Introduction to random zeros of holomorphic sections
Part 4: number variance and central limit theorem

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§0: Revisit: asymptotics of normalized Bergman kernels

Thm 5: $(L, h_L) \to (X, \omega)$

$$(H): \begin{cases} \omega = c_1(L, h_L) \\ g^{TX}(\cdot,\cdot) = \omega(\cdot, J\cdot) \text{ complete} \\ \mathrm{Ric}_\omega \geq -C\omega \end{cases}$$

$B_p \rightsquigarrow (L^p, h_p)$

Then fix $U \subset\subset X$, $\exists\ \delta > 0$ s.t. $\begin{cases} \forall\ x \in U \\ Z, Z' \in T_x X \\ |Z|_{g^{TX}}, |Z'|_{g^{TX}} \leq \delta \end{cases}$

we have

$$\left| \frac{1}{p^n} B_p(\exp_x Z, \exp_x Z') - \sum_{r=0}^{N} J_r(\sqrt{p}Z, \sqrt{p}Z') \mathcal{B}(\sqrt{p}Z, \sqrt{p}Z') \kappa^{-1/2}(Z) \kappa^{-1/2}(Z') p^{-r/2} \right|_{\mathcal{C}^{\ell}_{Z,Z'}}$$

$$\leq C_{N,\ell}\, p^{-(N+1-\ell)/2} (1+\sqrt{p}|Z| + \sqrt{p}|Z'|)^{M(N,\ell)} \exp(-C\sqrt{p}|Z-Z'|) + O(p^{-\infty})$$

where $\begin{cases} \deg_{Z,Z'} J_r \leq 3r \\ J_0 \equiv 1 \end{cases}$ $\quad \kappa(0) = 1 \quad$ "$x^M e^{-x}$"

$Z, Z' \in T_x X \simeq \mathbb{R}^{2n}$, $\quad z_j = Z_{2j-1} + \sqrt{-1} Z_{2j}$

$$\mathcal{B}(Z, Z') = \exp\left(-\frac{\pi}{2} \sum_j (|z_j|^2 + |z'_j|^2 - 2 z_j \bar{z}'_j)\right)$$

$$\rightsquigarrow |\mathcal{B}(Z, Z')| = \exp\left(-\frac{\pi}{2}|Z - Z'|^2\right)$$

Recall **Cor 1**: $\forall\ \delta > 0$, $\forall\ \ell > 0$, $U \subset\subset X$

$$|B_p(x, y)| \leq C_\ell\, p^{-\ell} \quad \forall\ \begin{matrix} x, y \in U \\ d(x,y) > \delta \end{matrix}$$

Today, we firstly prove Thm 7 (from Lecture 3)

Thm 7: $U \subset\subset X$, for $k \geq 1$, $b \geq \sqrt{6k/\pi}$, we have $\forall\, p \gg 0$

$$N_p(x,y) := \frac{|B_p(x,y)|_{h_{p,x}\otimes h^*_{p,y}}}{\underbrace{\sqrt{B_p(x)}\sqrt{B_p(y)}}_{p^n}} = \begin{cases} O(p^{-k}), & \forall\, x,y\in U,\ d(x,y)\geq b\sqrt{\frac{\log p}{p}} \\ (1+R_p(x,y))\exp\left(-\frac{p\pi}{2}d(x,y)^2\right) & \forall\, x,y\in U,\ d(x,y)\leq b\sqrt{\frac{\log p}{p}} \end{cases}$$

step 1

and $\forall\, x\in U$, $Z\in T_xX$, $|Z|_{g^{TX}}\leq b\sqrt{\log p}$,

for any $\varepsilon\in(0,\frac{1}{4})$ $\quad \left|R_p\left(x,\exp_x\frac{Z}{\sqrt{p}}\right)\right| \leq C p^{-\frac{1}{2}+\varepsilon}|Z|^2_{g^{TX}} \longrightarrow 0$

Hence $\sup\limits_{\substack{x,y\in U\\ d(x,y)\leq b\sqrt{\frac{\log p}{p}}}} |R_p(x,y)| \longrightarrow 0$ as $p\to+\infty$

§1 Proof of Thm 7 using Thm 5

step 1 $\quad$ Fix $k\geq 1$

By Cor 1 $\quad$ if $(x,y)\in U\times U$ $d(x,y)>\delta$

$$|B_p(x,y)| \leq C_k p^{-k}$$

$$\rightsquigarrow \quad N_p(x,y) \lesssim p^{-k}$$

$b>0$

$$b\sqrt{\frac{\log p}{p}} \leq d(x,y) \leq \delta$$

Taking $N=2k$, $Z=0$, $y=\exp_x Z'$

$$d(x,y) = |Z'|_{g^{TX}}$$

$$B_p(x,y) = p^n \mathcal{B}(0,\sqrt{p}Z')\,\kappa^{-1/2}(Z') + \sum_{r=1}^{2k} p^{n-r/2} J_r(0,\sqrt{p}Z')\,\mathcal{B}(0,\sqrt{p}Z')\,\kappa^{-1/2}(Z') + O(p^{-\frac{2k+1}{2}})$$

$y = \exp_x Z'$

$\underline{|Z'|_{g^{TX}} \geq b\sqrt{\frac{\log p}{p}}}$

$$|\mathcal{B}(0,\sqrt{p}Z')| = \exp\left(-\frac{\pi}{2}p|Z'|^2_{g^{TX}}\right) = \exp\left(-\frac{\pi}{2}p\, \underbrace{d(x,y)^2}\right) \leq \underline{\exp\left(-\frac{\pi}{2}b^2\log p\right)}$$

$\underline{r = 0, \dots, 2k}$

$$\left|J_r(0,\sqrt{p}Z')\,\underline{\mathcal{B}(0,\sqrt{p}Z')}\,\underbrace{\kappa^{-1/2}(Z')}\,p^{n-r/2}\right|$$

$\underline{|Z'| < \delta}$

$$\lesssim |\sqrt{p}Z'|^{3r} p^{n-r/2} \exp\left(-\frac{\pi}{2}b^2\log p\right)$$

$$\lesssim \underbrace{p^{n+r}\exp\left(-\frac{\pi}{2}b^2\log p\right)}$$

we want $\leq p^{n-k}$

$$\Longleftrightarrow \quad p^{2k}\exp\left(-\frac{\pi}{2}b^2\log p\right) \leq p^{-k}$$

The condition for b: $\frac{\pi}{2}b^2 \geq 3k$

<u>Take</u> $\underline{b \geq \sqrt{\frac{6k}{\pi}}}$

b fixed $\geq \sqrt{\frac{6k}{\pi}}$

Step 2 $\quad d(x, y) \leq b\sqrt{\frac{\log p}{p}}$ $\qquad p \gg 0$

Take : $\begin{cases} Z = 0 \\ \exp_x Z = x \end{cases}$ $\qquad Z'$, $\; y = \exp_x Z'$ $\qquad |Z'| \leq b\sqrt{\frac{\log p}{p}}$

$$\underbrace{\exp\left(\frac{\pi p}{2}|Z'|^2\right)}_{\Delta} \left| J_r(0, \sqrt{p}Z') \, \mathcal{B}(0, \sqrt{p}Z') \, \kappa^{-1/2}(Z') \, p^{-r/2} \right|$$

$$= \begin{cases} \kappa^{-1/2}(Z') = \underset{\Delta}{1} + O(|Z'|) = 1 + O\left(\sqrt{\frac{\log p}{p}}\right) & r = 0 \\ (1 + \underbrace{\sqrt{p}|Z'|}_{\sqrt{\log p}})^{3r} p^{-r/2} \leq (\log p)^N p^{-1/2} \sim p^{-1/2+\varepsilon} & r \geq 1 \end{cases}$$

$$\exp\left(\frac{\pi}{2}|Z'|^2\right) \leq \exp\left(\frac{\pi b^2}{2}\log p\right)$$
$$\leq p^{\left[\frac{\pi b^2}{2}\right] + 1}$$

Thm 5 : $N = [\pi b^2] + 2$

$$\text{Error term} = p^{-(N+1)/2}$$
$$= p^{-\left(\left[\frac{\pi b^2}{2}\right] + 1 + \frac{1}{2}\right)}$$

$$\underbrace{\exp\left(\frac{\pi}{2}|Z'|^2\right)\cdot \text{Error term}}_{\Delta} \leq p^{-\frac{1}{2}}.$$

$\varepsilon \in (0, \frac{1}{4})$

$$\frac{\exp\left(\frac{\pi}{2}d(x,y)^2\right) B_p(x,y)}{p^n} = \underset{\Delta}{1} + \underbrace{O(p^{-1/2+\varepsilon})}_{R_p(x,y)}$$

$$\sim \sqrt{B_p(x) B_p(y)}$$

For : $R_p(x, y)$ $\quad$ Exercise $\qquad$ #

§2 Number variance of $[Z(S(L^p))]$

In Thm 4 we prove

$$Y_p = \langle \frac{1}{p}[Z(S(L^p))] - \frac{1}{p}\gamma_{FS}(L^p, h_p), \varphi \rangle$$

$$\longrightarrow \mathbb{E}[|Y_p|^2] = O(\frac{1}{p^2})$$

{ Bleher - Shiffman - Zelditch
{ Shiffman - Zelditch GAFA 2008

$\mathrm{Var}[Z(S(L^p))]$ current

$$\begin{tikzcd} & X \times X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & X \end{tikzcd}$$

β_1, β_2 $(1,1)$-currents on X

\boxtimes

$\beta_1 \boxtimes \beta_2 \in (2,2)$-currents on $X \times X$

$:= \pi_1^* \beta_1 \wedge \pi_2^* \beta_2$ well-defined.

$$\partial = \partial_1 + \partial_2 \qquad \bar\partial = \bar\partial_1 + \bar\partial_2$$

Def: The variance current of $[Z(S(L^p))]$ is $(2,2)$-current on $X \times X$ defined as

$$\mathrm{Var}[Z(S(L^p))] := \mathbb{E}\Big[[Z(S(L^p))] \boxtimes [Z(S(L^p))]\Big] - \mathbb{E}[[Z(S(L^p))]] \boxtimes \mathbb{E}[[Z(S(L^p))]]$$

$\forall\ \varphi \in \Omega_0^{n-1,n-1}(X, \mathbb{R})$ real test form

$$\mathrm{Var}\left[\langle [Z(S(L^p))], \varphi\rangle\right]$$
$$= \langle \mathrm{Var}[Z(S(L^p))], \varphi \boxtimes \varphi \rangle$$

Rk: cpt Kähler Shiffman-Zelditch (2008, 2010)

Their method extends to our setting.

Lemma (Bleher-Shiffman-Zelditch 2000, Shiffman-Zelditch 2008)

If (η_1, η_2) is a joint Gaussian vector in $\mathbb{C}^2$

$$\begin{cases} \eta_j \sim \mathcal{N}_{\mathbb{C}}(0,1) \\ \left|\mathbb{E}[\eta_1\bar{\eta}_2]\right| = \cos\theta > 0 \end{cases}$$

Then $\mathbb{E}\left[\log|\eta_1| \log|\eta_2|\right] = \pi^2 G(\cos\theta) + \frac{\gamma^2}{4}$

γ = Euler's constant

$t \in [0,1]$

$$G(t) := -\frac{1}{4\pi^2}\int_0^{t^2} \frac{\log(1-s)}{s}\, ds$$
$$= \frac{1}{4\pi^2}\sum_{j=1}^{+\infty} \frac{t^{2j}}{j^2}$$

Thm 8: Assume (H)

Set $Q_p(x,y) = G(N_p(x,y))$

Then

$$\mathrm{Var}[Z(S(L^p))] = -\partial_1\bar{\partial}_1\partial_2\bar{\partial}_2 Q_p \quad \text{on } X \times X$$

(2,2)-current

Fix $\varphi \in \Omega_{0}^{n-1,n-1}(X,\mathbb{R})$, $\forall p \gg 0$

$$\mathrm{Var}(\langle [Z(S(L^p))], \varphi\rangle) = p^{-n}\left(\frac{\zeta(n+2)}{4\pi^2}\int_X |L(\varphi)|^2 dV(g) + O(p^{-\frac{1}{2}+\varepsilon})\right)$$

Shiffman 2021

for $n=1$ we get $\Delta\varphi$

$$\zeta(n+2) = \sum_{k=1}^{+\infty}\frac{1}{k^{n+2}}$$

$$\underset{\substack{\downarrow \\ C_0^\infty(X,\mathbb{R})}}{L(\varphi)}\underbrace{\frac{c_1(L,h_L)^n}{n!}}_{dV(g)} := \sqrt{-1}\partial\bar\partial\varphi$$

Pf: As in the proof of Thm 4

$$\mathrm{Var}(\langle [Z(S(L^p))], \varphi\rangle)$$

$$= -\frac{1}{\pi^2}\int_{\substack{X\times X \\ (x,y)}} \partial_1\bar\partial_1\varphi(x)\wedge\partial_2\bar\partial_2\varphi(y)\,\underbrace{\mathbb{E}\left[\log\left|\frac{S(L^p)(x)}{\sqrt{B_p(x)}}\right|_{h_p}\log\left|\frac{S(L^p)(y)}{\sqrt{B_p(y)}}\right|_{h_p}\right]}_{\pi^2 G(N_p(x,y)) + \frac{\gamma^2}{4}}$$

$$Q_p(x, y=\exp_x(\tfrac{Z}{\sqrt{p}})) = \begin{cases} G(\exp(-\frac{\pi}{2}|Z|^2)) + O(p^{-\frac{1}{2}+\varepsilon}) & |Z|\le b\sqrt{\log p} \\ O(p^{-k}) & |Z|\ge b\sqrt{\log p}\end{cases}$$

$$\operatorname{Var}\Big[\underset{\substack{\uparrow\\ \varphi}}{\ }\Big] = \int_{x\in X} \overbrace{\sqrt{-1}\,\partial_1\bar\partial_1\varphi(x)}^{L(\varphi)(x)\,dV(x)} \underbrace{\int_{y\in X} Q_p(x,y)\,\underbrace{\sqrt{-1}\,\partial_2\bar\partial_2\varphi(y)}_{L(\varphi)(y)\,dV(y)}}$$

fixed

$$I_\varphi(x) = \int_{\substack{Z\in T_xX\\ |Z|\le b\sqrt{\frac{\log p}{p}}}} Q_p(x,\exp_x Z)\,L(\varphi)(\exp_x Z)\,dV^X(Z) + O(p^{-n-1})$$

$$Z \leftrightarrow \frac{Z}{\sqrt{p}}$$

$$= \frac{1}{p^n}\int_{|Z|\le b\sqrt{\log p}} \underbrace{Q_p\left(x,\exp_x\left(\frac{Z}{\sqrt p}\right)\right)}\,L(\varphi)\left(\exp_x\frac{Z}{\sqrt p}\right)dV^X\left(\frac{Z}{\sqrt p}\right) + O(p^{-n-1})$$

$$= \frac{L(\varphi)(x)}{p^n}\left(\underbrace{\int_Z G\left(\exp\left(-\frac{\pi}{2}|Z|^2\right)\right)dV^{T_xX}(Z)}_{\substack{\|\\ \frac{\xi(n+2)}{4\pi^2}}} + O(p^{-1/2+\varepsilon})\right)$$

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§3 CLT (central limit theorem)

Theorem 9: Assume (H), $\varphi \in \Omega_0^{n-1,n-1}(X, \mathbb{R})$
$\partial\bar\partial\varphi \not\equiv 0$

Then

$$\frac{\langle [Z(S(L^p))], \varphi \rangle - \mathbb{E}[\langle [Z(S(L^p))], \varphi \rangle]}{\sqrt{\mathrm{Var}(\langle [Z(S(L^p))], \varphi \rangle)}}$$

$\Big\downarrow$ converges in distribution

$$\mathcal{N}_{\mathbb{R}}(0, 1)$$

[Sodin - Tsirelson 2004
Shiffman - Zelditch 2010.

Cor 3: (H) fix $\varphi \in \Omega_0^{n-1,n-1}(X, \mathbb{R})$

$$\sigma(\varphi) \overset{\triangle}{=} \frac{\zeta(n+2)}{4\pi^2} \int_X |L(\varphi)|^2(x)\,dV(x) \neq 0$$

Then

$$p^{n/2} \langle [Z(S(L^p))] - p\,c_1(L, h_L), \varphi \rangle \xrightarrow{\text{in distribution}} \mathcal{N}_{\mathbb{R}}(0, \sigma(\varphi)).$$

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