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Introduction to random zeros of holomorphic sections:
Part 1: construction of random sections

Central objective: zeros of random holomorphic sections on a complex manifold and their semi-classic limit

1st: construction of Gaussian (random) holomorphic sections
How to study their zeros?

2nd: Introduce the semi-classical setting
Prove the equidistribution results for random zeros.

3th: Large Deviation Estimates (LDEs)
⇒ Hole probability

4th: Variance of "random zeros"
& Central Limit Theorem (CLT)

Basic tools: ① Probability theory

② complex geometry/analysis { Bergman kernel, Subharmonic function, positive currents, ⋮ }

③ Techniques from local index theory ....

Main References:

compact Kähler mfds {

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non-compact setting {

A. DREWITZ, B. LIU, AND G. MARINESCU, Large deviations for zeros of holomorphic sections on punctured Riemann surfaces, Michigan Mathematical Journal, Advance Publication (2023), pp. 1–41.

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§0 Notation / Overview

X connected complex mfd (paracompact)

$\pi : L \to X$ hol. line bdle

$H^0(X, L) = \{ \text{(global) hol. sections} \}$

$\bar\partial^L \underset{\text{local}}{\simeq} \sum_j d\bar z_j \frac{\partial}{\partial \bar z_j}$ $(z_1, \dots, z_n)$ complex coord. $n = \dim_{\mathbb{C}} X$.

More : $\begin{cases} dV & \text{volume form}^X \\ h_L & \text{Hermitian metric on } L \end{cases}$ C^∞

$\mathcal{L}^2$ - inner product

$$\mathcal{L}^2(X, L) = \{ \|s\|^2_{\mathcal{L}^2(X,L)} := \int_X |s(x)|^2_{h_L} dV(x) < \infty \}$$

Def : $H^0_{(2)}(X, L) = H^0(X, L) \cap \mathcal{L}^2(X, L) \subset \mathcal{L}^2(X, L)$ closed

(separable) Hilbert space

$$d = \dim_{\mathbb{C}} H^0_{(2)}(X, L) = \begin{cases} \mathbb{N} \\ \infty & \text{"difficult"} \end{cases}$$

h_L Chern connection $\nabla^L \rightsquigarrow$ Chern curvature R^L

First Chern form $c_1(L, h_L) := \frac{\sqrt{-1}}{2\pi} R^L$

C^∞ $(1,1)$ - forms

s ~~hol.~~ section $Z(s) := \{ x \in X, \ s(x) = 0 \}$

$\approx$ complex submanifold of codim $= 1$

$\Rightarrow$ $[Z(s)]$ $(1,1)$ - current

Q : $[Z(s)] \overset{?}{\longleftrightarrow} G(L, h_L)$

$\neq$

For us: random $\qquad$ deterministic

§1 Preliminary

§1.1 Probability theory : Gaussian variables

$\eta \sim N_{\mathbb{C}}(0, \underset{\triangle}{1})$ standard complex Gaussian

$\eta = Re(\eta) + \sqrt{-1}\, Im(\eta)$

$N_{\mathbb{R}}(0, \frac{1}{2})$ independent.

Probability density fct : $\frac{1}{\pi} e^{-|z|^2} d\lambda(z)$ PDF on $\mathbb{C}$.

i.i.d : independent & identically distributed

Gaussianity : $\{\eta_j\}_{j=1}^{\infty}$ i.i.d $\sim N_{\mathbb{C}}(0,1)$

$\vec{a}, \vec{b} \in \ell^2(\mathbb{C})$

$\langle \vec{a}, \vec{\eta} \rangle, \quad \langle \vec{b}, \vec{\eta} \rangle$ Gaussian

$\wr$

$N_{\mathbb{C}}(0, |\vec{a}|^2_{\ell^2})$

If $\langle \vec{a}, \vec{b} \rangle_{\ell^2} = 0$, $\langle \vec{a}, \vec{\eta} \rangle, \langle \vec{b}, \vec{\eta} \rangle$ are independent

§1.2 Current theory

(1, 1)-current β : $\Omega_0^{n-1,n-1}(X) \longrightarrow \mathbb{C}$

test form

流动形 $\qquad \varphi \longmapsto \langle \beta, \varphi \rangle$

Thm (Lelong 1957) $Y \subset X$ analytic subset with pure codim $= 1$

$$\langle [Y], \varphi \rangle := \int_{Y^{reg}} \varphi|_{Y^{reg}}$$

$[Y]$ defines a closed positive $(1,1)$-current.
$(d[Y] = 0)$

$0 \not\equiv s \in H^0(X, L)$ $\quad [Z(s)]$ is closed positive $(1,1)$-current

With h_L

Thm (Poincaré-Lelong formula) $\quad s \not\equiv 0$

$$[Z(s)] = \frac{\sqrt{-1}}{2\pi} \partial\bar\partial \underbrace{\log |s|^2_{h_L}}_{\mathcal{L}^1_{loc}(X)} + c_1(L, h_L)$$

Weak topology on $(1,1)$-currents

$$\beta_p \longrightarrow \beta \quad \text{as } (1,1)\text{-currents}$$

$\iff \forall \varphi \in$ test forms

$$\langle \beta_p, \varphi \rangle \to \langle \beta, \varphi \rangle \text{ in } \mathbb{C}.$$

§2 Construction of Gaussian hol. sections

X cplx mfd $\quad dV$

(L, h_L) Hermitian hol. line bdle

$H^0(X, L) \neq \{0\}$

— Take $s_1, \cdots, s_d \in H^0(X, L)$ $\quad s_j \not\equiv 0$

$d \in \mathbb{N}_{\geq 1} \sqcup \{\infty\}$

Hypothesis :

$$B(x) := \sum_{j=1}^{\infty} |S_j(x)|^2_{h_L} < \infty \quad \forall x$$

locally uniformly convergent !

— $\{\eta_j\}_{j=1}^{d}$ i.i.d $\sim N_{\mathbb{C}}(0, 1)$

Def :

$$S_\eta := \sum_{j=1}^{d} \eta_j S_j$$

Thm 1 : (Cf. Kahane's book some random series of functions)

① S_η is almost surely (a.s.) a holomorphic section of L on X.

② If $\{S_j\}_{j=1}^{d} \subset H^0_{(2)}(X, L)$ is an $\mathcal{L}^2$-orthogonal family, $0 < C_1 < C_2 < +\infty$

$$C_1 < \|S_j\|^2_{\mathcal{L}^2(X,L)} < C_2 \quad \forall j.$$

Then

$$d < \infty \iff \int_X B(x)\, dV(x) < \infty \iff S_\eta \text{ is a.s. } \mathcal{L}^2 \text{ on } X$$

$$d = \infty \iff \int_X B(x)\, dV(x) = \infty \iff S_\eta \text{ is a.s. non-}\mathcal{L}^2 \text{ on } X$$

(almost never $\mathcal{L}^2$)

Proof : Assume $d = \infty$ $\quad U$ compact $\subset X$.

① It's enough to prove that : $U \subset\subset X$

$$(*) \quad \mathbb{P}\Big(\limsup_{k \to +\infty} \sup_{\ell \geq 1} \Big\| \sum_{j=1}^{\ell} \eta_{k+j} S_{k+j} \Big\|^2_{\mathcal{L}^2(U)} > \varepsilon \Big) = 0$$

Why ?

For holomorphic sections

local $\mathcal{L}^2$-norm $\gtrsim$ local C^0-norm

(*) holds, taking $\varepsilon = \frac{1}{N}$ $\quad N \to +\infty$

$\Rightarrow$ $S_\eta|_U$ is almost surely holomorphic

$\underline{U_j} \nearrow X \Rightarrow S_\eta$ is a.s. holomorphic on X.

Proof of (*): $Y_\ell^k(\cancel{U}) = \left\| \sum_{j=1}^{\ell} \eta_{k+j} S_{k+j} \right\|^2_{\mathcal{L}^2(U)}$

k, U fixed

$\underline{\{ Y_\ell^k(U) \}_{\ell=1}^N}$ is a submartingale w.r.t.

(下鞅)

$\mathcal{F}_\ell^k = \sigma(\langle \eta_{k+i} S_{k+i}, \eta_{k+j} S_{k+j} \rangle_{\mathcal{L}^2}$

$i, j \leq \ell)$

i.e. $\underline{\mathbb{E}[Y_{\ell+1}^k \mid \mathcal{F}_\ell^k] \geq Y_\ell^k}$

$$Y_{\ell+1}^k = Y_\ell^k + \|\eta_{k+\ell+1} S_{k+\ell+1}\|^2_{\mathcal{L}^2(U)}$$
$$+ 2\,\mathrm{Re}\, \langle \overset{\mathbb{E}[\ ]}{\eta_{k+\ell+1}} S_{k+\ell+1}, \sum_{j=1}^{\ell} \eta_{k+j} S_{k+j} \rangle_{\mathcal{L}^2(U)}$$

$$\mathbb{E}[Y_{\ell+1}^k \mid \mathcal{F}_\ell^k] = Y_\ell^k + \underbrace{\|S_{k+\ell+1}\|^2_{\mathcal{L}^2(U)}} + 0$$
$$> Y_\ell^k$$

Doob's submartingale inequality $\quad \underline{r > 0}$

Fix k

$$\mathbb{P}\Big(\underbrace{\sup_{\ell = 1, \cdots, N}} Y_\ell^k(U) > r \Big) \leq \frac{1}{r} \underbrace{\mathbb{E}[Y_N^k(U)]}$$

$$\underline{\mathbb{E}[Y_N^k(U)]} = \int_U \underbrace{\sum_{j=1}^{N} |S_{k+j}(x)|^2_{h_L}}_{\substack{\downarrow \\ \text{convergence as } N \to +\infty}} dV(x)$$

$\underline{N \to +\infty}$

$$\underbrace{\mathbb{P}\left(\sup_{\ell \geq 1} Y_\ell^k(\omega) > r \right)} \leq \frac{1}{r} \int_U \underbrace{\sum_{j=k+1}^{\infty} |S_j(x)|_{h_L}^2}_{\substack{\Delta \\ \downarrow k \to +\infty \\ 0}} \, d\mu(x)$$

$k \to +\infty$

$$\mathbb{P}\left(\limsup_{k \to +\infty} \sup_{\ell \geq 1} Y_\ell^k(\omega) > r \right) = 0 \quad \Rightarrow \quad (*) .$$

② $\underline{d = \infty} \iff \int_X B(x)\, d\mu(x) = \infty$

Komolgorov's strong law of large number

$$\underbrace{\frac{1}{N} \sum_{j \leq N} |\eta_j|^2 \|S_j\|_{\mathcal{L}^2(X)}^2}_{\substack{\Delta \\ \gtrsim N c_1}} - \underbrace{\frac{1}{N} \sum_{j \leq N} \|S_j\|_{\mathcal{L}^2(X)}^2}_{0 < c_1 \leq} \xrightarrow{a.s} 0 .$$

$$\mathbb{P}\left(\sum_{j \leq N} |\eta_j| \|S_j\|_{\mathcal{L}^2(X)}^2 \xrightarrow[N \to \infty]{} \infty \right) = 1 .$$

S_η almost $\underline{\text{never}}$ $\mathcal{L}^2$-integrable on X. #

§3. Expectation of zeros of S_η

$\underline{\text{Thm 1}}$: $(\Omega, \mathcal{P})$ probability where $\{\eta_j\}$ lives

$$\begin{aligned} S_\eta : \Omega &\longrightarrow H^0(X, L) \\ \omega &\longmapsto S_{\eta(\omega)} \end{aligned}$$

"measurable"

Fréchet space

semi-norms $\|s\|_{C^\ell(K)}$

$\underline{\text{Fact}}$: $[Z(S_\eta)] : \Omega \longrightarrow (1,1)$-currents

is "measurable"

i.e. $\forall\ \varphi$ test forms

$\langle [Z(S_\eta)], \varphi\rangle \in \mathbb{C}$ is measurable.

$$\begin{aligned}&\langle [Z(S_\eta)], \varphi\rangle \\ &\quad \| \\ &\lim_{N\to+\infty} \langle \underbrace{\frac{\sqrt{-1}}{2\pi}\log\left(|S_\eta(z)|^2_{h_L}+\frac{1}{N}\right)+c_1(L,h_L)}_{\text{measurable}}, \varphi\rangle\end{aligned}$$

Def: If β $(1,1)$-current s.t. $\forall\ \varphi$ test form

$$\mathbb{E}[\langle [Z(S_\eta)], \varphi\rangle] = \langle \beta, \varphi\rangle$$

Then β is expectation of $[Z(S_\eta)]$

$$\beta =: \mathbb{E}[[Z(S_\eta)]]$$

Thm 2 (Edelman-Kostlan formula / Probabilistic Poincaré-Lelong formula)

S_η as in Thm 1.

$$\mathbb{E}[[Z(S_\eta)]] = \underbrace{\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log B(x) + c_1(L,h_L)}_{\text{Fubini-Study current}} \geq 0$$

$\gamma_{FS}(\{S_j\}_j, h_L)$

analogue to $X \dashrightarrow \mathbb{CP}^{d-1}$

$z \longmapsto (S_j(z))$

Proof: φ test form

$$\mathbb{E}[\langle [Z(S_\eta)], \varphi\rangle] = \mathbb{E}\left[\int_X \frac{\sqrt{-1}}{2\pi}\log|S(x)|^2_{h_L}\partial\bar\partial\varphi(x)\right]$$

$$+ \langle c_1(L,h_L), \varphi\rangle$$

Using Fubini-Tonelli Thm to exchange the integrals.

$x \in X \setminus \{B(x)=0\}$ measure 0 (real codim ≥ 2)

$$S_\eta(x) = \sum \eta_j S_j(x)$$

$|e_L(x)|_{h_L} = 1 \qquad S_j(x) = f_j(x)\, e_L(x)$

$$B(x) = \sum_{j=1}^{\infty} |f_j(x)|^2 \neq 0$$

$$S_\eta(x) = \left(\boxed{\sum_{j=1}^{d} \eta_j \frac{f_j(x)}{\sqrt{B(x)}}} \cdot e_L(x) \right) \cdot \sqrt{B(x)}$$

$$\left(\frac{f_j(x)}{\sqrt{B(x)}} \right)_{j=1}^{d} \in \ell^2(\mathbb{C}) \quad \text{norm} = 1.$$

Gaussian $\sim \mathcal{N}_{\mathbb{C}}(0,1)$ independent of x

$$\mathbb{E}\left[\log |S_\eta(x)|^2_{h_L}\right] = \underbrace{\mathbb{E}\left[\log |\eta|^2\right]}_{\text{cst}} + \log B(x), \qquad \eta \sim \mathcal{N}_{\mathbb{C}}(0,1)$$

$$\mathbb{E}\left[\langle [Z(S_\eta)], \varphi \rangle\right] = \langle c_1(L,h_L), \varphi \rangle + \underbrace{\int_X \frac{\sqrt{-1}}{2\pi} \left(\cancel{\text{cst}} + \log B(x)\right) \partial\bar\partial \varphi}_{\langle \frac{\sqrt{-1}}{2\pi} \partial\bar\partial \log B(x),\ \varphi \rangle}$$

#

§4. Standard Gaussian hol. section

$(L, h_L) \to X \quad dV$

$\{S_j\}_{j=1}^{d}$ ONB $H^0_{(2)}(X, L)$

$d = \dim_{\mathbb{C}} H^0_{(2)}(X, L) \in \mathbb{N} \cup \{\infty\}$

Bergman kernel:

$$B : \mathcal{L}^2(X, L) \xrightarrow{\perp} H^0_{(2)}(X, L)$$

$B(x, y)$ Schwartz kernel

$\bar{\partial}^L$ elliptic $\Rightarrow$ $B(x,y)$ C^∞ on $X \times X$.

$$B(x, y) = \sum_{j=1}^{d} S_j(x) \otimes S_j(y)^* \in L_x \otimes L_y^*$$

locally uniformly convergent!

Bergman kernel function

$$\underline{\underline{B(x)}} := B(x, x) = \sum_{j=1}^{d} |S_j(x)|^2_{h_L}$$

satisfies Hypothesis for Thm 1.

<u>Def</u>: $S(L) := S_\eta$ from Thm 1.

called $\boxed{\text{standard}}$ Gaussian hol. section of L over X.

<u>Lemma</u>: ① $d < \infty \iff S(L)$ a.s. $\mathcal{L}^2(X)$

$d = \infty \iff S(L)$ a.s non-$\mathcal{L}^2(X)$.

② Uniqueness:

the <u>probability distribution</u> of $S(L)$

Gaussianity $\rightarrow$ does not depend on the choice of ONB $\{S_j\}$.

#

— Bargmann-Fock

$$X = \mathbb{C}$$

$$(L, h_L) = (\mathbb{C}, e^{-|z|^2})$$

$$H^0_{(2)}(X, L) = \mathrm{Span}\left\{ \frac{z^j}{\sqrt{j!}} \right\}$$

$$\dim = \infty$$

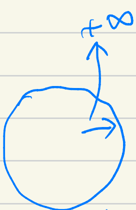
— $D \subset \mathbb{C}^n$

$$H^0_{(2)}(D) = \{ f \text{ hol}, \ \|f\|_{L^2(D)} < \infty \}$$

$$\downarrow \quad B(x)$$

$$\omega_B := \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log B(x) \qquad \text{Bergman metric.}$$

$$\| \quad \mathbb{E}[[Z(S_n)]]$$



$\Rightarrow$ ω_B is complete

hyperconvex

bded