Introduction to random zeros of holomorphic sections: Part 3: large deviation and hole probability

Notes available on webpage
http://www.mi.uni-koeln.de/~bxliu/

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§0 Revisit: equidistribution in semi-classical limit

X (connected) complex manifold

ω J-invariant Hermitian metric

$\rightsquigarrow \quad \underset{\text{Riemannian metric}}{g^{TX}(\cdot,\cdot)} = \omega(\cdot, J\cdot)$

(L, h_L) Hermitian holomorphic line bundle

(H): $\begin{cases} \omega = c_1(L, h_L) := \frac{\sqrt{-1}}{2\pi} R^L > 0 \\ g^{TX} \text{ (or } \omega) \text{ is complete.} \\ \mathrm{Ric}_\omega \geq -C\omega \quad \text{for some } C>0. \end{cases}$

Semi-classical setting

$H^0_{(2)}(X, L^p)$ dimension $d_p \in \mathbb{N} \cup \{\infty\}$

$d_p \gg 0$ as $p \nearrow$.

In Theorem 3 (Lecture 2), we have

$$B_p(x) = p^n + b_1(x) p^{n-1} + \cdots\cdots$$

Bergman kernel function

$\rightsquigarrow$ Equidistribution results in Theorem 4: $\forall \varphi \in \Omega_0^{n-1,n-1}(X)$

$$\left\langle \frac{1}{p}[Z(S(L^p))], \varphi \right\rangle \xrightarrow[a.s.]{} \langle c_1(L, h_L), \varphi \rangle.$$

In fact, we have shown more results on Bergman kernels

$$|B_p(x,y)| \sim p^n \exp\left(-\frac{\pi}{2} d(x,y)^2\right) \cdots\cdots$$

when x, y are nearby

$\rightsquigarrow$ ······ refine the above results!

§ 1 Large deviation estimates and hole probability

Def : $U \subset\subset X$, β a current on X, for $\alpha > 0$

$$\|\beta\|_{U,-\alpha} := \sup_{\substack{\varphi \text{ test form} \\ \operatorname{supp}\varphi \subset U \\ |\varphi|_{C^\alpha} \le 1}} |\langle \beta, \alpha \rangle|$$

$\alpha = 2$ For $0 \not\equiv S_p \in H^0(X, L^p)$ $\forall\, U \subset\subset X$

$$\|\underbrace{[Z(S_2)]}_{\partial\bar\partial \underbrace{\log|S_2|}_{L^1_{loc}}}\|_{U,-2} < +\infty$$

Thm 6 : $n = \dim_{\mathbb{C}} X$. $\forall\, \delta > 0$, $U \subset\subset X$,

$\exists\ C = C(\delta, U) > 0$ s.t.

$$\mathbb{P}\left(\left\| \frac{1}{p}[Z(S(L^p))] - c_1(L, h_L) \right\|_{U,-2} > \delta \right) \le e^{-Cp^{n+1}}$$

LDE

large deviation estimates

concentration of measure

$$\Rightarrow \quad \mathbb{P}\left(\lim_{p\to+\infty} \left\| \frac{1}{p}[Z(S(L^p))] - c_1(L, h_L) \right\|_{U,-2} = 0 \right) = 1.$$

(A bit stronger than Thm 4)

Rk : On $\mathbb{C}$ or $\mathbb{D}$, $\mathbb{C}^n$, GAF, by Offord, Sodin, Zrebiec

For cpt Kähler, Shiffman - Zelditch - Zrebiec 2008

Non-cpt, Drewitz - L. - Marinescu 2023

$Y \subset X$ $\quad\quad$ $U \subset\subset X$

codim 1.

$$\begin{cases} \mathrm{Vol}^{L}_{2n-2}(U \cap Y) = \int_{U \cap Y} \frac{c_1(L,h_L)^{n-1}}{(n-1)!} \\ \mathrm{Vol}^{L}_{2n}(U) = \int_{U} \frac{c_1(L,h_L)^{n}}{n!} \end{cases}$$

$$\mathrm{Vol}^{L}_{2n-2}(Z(S_p) \cap U) = \sum_{\substack{V \subset X \\ \text{hypersurface} \\ V \subset Z(S_p)}} \mathrm{ord}_V(S_p)\, \mathrm{Vol}^{L}_{2n-2}(V \cap U).$$

$\mathrm{ord}_V(S_p)$ — vanishing order

Cor 2: $U \subset\subset X$, ∂U measure zero w.r.t. $dV = \frac{\omega^n}{n!}$.

$$\mathbb{P}\left(\left| \frac{1}{p} \mathrm{Vol}^{L}_{2n-2}(Z(S(L^p)) \cap U) - n \mathrm{Vol}^{L}_{2n}(U) \right| > \delta \right) \le e^{-C_{\delta,U}\, p^{n+1}}.$$

$\rightsquigarrow$ Take $\delta = n \mathrm{Vol}^{L}_{2n}(U)$

$$e^{-C'_U p^{n+1}} \overset{?}{\le} \mathbb{P}\left(Z(S(L^p)) \cap U = \emptyset \right) \le e^{-C_U p^{n+1}}$$

$\uparrow$ the lower bounds exists in special cases or with more assumptions.

Hole Probability

Δ $\begin{cases} \mathbb{C}^n & \text{Zrebiec} \\ \text{cpt Kähler} & \end{cases}$

Rk: $\mathbb{C}, \mathbb{D}$ hole probability can be explicitly computed

$\begin{cases} \text{Peres-Virag} & 2005 \text{ Acta} \\ \text{Nishry} & 2010 \text{ IMRN} \end{cases}$

idea: ∂U zero measure

存在 $\varphi_2 \le \chi_U \le \varphi_1$

χ_U — indicator function

s.t. $\left| \int_X \varphi_j \frac{c_1(L,h_L)^n}{n!} - \mathrm{Vol}^{L}_{2n}(U) \right| < \frac{\delta}{2}$. Δ

Put $\varphi_j \dfrac{c_1(L, h_L)^{n-1}}{(n-1)!}$ test forms on LDE #

§2 A Key intermediate result to prove Thm 6

For $U \subset\subset X$ $\quad 0 \not\equiv s_p \in H^0(X, L^p)$

local sup norm $\quad \mathcal{M}_p^U(s_p) := \sup_{z \in U} |s_p(z)|_{h_p} > 0$

Prop 1: $\forall\, \delta > 0$, $\exists\, C = C(U, \delta) > 0$ s.t.

$$\mathbb{P}\left(\left| \log \mathcal{M}_p^U(S(L^p)) \right| > \delta p \right) \leq e^{-C p^{n+1}}$$

Sketched proof of Thm 6 via Prop 1.

By Prop 1 $\quad \exists\, E_p(\delta, U) \subset \Omega$ (probability space)

$$\mathbb{P}(E_p(\delta, U)) \leq e^{-Cp^{n+1}}$$

on $E_p(\delta, U)^c$: we can always find $x \in U$ s.t.

$$\left| \log |S(L^p)(x)|_{h_p} \right| \leq \delta \cdot p$$

hence $\Rightarrow \log |S(L^p)(x)|_{h_p} \geq -\delta p$ or $\leq \delta p$

A consequence :

$$\mathbb{P}\left(\int_U \left| \log |S(L^p)(z)|_{h_p} \right| dV(z) > \delta p \right) \leq e^{-Cp^{n+1}}$$

To get this :

① $\begin{cases} \log = \log^+ - \log^- \\ |\log| = \log^+ + \log^- \end{cases}$

$$\log^+ |S(L^p)(x)|_{h_p} \le \underline{\log |M_p^{L}(S(L^p))|}$$

$$\longrightarrow \quad \mathbb{P}\left(\int_{L} \log^+ > \delta p\right) \lesssim e^{-cp^{n+1}}$$

$$\log^- = \log^+ - \log$$

② We need to prove

(*) $$\mathbb{P}\left(-\int_L \log|S(L^p)(x)|_{h_p}\, dV(x) \ge K\delta p\right) \le e^{-cp^{n+1}}$$

large number > 0

we use <u>sub-mean inequality</u>:

$\log f$ subharmonic on $\mathbb{C}^n$

$$P_r(\xi, z) = r^{2n-2}\,\frac{r^2-|\xi|^2}{|\xi-z|^{2n}} \qquad |z| = r, \quad |\xi| < r$$

Poisson kernel for $\mathbb{D}(r) \subset \mathbb{C}^n$

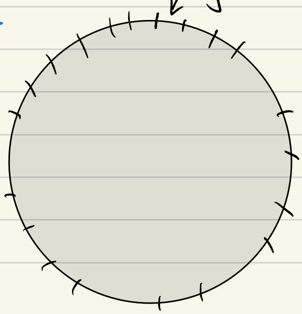
$\forall\, |\xi| < r$

1st estimate: $$\log f(\xi) \le \int_{|z|=r} P_r(\xi,z) \log f(z)\, d\sigma_r(z)$$

$$\le 2\,|\log f|_{\text{local sup}} - \int_{|z|=r} P_r(\xi,z)\,|\log f(z)|\, d\sigma_r(z)$$

These two are "bounded" by Prop 1

I_j



$\sqcup I_j \sim \partial\mathbb{D}(0,r)$

diameter$(I_j) \sim \delta^{2n+2}$

$\sum \sigma_r(I_j) \sim 1$

$\exists\, c>0, \quad \forall\, \xi_j \quad d(\xi_j, (r-\delta) I_j) < \delta^{2n+2}$

2nd estimate

$$-\int_{|z|=r} \log f(z)\, d\sigma_r(z) \le -\sum_j \sigma_r(I_j) \log f(\xi_j) \le \sum_j \sigma_r(I_j)\,\delta p$$

$$\sum_j \sigma_r(I_j) = 1$$

$$+\, c\delta \underbrace{\int_{|z|=r} |\log f(z)|\, d\sigma_r(z)}_{\le Kp \text{ by the 1st estimate}}$$

Consider $f(z) = S(L^p) + C^\infty\text{-fct}$

$\Rightarrow$ Except an even $\le \sum_{\text{finite}} e^{-cp^{n+1}} \sim e^{-c' p^{n+1}}$

we have $\forall j$

$$\begin{cases} \log |S(L^p)(\xi_j)|_{h_p} \ge -\delta p \\ \int_U |\log |S(L^p)|(z)|_{h_p}\, dV(z) \le Kp \end{cases} \Rightarrow (*)$$

Using $(*)$ and Poincaré-Lelong formula: $\operatorname{supp} \varphi \subset U \quad \varphi^{(n-1,n-1)}$

$$\left| \left\langle \frac{1}{p}[Z(S(L^p))] - c_1(L, h_L), \varphi \right\rangle \right|$$

$$= \left| \frac{\sqrt{-1}}{p\pi} \int_X \log |S(L^p)|(z)_{h_p}\, \partial\bar\partial \varphi \right|$$

$$\le \frac{|\varphi|_{C^2}}{p\pi} \underbrace{\int_U \left|\log |S(L^p)|\right| \frac{c_1^n}{n!}}_{\le \delta \cdot p \text{ except an even of probability } \le e^{-cp^{n+1}}}$$

$|\partial\bar\partial \varphi| \le |\varphi|_{C^2} \cdot dV(z)$

$$\Rightarrow \mathbb{P}\left(\left\| \frac{1}{p}[Z(S(L^p))] - c_1(L,h_L) \right\|_{U,-2} > \frac{\delta}{c} \right) \le e^{-cp^{n+1}}.$$

$\delta > 0$ arbitrary. #

§3 Proof of Prop 1 : normalized Bergman kernel.

(A) $\mathbb{P}\left(M_p^U(S(L^p)) \ge e^{\delta p} \right) \le e^{-cp^{n+1}}$

(B) $\mathbb{P}\left(M_p^U(S(L^p)) \le e^{-\delta p} \right) \le e^{-cp^{n+1}}$

(A) : local C^0-norm $\lesssim$ local L^2-norm

$$M_p^U(S(L^p))^{2p^n} \le C_p^{p^n} \left[\int_U |S(L^p)(z)|^2_{h_p} dV(z) \right]^{p^n}$$

U very small

e_L hol. frame on U

$\sup_U |e_L|_{h_L} = 1$ $\qquad \nu = \inf_U |e_L|_{h_L} \le 1$ ~ 1 if U sufficiently small.

$$C_p \simeq \frac{C_U}{\nu^{2p}}$$

$C > 0$

$$\mathbb{E}\left[\|S(L^p)\|^{2p^n}_{L^2(U)} \right] \le \frac{1}{\nu^{4p^{n+1}}} \exp(C p^n \log p)$$

$\forall x \in U$

$S(L^p)(x) \in L_x \simeq \mathbb{C}$

Gaussian r.v. of variance $B_p(x)$

$$\mathbb{E}\left[\|S(L^p)\|^{2p^n}_{L^2(U)} \right] \lesssim \mathrm{Vol}(U)^{p^n-1} \int_U \underbrace{\mathbb{E}\left[|S(L^p)(z)|^{2p^n} \right]}_{(2p^n)\text{-th moments of Gaussian random variable}} dV(z)$$

$$\mathbb{P}\big(\underbrace{\mathcal{M}_p^{U}(S(L^p))^{2p^n}} \geq e^{2\delta p^{n+1}} \big) \leq e^{-2\delta p^{n+1}} \mathbb{E}\left[\mathcal{M}_p^{U}(S(L^p))^{2p^n} \right]$$

$$\leq e^{-2\delta p^{n+1}} \frac{C_U^{p^n}}{\nu^{2p^{n+1}}} \underbrace{\mathbb{E}\left[\| S(L^p) \|_{L^2(U)}^{2p^n} \right]}$$

$$\leq e^{-2\delta p^{n+1} \underline{-6\log\nu}\, p^{n+1} + Cp^n \log p}$$

When U is small, we can take $\underline{-6\log\nu < \delta}$

$$\lesssim e^{-\delta p^{n+1} + Cp^n \log p}$$

$\Rightarrow$ (A)

$=$

(B) Def: normalized Bergman kernel

$$\underline{N_p(x,y)} = \frac{|B_p(x,y)|_{h_{p,x} \otimes h_{p,y}^*}}{\sqrt{B_p(x)}\sqrt{B_p(y)}} \in [0,1]$$

$S(L^p)$ holomorphic Gaussian field.

N_p correlation function of $S(L^p)$

Thm 7: $U \subset\subset X$, for $k \geq 1$, $b \geq \sqrt{6k/\pi}$

Δ we have for $p \gg 0$

$x, y \in U$

$$N_p(x,y) = \begin{cases} \underline{O(p^{-k})} & \underline{d(x,y)} \geq b\sqrt{\frac{\log p}{p}} \\ (1+R_p(x,y)) \exp\left(-\frac{p\pi}{2} d(x,y)^2\right) & d(x,y) \leq b\sqrt{\frac{\log p}{p}} \end{cases}$$

the proof will be given in next lecture

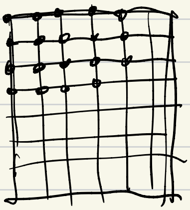
$$\sup_{\substack{x,y \in U \\ d(x,y) \leq b\sqrt{\frac{\log p}{p}}}} |R_p(x,y)| \longrightarrow 0 \quad \text{as } p \to +\infty$$

How to prove (B) from Thm 7.

$F_p \subset W$ "subset"

$$\mathbb{P}\left(\mu_p^W(S(L^p)) \le e^{-\delta p}\right) \le \underbrace{\mathbb{P}\left(\sup_{x\in F_p} |S(L^p)(x)|_{h_p} \le e^{-\delta p}\right)}_{\text{(B)} \approx \le e^{-cp^{n+1}}}$$

$\subset W$



$\subset \mathbb{C}^n$ local chart

to be determined

$$F_p = \left\{ x_p(\nu) = \frac{c_0}{\sqrt{p}}\nu \; : \; \nu = (\nu_1, \dots, \nu_{2n}) \in \mathbb{Z}^{2n}, \; |\nu_j| \le \varepsilon\sqrt{p} \right\}$$

ε small >0

$\# F_p \sim p^n$

$|\cdot|_{h_p} = 1$

Def: $$\xi(x_p(\nu)) = \frac{\langle S(L^p)(x_p(\nu)), \lambda_p(x_p(\nu)) \rangle_{h_p}}{\sqrt{B_p(x_p(\nu))}} \sim \mathcal{N}_{\mathbb{C}}(0,1)$$

$$\{\xi(x_p(\nu))\}_{x_p(\nu)\in F_p}$$

Covariance

$$\Delta_p(u,\nu) = \mathbb{E}\left[\xi(x_p(u))\overline{\xi(x_p(\nu))}\right]$$

$$= N_p(x_p(u), x_p(\nu))$$

$$= \begin{cases} \le 2\exp\left(-\frac{c_0^2\pi}{2}|u-\nu|^2\right) & \frac{c_0}{\sqrt{p}}|u-\nu| \le b\sqrt{\frac{\log p}{p}} \\ O(p^{-n-1}) & \frac{c_0}{\sqrt{p}}|u-\nu| \gtrsim b\sqrt{\frac{\log p}{p}} \end{cases}$$

$k = n+1$

$b = \sqrt{\frac{6(n+1)}{\pi}}$

Take $C_0 >> 0$

$$\Delta_p(u,u) = 1$$

$$\sum_{v\neq u} |\Delta_p(u,v)| \le 1/2$$

$$\Delta_p = \begin{bmatrix} 1 & & \sim 0 \\ & \ddots & \\ \sim 0 & & 1 \end{bmatrix}$$

Δ_p is a positive invertible matrix

$$\max \operatorname{spec}(\Delta_p^{-1}) \le 2$$

$\#F_p$ = the length of this vector

$$\Delta_p^{-1/2}\big(\xi(x_p(u))\big)_{x_p(u)\in F_p} = (\eta_1, \dots, \eta_{\#F_p}) \rightsquigarrow \text{i.i.d} \sim N_{\mathbb{C}}(0,1)$$

↑ Gaussianity

$$\mathbb{P}\Big(\max_{F_p} |\xi(x_p(u))| \le e^{-\delta p}\Big)$$

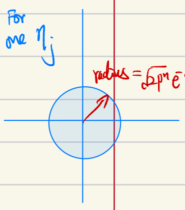
$\sqrt{2}\cdot\sqrt{\#F_p}$
SS

$$\le \mathbb{P}\Big(\max_j |\eta_j| \le \sqrt{2p^n}\, e^{-\delta p}\Big)$$

$$\lesssim \Big[\pi\big(\sqrt{2p^n}\, e^{-\delta p}\big)^2\Big]^{\#F_p}$$

$$\lesssim C e^{-c'\delta p^{n+1}} \rightsquigarrow (B)$$

$\#F_p \sim p^n$



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