Random holomorphic sections on noncompact complex manifolds

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Quantization in Geometry

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EQUIDISTRIBUTION OF RANDOM ZEROS

- (X, ω) compact Kähler manifold with Hermitian line bundle (L, h),

$$\omega = c_1(L, h).$$

- Υ_p Gaussian probability measure on $H^0(X, L^p)$ w.r.t. $\mathcal{L}^2$-metric.
- Shiffman-Zelditch (1999): $(s_p)_{p\in\mathbb{N}} \in \prod_p (H^0(X, L^p), \Upsilon_p)$, with probability one (almost surely), we have weak convergence of $(1,1)$-currents

$$\frac{1}{p}[\mathrm{Div}(s_p)] \to c_1(L, h) := \frac{i}{2\pi} R^L, \text{ as } p \to \infty.$$

- Ingredients in proof:
  - Poincaré-Lelong formula, as $(1,1)$-currents on X,

$$[\mathrm{Div}(s_p)] = \frac{i}{2\pi}\partial\overline{\partial}\log|s_p|^2_{h^p} + pc_1(L, h).$$

  - Bergman kernel: $B_p(x, x) \simeq p^{\dim_{\mathbb{C}} X} + \text{lower order terms}.$

Example: $\mathrm{SU}(2)$-polynomial

Bogomolny-Bohigas-Leboeuf '96, Shiffman-Zelditch '99:

- Kähler manifold $(\mathbb{CP}^1, \omega_{\mathrm{FS}})$, prequantum line bundle $(L, h) = (\mathcal{O}(1), h_{\mathrm{FS}})$.
- $H^0(\mathbb{CP}^1, \mathcal{O}(p) = \mathcal{O}(1)^{\otimes p}) \simeq \mathbb{C}_p[z]$.
- Random $\mathrm{SU}(2)$-polynomial:

$$f_p(z) = \sum_{j=0}^{p} \eta_j \sqrt{(p+1)\binom{p}{j}}\, z^j.$$

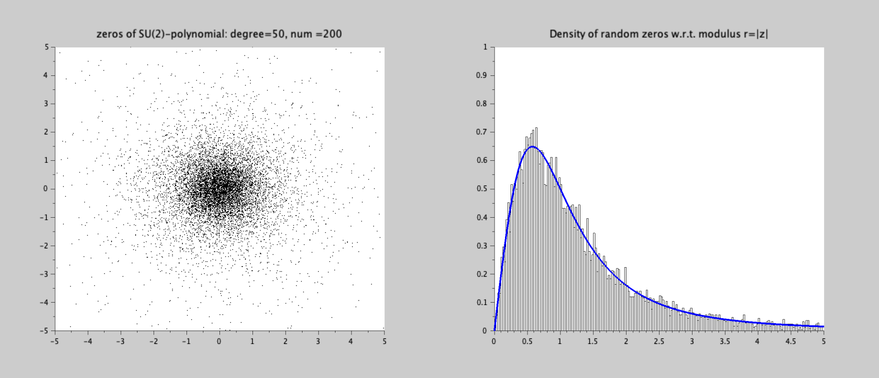
- $\eta_0, \eta_1, \eta_2, \cdots$ i. i. d. (independent and identically distributed) standard complex Gaussian random variables.
- Random zeros are *asymptotically* uniformly distributed on $\mathbb{CP}^1$, as $p \to +\infty$.

Remark

Quantum chaotic dynamics on torus: Nonnenmacher-Voros (1998).

Zeros of SU(2)-polynomial

Simulation of random zeros on local chart $U_0 \cong \mathbb{C}$, RIGHT = density histogram w.r.t. modulus of zeros.



Blue line = radial density function $\frac{2r}{(1+r^2)^2}$, representing the limit distribution $\omega_{\mathrm{FS}} = \frac{i dz \wedge d\bar{z}}{2\pi(1+|z|^2)^2}$.

EQUIDISTRIBUTION OF RANDOM ZEROS: GENERALIZATIONS

Extensions (for random holomorphic sections) on compact complex manifolds or normal Kähler spaces:

- ... via meromorphic transformation: Dinh-Sibony (2006), with convergence speed.
- Singular Hermitian line bundle, normal Kähler space, general sequence of line bundles ... : Coman-Marinescu (2015, 2020), Dinh-Ma-Marinescu (2016), Coman-Ma-Marinescu (2017), Coman-Marinescu-Nguyên (2018), Coman-Lu-Ma-Marinescu (2020), etc.
- ... general class of probability measures on holomorphic sections: Bayraktar-Coman-Marinescu (2020), etc.
- Linear statistics/Variance of random zeros : Bleher-Shiffman-Zelditch (2000), Shiffman-Zelditch (2008, 2010), Shiffman (2021), etc.

Large deviation & Hole probability

- Shiffman-Zelditch-Zrebiec (2008): **Large Deviation Estimates**.
- Compact Kähler manifold (X, ω) with (L, h) and $\omega = c_1(L, h)$.
- Gaussian probability measure Υ_p on $H^0(X, L^p)$.
- Set $m = \dim_{\mathbb{C}} X$, for $\varphi \in \Omega^{(m-1,m-1)}(X)$, $\delta > 0$, as p large,

$$\Upsilon_p\left(\left\{s_p \ : \ \left|\left(\frac{1}{p}[\mathrm{Div}(s_p)] - c_1(L,h), \varphi\right)\right| > \delta\right\}\right) \leq e^{-C_{\varphi,\delta}\, p^{m+1}}.$$

- **Hole Probability**: fix $U \subset X$ open

$$\Upsilon_p(\{s_p \text{ has no zeros in } U\}) \leq e^{-C_U\, p^{m+1}}.$$

- With a further assumption for small open set U,

$$\Upsilon_p(\{s_p \text{ has no zeros in } U\}) \geq e^{-C'_U\, p^{m+1}}.$$

QUESTION: FOR NONCOMPACT COMPLEX MANIFOLDS

- Compact $\Rightarrow$ noncompact ?
- (X, ω) noncompact complete Hermitian manifold, *bounded geometry*.
- (L, h) uniformly positive ...
- $H^0_{(2)}(X, L)$: separable Hilbert space of $\mathcal{L}^2$-holomorphic sections,

$$d_p := \dim_{\mathbb{C}} H^0_{(2)}(X, L^p) \in \mathbb{N}_0 \cup \{\infty\}.$$

- If $d_p = \mathcal{O}(p^m)$, e.g. cusp forms on arithmetic surface (with cusps)
  - By Dinh-Marinescu-Schmidt (2012), Drewitz-L.-Marinescu (2021), etc.
  - ... not just Gaussian probability measures.
  - Bergman kernel expansions and estimates: Ma-Marinescu (2007), Auvray-Ma-Marinescu (2016, 2021).
- What happens if $d_p = \infty$?

OUTLINE

COMPLETE HERMITIAN MANIFOLD....

GEOMETRIC SETTING

- (X^m, J, ω) noncompact such that $g^{TX}(\cdot,\cdot) = \omega(\cdot, J\cdot)$ is complete,

$$iR^{\det(T^{(1,0)}X)} > -C_0\omega, \quad |\partial\omega|_{g^{TX}} < C_0.$$

- (L, h) holomorphic line bundle such that

$$iR^L > \varepsilon_0\omega.$$

- $d_p = \dim_{\mathbb{C}} H^0_{(2)}(X, L^p) \in \mathbb{N}_0 \cup \{\infty\}$.

- Bergman projection $B_p : \mathcal{L}^2(X, L^p) \to H^0_{(2)}(X, L^p)$.

- Ma-Marinescu (2007): Bergman kernel expansion uniformly on compact subsets...

Bergman kernel expansions

- $S_p = \{S_j^p\}_{j=1}^{d_p}$ ONB of $(H^0_{(2)}(X, L^p), \langle\cdot,\cdot\rangle_{\mathcal{L}^2})$.
- Bergman function: $B_p(x,x) = \sum_{j=1}^{d_p} |S_j^p(x)|^2_{h^p}$.

Bergman kernel expansion

On any given compact subset of X, and in any local $\mathscr{C}^\ell$-norm, we have

$$B_p(x,x) = \sum_{r=0}^{\infty} b_r(x) p^{n-r} + \mathcal{O}(p^{-\infty}).$$

1. $b_0(x) = \det \frac{c_1(L,h_L)}{\omega} \geq \left(\frac{\varepsilon_0}{2\pi}\right)^n > 0.$
2. on any compact subset, for $p \gg 0$,

$$\partial\overline{\partial} \log B_p(x,x) = \mathcal{O}(1).....$$

3. if prequantum condition also holds, i.e. $\omega = c_1(L, h_L)$, then

$$\partial\overline{\partial} \log B_p(x,x) = \mathcal{O}(\frac{1}{p}).....$$

Gaussian random holomorphic sections (I)

- $\eta = \{\eta_j\}_{j=1}^{d_p}$ i.i.d. standard complex Gaussian random variables.

Gaussian random holomophic section

Define

$$\Psi_\eta^{S_p}(x) = \sum_{j=1}^{d_p} \eta_j S_j^p(x).$$

1. If $d_p < \infty$, it is equivalent to equip $H^0_{(2)}(X, L^p)$ with Gaussian probability measure.
2. Focus on case $d_p = \infty$...
3. ... from random functions or random power series: Littlewood-Offord, Offord, Kahane, Edelman-Kostlan, Sodin-Tsirelson, Peres-Virág, etc.

Gaussian random holomorphic sections (II)

When dimension $d_p = \infty$ ...

$$\Psi_{\eta}^{S_p}(x) = \sum_{j=1}^{\infty} \eta_j S_j^p(x).$$

We have:

- $\Psi_{\eta}^{S_p}$ is almost surely a holomorphic section of L^p over X
- Uniqueness: probability distribution of $\Psi_{\eta}^{S_p}$ is independent of choices of ONB S_p.
- $\Psi_{\eta}^{S_p}$ is almost never $\mathcal{L}^2$ integrable on X.
- ...... $\mathrm{Prob}(\sum_{j=1}^{\infty} |\eta_j|^2 = \infty) = 1$.
- As $(1,1)$-currents, we have

$$\mathbb{E}[[\mathrm{Div}(\Psi_{\eta}^{S_p})]] = \gamma_{\mathrm{FS}}(X, L^p) := pc_1(L,h) + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log B_p(x,x).$$

MAIN RESULTS: LARGE DEVIATION & HOLE PROBABILITY

THEOREM 1 (DREWITZ-L.-MARINESCU, 2023)

We have the weak convergence of $(1,1)$-currents on X as $p \to \infty$,

$$\frac{1}{p}\mathbb{E}[[\mathrm{Div}(\Psi_\eta^{S_p})]] \to c_1(L,h).$$

For φ test form with compact support, almost surely,

$$\frac{1}{p}\langle[\mathrm{Div}(\Psi_\eta^{S_p})],\varphi\rangle \to \langle c_1(L,h),\varphi\rangle.$$

Large Deviation Estimate:

$$\mathrm{Prob}(\{\left|(\frac{1}{p}[\mathrm{Div}(\Psi_\eta^{S_p})] - c_1(L,h),\varphi)\right| > \delta\}) \leq e^{-C_{\varphi,\delta}\, p^{m+1}}.$$

PROPOSITION 1 (DREWITZ-L.-MARINESCU, 2023)

Hole Probability: for U relatively compact (with ∂U having measure zero),

$$\mathrm{Prob}(\{\Psi_\eta^{S_p} \text{ has no zeros in } U\}) \leq e^{-C_U\, p^{m+1}}.$$

MAIN RESULTS (II)

- For U open (non-empty) subset of X, $s_p \in H^0(X, L^p)$, define

$$\mathcal{M}_p^U(s_p) = \sup_{x \in U} |s_p(x)|_{h^p}.$$

THEOREM 2 (DREWITZ-L.-MARINESCU, 2023)

For U relatively compact, for $\delta > 0$, there exists constant $C_{U,\delta} > 0$ such that for $p \in \mathbb{N}$,

$$\mathrm{Prob}(|\log \mathcal{M}_p^U(\Psi_\eta^{S_p})| \geq \delta p\}) \leq e^{-C_{U,\delta} p^{m+1}}.$$

Applying Poincaré-Lelong formula, Theorem 1 follows as consequence of Theorem 2.

$$[\mathrm{Div}(\Psi_\eta^{S_p})] = \frac{i}{2\pi} \partial \overline{\partial} \log |\Psi_\eta^{S_p}|_{h^p}^2 + p c_1(L, h).$$

Example: Bargmann-Fock space

- Bargmann-Fock space $H^0_{(2)}(X, L)$:

$$X = \mathbb{C}^m,\ \omega_{\text{flat}} = \frac{i}{2\pi}\sum_{j=1}^m dz_j \wedge d\bar{z}_j,\ L = \mathbb{C},\ |1|^2_h(z) = e^{-|z|^2}.$$

- $H^0_{(2)}(\mathbb{C}^m, L)$ has a canonical ONB: $\alpha \in \mathbb{N}^m$,

$$S_\alpha(z) = \frac{z^\alpha}{\sqrt{\alpha!}}.$$

- Flat Gaussian random holomorphic function (or flat G.A.F. or G.E.F.) on $\mathbb{C}^m$:

$$F_\eta(z) = \sum_{\alpha\in\mathbb{N}^m} \eta_\alpha S_\alpha(z).$$

- It is almost surely a holomorphic function on $\mathbb{C}^m$, but almost never $\mathcal{L}^2$-integrable.

RESULTS OF SODIN-TSIRELSON & ZREBIEC ON HOLE PROBABILITY

- How are zeros of F_η distributed in $\mathbb{C}^m$?
- Offord (1965, 1967), Edelman-Kostlan (1995), etc: as $(1, 1)$-currents on $\mathbb{C}^m$,

$$\mathbb{E}[[\mathrm{Div}(F_\eta)]] = c_1(L, h) = \omega_{\mathrm{flat}}.$$

THEOREM (SODIN-TSIRELSON 2005, ZREBIEC 2007)

There are constants $c > c' > 0$ such that for $r \gg 0$,

$$e^{-cr^{2m+2}} \leq \mathrm{Prob}(\{F_\eta \text{ has no zeros in ball } \mathbb{B}(0, r)\}) \leq e^{-c' r^{2m+2}}.$$

- Nishry (2010, 2011, 2012): explicit asymptotics on hole probabilities as $r \to +\infty$ for $\mathbb{C}$ $(m = 1)$.

CONNECTION TO OUR RESULTS...

- How to understand their results in our setting?
- For integer $p > 0$, scaled Bargmann-Fock space $H^0_{(2)}(\mathbb{C}^m, L^p)$ has ONB:

$$p^{m/2} S_\alpha(\sqrt{p}z), \alpha \in \mathbb{N}^m.$$

- Gaussian random holomorphic functions:

$$\Psi^{S_p}_\eta(z) = p^{m/2} F_\eta(\sqrt{p}z).$$

- Our results read, fixing $r_0 > 0$,

$$e^{-c_1 p^{m+1}} \leq \mathrm{Prob}(\{\Psi^{S_p}_\eta \text{ has no zeros in } \mathbb{B}(0, r_0)\}) \leq e^{-c_2 p^{m+1}}.$$

- ... is equivalent to

$$e^{-c_1 \sqrt{p}^{2m+2}} \leq \mathrm{Prob}(\{F_\eta \text{ has no zeros in } \mathbb{B}(0, \sqrt{p}r_0)\}) \leq e^{-c_2 \sqrt{p}^{2m+2}}.$$

- Approximating r by $\sqrt{p}r_0$, we recover the results of Sodin-Tsirelson & Zrebiec.

OUTLINE

INGREDIENTS IN THE PROOF, I

- **Theorem 2**: $\mathrm{Prob}(\{|\log \mathcal{M}_p^U(\Psi_\eta^{S_p})| \geq \delta p\}) \leq e^{-C_{U,\delta} p^{m+1}}$.
- First step: $\mathrm{Prob}(\{\mathcal{M}_p^U(\Psi_\eta^{S_p}) \geq e^{\delta p}\}) \leq e^{-C_{U,\delta} p^{m+1}}$.
- Basic idea: $U \subset U' \subset X$, f holomorphic functions on U', then

$$||f||_{\mathcal{L}^\infty(U)} \leq C_U ||f||_{\mathcal{L}^2(U')}.$$

- For $s_p \in H^0(X, L^p)$, $\mathcal{M}_p^U(s_p) \leq C_{U,p} ||s_p||_{\mathcal{L}^2(U')}$.
- Take $s_p = \Psi_\eta^{S_p}$:

$$\mathbb{E}[\mathcal{M}_p^U(\Psi_\eta^{S_p})^{2p^m}] \leq (C_{U,p})^{2p^m} p^m! \int_{U'} B_p(x,x)^{p^m} \mathrm{dV}(x)$$
$$\leq (C_{U,p})^{2p^m} e^{C'_U p^m \log p}, \quad C'_U > 0.$$

- Key technique: control the dependence of $C_{U,p}$ on $p \gg 0$.
- e_L holomorphic frame of L on U' such that

$$\sup_{x \in U'} |e_L(x)|_h = 1, \ \nu = \inf_{x \in U'} |e_L(x)|_h \leq 1.$$

- Then $C_{U,p} = \frac{C_U}{\nu^p}$, for U' sufficiently small, $\frac{1}{\nu} \leq e^{\frac{\delta}{6}}$.
- By Chebyshev inequality: $\mathrm{Prob}\big(\{\mathcal{M}_p^U(\Psi_\eta^{S_p})^{2p^m} \geq e^{2\delta p^{m+1}}\}\big) \leq e^{-\delta p^{m+1} + C'_U p^m \log p}$.

INGREDIENTS IN THE PROOF, II

- Second step: $\mathrm{Prob}\left(\left\{\mathcal{M}_p^U(\Psi_\eta^{S_p}) \leq e^{-\delta p}\right\}\right) \leq e^{-C_{U,\delta} p^{m+1}}$.
- ... arguments from Shiffman-Zelditch-Zrebiec (2008), Drewitz-L.-Marinescu (2021).
- ... near-diagonal expansions of $B_p(x,y)$: Ma-Marinescu (2007)
- Fix $k \in \mathbb{N}_{>0}$, $b > \sqrt{16k/\varepsilon_0}$, as $p \to \infty$, for $\mathrm{d}(x,y) \leq b\sqrt{\frac{\log p}{p}}$,

$$N_p(x,y) := \frac{|B_p(x,y)|_{h_x^p \otimes h_y^{p,*}}}{\sqrt{B_p(x,x)}\sqrt{B_p(y,y)}} \simeq (1+o(1)) \exp\left(-\frac{a(x)p}{4}\mathrm{d}(x,y)^2\right).$$

- $a(x) \geq \varepsilon_0 > 0$.
- Gaussian field $U \ni x \mapsto \Psi_\eta^{S_p}(x) \in L_x^p$ ... correlation function $N_p(x,y)$.
- Consider lattice $\{z_\nu^p\}_\nu$ (of points) in U of mesh $\sim c/\sqrt{p}$ with large $c > 0$...
- Values of $\Psi_\eta^{S_p}$ at z_ν^p are asymptotically uncorrelated/independent, which are a finite family of Gaussian random varaibles.
- $\mathrm{Prob}(\{\sup_\nu |\Psi_\eta^{S_p}(z_\nu^p)|_{h^p} \leq e^{-\delta p}\}) \leq e^{-Cp^{m+1}}$.

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   - Bargmann-Fock space and result of Sodin-Tsirelson & Zrebiec
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How to randomize $\mathcal{L}^2$-holomorphic sections?

- Recall: when $d_p = \infty$, a.s. $\Psi_\eta^{S_p} \notin H^0_{(2)}(X, L^p)$.
- How to find *geometric* way to randomize $\mathcal{L}^2$-holomorphic sections of L^p, for all large p ?
- One construction: for $c = (c_j)_{j\in\mathbb{N}} \in \ell^2(\mathbb{C})$, set

$$\Psi_{c,\eta}^{S_p} = \sum_{j=1}^{d_p} c_j \eta_j S_j^p.$$

- Abstract Wiener space by L. Gross (1965):

$$c = (c_j)_{j\in\mathbb{N}} \Leftrightarrow \text{Eigenvalues of Hilbert-Schmidt operator } T \text{ on } H^0_{(2)}(X, L^p)$$

$$\Psi_{c,\eta}^{S_p} = T\Psi_\eta^{S_p}.$$

- Construct in a *canonical* way a family of such operators T_p for all large p .....
- Toeplitz operators associated to a given suitable function f on X.
- Bordemann-Meinrenken-Schlichenmaier (1994).... they quantize the function f, classical observable.

Random $\mathcal{L}^2$-holomorphic sections: Hilbert-Schmidt operator

- f smooth bounded function on X: $T_{f,p} = B_p M_f B_p \in \mathrm{End}(H^0_{(2)}(X, L^p))$.
- Take $f \geq 0$, not identically zero, then $T_{f,p}$ is injective.

Proposition

If f satisfies

$$\int_X f(x) B_p(x,x) \mathrm{dV}(x) < \infty,$$

then $T_{f,p}$ is Hilbert-Schmidt.

- True for $f \geq 0$ with compact support ...

RANDOM $\mathcal{L}^2$-HOLOMORPHIC SECTIONS: ABSTRACT WIENER SPACE

- Abstract Wiener space (L. Gross 1965):
  - $\mathcal{B}_f(X, L^p)$=Hilbert space as completion of $H^0_{(2)}(X, L^p)$ under norm $\|T_{f,p} \cdot\|_{\mathcal{L}^2}$;
  - there exists a unique Gaussian probability measure $\mathcal{P}_{f,p}$ on $\mathcal{B}_f(X, L^p)$ such that ...
  - ... it extends Gaussian measures of any finite dimensional subspaces of $H^0_{(2)}(X, L^p)$ w.r.t. $\mathcal{L}^2$-metric.
- $T_{f,p} : \mathcal{B}_f(X, L^p) \to H^0_{(2)}(X, L^p)$ isometrie of Hilbert spaces.
- ... we obtain a Gaussian probability measure $\mathbb{P}_{f,p}$ on $H^0_{(2)}(X, L^p)$.
- $(H^0_{(2)}(X, L^p), \mathbb{P}_{f,p}) =$ probability distribution of $T_{f,p}\Psi^{S_p}_{\eta}$, as in Ancona-Le Floch (2022, for compact Kähler manfiolds).

ZEROS OF RANDOM $\mathcal{L}^2$-HOLOMORPHIC SECTIONS ON SUPPORT OF f, (I)

- Assume $f \geq 0$ smooth and compactly supported.
- $\mathbf{s}_{f,p} \in (H^0_{(2)}(X, L^p), \mathbb{P}_{f,p})$.

THEOREM 3 (DREWITZ-L.-MARINESCU, 2023)

As currents on open subset $U \subset \{f > 0\}$, we have

$$\frac{1}{p}\mathbb{E}^{\mathbb{P}_{f,p}}[[\mathrm{Div}(\mathbf{s}_{f,p})]|_U] \to c_1(L, h^L)|_U.$$

For a test form φ with $f > 0$ on $\operatorname{supp}\varphi$, then almost surely,

$$\frac{1}{p}\langle[\mathrm{Div}(\mathbf{s}_{f,p})], \varphi\rangle \to \langle c_1(L, h), \varphi\rangle.$$

- Expansion of Berezin-Toeplitz kernel $T^2_{f,p}(x, x)$:

$$T^2_{f,p}(x, x) = f(x)^2\mathbf{b}_0(x)p^m + \mathcal{O}(p^{m-1}).$$

- When X is compact, $d_p < \infty$, Ancona-Le Floch (2022, arXiv:2206.15112).

ZEROS OF RANDOM $\mathcal{L}^2$-HOLOMORPHIC SECTIONS ON SUPPORT OF f, (II)

- f can vanish on its support, the least possible vanishing order is 2 at a vanishing point x_0 since $f \geq 0$: $f(Y) = \sum_j c_j(x_0) y_j^2 + \mathcal{O}(|Y|^3)$, centered at x_0.
- Assume prequantum condition: $\omega = c_1(L, h)$. Then we have (by Ma-Marinescu 2012)

$$T_{f,p}^2(x,x) = f(x)^2 p^m + b_1(f,f)(x) p^{m-1} + b_2(f,f)(x) p^{m-2} + \mathcal{O}(p^{m-3})$$

with $b_1(f,f)(x) \geq 0$ for x near x_0 and

$$b_2(f,f)(x_0) = \frac{1}{4\pi^2}\Big(\sum_j c_j(x_0)\Big)^2 + \frac{1}{8\pi^2}|D^{0,1}\overline{\partial} f(x_0)|^2_{g^{T^*X\otimes T^*X}} > 0.$$

THEOREM 3 (DREWITZ-L.-MARINESCU, 2023)

Assume $\omega = c_1(L, h)$, for open subset U where f vanishes up to order 2, we have

$$\frac{1}{p}\mathbb{E}^{\mathbb{P}_{f,p}}[[\mathrm{Div}(\mathbf{s}_{f,p})]|_U] \to c_1(L, h^L)|_U.$$

For a test form φ with $\operatorname{supp}\varphi \subset U$, then almost surely,

$$\frac{1}{p}\langle[\mathrm{Div}(\mathbf{s}_{f,p})], \varphi\rangle \to \langle c_1(L,h), \varphi\rangle.$$

Thank you.