TOEPLITZ OPERATORS AND SQUARE-INTEGRABLE GAUSSIAN HOLOMORPHIC SECTIONS

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KÄHLER MANIFOLD AND ......

- (X, ω) (connected) Kähler manifold of complex dimension n... without boundary.
- X locally is an open subset of $\mathbb{C}^n$, called a local chart, different local charts glued together by biholomorphic diffeomorphisms.

- ω Kähler form, a real differential 2-form on X ... in local coordinates $(z_1, \ldots, z_n) \in \mathbb{C}^n$

$$\omega = \sqrt{-1} \sum_{i,j} g_{i,j}(z) dz_i \wedge d\bar{z}_j,$$

  where $(g_{i,j}(z))_{i,j=1}^n$ is Hermitian, positive definite, depending smoothly on z.
- Kähler condition: $d\omega = 0$ on X ... a conservative system ...
- ... it defines volume form on X

$$\mathrm{dV} = \frac{\omega^n}{n!} = \frac{1}{n!} \underbrace{\omega \wedge \ldots \wedge \omega}_{n \text{ copies}}.$$

Photo of Erich Kähler by L. Reidemeister from Oberwolfach Photo Collection.

EXAMPLES OF KÄHLER MANIFOLD

- Ex 1: Riemann sphere $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\} \simeq \mathbb{S}^2$, gluing two copies $\mathbb{C}_1$ & $\mathbb{C}_2$ of $\mathbb{C}$ via the biholomorphic diffeomorphism

$$\mathbb{C}_1 \setminus \{0\} \ni z \mapsto \frac{1}{z} =: w \in \mathbb{C}_2 \setminus \{0\}.$$

- Fubini-Study metric

$$\omega_{\mathrm{FS}} = \frac{\sqrt{-1}}{2\pi} \frac{dz \wedge d\bar{z}}{(1+|z|^2)^2} = \frac{\sqrt{-1}}{2\pi} \frac{dw \wedge d\bar{w}}{(1+|w|^2)^2}.$$

- Ex 2: complex vector space $\mathbb{C}^n$ with the flat Kähler metric

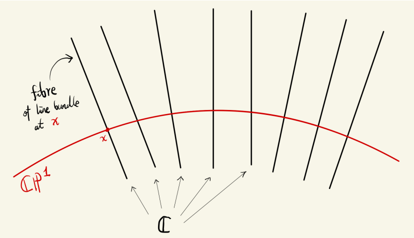
$$\omega_{\mathrm{flat}} = \frac{\sqrt{-1}}{2\pi} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j.$$

- Ex 3: Poincaré disc $\mathbb{D} = \{|z| < 1\}$
- .... hyperbolic metric

$$\omega_{\mathrm{Poinc}} = \frac{\sqrt{-1}}{2\pi} \frac{dz \wedge d\bar{z}}{(1-|z|^2)^2}.$$

...... AND HOLOMORPHIC LINE BUNDLE

- Holomorphic line bundle L on X is a holomorphic family of $\mathbb{C}$ parametrized by $x \in X$ ... denote as $L \to X$.
- ... such as the trivial line bundle $\underline{\mathbb{C}} := X \times \mathbb{C} \to X$ ...
- another example, for $v \in \mathbb{C}^2 \setminus \{0\}$, $[v] \in \mathbb{CP}^1$, line bundle $\mathscr{O}(1)_{[v]} := \mathbb{C}v^*$.
- Hermitian metric h on L: locally $|1|_h^2 = e^{-\phi}$. Local potential ϕ is real function.



- First Chern form $c_1(L, h) \in \Omega^{(1,1)}(X, \mathbb{R})$ locally $= \frac{\sqrt{-1}}{2\pi} \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j$.
- a global holomorphic section s of L on X is a holomorphic map $s : X \to L$ such that for every $x \in X$, $s(x) \in L_x \simeq \mathbb{C}$.
- ... holomorphic section s is locally regarded as a holomorphic function.

Holomorphic sections

- $H^0(X, L)$ = space of (global) holomorphic sections of L.
- Ex 1: $H^0(X, \underline{\mathbb{C}})$ = global holomorphic functions on X.
- If X is compact (without boundary), the holomorphic functions are always constant, but one can have many nontrivial holomorphic sections ...
- Ex 2: for integer p, $\mathscr{O}(p) := \mathscr{O}(1)^{\otimes p}$ holomorphic line bundle on $\mathbb{CP}^1$,

$$H^0(\mathbb{CP}^1, \mathscr{O}(p)) = \{\text{polynomials in } z \text{ of degree up to } p\}.$$

- Square-integrable (or simply $\mathcal{L}^2$) inner product:

$$\langle s_1, s_2 \rangle_{\mathcal{L}^2} := \int_X h(s_1(x), s_2(x)) \mathrm{dV}(x).$$

- Square-integrable holomorphic sections:

$$H^0_{(2)}(X, L) := H^0(X, L) \cap \mathcal{L}^2(X, L).$$

- ... it is a Hilbert space with the $\mathcal{L}^2$-inner product.

GEOMETRIC QUANTIZATION IN KÄHLER GEOMETRY

- (X, ω) Kähler manifold, (L, h) Hermitian holomorphic line bundle ...
- Prequantum condition: $\omega = c_1(L, h)$.
- In this case, (L, h) is uniformly positive ... guaranteeing existence of many holomorphic sections ...
- Planck constant $\hbar \simeq 1/p$, $p \in \mathbb{N}$ corresponding to tensor power of L,

$$(L^p, h_p) := (L^{\otimes p}, h^{\otimes p}).$$

- Space of Quantum States of level p is the Hilbert space

$$\mathcal{H}_\hbar = H^0_{(2)}(X, L^p).$$

- $d_p := \dim_{\mathbb{C}} H^0_{(2)}(X, L^p) \in \mathbb{N} \cup \{+\infty\}$.
- If X compact, $d_p = \mathcal{O}(p^n)$.
- A fundamental principle states that quantum mechanics contains the classical one as the limiting case $\hbar \to 0$, represented by the Kähler form ω.
- Semi-classical limit: $p \to +\infty$.

THE GENERAL QUESTIONS ON RANDOM ZEROS

- ... aim to study the random quantum states and their related quantities under semi-classical limit ...

- More precisely ... we focus on ...

1. Define the random holomorphic section $\boldsymbol{S}_p$ from/valued in each $H^0_{(2)}(X, L^p)$, such as defining some Gaussian ensembles $\{\boldsymbol{S}_p\}_p$.

2. Study the asymptotic behaviours of the zeros of $\boldsymbol{S}_p$ as $p \to +\infty$.

THREE BASIC EXAMPLES IN DIMENSION ONE, I

For $p = 1, 2, 3, \cdots$, and η_j, $j = 1, 2, 3, \cdots$ i.i.d. standard complex Gaussian random variables.

BOGOMOLNY-BOHIGAS-LEBOEUF '96, LEBOEUF '99, SODIN-TSIRELSON '04,'05, ZREBIEC '07 ....

- Elliptic model (known as $\mathrm{SU}(2)$-polynomial): on Riemann sphere $\mathbb{CP}^1 \simeq \mathbb{C} \cup \{\infty\}$

$$\boldsymbol{S}_p^{\mathrm{ell}}(z) = \sum_{j=0}^{p} \eta_j \sqrt{(p+1)\binom{p}{j}} z^j.$$

- Flat model (Flat Gaussian Analytic Function, flat GAF): on $\mathbb{C}$

$$\boldsymbol{S}_p^{\mathrm{flat}}(z) = \sum_{j=0}^{\infty} \eta_j \sqrt{\frac{p^{j+1}}{j!}} z^j.$$

- Hyperbolic model: on unit disc $\mathbb{D} = \{|z| < 1\} \subset \mathbb{C}$

$$\boldsymbol{S}_p^{\mathrm{hyp}}(z) = \sum_{j=0}^{\infty} \eta_j \sqrt{(p-1)\binom{p+j-1}{j}} z^j.$$

- Zeros set $Z(\boldsymbol{S}_p^{\bullet})$ of $\boldsymbol{S}_p^{\bullet}$ are isolated points in the space $X_{\bullet} = \mathbb{CP}^1, \mathbb{C}, \mathbb{D}$.
- View $Z(\boldsymbol{S}_p^{\bullet})$ as a measure on $X_{\bullet}$, for test function h: $\langle [Z(\boldsymbol{S}_p^{\bullet})], h \rangle = \sum_{\boldsymbol{S}_p^{\bullet}(z)=0} h(z)$.

THREE BASIC EXAMPLES IN DIMENSION ONE, II

Models	X	$\omega = c_1(L,h)$	L	$h = e^{-\phi}$
Elliptic	$\mathbb{CP}^1 \simeq \mathbb{C} \cup \{\infty\}$	$\omega_{\mathrm{FS}} = \frac{\sqrt{-1}}{2\pi} \frac{dz\wedge d\bar{z}}{(1+\|z\|^2)^2}$	$\mathscr{O}(1)$	h_{FS}
Flat*	$\mathbb{C}$	$\omega_{\mathrm{flat}} = \frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z}$	trivial $\underline{\mathbb{C}}$	$\|1\|_h^2 = e^{-\|z\|^2}$
Hyperbolic	$\mathbb{D} = \{\|z\| < 1\}$	$\omega_{\mathrm{Poinc}} = \frac{\sqrt{-1}}{2\pi} \frac{dz\wedge d\bar{z}}{(1-\|z\|^2)^2}$	trivial $\underline{\mathbb{C}}$	$\|1\|_h^2 = 1 - \|z\|^2$

* also known as Bargmann-Fock model.

Models	ONB of $H^0_{(2)}(X, L^p)$	$d_p = \dim_{\mathbb{C}} H^0_{(2)}(X, L^{\otimes p})$
Elliptic	$S_j^p = \sqrt{(p+1)\binom{p}{j}} z^j,\ j = 0, \ldots, p$	$d_p = p + 1$
Flat	$S_j^p = \sqrt{\frac{p^{j+1}}{j!}} z^j,\ j \in \mathbb{N}$	$d_p = +\infty$
Hyperbolic	$S_j^p = \sqrt{(p-1)\binom{p+j-1}{j}} z^j,\ j \in \mathbb{N}$	$d_p = +\infty$

- Gaussian holomorphic functions/sections in terms of orthonormal basis $\{S_j^p\}_j$:

$$\boldsymbol{S}_p^{\bullet} = \sum_j \eta_j S_j^p .$$

- Random zeros as measures on $X_\bullet$:

$$\frac{1}{p}\mathbb{E}[[Z(\boldsymbol{S}_p^{\bullet})]] = c_1(L,h) = \text{ locally } \frac{\sqrt{-1}}{2\pi} \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j .$$

- Large deviation estimates (concentration of measure) and central limit theorem also hold.

Standard Gaussian holomorphic sections

- $\{S_j^p\}_{j=1}^{d_p}$ ONB of $(H^0_{(2)}(X, L^p), \langle\cdot,\cdot\rangle_{\mathcal{L}^2})$.
- $\eta = \{\eta_j\}_{j=1}^{d_p}$ i.i.d. standard complex Gaussian random variables.

Standard Gaussian holomorphic section

$$\boldsymbol{S}_p(x) := \sum_{j=1}^{d_p} \eta_j S_j^p(x).$$

1. If $d_p < \infty$, it is equivalent to equip $H^0_{(2)}(X, L^p)$ with Gaussian probability measure.
2. When $d_p = \infty$ ... from random functions or random power series: Littlewood-Offord, Offord, Kahane, Edelman-Kostlan, Sodin-Tsirelson, etc.
3. Due to the ellipticity of $\bar{\partial}$-operator, we have the locally uniform convergence of Bergman kernel function
$$B_p(x) = \sum_{j=1}^{\infty} |S_j^p(x)|^2_{h_p} < \infty.$$
4. $\boldsymbol{S}_p$ is almost surely a holomorphic section of L^p over X
5. Uniqueness: distribution of $\boldsymbol{S}_p$ is independent of choices of ONB $\{S_j^p\}_j$.

ZEROS OF HOLOMORPHIC SECTIONS

- s_p holomorphic section of holomorphic line bundle L^p.
- ... zeros set $Z(s_p)$ is a complex submanifold of X with dimension $(n-1)$, locally think of

$$\mathbb{C}^{n-1} \subset \mathbb{C}^n.$$

- $\varphi \in \Omega^{(n-1,n-1)}_{\mathrm{comp}}(X)$ a test form, we study $Z(s_p)$ via the functional

$$\langle [Z(s_p)], \varphi \rangle := \int_{Z(s_p)} \varphi|_{Z(s_p)} \in \mathbb{C}.$$

  this functional $[Z(s_p)]$ is called $(1,1)$-current on X.
- Poincaré-Lelong formula, as $(1,1)$-currents,

$$[\mathrm{Z}(s_p)] = \frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} \log |s_p|^2_{h_p} + pc_1(L,h).$$

- ... means that for $\varphi \in \Omega^{(n-1,n-1)}_{\mathrm{comp}}(X)$,

$$\begin{aligned}\langle [\mathrm{Z}(s_p)], \varphi \rangle &= \frac{\sqrt{-1}}{2\pi} \int_X \log |s_p|^2_{h_p} \partial\overline{\partial}\varphi + p \int_X c_1(L,h) \wedge \varphi \\ &=: \langle \frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} \log |s_p|^2_{h_p}, \varphi \rangle + p\langle c_1(L,h), \varphi \rangle.\end{aligned}$$

EQUIDISTRIBUTION AND LARGE DEVIATION RESULTS FOR ZEROS OF $\boldsymbol{S}_p$

- (X,ω) Kähler manifold with prequantum line bundle (L,h) ...
- If X is non-compact, we need assumptions of *bounded geometry*: $g^{TX}(\cdot,\cdot)=\omega(\cdot,J\cdot)$ is complete and $\sqrt{-1}R^{\det(T^{(1,0)}X)}>-C_0\omega$.

THEOREM 0 (DREWITZ-L.-MARINESCU, 2021, 2023)

Given a test form $\varphi\in\Omega^{(n-1,n-1)}_{\mathrm{comp}}(X)$, almost surely, as $p\to\infty$

$$\frac{1}{p}\langle[Z(\boldsymbol{S}_p)],\varphi\rangle\longrightarrow\langle c_1(L,h),\varphi\rangle:=\int_X c_1(L,h)\wedge\varphi.$$

Weak convergence of $(1,1)$-currents on X: $\frac{1}{p}\mathbb{E}[[Z(\boldsymbol{S}_p)]]\longrightarrow c_1(L,h)$.

Large Deviation Estimate (Concentration of measure):

$$\mathbb{P}\left(\left|\left(\frac{1}{p}[Z(\boldsymbol{S}_p)]-c_1(L,h),\varphi\right)\right|>\delta\right)\leq e^{-C_{\varphi,\delta}\,p^{n+1}}.$$

Upper bound on hole probability: U relatively compact domain, and assuming ∂U to be negligible,

$$\mathbb{P}(\boldsymbol{S}_p\text{ has no zeros in }U)\leq e^{-C_U\,p^{n+1}}.$$

About the zeros of random holomorphic sections

- For compact X, these results were already known: Nonnenmacher-Voros (1998) on torus, Shiffman-Zelditch (1999), Shiffman-Zelditch-Zrebiec (2008) ... Dinh-Sibony (2006, convergence speed of random zeros) ...
- For non-compact X: Dinh-Marinescu-Schmidt (2012), Drewitz-L.-Marinescu (2021, 2023).
- Other extensions:
  - General probability measures (universality results): Bayraktar-Coman-Marinescu (2020), Drewitz-L.-Marinescu (2021) ...
  - Semi-positive line bundle on Riemann surfaces: Marinescu-Savale (2023, 2024), L.-Zielinski (2024) ...
  - General sequence of line bundles (L_p, h_p) instead of tensor powers: Coman-Ma-Marinescu (2017), Coman-Lu-Ma-Marinescu (2023) ... Bojnik-Günyüz (2024) ...

Large deviation principle for hole probability

Large Deviation Principle for hole probability:

$$\lim_{p \to +\infty} \frac{1}{p^{n+1}} \log \mathbb{P}\big(\boldsymbol{S}_p \text{ has no zeros in } U\big)$$

or let the domain grow to *infinity* or *boundary of defining domain* ....

$$\lim_{r \to +\infty \text{ or boundary value}} \frac{1}{r^{2n+2}} \log \mathbb{P}\big(\boldsymbol{S} \text{ has no zeros in } U_r\big)$$

- Hyperbolic case with $p = 1$ or limiting Kac poly. on $\mathbb{D}$: Peres-Virág (Acta, 2005).
- GAF on $\mathbb{C}$ and Gaussian power series: Nishry (2010, 2011) ... Ghosh-Zeitouni (2016) ...
- Compact Riemann surfaces: Zeitouni-Zelditch (2010), Zelditch (2013); Dinh-Ghosh-Hao Wu (2024), Hao Wu - Songyan Xie (2024), ...
- $\mathrm{SU}(n+1)$-polynomial on $\mathbb{CP}^n$: Junyan Zhu (2014).
- General higher dimensional Kähler manifolds: still open.

Question: randomize the quantum states in $H^0_{(2)}(X, L^p)$

- When $d_p < \infty$ (e.g., X is compact), Gaussian holomorphic section $\boldsymbol{S}_p$ is a *good* random quantum states in $H^0_{(2)}(X, L^p)$.

- However, if $d_p = \infty$ ...

- ... $\boldsymbol{S}_p := \sum_{j=1}^{\infty} \eta_j S_j^p$ is almost never square-integrable on X, that is

$$\boldsymbol{S}_p \notin H^0_{(2)}(X, L^p), \text{ a. s.}$$

- Why? $\mathbb{P}(\sum_{j=1}^{\infty} |\eta_j|^2 = \infty) = 1$.

- Is there a *good* way to randomize the quantum states in $H^0_{(2)}(X, L^p)$ in the framework of geometric quantization?

OUTLINE

BEREZIN-TOEPLITZ QUANTIZATION, I

- Classical mechanics has a reformulation as Hamiltonian mechanics.
- ... closely related to geometry such as symplectic geometry and Poisson structures, in particular, Kähler manifold (X, ω) as a special case.
- ... dynamics of the system governed by the Hamiltonian flows associated to real functions on X, known as Hamiltonian.
- Values of a Hamiltonian are interpreted as the total energy of system, classical observable.
- To quantize the Hamiltonian mechanics is to associate each classical observable, the function f, with self-adjoint linear operators $T_{f,\hbar} \in \mathrm{End}(\mathcal{H}_\hbar)$.
- $T_{f,\hbar}$ is the quantum observable... the spectrum of $T_{f,\hbar}$ is the quantization of the values of f at level $\hbar$.
- ... a good quantization will be compatible with Poisson structure in semi-classical limit $\hbar \to 0$.
- In context of Kähler manifolds with prequantum line bundles, Berezin-Toeplitz quantization is a good one.

BEREZIN-TOEPLITZ QUANTIZATION, II

DEFINITION

Given (X, ω) Kähler manifold with prequantum line bundle (L, h). The Berezin-Toeplitz quantization is the linear map

$$\mathcal{L}^\infty(X, \mathbb{R}) \ni f \mapsto (T_{f,p})_p \in \Pi_p \mathrm{End}(H^0_{(2)}(X, L^p)).$$

- Bergman projection $B_p : \mathcal{L}^2(X, L^p) \to H^0_{(2)}(X, L^p)$.
- $T_{f,p}$: Toeplitz operator with symbol f and of level p ...
- $T_{f,p} := B_p M_f B_p$, that is, for $s_p \in H^0_{(2)}(X, L^p)$,

$$T_{f,p} s_p := B_p(f s_p) \in H^0_{(2)}(X, L^p).$$

- $T_{f,p}(x, y)$ smooth integral kernel of $T_{f,p}$ such that

$$(T_{f,p} s_p)(x) = \int_X T_{f,p}(x, y) s_p(y) \mathrm{dV}(y).$$

Photo of Otto Toeplitz Source: archives of P. Roquette, Heidelberg.

TOEPLITZ KERNEL EXPANSIONS (FOLLOWING MA-MARINESCU 2007)

- (X, ω) Kähler manifold with prequantum line bundle (L, h) ...
- Assume $g^{TX}(\cdot,\cdot) = \omega(\cdot, J\cdot)$ to be complete and $\sqrt{-1}R^{\det(T^{(1,0)}X)} > -C_0\omega$.
- Suppose that $f \in \mathcal{L}^\infty_{\mathrm{comp}}(X, \mathbb{R})$, $T_{f,p}$ is self-adjoint linear operator of trace class.
- For integer $k \geq 1$, set $T^k_{f,p} := \underbrace{T_{f,p} \circ \ldots \circ T_{f,p}}_{k \text{ times}}$ with $\mathscr{C}^\infty$ integral kernel $T^k_{f,p}(x, y)$.

TOEPLITZ KERNEL EXPANSION

Then as $p \to +\infty$, that is, taking semi-classical limit,

1. When f is smooth on open subset $U \subset X$, locally uniformly on U, we have
$$T^k_{f,p}(x,x) = f(x)^k p^n + \mathcal{O}(p^{n-1}).$$
2. If f vanishes at x with order ∞,
$$T^k_{f,p}(x,x) = \mathcal{O}(p^{-\infty}).$$
3. Then we conclude: $T_{f,p}$'s spectrum quantizes the values of f
$$\underbrace{\frac{1}{p^n}\mathrm{Tr}[T^k_{f,p}]}_{\text{spectral distribution of } T_{f,p}} \longrightarrow \underbrace{\int_X f(x)^k \mathrm{dV}(x)}_{\text{value distribution of } f}.$$

Probabilistic Berezin-Toeplitz quantization

- $f \in \mathcal{L}^\infty_{\mathrm{comp}}(X, \mathbb{R})$, a probabilistic model for Berezin-Toeplitz quantization consists of the random sections

$$\boldsymbol{S}_{f,p} := T_{f,p}\boldsymbol{S}_p.$$

- $\boldsymbol{S}_p$ is the standard Gaussian holomorphic section ...
- If $d_p < \infty$, e.g., when X is compact, then $\boldsymbol{S}_{f,p}$ is a well-defined square-integrable Gaussian holomorphic section on X (also by Ancona-Le Floch, 2022).
- However, if $d_p = \infty$ (when X is non-compact), $T_{f,p}$ does NOT act on $\boldsymbol{S}_p$ by definition.
- ... since $\boldsymbol{S}_p$ is almost never square-integrable on X.
- Abstract Wiener space by L. Gross (1965) to make a rigorous definition.

GAUSSIAN $\mathcal{L}^2$-HOLOMORPHIC SECTIONS: ABSTRACT WIENER SPACE

- For simplicity's sake, $f \in \mathcal{L}^{\infty}_{\mathrm{comp}}(X, \mathbb{R})$, we assume $f \geq 0$ is nontrivial and f is smooth on X except on a closed subset of null measure.
- Ex: $f = \mathbf{1}_B$, indicator function of a geodesic ball $B \subset X$.
- $T_{f,p}$ injective, positive, trace-class (hence Hilbert-Schmidt) operator on $H^0_{(2)}(X, L^p)$.
- Abstract Wiener space (L. Gross 1965):
  - $\mathcal{B}_f(X, L^p)$=Hilbert space as completion of $H^0_{(2)}(X, L^p)$ under norm $\|T_{f,p} \cdot \|_{\mathcal{L}^2}$;
  - there exists a unique Gaussian probability measure $\mathcal{P}_{f,p}$ on $\mathcal{B}_f(X, L^p)$ such that ...
  - ... it extends Gaussian measures of any finite dimensional subspaces of $H^0_{(2)}(X, L^p)$ w.r.t. $\mathcal{L}^2$-metric.
- $T_{f,p} : \mathcal{B}_f(X, L^p) \to H^0_{(2)}(X, L^p)$ isometry of Hilbert spaces.
- ... we obtain a Gaussian probability measure $\mathbb{P}_{f,p}$ on $H^0_{(2)}(X, L^p)$.
- Gaussian $\mathcal{L}^2$-holomorphic sections: $\boldsymbol{S}_{f,p} \sim (H^0_{(2)}(X, L^p), \mathbb{P}_{f,p})$, which is the rigorous version for the action of $T_{f,p}$ on $\boldsymbol{S}_p$.

EQUIDISTRIBUTION AND LARGE DEVIATION ON THE SUPPORT

THEOREM 1 (DREWITZ-L.-MARINESCU, 2023, 2024)

If $U \subset \mathrm{supp}(f)$, then we have the weak convergence of $(1,1)$-currents as $p \to \infty$,

$$\frac{1}{p}\mathbb{E}[[Z(\boldsymbol{S}_{f,p})]|_U] \to c_1(L,h)|_U.$$

and almost surely,

$$\frac{1}{p}[Z(\boldsymbol{S}_{f,p})]|_U \to c_1(L,h)|_U.$$

THEOREM 2 (DREWITZ-L.-MARINESCU, 2024)

If $U \subset \mathrm{supp}(f)$, then we have

Large Deviation Estimate: for a test form $\varphi \in \Omega^{(n-1,n-1)}_{\mathrm{comp}}(U)$,

$$\mathbb{P}\left(\left|\langle\frac{1}{p}[Z(\boldsymbol{S}_{f,p})] - c_1(L,h),\varphi\rangle\right| > \delta\right) \leq e^{-C_{\varphi,\delta}\,p^{n+1}}.$$

Hole Probability: assuming in addition ∂U to be negligible,

$$\mathbb{P}(\,\boldsymbol{S}_{f,p} \text{ has no zeros in } U\,) \leq e^{-C_U\,p^{n+1}}.$$

CENTRAL LIMIT THEOREM ON THE SUPPORT

THEOREM 3 (DREWITZ-L.-MARINESCU, 2024)

Fix $f \in \mathscr{C}^\infty_c(X, \mathbb{R}_{\geq 0})$ which is not identically zero, and let U be an open subset of X such that $\overline{U} \subset \{f \neq 0\}$. Let φ be a real $(n-1, n-1)$-form on X with $\mathscr{C}^3$-coefficients such that $\operatorname{supp}\varphi \subset U$ and $\partial\overline{\partial}\varphi \not\equiv 0$, set

$$Z_{f,p}(\varphi) := \langle [Z(\boldsymbol{S}_{f,p})], \varphi \rangle,$$

then the distribution of the real random variable

$$p^{n/2}\langle [Z(\boldsymbol{S}_{f,p})] - pc_1(L, h_L), \varphi \rangle$$

converges weakly to $\mathcal{N}_{\mathbb{R}}(0, \sigma(\varphi))$ as $p \to +\infty$.

- $\sigma(\varphi) := \frac{\zeta(n+2)}{4\pi^2} \int_U |L(\varphi)(x)|^2 \mathrm{dV}(x)$, where ...
- $\zeta(n+2) = \sum_{k=1}^\infty \frac{1}{k^{n+2}}$, and $L(\varphi)$ is a function on X given by

$$\sqrt{-1}\partial\overline{\partial}\varphi = L(\varphi)\frac{c_1(L, h_L)^n}{n!}.$$

- ... this extends a result of Shiffman-Zelditch (2010) for compact Kähler manifolds, the essential step is given in a seminal work of Sodin-Tsirelson (2004).

Outline

Why on the support of function? An example $f = \mathbf{1}_U$

Let us consider the zeros of *classical* Gaussian random section $\boldsymbol{S}_p$ in the relative compact domain U:

- Take the indicator function : $\mathbf{1}_U(x) = \begin{cases} 1 & \text{if } x \in U; \\ 0 & \text{if else}. \end{cases}$
- It is equivalent to study the zeros of the random section $\mathbf{1}_U \boldsymbol{S}_p$ in U ...
- $\mathbf{1}_U \boldsymbol{S}_p$ is $\mathcal{L}^2$-integrable, but it is not globally holomorphic on $\mathbb{CP}^1$ ....
- ... apply Bergman projection: consider new random $\mathcal{L}^2$-holomorphic section $B_p(\mathbf{1}_U \boldsymbol{S}_p)$ on X ...
- it is exactly our Gaussian $\mathcal{L}^2$-holomorphic section $\boldsymbol{S}_{f,p}$ with $f = \mathbf{1}_U$.
- **In semi-classical limit**: zeros of $\boldsymbol{S}_{f,p}$ in $\overline{U} \simeq$ zeros of $\boldsymbol{S}_p$ in $\overline{U} = \operatorname{supp} f$,
- ... which means: we expect everything for the zeros of $\boldsymbol{S}_p$ in $\overline{U}$ to hold in the same way for the zeros of $\boldsymbol{S}_{f,p}$ in $\overline{U} = \operatorname{supp} f$.

Sketched proof of convergence of expectations

- ... bounded $f \geq 0$ is smooth on X except on a closed subset of null measure.
- To prove $\frac{1}{p}\mathbb{E}[[Z(\boldsymbol{S}_{f,p})]|_U] \to c_1(L,h)|_U$ ...
- Poincaré-Lelong formula says $[Z(\boldsymbol{S}_{f,p})] = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log|\boldsymbol{S}_{f,p}|^2_{h_p} + pc_1(L,h)$.
- Then we get (also known as Edelman-Kostlan formula)

$$\frac{1}{p}\mathbb{E}[[Z(\boldsymbol{S}_{f,p})]|_U] - c_1(L,h)|_U = \frac{\sqrt{-1}}{2\pi p}\partial\overline{\partial}\log T^2_{f,p}(x,x).$$

- Recall $T^2_{f,p}(x,x) = f(x)^2 p^n + \mathcal{O}(p^{n-1})$, hence

$$\frac{\sqrt{-1}}{2\pi p}\partial\overline{\partial}\log T^2_{f,p}(x,x) = \mathcal{O}(\frac{\log p}{p})$$ as currents near nonvanishing point of f.

- However, when $U \subset \mathrm{supp}(f)$, f still can vanish inside U.
- We use the techniques from theory of pluri-subharmonic functions to conclude the convergence on whole U.

ON THE PROOFS OF THEOREMS 1 AND 2

- The almost sure convergence of $\frac{1}{p}[Z(\boldsymbol{S}_{f,p})]|_U$ to $c_1(L,h)|_U$ as well as the hole probability are the consequences of Large Deviation Estimate in Theorem 2.
- ... and Theorem 2 follows from the proposition below together with the Poincaré-Lelong formula.

PROPOSITION 1 (DREWITZ-L.-MARINESCU, 2024)

As in Theorem 2, $U \subset \mathrm{supp}(f)$, define $\mathcal{M}_p^U(\boldsymbol{S}_{f,p}) = \sup_{x\in U} |\boldsymbol{S}_{f,p}(x)|_{h_p}$. Then for $p \in \mathbb{N}$,

$$\mathbb{P}\left(\left|\log \mathcal{M}_p^U(\boldsymbol{S}_{f,p})\right| \geq \delta p\right) \leq e^{-C_{U,\delta}p^{n+1}} .$$

- ... near-diagonal expansions of $T_{f,p}^2(x,y)$: Ma-Marinescu (2007)
- For $\mathrm{dist}(x,y) \lesssim \sqrt{\frac{\log p}{p}}$ and for $f(x) \neq 0$, $f(y) \neq 0$,

$$N_{f,p}(x,y) := \frac{|T_{f,p}^2(x,y)|_{h_{p,x}\otimes h_{p,y}^*}}{\sqrt{T_{f,p}^2(x,x)}\sqrt{T_{f,p}^2(y,y)}} \simeq (1+o(1))\exp\left(-\frac{p\pi}{2}\mathrm{dist}(x,y)^2\right).$$

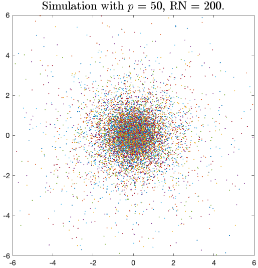
- Gaussian field $U \ni x \mapsto \boldsymbol{S}_{f,p}(x) \in L_x^p$ ... correlation function $N_{f,p}(x,y)$.

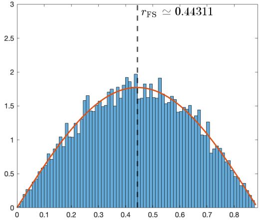
WHAT HAPPENS OUTSIDE THE SUPPORT OF f?

Simulation of zeros of $\boldsymbol{S}_p^{\mathrm{ell}}(z)$ on local chart $U_0 \cong \mathbb{C}$ of $\mathbb{CP}^1$, note that $\boldsymbol{S}_p^{\mathrm{ell}} = \boldsymbol{S}_{f,p}$ with $f \equiv 1$ on $\mathbb{CP}^1$.

$$\boldsymbol{S}_p^{\mathrm{ell}}(z) = \sum_{j=0}^{p} \eta_j \sqrt{(p+1)\binom{p}{j}} z^j.$$

RIGHT = density histogram w.r.t. Fubini-Study modulus of zeros.





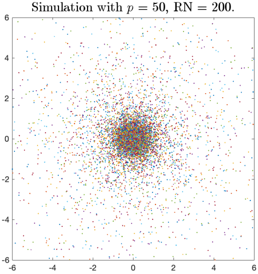
Red curve on right = radial density function $\psi(r_{\mathrm{FS}}) = \sqrt{\pi}\sin\left(2\sqrt{\pi}r_{\mathrm{FS}}\right)$, representing the limiting distribution $\omega_{\mathrm{FS}} = \frac{\sqrt{-1}dz \wedge d\bar{z}}{2\pi(1+|z|^2)^2}$.

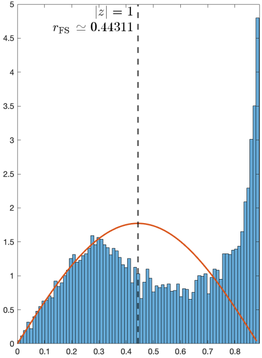
WHAT HAPPENS OUTSIDE THE SUPPORT OF f?

Simulation of zeros of $\boldsymbol{S}_{f,p}(z)$ on local chart $U_0 \cong \mathbb{C}$ of $\mathbb{CP}^1$, where ...

$$f(z) = \mathbf{1}_{\mathbb{D}}$$

is the indicator function of unit $\mathbb{D} \subset \mathbb{C}$, on Riemann sphere, $\mathbb{D}$ represents the southern hemisphere. $\mathrm{supp}(f) = \mathbb{D} = \{r_{\mathrm{FS}} \leq \frac{\sqrt{\pi}}{4} \simeq 0.44311\ldots\}$





PARTIAL BERGMAN KERNELS: ZELDITCH-PENG ZHOU (2019)

- Still consider $(\mathbb{CP}^1, \mathcal{O}(p))$ and a local chart $U_0 \simeq \mathbb{C}$.
- Function $f := \mathbf{1}_{\mathbb{D}}$ on $\mathbb{CP}^1$, note that $\mathrm{Vol}_{\mathrm{FS}}(\mathbb{D}) = 1/2$ ...
- Toeplitz spectra quantize values of f:
  For $p \gg 1$, half number of eigenvalues of $T_{f,p} \simeq 1$, and another half $\simeq 0$.
- Fix $a = 0.8 \leq 1 = \max f$, let $H^0(\mathbb{CP}^1, \mathcal{O}(p))_{\geq 0.8}$ be subspace of $H^0(\mathbb{CP}^1, \mathcal{O}(p))$ spanned by the eigensections of $T_{f,p}$ associated with eigenvalues ≥ 0.8 ....
- ... we have $d'_p := \dim H^0(\mathbb{CP}^1, \mathcal{O}(p))_{\geq 0.8} \simeq \frac{1}{2} \dim H^0(\mathbb{CP}^1, \mathcal{O}(p)) \simeq \frac{p}{2}$.
- Partial Bergman projection:

$$B_{p,f,\geq 0.8} : \mathcal{L}^2(\mathbb{CP}^1, \mathcal{O}(p)) \to H^0(\mathbb{CP}^1, \mathcal{O}(p))_{\geq 0.8},$$

- .... with partial Bergman kernel function $B_{p,f,\geq 0.8}(x) = \sum_{j=1}^{d'_p} |S_j^p(x)|^2_{h_p}$, where $\{S_j^p\}_{j=1}^{d'_p}$ is an ONB of $H^0(\mathbb{CP}^1, \mathcal{O}(p))_{\geq 0.8}$.

THEOREM (ZELDITCH-ZHOU, 2019, WITH PROPER ASSUMPTIONS FOR GENERAL f)

For the above example, as $p \to +\infty$, we have on $\mathbb{CP}^1 \setminus \partial\mathbb{D}$ (since $\mathbb{D} = \{f \geq 0.8\}$)

$$\frac{B_{p,f,\geq 0.8}(x)}{B_p(x)} \to \mathbf{1}_{\mathbb{D}}$$

Zeros of *partial* Gaussian holomorphic section

- Define the *partial* Gaussian holomorphic section:

$$\boldsymbol{S}_{p,f,\geq 0.8} := \sum_{j=1}^{d'_p} \eta_j S_j^p,$$

where $\{S_j^p\}_{j=1}^{d'_p}$ is an ONB of $H^0(\mathbb{CP}^1, \mathcal{O}(p))_{\geq 0.8}$.

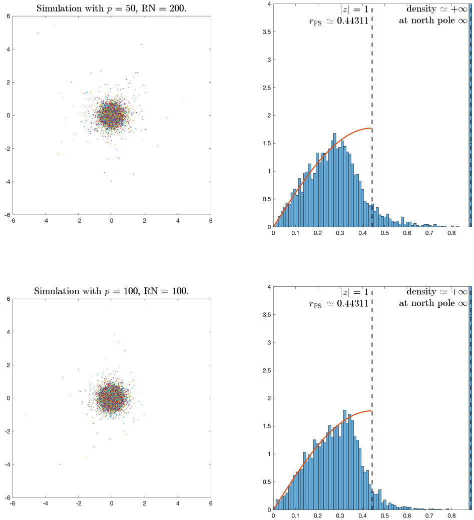
- Question: what is the asymptotic distribution of $Z(\boldsymbol{S}_{p,f,\geq 0.8})$ on $\mathbb{CP}^1$ as $p \to +\infty$?
- With certain assumptions: limiting distribution given by Zelditch-Zhou (2019).
- Roughly speaking: in our setting $\boldsymbol{S}_{p,f,\geq 0.8}$ is a random polynomial of degree $d'_p \simeq \frac{1}{2}p$ as a section of $\mathcal{O}(p)$ on $\mathbb{CP}^1$, that is

$$\boldsymbol{S}_{p,f,\geq 0.8} \simeq \sum_{j=0}^{d'_p \simeq p/2} \eta_j \sqrt{(p+1)\binom{p}{j}} z^j,$$

- ... as section of $\mathcal{O}(p)$, it vanishes at north pole ∞ with multiplicity $p - d'_p \simeq p/2$ ...
- ... half number of roots of $\boldsymbol{S}_{p,f,\geq 0.8}$ are uniformly distributed in southern hemisphere of $\mathbb{CP}^1$, that corresponding to $\mathbb{D} = \operatorname{supp} f$ in local chart $U_0 \simeq \mathbb{C}$, another half of roots concentrate at the north pole $\infty \in \mathbb{CP}^1$, the north pole is the farthest point from $\operatorname{supp} f$.

SIMULATION FOR ZEROS OF *partial* GAUSSIAN HOLOMORPHIC SECTION

Limiting distribution of zeros of $\boldsymbol{S}_{p,f,\geq 0.8} \simeq \omega_{\mathrm{FS}}|_{\mathbb{D}} + \frac{1}{2}\delta_{\infty}$.



Thank you.