambda_p\}) \lesssim e^{-d_p \log \lambda_p + C d_p \log p}. \tag{3.24}$$

3.2. *Uniform Bound on the Marginal Density Function*

In this subsection, we prove an important consequence of (1.8), that is, an upper bound on the marginal densities of Υ_p. We now fix $p \in \mathbb{N}$. Let $V \subset \mathbb{C}^{d_p}$ be a $\mathbb{C}$-subspace of dimension $n \leq d_p$, and let $V^\perp \subset \mathbb{C}^{d_p}$ denote its orthogonal subspace with respect to the standard Hermitian metric on $\mathbb{C}^{d_p}$.

If $v \in \mathbb{C}^{d_p}$, let $v = v_0 + v_1$, $v_0 \in V$, $v_1 \in V^\perp$ denote the orthogonal decomposition of v. Let dV_0, dV_1 denote the standard Lebesgue volume elements on V, $V^\perp$ respectively such that

$$\mathrm{dVol}_p(v) = \mathrm{dV}_0(v_0)\, \mathrm{dV}_1(v_1). \tag{3.25}$$

PROPOSITION 3.7. *For $v_0 \in V$, set*

$$g_V^p(v_0) = \int_{v_1\in V^\perp} f^p(v_0+v_1)\, \mathrm{dV}_1(v_1). \tag{3.26}$$

Then g_V^p is a probability density function on V such that

$$\sup_{v_0 \in V} g_V^p(v_0) \le M_0^n \binom{d_p}{n}, \tag{3.27}$$

where M_0 is the constant in (1.8), *and* $\binom{d_p}{n} = \frac{d_p!}{n!\,(d_p-n)!}$.

Proof. If $n = d_p$ or 0, then the proposition trivially holds true. Hence, without loss of generality we can and do assume $n < d_p$ from now on for the rest of the proof. Since p is fixed, we simply set $d = d_p$, $k = d - n > 0$, and we let $E_1, \dots, E_k$ be an orthonormal basis of $V^\perp$. Writing $e_1, \dots, e_d$ for the standard orthonormal basis of $\mathbb{C}^d$, this corresponds exactly to the sections S_j^p under the identification $H^0_{(2)}(\Sigma, L^p) \simeq \mathbb{C}^{d_p}$. Write for $i = 1, \dots, k$,

$$E_i = \sum_{j=1}^{d} a_i^j e_j, \quad a_i^j \in \mathbb{C}. \tag{3.28}$$

Let W_p denote the matrix (a_i^j) of size $k \times d$, and denote by W_p^* its complex adjoint matrix. The orthonormality of the basis implies

$$W_p W_p^* = \mathrm{Id}_{k\times k}. \tag{3.29}$$

Let $I(d,k)$ denote all the subsets of $\{1, \dots, d\}$ of cardinality k, then $|I(d,k)| = \binom{d}{n}$. If $S \in I(d,k)$, let $W_{p,S}$ denote the square matrix consisting of the k columns of W_p indexed by S (in the order induced by S), and let $W_p^{*,S}$ denote the square matrix consisting of k rows of W_p^* indexed by S (in the order induced by S). It is clear that $W_p^{*,S}$ is exactly the complex adjoint matrix of $W_{p,S}$. Then, due to the Cauchy–Binet formula (i.e., a generalized Pythagorean or Gougu theorem), we have

$$1 = \det W_p W_p^* = \sum_{S \in I(d,k)} \det W_{p,S} W_p^{*,S}. \tag{3.30}$$

Now observe that $\det W_{p,S} W_p^{*,S} \ge 0$, and hence due to (3.30) there exists $S_V \in I(d,k)$ such that

$$\det W_{p,S_V} W_p^{*,S_V} \ge \frac{1}{|I(d,k)|}. \tag{3.31}$$

In particular, W_{p,S_V} is an invertible square matrix.

We now prove (3.27). Let $t = (t_1, \dots, t_k)$ denote the complex coordinates of $V^\perp$ with respect to the basis E_i, $i = 1, \dots, k$. Then we can write

$$v_1 = \sum_{i=1}^{k} t_i E_i \in V^\perp, \qquad \mathrm{dV}_1(t) = \prod_{i=1}^{k} \frac{\sqrt{-1}}{2} dt_i \wedge d\bar{t}_i. \tag{3.32}$$

Let $s = (s_j)_{j \in S_V} \in \mathbb{C}^k$ be another complex coordinate system of $V^\perp$ such that $s_j = \sum_{i=1}^{k} t_i a_i^j$. Then W_{p,S_V} represents exactly the Jacobian matrix for the holomorphic coordinate change from t to s, so that the Jacobian determinant for the real coordinate change is given by $\det W_{p,S_V} W_p^{*,S_V}$. Then, for any integrable function F on $\mathbb{C}^k$, we have

$$\int_{t\in\mathbb{C}^k} F\left(\sum_{i=1}^k t_i a_i^j,\, j\in S_V\right) \mathrm{dV}_1(t) = \int_{s\in\mathbb{C}^k} F(s) \frac{1}{\det W_{p,S_V} W_p^{*,S_V}} \mathrm{dV}_1(s). \tag{3.33}$$

Writing $v_0 = (v_0^1, \dots, v_0^d) \in V \subset \mathbb{C}^d$, we infer that

$$\begin{aligned} f^p(v_0+v_1) &= \prod_{j\notin S_V} f_j^p\left(v_0^j + \sum_{i=1}^k t_i a_i^j\right) \cdot \prod_{j\in S_V} f_j^p\left(v_0^j + \sum_{i=1}^k t_i a_i^j\right) \\ &\le M_0^n \cdot \prod_{j\in S_V} f_j^p\left(v_0^j + \sum_{i=1}^k t_i a_i^j\right), \end{aligned} \tag{3.34}$$

where the last inequality follows from (1.8). Now, applying formula (3.33), we deduce

$$\begin{aligned} g_V^p(v_0) &\le \frac{M_0^n}{\det W_{p,S_V} W_p^{*,S_V}} \int_{s\in\mathbb{C}^k} \prod_{j\in S_V} f_j^p(v_0^j + s_j)\, \mathrm{dV}_1(s) \\ &= \frac{M_0^n}{\det W_{p,S_V} W_p^{*,S_V}}. \end{aligned} \tag{3.35}$$

Combining (3.31) with (3.35), we get (3.27). This completes our proof. □

3.3. *Expectation and Concentration: Proof of Theorem 1.4*

We prove here Theorem 1.4 about the expected value and concentration of the random supremum norm.

Note that inequality (1.16) follows from Proposition 3.3. By Remark 3.4, the lower bound in (1.16) can be improved to $\frac{1}{C_U} p^{-7/4-\epsilon}$ for any given $\epsilon > 0$. Moreover, as a consequence of Theorem 2.1, the constant $\frac{1}{C_U}$ in this lower bound can be made more explicit such as $c_0 a(x_0)$ with any point $x_0 \in U$ and for sufficiently large p.

Now we start to prove (1.17). Note that

$$\{s_p : |\log \mathcal{M}_p^U(s_p)| \ge \delta p\} \subset \{s_p : \mathcal{M}_p^U(s_p) \ge e^{\delta p}\} \cup \{s_p : \mathcal{M}_p^U(s_p) \le e^{-\delta p}\}. \tag{3.36}$$

Upon choosing $\lambda_p = e^{\delta p}$ in (3.24) this entails by $d_p \sim p$,

$$\Upsilon_p(\{s_p : \mathcal{M}_p^U(s_p) \ge e^{\delta p}\}) \le e^{-C_{U,\delta} p^2}, \quad \forall p \gg 0. \tag{3.37}$$

Now we consider the probability of $\{\mathcal{M}_p^U(s_p) \le \lambda_p\}$ for arbitrary sequences $\{\lambda_p\}_{p\in\mathbb{N}}$ of positive numbers less than 1. We claim that there exist constants $C>0$, $C'>0$ such that for $p \gg 0$,

$$\Upsilon_p(\{s_p : \mathcal{M}_p^U(s_p) \le \lambda_p\}) \le e^{C d_p \log \lambda_p + C' d_p \log p}. \tag{3.38}$$

If we take $\lambda_p = e^{-\delta p}$ in (3.38), then we get, with a constant $C_{U,\delta} > 0$,

$$\Upsilon_p(\{s_p : \mathcal{M}_p^U(s_p) \le e^{-\delta p}\}) \le e^{-C_{U,\delta} p^2}, \quad \forall p \gg 0; \tag{3.39}$$

then inequality (1.17) follows.

Therefore, in the sequel, we focus on proving (3.38), which is clearly a more general statement than that we actually need. For $U' \subset U$, a smaller open subset that is relatively compact in Σ, we have

$$\Upsilon_p(\{s_p : \mathcal{M}_p^U(s_p) \le \lambda_p\}) \le \Upsilon_p(\{s_p : \mathcal{M}_p^{U'}(s_p) \le \lambda_p\}). \tag{3.40}$$

Fix a point $x_0 \in U'$ and a 2-cube $[-t,t]^2$ in $\mathbb{R}^2 \simeq T_{x_0}\Sigma$. We choose $t > 0$ sufficiently small so that

$$F_t := \exp_{x_0}([-t,t]^2) \subset U',$$

and that

$$\frac{1}{2}\|v-u\| \le \operatorname{dist}(\exp_{x_0}(v), \exp_{x_0}(u)) \le 2\|v-u\| \quad \text{for all } v, u \in [-t,t]^2. \tag{3.41}$$

Instead of proving directly (3.38), it is enough to prove the following estimate:

$$\Upsilon_p(\{s_p : \mathcal{M}_p^{F_t}(s_p) \le \lambda_p\}) \le e^{C d_p \log \lambda_p + C' d_p \log p} \quad \text{for } p \gg 0. \tag{3.42}$$

The uniform estimates in Theorem 1.8 hold for the open set U'. Although our proof of (3.42) is inspired by the arguments in [25, Section 3.2], some new computational techniques such as Proposition 3.7 are needed since we are concerned with non-Gaussian ensembles of random variables.

Let $d > 0$ be a constant to be determined later. For each $p > 0$, we consider the lattice points

$$\Gamma_p := \left\{ (v_1, v_2) \in \mathbb{Z}^2 : |v_j| \le \frac{t\sqrt{p}}{d} \right\}, \tag{3.43}$$

and for $v \in \Gamma_p$ we define lattice points on the surface by

$$x_v^p = \exp_{x_0}\left(\frac{d}{\sqrt{p}} v\right) \in F_t. \tag{3.44}$$

The number of lattice points is given by

$$n_p := \sharp\Gamma_p = \left(2\left[\frac{t\sqrt{p}}{d}\right] + 1\right)^2 = \frac{4t^2}{d^2} p + \mathcal{O}(\sqrt{p}) \simeq d_p. \tag{3.45}$$

For $v \in \Gamma_p$, we fix some $\lambda_v \in L_{x_v^p}$ with $|\lambda_v|_h = 1$ and set

$$\xi_v = \frac{\langle \lambda_v^{\otimes p}, s_p(x_v^p)\rangle_{h^p}}{B_p(x_v^p, x_v^p)^{1/2}}. \tag{3.46}$$

Then ξ_v is a complex-valued random variable. By Theorem 2.1, for $x_v^p \in F_t$, we have the following uniform estimate for $p \ge 1$ and $v \in \Gamma_p$:

$$B_p(x_v^p, x_v^p) = p\frac{a(x_v^p)}{2\pi} + \mathcal{O}(1). \tag{3.47}$$

Then (3.42) follows from the claim

$$\Upsilon_p\Big(\Big\{\max_{v\in\Gamma_p}|\xi_v|\le\lambda_p\Big\}\Big)\le e^{Cd_p\log\lambda_p+C'd_p\log p},\quad \forall p\gg 0. \tag{3.48}$$

Note that $s_p=\sum_j\eta_j^pS_j^p$, so we have

$$\xi_v=\sum_j\eta_j^p\frac{\langle\lambda_v^{\otimes p},S_j^p(x_v^p)\rangle_{h^p}}{B_p(x_v^p,x_v^p)^{1/2}}. \tag{3.49}$$

Recall that η_j^p, $j=1,\dots,d_p$, are independently distributed random variables with expectation $\mathbb{E}[\eta_j^p]=0$ and uniformly bounded variance σ_p^2 as in (1.9), (3.2). Then

$$c_0\le\mathbb{E}[|\xi_v|^2]=\sigma_p^2\le K_0d_p^2. \tag{3.50}$$

Let $\Delta_{uv}:=\mathbb{E}[\xi_u\overline{\xi}_v]$ denote the covariance of ξ_u and ξ_v for $u,v\in\Gamma_p$, and let $\Delta=(\Delta_{uv})_{u,v\in\Gamma_p}$ denote the covariance matrix. Then, by (1.27), (3.49),

$$|\mathbb{E}[\xi_u\overline{\xi}_v]|=\sigma_p^2P_p(x_u^p,x_v^p). \tag{3.51}$$

For $b=\sqrt{32/\varepsilon_0+1}$, we get by Theorem 1.8 that for $p\gg 0$,

$$|\Delta_{uv}|\le\begin{cases}2\sigma_p^2\exp(-\frac{a(x_u^p)p}{4}\operatorname{dist}(x_u^p,x_v^p)^2) & \text{if }\operatorname{dist}(x_u^p,x_v^p)\le b\sqrt{\frac{\log p}{p}},\\ \sigma_p^2\mathcal{O}(p^{-2}) & \text{if }\operatorname{dist}(x_u^p,x_v^p)\ge b\sqrt{\frac{\log p}{p}},\end{cases} \tag{3.52}$$

where the constant defining $\mathcal{O}(\cdot)$ is independent of p.

Fix $u\in\Gamma_p$, then by (3.45) and the second estimate in (3.52), we have

$$\frac{1}{\sigma_p^2}\sum_{v\in\Gamma_p,v\ne u}|\Delta_{uv}|\le\sum_{\text{near}}+\mathcal{O}(p^{-1}), \tag{3.53}$$

where

$$\sum_{\text{near}}=\sum\Big\{\frac{1}{\sigma_p^2}|\Delta_{uv}|:0<\operatorname{dist}(x_u^p,x_v^p)\le b\sqrt{\frac{\log p}{p}}\Big\}. \tag{3.54}$$

Noting that

$$a(x_u^p)\operatorname{dist}(x_u^p,x_v^p)^2>\varepsilon_0\frac{d^2}{4p}\|u-v\|^2, \tag{3.55}$$

the first estimate in (3.52) supplies us with

$$\begin{aligned}\sum_{\text{near}}&\le 2\sum_{v\ne u}e^{-\frac{\varepsilon_0d^2}{16}\|u-v\|^2}\le 2\sum_{v\in\mathbb{Z}^2,v\ne 0}e^{-\frac{\varepsilon_0d^2}{16}\|v\|^2}\\ &\le b'\int_{x\in\mathbb{R}^2,\|x\|\ge 2/3}e^{-\frac{\varepsilon_0d^2}{64}\|x\|^2}\,\mathrm{d}x\le b''e^{-\frac{\varepsilon_0d^2}{144}}\le\frac{1}{3},\end{aligned} \tag{3.56}$$

where the constants b', $b''>0$ are independent of $p\gg 0$, and $d>0$ is chosen large enough so as to guarantee the last inequality.

We denote the ℓ^∞-norm of $\eta \in \mathbb{C}^n$ by $\|\eta\|_\infty = \max_v |\eta_v|$. Furthermore, we write I_n for the $n \times n$ identity matrix, as well as $\Delta = \sigma_p^2 I_n + A$, where A has zero diagonal entries. By (3.53) and (3.56), for $p \gg 0$,

$$\|A\eta\|_\infty \leq \frac{\sigma_p^2}{2}\|\eta\|_\infty, \quad \eta \in \mathbb{C}^n. \tag{3.57}$$

Then

$$\|\Delta\eta\|_\infty \geq \|\sigma_p^2\eta\|_\infty - \|A\eta\|_\infty \geq \frac{\sigma_p^2}{2}\|\eta\|_\infty \geq \frac{c_0}{2}\|\eta\|_\infty. \tag{3.58}$$

As a Hermitian square matrix, Δ is invertible and the eigenvalues of Δ^{-1} are bounded above by $2/\sigma_p^2$. Now set

$$\zeta = (\zeta_v) := \Delta^{-1/2}\xi \in \mathbb{C}^n, \tag{3.59}$$

so that the coordinates ζ_v are random variables centered with finite variance, but generally they are not independently distributed. Moreover, we have

$$\mathbb{E}[\zeta_u\overline{\zeta}_v] = \delta_{uv}, \quad \forall u, v \in \Gamma_p. \tag{3.60}$$

Next, note that each ζ_v is a linear combination of the (η_j^p), that is,

$$\zeta_v = \sum_j \eta_j^p \beta_j(v), \tag{3.61}$$

where $\beta_j(v) \in \mathbb{C}$ are constants. To apply directly Proposition 3.7, we normalize the random variables η_j^p, $j = 1, \ldots, d_p$, as follows:

$$\tilde{\eta}_j^p = \frac{1}{\sigma_p}\eta_j^p. \tag{3.62}$$

Then the PDF of $\tilde{\eta}_j^p$ on $\mathbb{C}$, with respect to the Lebesgue measure, is given by

$$\tilde{f}_j^p(z) = f_j^p(\sigma_p z)\sigma_p^2. \tag{3.63}$$

By (1.8) and (1.9), we have that for all p, $j \in \{1, \ldots, d_p\}$,

$$\sup_{z\in\mathbb{C}} |\tilde{f}_j^p(z)| \leq M_0\sigma_p^2 < \infty. \tag{3.64}$$

As in (1.12), we denote by $\tilde{f}^p$ the joint probability density function of the random vector $(\tilde{\eta}_j^p)_{j=1}^{d_p} \in \mathbb{C}^{d_p}$. For $v \in \Gamma_p$, set

$$E_v = (\sigma_p\overline{\beta}_1(v), \ldots, \sigma_p\overline{\beta}_{d_p}(v)) \in \mathbb{C}^{d_p}. \tag{3.65}$$

By (3.60), $\{E_v\}_{v\in\Gamma_p}$ forms an orthonormal set in $\mathbb{C}^{d_p}$, let $V_p \subset \mathbb{C}^{d_p}$ denote the $\mathbb{C}$-subspace spanned by $\{E_v\}_{v\in\Gamma_p}$. Then

$$n = \dim_{\mathbb{C}} V_p \leq d_p. \tag{3.66}$$

Let K_p denote the $d_p \times n$-matrix whose columns are the column vectors E_v, $v \in \Gamma_p$. Then

$$K_p^* K_p = I_n. \tag{3.67}$$

Set $Q_p = K_p K_p^*$, then Q_p is exactly the square matrix defining the orthogonal projection from $\mathbb{C}^{d_p}$ onto V_p in $\mathbb{C}^{d_p}$. Let $V_p^\perp = \mathrm{Im}(1 - Q_p)$ be the orthogonal complement of V_p. We also identify the vector $(\zeta_v)_{v\in\Gamma_p} \in \mathbb{C}^n$ with $\sum_{v\in\Gamma_p} \zeta_v E_v \in V_p$.

Considering $\zeta = (\zeta_v)_{v\in\Gamma_p}$ and $\tilde{\eta}^p = (\tilde{\eta}_j^p)_{j=1}^{d_p}$ as column vectors, (3.61) is equivalent to the relation

$$Q_p \tilde{\eta}^p = K_p \zeta. \tag{3.68}$$

As in (3.26), define for $\zeta \in \mathbb{C}^n$

$$g_{V_p}^p(\zeta) = \int_{\eta\in V_p^\perp} \tilde{f}^p(K_p\zeta + \eta)\,\mathrm{dV}_1(\eta). \tag{3.69}$$

Then $g_{V_p}^p$ is exactly the probability density function on $\mathbb{C}^n$ (with respect to the standard Lebesgue measure) for the random vector $(\zeta_v)_{v\in\Gamma_p}$ defined in (3.59). By Proposition 3.7 and (3.64), we get

$$\sup_{\zeta\in\mathbb{C}^n} g_{V_p}^p(\zeta) \le (M_0\sigma_p^2)^n \binom{d_p}{n}. \tag{3.70}$$

By (3.59), for $p \gg 0$,

$$\max_v |\zeta_v| \le \sqrt{\frac{2n}{\sigma_p^2}} \max_v |\xi_v|. \tag{3.71}$$

As a consequence of the above, for $p \gg 0$,

$$\begin{aligned}\Upsilon_p\Big(\Big\{\max_v |\xi_v| \le \lambda_p\Big\}\Big) &\le \Upsilon_p\Big(\Big\{\max_v |\zeta_v| \le \sqrt{\frac{2n}{\sigma_p^2}}\lambda_p\Big\}\Big)\\ &\le \int_{(\zeta_v)\in\mathbb{C}^n, |\zeta_v|\le\sqrt{\frac{2n}{\sigma_p^2}}\lambda_p} (M_0\sigma_p^2)^n \binom{d_p}{n}\,\mathrm{dVol}(\zeta)\\ &= (2\pi M_0 n)^n \lambda_p^{2n} \binom{d_p}{n}.\end{aligned} \tag{3.72}$$

Note that in the previous computations, $n \simeq d_p$. Using $\binom{d_p}{n} \le d_p!$, we get the desired inequality (3.48), hence (3.38) holds. This completes the proof.

REMARK 3.8. By examining the proofs to (3.24) and to (3.38), Theorem 1.4 still holds if condition (1.8) is replaced by a milder one: there exist $k_0 \in \mathbb{N}$, $M_0 > 0$ such that for $p \gg 0$,

$$\sup_{z\in\mathbb{C}} |f_j^p(z)| \le M_0 p^{k_0}. \tag{3.73}$$

3.4. Logarithm of the Modulus: Proof of Proposition 1.9

We start by showing the integrability of the function $|\log|s_p|_{h^p}|$ for nonzero $s_p \in H_{(2)}^0(\Sigma, L^p)$ and for $p \ge 2$ on each V_j (as in assumption (α)) with respect to ω^Σ.

We just consider an open subset $\mathbb{D}_r^* \subset V_j$ for some $r \in (0,1)$. Let 1 denote the canonical holomorphic frame of L over $\mathbb{D}_r^*$ so that

$$|1|_h^2(z) = |\log(|z|^2)|. \tag{3.74}$$

Then the section s_p, restricting on $\mathbb{D}_r^*$, can be written as

$$s_p(z) = z^k f(z) 1^{\otimes p}(z), \tag{3.75}$$

where $f(z)$ is a holomorphic function on $\overline{\mathbb{D}}$ with $f(0) \neq 0$, and $k \geq 1$ is the vanishing order of s_p at a_j. Then, for $z \in \mathbb{D}_r^*$, we have

$$\log(|s_p|_{h^p}^2) = 2k \log|z| + \log(|f|^2) + p \log|\log(|z|^2)|. \tag{3.76}$$

Note that

$$\int_0^r \log t \frac{2t\, dt}{t^2 \log^2 t} = \infty. \tag{3.77}$$

Comparing (1.1), the $\log|z|$-term in (3.76) with (3.77), we get that $\log(|s_p|_{h^p}^2)$ is not integrable with respect to the volume form $\omega_{\mathbb{D}^*}$ on $\mathbb{D}_r^*$.

As stated in Proposition 1.9, we only consider a relatively compact open subset U. The proof of Proposition 1.9 follows by combining the arguments in [25, Section 4.1] with Theorem 1.4. For $t > 0$, we introduce the following notation:

$$\log^+ t = \max\{\log t, 0\}, \log^- t := \log^+(1/t) = \max\{-\log t, 0\}. \tag{3.78}$$

Then

$$|\log t| = \log^+ t + \log^- t. \tag{3.79}$$

Let U be a relatively compact open subset in Σ, then, for nonzero s_p, $|\log|s_p|_{h^p}|$ is integrable on $\overline{U}$ with respect to ω_Σ. At first, we show the following claim:

$$\Upsilon_p\left(\left\{s_p : \int_U \log^+ |s_p|_{h^p} \omega_\Sigma \geq \frac{\delta}{2} p\right\}\right) \leq e^{-C_{U,\delta} p^2}. \tag{3.80}$$

Indeed, we have

$$\log^+ |s_p|_{h^p} \leq |\log \mathcal{M}_p^U(s_p)|. \tag{3.81}$$

Then

$$\begin{aligned} &\Upsilon_p\left(\left\{s_p : \int_U \log^+ |s_p|_{h^p} \omega_\Sigma \geq \frac{\delta}{2} p\right\}\right) \\ &\quad \leq \Upsilon_p\left(\left\{s_p : |\log \mathcal{M}_p^U(s_p)| \geq \frac{\delta}{2\mathrm{Area}(U)} p\right\}\right), \end{aligned} \tag{3.82}$$

where $\mathrm{Area}(U)$ denotes the area of U with respect to ω_Σ. Then (3.80) follows from Theorem 1.4. The next step is to prove that

$$\Upsilon_p\left(\left\{s_p : \int_U \log^- |s_p|_{h^p} \omega_\Sigma \geq \frac{\delta}{2} p\right\}\right) \leq e^{-C_{U,\delta} p^2}. \tag{3.83}$$

Suppose that U contains an annulus $B(2,3) := \{z \in \mathbb{C} : 2 < |z| < 3\}$ (after rescaling the coordinates), and the line bundle L on $B(1,4)$ (still contained in U) has a holomorphic local frame e_L. Set $\alpha(z) = \log|e_L(z)|_h^2$. For $s_p \in H^0_{(2)}(\Sigma, L^p)$, we can write

$$s_p = f_p e_L^{\otimes p}, \tag{3.84}$$

where f_p is a holomorphic function on $B(1,4)$. Then

$$\log|s_p|_{h^p} = \log|f_p| + \frac{p}{2}\alpha. \tag{3.85}$$

In the following estimates, each $K_\bullet$ denotes a sufficiently large positive constant. Then, by (3.79) and (3.82), we have

$$\Upsilon_p\left(\left\{s_p : \int_{B(2,3)} \log^+|f_p|\omega_\Sigma \geq K_1 p\right\}\right) \leq e^{-C_{U,K_1}p^2}. \tag{3.86}$$

Using the Poisson kernel and the submean inequality for $\log(|f_p|)$, we improve (3.86) as follows:

$$\Upsilon_p\left(\left\{s_p : \int_{B(2,3)} |\log|f_p||\omega_\Sigma \geq K_2 p\right\}\right) \leq e^{-C_{U,K_2}p^2}. \tag{3.87}$$

From this point we proceed as in [25, Section 4.1, p. 1992]. For $\delta \in (0, \frac{1}{2}]$, we fix a grid in the polar coordinate system of $B(2,3)$ so that, by enlarging a bit the grid cells, we obtain an open covering $\{U_j\}_{j=1}^q$ of $B(2,3)$ consisting of small boxes of diameters $\simeq \delta^4$. Then for a finite set of points $\{z_j\}_{j=1}^q$ with $z_j \in U_j$ and for all s_p, $p \in \mathbb{N}$, we have

$$\begin{aligned} &-\int_{B(2,3)} \log|s_p|_{h_p}\omega_\Sigma \\ &\quad\lesssim -\sum_{j=1}^q \mu_j \log|s_p(z_j)|_{h^p} + K_3\delta \int_{B(2,3)} |\log|f_p||\omega_\Sigma \\ &\qquad + p\delta K_3 \sup_{z\in B(2,3)} |d\alpha(z)|_{\omega_\Sigma}, \end{aligned} \tag{3.88}$$

where q and $\mu_j > 0$ only depend on δ, and we have $\sum_{j=1}^q \mu_j \simeq 1$. Applying Theorem 1.4 to each U_j, then for s_p outside the event described in (1.17), we can choose $z_j \in U_j$ with $\log|s_p(z_j)|_{h^p} \geq -\delta p$. Combining this with (3.87), we infer that

$$\Upsilon_p\left(\left\{s_p : -\int_{B(2,3)} \log|s_p|_{h^p}\omega_\Sigma \geq K_4\delta p\right\}\right) \leq e^{-C_{U,\delta}p^2}, \quad \forall p \gg 0. \tag{3.89}$$

Note that $\log^- = -\log + \log^+$ and that there exists a finite cover of U with sets biholomorphic to the annulus $B(2,3)$, we get (3.83) from (3.82) and (3.89).

3.5. *Large Deviation Estimate: Proof of Theorem 1.10*

As explained in the last part of Section 1.5, we need to control the vanishing order of holomorphic sections in $H^0_{(2)}(\Sigma, L^p)$ at the punctures.

For $a_j \in D$, set

$$H^0(\overline{\Sigma}, L^p)_{(a_j,k)} := \{s \in H^0(\overline{\Sigma}, L^p) : \mathrm{ord}_{a_j}(s) \geq k\}. \tag{3.90}$$

It is clear that $H^0(\overline{\Sigma}, L^p)_{(a_j,k)}$ is a vector subspace of $H^0(\overline{\Sigma}, L^p)$. We always view $H^0_{(2)}(\Sigma, L^p)$ as a subspace of $H^0(\Sigma, L^p)$, set

$$H^0_{(2)}(\Sigma, L^p)_{(a_j,k)} = H^0_{(2)}(\Sigma, L^p) \cap H^0(\overline{\Sigma}, L^p)_{(a_j,k)}. \tag{3.91}$$

LEMMA 3.9. *There exist $p_0 > 0$, $k_0 > 0$ such that for any $a_j \in D$ and any $p \geq p_0$, $k \geq k_0$,*

$$\dim H^0_{(2)}(\Sigma, L^p)_{(a_j,k)} \leq d_p - 1 \tag{3.92}$$

so that

$$\Upsilon_p(H^0_{(2)}(\Sigma, L^p)_{(a_j,k)}) = 0. \tag{3.93}$$

Proof. It is clear that (3.93) is a direct consequence of (3.92) since Υ_p has an integrable PDF on $H^0_{(2)}(\Sigma, L^p)$ with respect to the Lebesgue measure. We only need to prove (3.92) for fixed $a_j \in D$. Note that L is a positive holomorphic line bundle on $\overline{\Sigma}$ (since its degree is positive), so that for any sufficiently large p, there exists a nonzero section $s_{j,p} \in H^0(\overline{\Sigma}, L^p)$ such that

$$s_{j,p}(a_j) \neq 0. \tag{3.94}$$

We fix a sufficiently large $p_0 \in \mathbb{N}$ such that $H^0_{(2)}(\Sigma, L^{p_0})$ has a nonzero section f_{p_0}, and that if $p \geq 2p_0$, then

$$S_p := f_{p_0} \otimes s_{j,p-p_0} \in H^0_{(2)}(\Sigma, L^p) \tag{3.95}$$

has vanishing order

$$\mathrm{ord}_{a_j}(S_p) = \mathrm{ord}_{a_j}(f_{p_0}). \tag{3.96}$$

As a consequence, for $k \geq k_0 := \mathrm{ord}_{a_j}(f_{p_0}) + 1$,

$$0 \neq S_p \notin H^0_{(2)}(\Sigma, L^p)_{(a_j,k)}, \tag{3.97}$$

so that (3.92) holds. □

Fix k_0 in Lemma 3.9. For $p \geq p_0$, $k \geq k_0$, set

$$A^p_k = \bigcup_j H^0_{(2)}(\Sigma, L^p)_{(a_j,k)} \subset H^0_{(2)}(\Sigma, L^p). \tag{3.98}$$

Then $\Upsilon_p(A^p_k) = 0$. This way, we get Lemma 1.11 as mentioned in Section 1.5.

Let $\widetilde{U}$ be an open subset of $\overline{\Sigma}$, set $U = \widetilde{U} \backslash D \subset \Sigma$. Then, for $s_p \in H^0_{(2)}(\Sigma, L^p) \backslash A^p_{k_0}$, $p \geq p_0$, we have

$$|\mathcal{N}^U_p(s_p) - \mathcal{N}^{\widetilde{U}}_p(s_p)| \leq k_0 N, \tag{3.99}$$

the difference comes from the zeros of s_p at the punctures that are included in $\widetilde{U}$.

Proof of Theorem 1.10. We may assume that ϕ does not vanish identically on $\overline{\Sigma}$. Set $M_\phi = \max_{x \in \overline{\Sigma}} |\phi(x)| > 0$. Let $V_\phi \subset \overline{\Sigma}$ be an open neighborhood of D (with smooth boundary) on which ϕ is locally constant. In particular, $\partial\overline{\partial}\phi|_{V_\phi} \equiv 0$. Let $p_0' > 0$ be an integer such that

$$\frac{k_0 N M_\phi}{p_0'} \leq \frac{\delta}{3}. \tag{3.100}$$

The Lelong–Poincaré formula [18, Theorem 2.3.3] asserts that we have in the sense of currents on $\overline{\Sigma}$

$$\frac{\sqrt{-1}}{\pi}\partial\overline{\partial}\log|s_p|_{h^p} = [\mathrm{Div}_{\overline{\Sigma}}(s_p)] - pc_1(L,h). \tag{3.101}$$

Then

$$\begin{aligned}
&\left(\frac{1}{p}[\mathrm{Div}(s_p)],\phi\right) - \int_\Sigma \phi c_1(L,h)\\
&\quad= \left(\frac{1}{p}[\mathrm{Div}_{\overline{\Sigma}}(s_p)] - c_1(L,h),\phi\right) + \left(\frac{1}{p}([\mathrm{Div}_\Sigma(s_p)] - [\mathrm{Div}_{\overline{\Sigma}}(s_p)]),\phi\right)\\
&\quad= \frac{\sqrt{-1}}{p\pi}\int_\Sigma \log|s_p|_{h^p}\partial\overline{\partial}\phi + \left(\frac{1}{p}([\mathrm{Div}_\Sigma(s_p)] - [\mathrm{Div}_{\overline{\Sigma}}(s_p)]),\phi\right).
\end{aligned} \tag{3.102}$$

By (3.99)–(3.100), if $s_p \in H^0_{(2)}(\Sigma, L^p)\setminus A^p_{k_0}$, then

$$\left|\left(\frac{1}{p}([\mathrm{Div}(s_p)] - [\mathrm{Div}_{\overline{\Sigma}}(s_p)]),\phi\right)\right| \leq \frac{\delta}{3} \quad \text{for all } p \geq \max\{p_0, p_0'\}. \tag{3.103}$$

Since ω_Σ is smooth on $\Sigma\setminus V_\phi$, we can set

$$S_\phi = \max_{x\in\Sigma\setminus V_\phi}\left|\frac{\sqrt{-1}\partial\overline{\partial}\phi(x)}{\omega_{\Sigma,x}}\right|. \tag{3.104}$$

We only need to the consider the nontrivial case of $S_\phi > 0$. Then

$$\left|\frac{\sqrt{-1}}{p\pi}\int_\Sigma \log|s_p|_{h^p}\partial\overline{\partial}\phi\right| \leq \frac{S_\phi}{p\pi}\int_{\Sigma\setminus V_\phi}|\log|s_p|_{h^p}|\omega_\Sigma. \tag{3.105}$$

Therefore, we get that for $p \gg 0$ the following holds:

$$\begin{aligned}
&\left\{s_p : \left|\left(\frac{1}{p}[\mathrm{Div}(s_p)],\phi\right) - \int_\Sigma \phi c_1(L,h)\right| > \delta\right\}\\
&\qquad\subset \left\{s_p : \frac{S_\phi}{p\pi}\int_{\Sigma\setminus V_\phi}|\log|s_p|_{h^p}|\omega_\Sigma > \frac{2}{3}\delta\right\} \cup A^p_{k_0}.
\end{aligned} \tag{3.106}$$

Upon recalling that $\Upsilon_p(A^p_{k_0}) = 0$ and by applying Proposition 1.9 to (3.106), we get (1.30). This completes the proof of our theorem. □

3.6. Equidistribution and Large Deviation: Proof of Theorem 1.5

We first treat item (a) of Theorem 1.5. For $\phi \in \mathcal{C}^\infty_0(\Sigma)$, let $|\phi|_{\mathcal{C}^0}$ denote the $\mathcal{C}^0$-norm of ϕ on Σ. By (3.101), we get

$$\left|\left(\frac{1}{p}[\mathrm{Div}(s_p)],\phi\right)\right| \leq \left(\frac{1}{p}[\mathrm{Div}_{\overline{\Sigma}}(s_p)],1\right)|\phi|_{\mathcal{C}^0} = |\phi|_{\mathcal{C}^0}\int_\Sigma c_1(L,h). \tag{3.107}$$

By considering a countable $\mathcal{C}^0$-dense family of ϕ, it is enough to show that for fixed $\phi \in \mathcal{C}_0^\infty(\Sigma)$ we have Υ-a.s.

$$\lim_{p\to\infty}\left(\frac{1}{p}[\mathrm{Div}(s_p)], \phi\right) = \int_\Sigma \phi c_1(L,h). \tag{3.108}$$

Although this is a folklore consequence of Theorem 1.10 in probability theory, we provide the short deduction here for the sake of completeness. Write $Y_p = (\frac{1}{p}[\mathrm{Div}(s_p)], \phi)$ as well as $Y = \int_\Sigma \phi c_1(L,h)$. If there was not Υ-a.s. convergence, then by dominated convergence for $Z := \limsup_{p\to\infty} |Y_p - Y|$ there would exist $\delta > 0$ such that $\Upsilon(Z > \delta) > \delta$. Choosing $N_\delta \in \mathbb{N}$ such that $\sum_{p\geq N_\delta} \Upsilon(|Y_p - Y| > \delta) < \delta/2$ (which is possible due to Theorem 1.10) leads to a contradiction via $\Upsilon(Z > \delta) \leq \sum_{p\geq N_\delta} \Upsilon(|Y_p - Y| > \delta) < \delta/2$.

We prove now assertion (b) of Theorem 1.5. Note that for the open subset $U \subset \Sigma$ we have

$$\frac{\mathrm{Area}^L(U)}{2\pi} = \int_U c_1(L,h) < +\infty. \tag{3.109}$$

Here, we require no relative compactness for U.

Note that with respect to a smooth volume measure on Σ induced from $\overline{\Sigma}$, the measure Area^L is absolutely continuous. Then the assumption that ∂U has measure zero implies that $\mathrm{Area}^L(\partial U) = 0$. As a consequence, for any fixed $\delta > 0$ we can choose $\psi_1,\ \psi_2 \in \mathcal{C}^\infty(\overline{\Sigma})$ to be real-valued functions that take constant values near $a_j \in D$ such that

$$\begin{aligned} &0 \leq \psi_1 \leq \chi_U \leq \psi_2 \leq 1,\\ &\int_\Sigma \psi_1 c_1(L,h) \geq \frac{\mathrm{Area}^L(U)}{2\pi} - \delta,\\ &\int_\Sigma \psi_2 c_1(L,h) \leq \frac{\mathrm{Area}^L(U)}{2\pi} + \delta, \end{aligned} \tag{3.110}$$

where χ_U is the characteristic function of U on $\overline{\Sigma}$. By applying Theorem 1.10 for ψ_1, we get for s_p not in an exceptional set of probability less than $e^{-C_{\psi_1,\delta}p^2}$,

$$\begin{aligned} \mathcal{N}_p^U(s_p) &\geq (\mathrm{Div}_\Sigma(s_p), \psi_1) \geq p\int_\Sigma \psi_1 c_1(L,h) - p\delta\\ &\geq p\frac{\mathrm{Area}^L(U)}{2\pi} - (p+1)\delta. \end{aligned} \tag{3.111}$$

Similarly, if we proceed with ψ_2, we get that for s_p not in an exceptional set of probability less than $e^{-C_{\psi_2,\delta}p^2}$,

$$\mathcal{N}_p^U(s_p) \leq p\frac{\mathrm{Area}^L(U)}{2\pi} + (p+1)\delta. \tag{3.112}$$

Part (b) follows by combining (3.111) and (3.112).

3.7. Lower Bound on the Hole Probabilities: Proof of Proposition 1.7

Since we are concerned with Gaussian ensembles, using the fact that $\mathrm{Area}(\Sigma) := \int_\Sigma \omega_\Sigma < \infty$, the first part of Proposition 1.7 follows from the same arguments in [25, Section 4.2.4]. As for (1.26), we need a refined estimate for the norm of a holomorphic section near the punctures, which is explained as follows.

For $k_0 \geq 2$, the sections in $H^0_{(2)}(\Sigma, L^{k_0})$ are exactly the ones in $H^0(\overline{\Sigma}, L^{k_0})$ that vanish at every $a_j \in D$. Since the zeros of a nontrivial holomorphic section of L^{k_0} are isolated points in $\overline{\Sigma}$, we get the existence of $r_j \in]0, \frac{1}{2}[$ and τ_j as wanted.

We may and we always rescale τ_j by a nonzero constant so that $\sup_{x\in\overline{\Sigma}} |\tau_j(x)|_{h^{k_0}} = 1$. The following lemma is elementary.

LEMMA 3.10. *For $r \in (0, r_j)$, set*

$$b(r) = -\log\Big(\inf_{z\in\mathbb{D}(r,r_j)} |\tau_j(z)|_{h^{k_0}}\Big) > 0. \tag{3.113}$$

Then there exists $C_j > 0$ such that

$$b(r) \leq C_j |\log r|, \quad r \in (0, r_j). \tag{3.114}$$

Proof. Locally, we can write for $z \in \mathbb{D}^*_{2r_j}$,

$$\tau_j(z) = z^{m_j} g(z) 1^{\otimes k_0}(z), \tag{3.115}$$

where $m_j \in \mathbb{N}_{\geq 1}$ is the vanishing order of τ_j at a_j, and g is a holomorphic function such that $g(0) \neq 0$. Set $v_j = \inf_{z\in\overline{\mathbb{D}}_{r_j}} |g(z)| > 0$, then

$$1 \geq \inf_{z\in\mathbb{D}(r,r_j)} |\tau_j(z)|_{h^{k_0}} \geq r^{m_j} v_j |\log(r_j^2)|^{k_0/2}. \tag{3.116}$$

Then (3.114) follows easily. □

Proof of (1.26). Set

$$E_1^{pk_0} = \|\tau_j^{\otimes p}\|^{-1}_{\mathcal{L}^2} \tau_j^{\otimes p} \in H^0_{(2)}(\Sigma, L^{pk_0}). \tag{3.117}$$

Note that

$$\|\tau_j^{\otimes p}\|_{\mathcal{L}^2} \leq \mathrm{Area}(\Sigma)^{1/2}. \tag{3.118}$$

We then complete $\{E_1^{pk_0}\}$ to an orthonormal basis $\{E_1^{pk_0}, E_2^{pk_0}, \ldots, E_{d_{pk_0}}^{pk_0}\}$ of $H^0_{(2)}(\Sigma, L^{pk_0})$.

Since Υ_p is defined by i.i.d. standard random complex Gaussian variables, it does not depend on the choice of basis O_p. The random section s_{pk_0} is given by

$$s_{pk_0} = \sum_j \xi_j E_j^{pk_0} = \xi_1 E_1^{pk_0} + s'_{pk_0}, \tag{3.119}$$

where ξ_j, $j = 1, \ldots, d_{pk_0}$, are i.i.d. standard random complex Gaussians. Set $\xi' = (\xi_2, \ldots, \xi_{d_{pk_0}}) \in \mathbb{C}^{d_{pk_0}-1}$. Similar to (3.1) and (3.12), we have for $p \gg 0$ and $x \in \Sigma$,

$$|s'_{pk_0}(x)|_{h^{pk_0}} \leq C\|\xi'\| p^{3/4}, \tag{3.120}$$

where the constant $C > 0$ is independent of p and z. By Lemma 3.10 and (3.117), for $r \in]0, r_j[$, we have

$$\inf_{z\in\mathbb{D}(r,r_j)} |E_1^{pk_0}(z)|_{h^{pk_0}} \geq \frac{e^{-pb(r)}}{\mathrm{Area}(\Sigma)^{1/2}}. \tag{3.121}$$

Set

$$t_p(r) = \frac{e^{-pb(r)}}{C\,\mathrm{Area}(\Sigma)^{1/2} p^{3/4}\sqrt{d_{pk_0}}} > 0. \tag{3.122}$$

Note that $\|\xi'\| \leq \sqrt{d_{pk_0}} \max_{j\geq 2} |\xi_j|$, then for $r \in]0, r_j[$, we have

$$\begin{aligned} &\Big\{ s_{pk_0} = \sum_j \xi_j E_j^{pk_0} : |\xi_1| > 1, |\xi_j| < t_p(r), j \geq 2 \Big\} \\ &\quad \subset \{ s_{pk_0} : \mathcal{N}_{pk_0}^{\mathbb{D}(r,r_j)}(s_{pk_0}) = 0 \}. \end{aligned} \tag{3.123}$$

Therefore, for $r \in]0, r_j[$, we have

$$\Upsilon_{pk_0}(\{ s_{pk_0} : \mathcal{N}_{pk_0}^{\mathbb{D}(r,r_j)}(s_{pk_0}) = 0 \}) \geq e^{-1}\left(\frac{t_p(r)^2}{2} \right)^{d_{pk_0}-1}. \tag{3.124}$$

By (3.114) there exists $c_j > 0$ such that for $r \in]0, r_j[$, $p \gg 0$,

$$e^{-1}\left(\frac{t_p(r)^2}{2} \right)^{d_{pk_0}-1} \geq e^{-c_j|\log r|p^2}. \tag{3.125}$$

This completes our proof of (1.26). □

4. Cusp Forms on Hyperbolic Surfaces of Finite Volume

We give an important example where our results apply. Let $\overline{\Sigma}$ be a compact Riemann surface of genus g and consider a finite set $D = \{a_1, \ldots, a_N\} \subset \overline{\Sigma}$. We also denote by D the divisor $\sum_{j=1}^N a_j$ and let $\mathscr{O}_{\overline{\Sigma}}(D)$ be the associated line bundle. Let $K_{\overline{\Sigma}}$ be the canonical line bundle of $\overline{\Sigma}$. The following conditions are equivalent:

(i) $\Sigma = \overline{\Sigma} \smallsetminus D$ admits a complete Kähler–Einstein metric ω_Σ with $\mathrm{Ric}_{\omega_\Sigma} = -\omega_\Sigma$,

(ii) $2g - 2 + N > 0$,

(iii) the universal cover of Σ is the upper half-plane $\mathbb{H}$,

(iv) $L = K_{\overline{\Sigma}} \otimes \mathscr{O}_{\overline{\Sigma}}(D)$ is ample.

This follows from the uniformization theorem [15, Chapter IV] and the fact that the Euler characteristic of Σ equals $\chi(\Sigma) = 2 - 2g - N$ and the degree of L is

$$\deg L = 2g - 2 + N = -\chi(\Sigma).$$

If one of these equivalent conditions is satisfied, the Kähler–Einstein metric ω_Σ is induced by the Poincaré metric on $\mathbb{H}$; (Σ, ω_Σ) and the formal square root of (L, h) satisfy conditions (α) and (β), see [3, Lemma 6.2]. Theorem 1.5, Corollary 1.6, and Proposition 1.7 hence apply to this context.

Let Γ be the Fuchsian group associated with the above Riemann surface Σ, that is, $\Sigma \cong \Gamma\backslash\mathbb{H}$. Then Γ is a geometrically finite Fuchsian group of the first kind without elliptic elements. Conversely, if Γ is such a group, then $\Sigma := \Gamma\backslash\mathbb{H}$ can by compactified by finitely many points $D = \{a_1, \dots, a_N\}$ into a compact Riemann surface $\overline{\Sigma}$ such that the equivalent conditions (i)–(iv) are fulfilled.

The space $\mathcal{M}^{\Gamma}_{2p}$ of Γ-modular forms of weight $2p$ is by definition the space of holomorphic functions $f \in \mathscr{O}(\mathbb{H})$ satisfying the functional equation

$$f(\gamma z) = (cz + d)^{2p} f(z), \quad z \in \mathbb{H}, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \tag{4.1}$$

and which extend holomorphically to the cusps of Γ (fixed points of the parabolic elements). If $f \in \mathscr{O}(\mathbb{H})$ satisfies (4.1), then $f\,dz^{\otimes p} \in H^0(\mathbb{H}, K^p_{\mathbb{H}})$ descends to a holomorphic section $\Phi(f)$ of $H^0(\Sigma, K^p_{\Sigma}) \cong H^0(\Sigma, L^p)$. By [19, Propositions 3.3, 3.4(b)], Φ induces an isomorphism $\Phi : \mathcal{M}^{\Gamma}_{2p} \to H^0(\overline{\Sigma}, L^p)$.

The subspace of $\mathcal{M}^{\Gamma}_{2p}$ consisting of modular forms vanishing at the cusps is called the space of *cusp forms* (Spitzenformen) of weight $2p$ of Γ, denoted by $\mathcal{S}^{\Gamma}_{2p}$. The space of cusps forms is endowed with the Petersson scalar product

$$\langle f, g\rangle := \int_U f(z)\overline{g(z)}(2y)^{2p}\, dv_{\mathbb{H}}(z),$$

where U is a fundamental domain for Γ and $dv_{\mathbb{H}} = \frac{1}{2}y^{-2}\, dx \wedge dy$ is the hyperbolic volume form.

Under the above isomorphism, $\mathcal{S}^{\Gamma}_{2p}$ is identified to the space $H^0(\overline{\Sigma}, L^p \otimes \mathscr{O}_{\overline{\Sigma}}(D)^{-1}) = H^0(\overline{\Sigma}, K^p_{\overline{\Sigma}} \otimes \mathscr{O}_{\overline{\Sigma}}(D)^{p-1})$ of holomorphic sections of L^p over $\overline{\Sigma}$ vanishing on D.

If we endow $K_{\mathbb{H}}$ with the Hermitian metric induced by the Poincaré metric on $\mathbb{H}$, the scalar product of two elements $u\, dz^{\otimes p}, v\, dz^{\otimes p} \in K^p_{\mathbb{H},z}$ is $\langle u\, dz^{\otimes p}, v\, dz^{\otimes p}\rangle = u\overline{v}(2y)^{2p}$. Hence, the Petersson scalar product corresponds to the $\mathcal{L}^2$ inner product of pluricanonical forms on Σ,

$$\langle f, g\rangle = \int_{\Sigma} \langle \Phi(f), \Phi(g)\rangle \omega_{\Sigma}, \quad f, g \in \mathcal{S}^{\Gamma}_{2p}.$$

The isomorphism Φ gives thus an isometry (see also [11, Section 6.4])

$$\mathcal{S}^{\Gamma}_{2p} \cong H^0(\overline{\Sigma}, L^p \otimes \mathscr{O}_{\overline{\Sigma}}(D)^{-1}) \cong H^0_{(2)}(\Sigma, K^p_{\Sigma}) \cong H^0_{(2)}(\Sigma, L^p), \tag{4.2}$$

where $H^0_{(2)}(\Sigma, L^p)$ is the space of holomorphic sections of L^p that are $\mathcal{L}^2$-integrable with respect to the volume form ω_{Σ} and the metric h^p on L^p, with h introduced in [3, Lemma 6.2]. Moreover, $H^0_{(2)}(\Sigma, K^p_{\Sigma})$ is the space of $\mathcal{L}^2$-pluricanonical sections with respect to the metric $h^{K^p_{\Sigma}}$ and the volume form ω_{Σ}, where we denote by $h^{K_{\Sigma}}$ the Hermitian metric induced by ω_{Σ} on K_{Σ}. We thus identify the space of cusp forms $\mathcal{S}^{\Gamma}_{2p}$ to a subspace of holomorphic sections of L^p by (4.2).

COROLLARY 4.1. *Let $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ be a geometrically finite Fuchsian group of the first kind without elliptic elements. Then the assertions of Theorems 1.4 and 1.5, Corollary 1.6, and Proposition 1.7 hold for the zeros of cusp forms $s_p \in \mathcal{S}^{\Gamma}_{2p}$.*

5. Higher Dimensional Complex Hermitian Manifolds

In this section, we consider the extension of the above results to the noncompact complete complex Hermitian manifold of higher dimension. Our geometric setting is described in Section 1.6. At first, we recall the Bergman kernel expansion under this setting.

By [18, Theorems 4.2.1 and 6.1.1], the Bergman kernel expansions described in Section 2, for both on-diagonal and off-diagonal, still hold. More precisely, there exist coefficients $a_r \in C^\infty(X)$, $r \in \mathbb{N}$ such that the following asymptotic expansion

$$B_p(x,x) = \sum_{r=0}^{\infty} a_r(x) p^{m-r} \tag{5.1}$$

holds for any $\mathcal{C}^\ell$-topology on compact sets of X. In particular, let $\dot{R}^L \in \mathrm{End}(T^{(1,0)}X)$ such that for $u, v \in T_x^{(1,0)}X$,

$$R^L(u,v) = g^{TX}(\dot{R}^L u, v), \tag{5.2}$$

then

$$a_0(x) = \det\left(\frac{\dot{R}^L}{2\pi}\right) > \left(\frac{\varepsilon_1}{2\pi}\right)^m. \tag{5.3}$$

In particular, if $K \subset X$ is compact, then there exists $C_K > 0$ such that for $p \gg 0$,

$$\max_{x\in K} B_p(x,x) \le C_K p^m. \tag{5.4}$$

If X is noncompact, then the existence of a complete metric ω with $iR^L > \varepsilon_1 \omega$ ($\varepsilon_1 > 0$) is equivalent to saying that iR^L defines a complete Kähler metric. Recall that the volume $\mathrm{Vol}^L_{2m}(\cdot)$ is defined in (1.47). As in [14, Corollary 2.2], under assumption (1.38), we have

$$0 < \frac{1}{m!}\int_X c_1(L,h)^m \le \liminf_{p\to\infty} p^{-m} d_p < \infty. \tag{5.5}$$

As a consequence, we get

$$d_p \simeq p^m, \qquad \mathrm{Vol}^L_{2m}(X) < \infty. \tag{5.6}$$

Then, for any open subset $U \subset X$, we have $\mathrm{Vol}^L_{2m}(U) < \infty$.

Furthermore, the off-diagonal and near-diagonal expansions as in Proposition 2.3 also hold (with suitable change according to the dimension m). For a precise statement on the near-diagonal expansion, we need to introduce the complex coordinates for the real tangent space T_xX, $x \in X$.

Fix a point $x \in X$. Let $\{\mathbf{f}_j\}_{j=1}^m$ be an orthonormal basis of $(T_x^{1,0}X, g_x^{TX}(\cdot,\bar{\cdot}))$ such that

$$\dot{R}^L_x \mathbf{f}_j = \mu_j(x)\mathbf{f}_j, \tag{5.7}$$

where $\mu_j(x)$, $j = 1, \dots, m$, are the eigenvalues of $\dot{R}^L_x$. We have

$$\mu_j(x) > \varepsilon_1, \quad a_0(x) = \prod_{j=1}^m \frac{\mu_j(x)}{2\pi}. \tag{5.8}$$

Set $\mathbf{e}_{2j-1} = \frac{1}{\sqrt{2}}(\mathbf{f}_j + \bar{\mathbf{f}}_j)$, $\mathbf{e}_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(\mathbf{f}_j - \bar{\mathbf{f}}_j)$, $j = 1, \dots, m$. Then they form an orthonormal basis of the (real) tangent vector space $(T_x X, g_x^{TX})$. Now we introduce the complex coordinate for $T_x X$. If $v = \sum_{j=1}^{2m} v_j \mathbf{e}_j \in T_x X$, then we can write

$$v = \sum_{j=1}^{m} \big(v_{2j-1} + \sqrt{-1} v_{2j}\big) \frac{1}{\sqrt{2}} \mathbf{f}_j + \sum_{j=1}^{m} \big(v_{2j-1} - \sqrt{-1} v_{2j}\big) \frac{1}{\sqrt{2}} \bar{\mathbf{f}}_j. \tag{5.9}$$

Set $z = (z_1, \dots, z_m)$ with $z_j = v_{2j-1} + \sqrt{-1} v_{2j}$, $j = 1, \dots, m$. We call z the complex coordinate of $v \in T_x X$. Then, by (5.9),

$$\frac{\partial}{\partial z_j} = \frac{1}{\sqrt{2}} \mathbf{f}_j, \qquad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{\sqrt{2}} \bar{\mathbf{f}}_j \tag{5.10}$$

so that

$$v = \sum_{j=1}^{m} \left(z_j \frac{\partial}{\partial z_j} + \overline{z}_j \frac{\partial}{\partial \overline{z}_j} \right). \tag{5.11}$$

Note that $|\frac{\partial}{\partial z_j}|^2_{g^{TX}} = |\frac{\partial}{\partial \overline{z}_j}|^2_{g^{TX}} = \frac{1}{2}$. For $v, v' \in T_x X$, let z, z' denote the corresponding complex coordinates. Define

$$\mathcal{P}_x(v, v') = \prod_{j=1}^{m} \frac{\mu_j(x)}{2\pi} \exp\left(-\frac{1}{4} \sum_{j=1}^{m} \mu_j(x) (|z_j|^2 + |z'_j|^2 - 2 z_j \overline{z}_j) \right). \tag{5.12}$$

Define a weighted distance function $\Phi_x^{TX}(v, v')$ as follows:

$$\Phi_x^{TX}(v, v')^2 = \sum_{j=1}^{m} \mu_j(x) |z_j - z'_j|^2. \tag{5.13}$$

Then

$$|\mathcal{P}_x(v, v')| = \prod_{j=1}^{m} \frac{\mu_j(x)}{2\pi} \exp\left(-\frac{1}{4} \Phi_x^{TX}(v, v')^2 \right). \tag{5.14}$$

For sufficiently small $\delta_0 > 0$, we identify the small open ball $B^X(x, 2\delta_0)$ in X with the ball $B^{T_x X}(0, 2\delta_0)$ in $T_x X$ via the geodesic coordinate. Let $\operatorname{dist}(\cdot, \cdot)$ denote the Riemannian distance of (X, g^{TX}). There exists $C_2 > 0$ such that for $v, v' \in B^{T_x X}(0, 2\delta_0)$ we have

$$C_2 \operatorname{dist}(\exp_x(v), \exp_x(v')) \geq \Phi_x^{TX}(v, v') \geq \frac{1}{C_2} \operatorname{dist}(\exp_x(v), \exp_x(v')). \tag{5.15}$$

In particular,

$$\Phi_x^{TX}(0, v) \geq \varepsilon_1^{1/2} \operatorname{dist}(x, \exp_x(v)). \tag{5.16}$$

Moreover, if we consider a compact subset $K \subset X$, then the constants δ_0 and C_1 can be chosen uniformly for all $x \in K$.

We trivialize the line bundle L on $B^{T_x X}(0, 2\delta_0)$ using the parallel transport with respect to ∇^L along the curve $[0, 1] \ni t \mapsto tv$, $v \in B^{T_x X}(0, 2\delta_0)$. Under this trivialization, for $v, v' \in B^{T_x X}(0, 2\delta_0)$,

$$B_p(\exp_x(v), \exp_x(v')) \in \operatorname{End}(L_x) = \mathbb{C}. \tag{5.17}$$

By [18, Theorems 4.2.1 and 6.1.1], for any compact subset $K \subset X$ and for any $N \in \mathbb{N}$, there exist $\delta > 0$ and constants $C, C' > 0$ such that for $x \in K$, $v, v' \in T_x X$, $\|v\|, \|v'\| \leq 2\delta$, instead of (2.5), we have

$$\left| \frac{1}{p^m} B_p(\exp_x(v), \exp_x(v')) - \sum_{r=0}^{N} \mathcal{F}_r\big(\sqrt{p}v, \sqrt{p}v'\big)\kappa^{-1/2}(v)\kappa^{-1/2}(v')p^{-r/2} \right| \\ \leq Cp^{-(N+1)/2}\big(1 + \sqrt{p}\|v\| + \sqrt{p}\|v'\|\big)^{2(N+m)+4} \exp\big(-C'\sqrt{p}\|v - v'\|\big) \\ + \mathcal{O}(p^{-\infty}). \tag{5.18}$$

The functions $\mathcal{F}_r$, $r \in \mathbb{N}$, are given as follows:

$$\mathcal{F}_r(v, v') = \mathcal{P}_x(v, v')\mathcal{J}_r(v, v'), \tag{5.19}$$

where $\mathcal{J}_r(v, v')$ is a polynomial in v, v' of degree $\leq 3r$, whose coefficients are smooth in $x \in X$. In particular,

$$\mathcal{J}_0 = 1. \tag{5.20}$$

The normalized Bergman kernel $P_p(x, y)$ for $x, y \in X$ is defined as in (1.27). Then, by exactly the same arguments in Section 2.3, we get a version of Theorem 1.8 for P_p as follows.

THEOREM 5.1. *Let U be a relatively compact open subset of X, then the following uniform estimates on the normalized Bergman kernel hold for $x, y \in U$: fix $k \geq 1$ and $b > \sqrt{16k/\varepsilon_1}$, then we have*

$$P_p(x, y) = \begin{cases} (1 + o(1))\exp(-\frac{p}{4}\Phi_x(0, v')^2), \\ \quad \textit{uniformly for } \operatorname{dist}(x, y = \exp_x(v')) = \|v'\| \leq b\sqrt{\frac{\log p}{p}}, \\ \mathcal{O}(p^{-k}), \quad \textit{uniformly for } \operatorname{dist}(x, y) \geq b\sqrt{\frac{\log p}{p}}. \end{cases} \tag{5.21}$$

Now we start to give the proofs to the theorems in Section 1.6. Most of the arguments are exactly the same as given in Section 3.

Proof of Theorem 1.13. Since U is relatively compact in X, the Bergman kernel $B_p(x, x)$, $x \in U$, is uniformly bounded above by $C_{\overline{U}}p^m$ by (5.4). By (5.6), $d_p \simeq p^m$. The proof of (1.43) follows exactly by the arguments in the proof of Proposition 3.3. In particular, we have for $p \gg 0$,

$$\mathbb{E}[|\mathcal{M}_p^U(s_p)|^{d_p}] \lesssim d_p^{2d_p+3/2} \leq p^{Cp^m}, \tag{5.22}$$

where the constant $C > 0$ is sufficiently large.

Now we prove (1.44). At first, by (5.22), as in Corollary 3.6, there exists a constant $C > 0$ such that for any sequence $\{\lambda_p\}_{p \in \mathbb{N}}$ of strictly positive numbers we have

$$\Upsilon_p(\{s_p : \mathcal{M}_p^U(s_p) \geq \lambda_p\}) \lesssim e^{-d_p \log \lambda_p + Cd_p \log p}. \tag{5.23}$$

Secondly, we apply the same arguments in Section 3.3 by taking the lattice points x_v^p, $v \in \Gamma_p \subset \mathbb{Z}^m$, near a fixed point $x_0 \in U$, where we identify $(T_{x_0}X, g^{T_{x_0}X})$ with $\mathbb{R}^m$. Note that $n := \sharp\Gamma_p \simeq d_p$. As in (3.41),

$$\frac{d^2}{4p}\|u-v\|^2 \leq \mathrm{dist}(x_u^p, x_v^p)^2 \leq \frac{4d^2}{p}\|u-v\|^2. \tag{5.24}$$

Let $\xi(u,v) \in T_{x_u^p}X$ be the unique vector (with small norm) such that $\exp_{x_u^p}(\xi(u, v)) = x_v^p$. By (5.16), (5.24),

$$\Phi_{x_u^p}(0, \xi(u,v))^2 \geq \frac{\varepsilon_1 d^2}{4p}\|u-v\|^2. \tag{5.25}$$

This is an analogue of (3.55). Then using instead Theorem 5.1 and proceeding as in (3.49)–(3.72), we get that, for a sequence $\{\lambda_p\}_{p\in\mathbb{N}}$ of positive numbers less than 1, there exist constants $C > 0$, $C' > 0$ such that for $\forall p \gg 0$,

$$\Upsilon_p(\{s_p : \mathcal{M}_p^U(s_p) \leq \lambda_p\}) \leq e^{Cd_p \log\lambda_p + C' d_p \log p}. \tag{5.26}$$

Taking $\lambda_p = e^{\delta p}$ in (5.23) and $\lambda_p = e^{-\delta p}$ in (5.26), we get (1.44) upon using $d_p \simeq p^m$. This completes our proof. □

Proof of Theorem 1.14. The first part of this theorem is an analogue of Theorem 1.10, and its proof will follow the arguments as explained in Sections 3.4 and 3.5, by using instead Theorem 1.13. Recall that $\mathrm{dV} = \frac{\omega^m}{m!}$ is the volume element on X.

Indeed, the sketched proof in Section 3.4 also proves that, for the relative compact open subset U and for any $\delta > 0$, there exists $C_{U,\delta} > 0$ such that

$$\Upsilon_p\left(\left\{s_p : \int_U |\log|s_p|_{h^p}|\,\mathrm{dV} \geq \delta p\right\}\right) \leq e^{-C_{U,\delta}p^{m+1}}, \quad \forall p \gg 0. \tag{5.27}$$

To get (1.45), we apply the Lelong–Poincaré formula for $\phi \in \Omega_0^{m-1,m-1}(U)$,

$$([\mathrm{Div}(s_p)] - pc_1(h,L), \phi) = \frac{i}{\pi}\int_U \log|s_p|_{h_p}\partial\overline{\partial}\phi. \tag{5.28}$$

Then

$$\begin{aligned}\left|\left(\frac{1}{p}[\mathrm{Div}(s_p)] - c_1(h,L), \phi\right)\right| &\leq \frac{1}{\pi p}\left|\int_U \log|s_p|_{h_p}\partial\overline{\partial}\phi\right| \\ &\leq \frac{1}{\pi p}\sup_{x\in X}\left|\frac{\partial\overline{\partial}\phi(x)}{\mathrm{dV}(x)}\right| \cdot \int_U |\log|s_p|_{h^p}|\,\mathrm{dV}.\end{aligned} \tag{5.29}$$

Then, by (5.27), we get (1.45). As a consequence of (1.45), the proof to (1.46) follows exactly from the same arguments as in the part (a) of Section 3.6. This completes our proof. □

Proof of Theorem 1.15. We only need to prove (1.49), and (1.50) is just its direct consequence. Due to the results in Theorem 1.14 and that U is relatively compact, the proof of this theorem is quite routine as in Section 3.6.

Let $\delta > 0$ be arbitrary, and we take $\psi_1, \psi_2 \in \mathcal{C}_0^\infty(X, \mathbb{R}_{\geq 0})$ such that

$$\begin{aligned} 0 \leq \psi_1 \leq \chi_U \leq \psi_2 \leq 1, \\ \int_X \psi_1 \frac{c_1(L,h)^m}{m!} \geq \mathrm{Vol}^L_{2m}(U) - \delta, \\ \int_X \psi_2 \frac{c_1(L,h)^m}{m!} \leq \mathrm{Vol}^L_{2m}(U) + \delta. \end{aligned} \tag{5.30}$$

Set $\phi_j = \frac{\psi_j}{(m-1)!} c_1(L,h)^{m-1}$, $j = 1, 2$. By applying Theorem 1.14 to ϕ_j separately, we get (1.49), and thus the proof is completed. □

Remark 5.2. Assume that $\{\Upsilon_p\}_{p\in\mathbb{N}}$ is defined as in Example 1.2 with $\sigma_p = 1$. Then, similar to Proposition 1.7 and the second part of [25, Theorem 1.4], we can also give a lower bound ($\simeq e^{-C_U p^{m+1}}$) for the hole probabilities for a relative compact nonempty open subset $U \subset X$, provided there exists a nowhere vanishing section on $\overline{U}$.

We exhibit now two classes of manifolds for which Theorem 1.14 applies, each of them has its own interests in various fields of complex geometry.

Example 5.3. Let M be a compact complex manifold of dimension m, Σ is an analytic subvariety of M, $X := M \setminus \Sigma$. We assume that X admits a complete Kähler metric ω such that $\mathrm{Ric}_\omega \leq -\lambda\omega$ for some constant $\lambda > 0$. Assume, moreover, that $\dim \Sigma \leq m - k$, $k \geq 2$. Then $H^0_{(2)}(X, K_X^p) \subset H^0(M, K_M^p)$ and $d_p = \dim H^0_{(2)}(X, K_X^p) = \mathcal{O}(p^m)$ as $p \to \infty$.

Example 5.4. Let D be a bounded symmetric domain in $\mathbb{C}^m$, and let Γ be a neat arithmetic group acting properly discontinuously on D (see [19, p. 253]). Then $U := D/\Gamma$ is a smooth quasi-projective variety, called an arithmetic variety. By [2], U admits a smooth toroidal compactification X. In particular, $\Sigma := X \setminus U$ is a divisor with normal crossings. The Bergman metric $\omega_D^{\mathcal{B}}$ on D descends to a complete Kähler metric $\omega := \omega_U^{\mathcal{B}}$ on U. Moreover, ω is Kähler–Einstein with $\mathrm{Ric}_\omega = -\omega$ (since the metric $\omega_D^{\mathcal{B}}$ has this property). We denote by h^{K_U} the Hermitian metric induced by ω on K_U. Then our results pertain to the spaces $H^0_{(2)}(U, K_U^p)$ of $\mathcal{L}^2$-pluricanonical sections with respect to the metric $h^{K_U^p}$ and the volume form ω^m.

Acknowledgments. We gratefully acknowledge the support of DFG Priority Program 2265 ‘Random Geometric Systems’. The authors thank Dominik Zielinski for useful discussions and the referee for carefully reading the paper.

References

[1] R. J. Adler and J. E. Taylor, *Random fields and geometry,* Springer Monogr. Math., 2007.

[2] A. Ash, D. Mumford, M. Rapoport, and Y. S. Tai, *Smooth compactifications of locally symmetric varieties,* Cambridge Math. Lib., 2nd edition, Cambridge University Press, Cambridge, 2010. With the collaboration of Peter Scholze.

[3] H. Auvray, X. Ma, and G. Marinescu, *Bergman kernels on punctured Riemann surfaces,* Math. Ann. 379 (2021), 951–1002.

[4] ———, *Quotient of Bergman kernels on punctured Riemann surfaces,* Math. Z. 301 (2022), no. 3, 2339–2367.

[5] J.-M. Azaïs and M. Wschebor, *Level sets and extrema of random processes and fields,* John Wiley & Sons, Inc., Hoboken, NJ, 2009.

[6] T. Bayraktar, *Equidistribution of zeros of random holomorphic sections,* Indiana Univ. Math. J. 65 (2016), 1759–1793.

[7] ———, *Zero distribution of random sparse polynomials,* Michigan Math. J. 66 (2017), 389–419.

[8] T. Bayraktar, D. Coman, H. Herrmann, and G. Marinescu, *A survey on zeros of random holomorphic sections,* Dolomit. Res. Notes Approx. 11 (2018), 1–19, Special Issue Norm Levenberg.

[9] T. Bayraktar, D. Coman, and G. Marinescu, *Universality results for zeros of random holomorphic sections,* Trans. Amer. Math. Soc. 373 (2020), no. 6, 3765–3791.

[10] T. Bloom and N. Levenberg, *Random polynomials and pluripotential-theoretic extremal functions,* Potential Anal. 42 (2015), 311–334.

[11] D. Coman and G. Marinescu, *Equidistribution results for singular metrics on line bundles,* Ann. Sci. Éc. Norm. Supér. (4) 48 (2015), no. 3, 497–536.

[12] T.-C. Dinh, X. Ma, and G. Marinescu, *Equidistribution and convergence speed for zeros of holomorphic sections of singular Hermitian line bundles,* J. Funct. Anal. 271 (2016), no. 11, 3082–3110.

[13] T.-C. Dinh and N. Sibony, *Distribution des valeurs de transformations méromorphes et applications,* Comment. Math. Helv. 81 (2006), 221–258.

[14] T. C. Dinh, G. Marinescu, and V. Schmidt, *Equidistribution of zeros of holomorphic sections in the non compact setting,* J. Stat. Phys. 148 (2012), no. 1, 113–136.

[15] H. M. Farkas and I. Kra, *Riemann surfaces,* Grad. Texts in Math., 71, p. xi+337, Springer-Verlag, New York-Berlin, 1980.

[16] J. S. Friedman, J. Jorgenson, and J. Kramer, *Uniform sup-norm bounds on average for cusp forms of higher weights,* Arbeitstagung Bonn 2013, Progr. Math., 319, pp. 127–154, 2016.

[17] M. Krishnapur, *Overcrowding estimates for zeroes of planar and hyperbolic Gaussian analytic functions,* J. Stat. Phys. 124 (2006), 1399–1423.

[18] X. Ma and G. Marinescu, *Holomorphic Morse inequalities and Bergman kernels,* Progr. Math., 254, p. xiv+422, Birkhäuser Verlag, Basel, 2007.

[19] D. Mumford, *Hirzebruch's proportionality theorem in the noncompact case,* Invent. Math. 42 (1977), 239–272.

[20] A. C. Offord, *The distribution of zeros of power series whose coefficients are independent random variables,* Indian J. Math. 9 (1967), 175–196.

[21] Y. Peres and B. Virág, *Zeros of the i.i.d. Gaussian power series: a conformally invariant determinantal process,* Acta Math. 194 (2005), 1–35.

[22] Z. Rudnick, *On the asymptotic distribution of zeros of modular forms,* Int. Math. Res. Not. 34 (2005), 2059–2074.

[23] B. Shiffman and S. Zelditch, *Distribution of zeros of random and quantum chaotic sections of positive line bundles,* Comm. Math. Phys. 200 (1999), 661–683.

[24] ———, *Number variance of random zeros on complex manifolds,* Geom. Funct. Anal. 18 (2008), no. 4, 1422–1475.

[25] B. Shiffman, S. Zelditch, and S. Zrebiec, *Overcrowding and hole probabilities for random zeros on complex manifolds,* Indiana Univ. Math. J. 57 (2008), no. 5, 1977–1997.

[26] M. Sodin, *Zeros of Gaussian analytic functions,* Math. Res. Lett. 7 (2000), 371–381.

[27] M. Sodin and B. Tsirelson, *Random complex zeroes. I. Asymptotic normality,* Israel J. Math. 144 (2004), 125–149.

[28] ———, *Random complex zeroes. III. Decay of the hole probability,* Israel J. Math. 147 (2005), 371–379.

[29] J. Zhu, *Hole probabilities of* $\mathrm{SU}(m+1)$ *Gaussian random polynomials,* Anal. PDE 7 (2014), no. 8, 1923–1968.

[30] S. Zrebiec, *The zeros of flat Gaussian random holomorphic functions on* $\mathbb{C}^n$*, and hole probability,* Michigan Math. J. 55 (2007), 269–284.

A. Drewitz
Department Mathematik/Informatik
Universität zu Köln
Weyertal 86-90
50931 Köln
Germany

adrewitz@uni-koeln.de

B. Liu
Department Mathematik/Informatik
Universität zu Köln
Weyertal 86-90
50931 Köln
Germany

bingxiao.liu@uni-koeln.de

G. Marinescu
Department Mathematik/Informatik
Universität zu Köln
Weyertal 86-90
50931 Köln
Germany

gmarines@math.uni-koeln.de