Stochastic Interacting Particle Systems and Quantum Spin Chains

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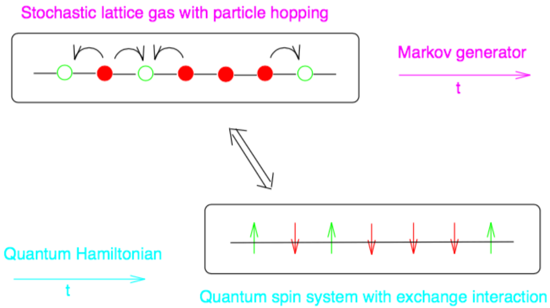
Overview: Classical stochastic many-body systems and quantum spin chains (Projector)

1. Markov Processes, master equation, and quantum systems
2. Matrix product states
3. Symmetry and duality
4. Large deviations and non-hermitian Hamiltonians

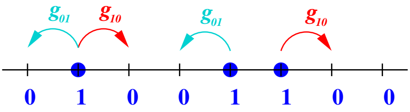
Outlook: Brief comments on large-scale behaviour (Projector)

Overview: Classical stochastic many-body systems and quantum spin chains

- Unpredictable time evolution of states
- Deterministic dynamics of probability (amplitude)
- No memory (interactions local in time)



Paradigmatic example: (A)symmetric Simple Exclusion Process



- Generator (ASEP)

$$\mathcal{L}f(\mathbf{n}) = \sum_{\ell=1}^{L} [g_{10} n_\ell (1 - n_{\ell+1}) + g_{01} n_{\ell+1}(1 - n_\ell)] \left[f(\mathbf{n}^{\ell,\ell+1}) - f(\mathbf{n}) \right]$$

- Quantum Hamiltonian (XXZ chain)

$$H = -\frac{1}{2} \sum_{\ell=1}^{L} \left[\sigma_\ell^x \sigma_{\ell+1}^x + \sigma_\ell^y \sigma_{\ell+1}^y + \sigma_\ell^z \sigma_{\ell+1}^z - \mathbf{1} + ig \left(\sigma_\ell^x \sigma_{\ell+1}^y - \sigma_\ell^y \sigma_{\ell+1}^x \right) \right]$$

$(g_{10} = 1 + g, g_{01} = 1 - g)$

ASEP: $\boldsymbol{\eta} = (n_1, \ldots, n_L)$, $n_\ell \in \{0, 1\}$

Observables: $A(\boldsymbol{\eta}_t)$ such as $n_\ell(t)$ and sums of products
Probability distribution: $P(\boldsymbol{\eta}, t) = \mathrm{Prob}\,[\,\boldsymbol{\eta}_t = \boldsymbol{\eta}\,]$
Expectations: $\langle\, A \,\rangle_{P(t)} = \sum_{\boldsymbol{\eta}} A(\boldsymbol{\eta}) P(\boldsymbol{\eta}, t)$
Stationary distribution(s): $P^*(\boldsymbol{\eta})$

XXZ: Choose $\boldsymbol{\sigma} = (\sigma_1^z, \ldots, \sigma_L^z)$, $\sigma_\ell^z \in \{1, -1\}$

Observables: $A(t)$ such as $\sigma_\ell^\alpha(t)$ and hermitian sums of products
Probability amplitude: $\Psi(\boldsymbol{\sigma}, t)$ with $\mathrm{Prob}\,[\,\boldsymbol{\sigma}_t = \boldsymbol{\sigma}\,] = |\Psi(\boldsymbol{\sigma}, t)|^2$
Expectations: $\langle\, A \,\rangle_{\Psi(t)} = \sum_{\boldsymbol{\sigma}} A(\boldsymbol{\sigma}) |\Psi(\boldsymbol{\sigma}, t)|^2$
Stationary distribution(s): Ground state(s) $\Psi^a st(\boldsymbol{\sigma})$

More general:

Density matrix: $\rho(t)$ (= above for $\rho(t) = |\, \Psi(t)\,\rangle\langle\, \Psi(t)\,|$)
Expectations: $\langle\, A \,\rangle_{\rho(t)} = \mathrm{Tr}\, A\rho(t)$
Stationary distributions: Any eigenvector, ρ^* any function of H

Equilibrium and non-equilibrium systems (I)

- Stochastic equilibrium processes:

- Boltzmann weight: $P^*(\boldsymbol{\eta}) \propto \exp\left(-\beta E(\boldsymbol{\eta})\right)$
- Reversibility (e.g. detailed balance $w_{\boldsymbol{\eta},\boldsymbol{\eta}'} P^*(\boldsymbol{\eta}') = w_{\boldsymbol{\eta}',\boldsymbol{\eta}} P^*(\boldsymbol{\eta})$)
- Spectrum of generator real

- Quantum systems in equilibrium:

- Density matrix: $\rho^*(\boldsymbol{\eta}) \propto \exp\left(-\beta H\right)$
- H is hermitian
- Spectrum real

Equilibrium and non-equilibrium systems (II)

- Stochastic non-equilibrium processes:

- Stationary distribution $P^*(\eta)$ not given in terms of energy
- No reversibility
- Spectrum usually complex

- Quantum systems out of equilibrium:

- Density matrix: $\rho^*(\eta) \neq \exp(-\beta H)$
- Generator is non-hermitian
- Spectrum usually complex

Markov and Quantum in a nutshell:

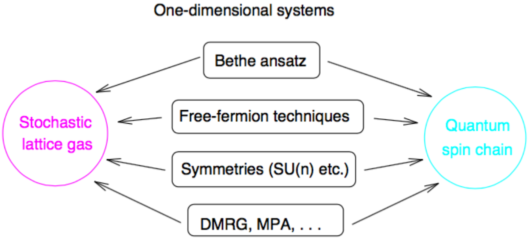
No direct correspondence on the level of the process, but some mathematical equivalences on the level of expectations!

A little dictionary for classical stochastic and quantum dynamics

Concept	Classical	Quantum
state	probability vector $\vert P(t)\rangle$ $\langle s \vert P(t)\rangle = 1$	wave function $\vert \Psi(t)\rangle$ $\langle \Psi(t) \vert \Psi(t)\rangle = 1$
time evolution	Master equation $\frac{d}{dt}\vert P(t)\rangle = -H\ \vert P(t)\rangle$ Markov generator H	Schrödinger equation $i\frac{d}{dt}\vert \Psi(t)\rangle = -H\ \vert \Psi(t)\rangle$ Quantum Hamiltonian H
observable	Diagonal operator A $\langle A(t)\rangle = \langle s \vert A \vert P(t)\rangle$	Hermitian operator A $\langle A(t)\rangle = \langle \Psi(t) \vert A \vert \Psi(t)\rangle$
stationary states	$\vert P^{*}\rangle = \lim_{t \to \infty} \vert P(t)\rangle$	ground state of H, eigenvectors
spectrum	(inverse) relaxation times	energy gaps

Common techniques and questions

Common techniques:



and more ...

Common questions: Many, a few will be discussed...

Brief comments on large-scale behaviour

Non-linear fluctuating hydrodynamics (Spohn, 2014)

- Stochastic non-linear PDE for coarse-grained fluctuation fields
- Universal tool for translation invariant 1-d systems when

– short-range interactions, local conservation laws and currents

– slow variables relevant for long-time behavior = long-wavelength Fourier components of the conserved densities

$\Longrightarrow$ Hamiltonian dynamics, anharmonic chains, stochastic lattice gases, ...

- Quadratic non-linear terms leading, cubic terms only marginally relevant (and only if quadratic terms are absent), quartic and higher order irrelevant in RG sense.

Starting point

- Conservation law, LLN, local stationarity, Onsager-type symmetry

$\Longrightarrow$ Euler scale: Hyperbolic system of conservation laws

$$\frac{\partial}{\partial t}\vec{\rho}(x,t) + \bar{\mathbf{J}}\frac{\partial}{\partial x}\vec{\rho}(x,t) = 0$$

with current Jacobian $\bar{\mathbf{J}}(x,t) = \mathbf{J}(\vec{\rho}(x,t))$

- Expand around stationary solution $\rho^\lambda(x,t) = \rho^\lambda + u^\lambda(x,t)$
- Transform to normal modes $\vec{\phi} = \mathbf{R}\vec{u}$ where $\mathbf{RJR}^{-1} = \mathrm{diag}(v_\alpha)$ for $\mathbf{J} \equiv \mathbf{J}(\vec{\rho})$ and R normalized such that $\mathbf{RKR}^T = \mathbb{1}$

$\Longrightarrow$ Coupled Burgers equations

$$\partial_t \phi^\alpha = -\partial_x \left(v_\alpha \phi^\alpha + \langle \vec{\phi}, G^\alpha \vec{\phi} \rangle \right)$$

First and second order in ϕ^α

- First order only: Travelling waves $\phi^\alpha(x,t) = \phi_0^\alpha(x - v_\alpha t)$ with initial data $\phi^\alpha(x,0) = \phi_0^\alpha(x)$

$\Longrightarrow$ v_α = cms velocity of fluctuation field α

- Second order nonlinearity:

Mode coupling matrices $\mathbf{G}^\alpha = \frac{1}{2}\sum_\lambda R_{\alpha\lambda}(\mathbf{R}^{-1})^T \mathbf{H}^\lambda \mathbf{R}^{-1}$

Mode-coupling coefficients $G^\alpha_{\beta\gamma}$ determined by the current Hessians $\mathbf{H}^\lambda$ with $\mathbf{H}^\lambda_{\alpha\beta} = \frac{\partial^2}{\partial\rho^\alpha\partial\rho^\beta} j^\lambda$

$\Longrightarrow$ Coarse-grained evolution equation fully determined by macroscopic stationary current and compressibility!

Fluctuations

- System of conservation laws deterministic
- Add phenomenological diffusion term and noise to current:

$\Longrightarrow$ Coupled noisy Burgers equations

$$\partial_t \phi^\alpha = -\partial_x \left(v_\alpha \phi^\alpha + \langle \vec{\phi}, \mathbf{G}^\alpha \vec{\phi} \rangle - \partial_x (\mathbf{D}\vec{\phi})^\alpha + \xi^\alpha \right)$$

- equivalent to coupled KPZ equations with $\phi^\alpha = \partial_x h^\alpha$

$\star\star$ Basis for discussing dynamical structure functions

$$S^{\alpha\beta}(x,t) = \langle\, \phi^\alpha(x,t)\phi^\beta(0,0)\,\rangle$$

in generic one-dimensional conservative systems with short-range interactions and local conservation laws and currents $\star\star$

Mode-coupling theory

- Postulate Gaussian stationary distribution for height variables (can be proved on discretized level for "trilinear relation" $G^{\alpha}_{\beta\gamma} = G^{\beta}_{\alpha\gamma}$) $\Longrightarrow$ Wick theorem
- Consider strictly hyperbolic case (non-degenerate $\mathbf{J}$)

$\Longrightarrow$ Off-diagonal $S^{\alpha\beta}$ as well as products $S^{\alpha\alpha}S^{\beta\beta}$ decay quickly

$\Longrightarrow$ One-loop mode coupling equation for $S_\alpha \equiv S^{\alpha\alpha}$

$$\partial_t S_\alpha(x,t) = \hat{D}_\alpha S_\alpha(x,t) + \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \, S_\alpha(x-y,t-s) M_\alpha(y,s)$$

Linear diffusion operator $\hat{D}_\alpha = -v_\alpha \partial_x + D_\alpha \partial_x^2$

Nonlinear memory kernel $M_\alpha(y,s) = 2\partial_y^2 \sum_\beta \left(G^{\alpha}_{\beta\beta} S_\beta(y,s) \right)^2$

Fibonacci universality classes

Define the set $\mathbb{I}_\alpha := \{\beta : G^\alpha_{\beta\beta} \neq 0\}$ of modes β that give rise to a non-linear term in the time-evolution of mode α

Theorem (Popkov, Schadschneider, Schmidt, GMS, 2016)

a) Scaling solution of mode coupling equation for $|p| > 0$:

- *Case 1:* $\mathbb{I}_\alpha = \emptyset \implies$ *Dynamical exponent* $z_\alpha = 2$, *Scaling form* $\hat{S}_\alpha(p,t) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-iv_\alpha pt - D_\alpha p^2 t}$ *(Diffusion)*
- *Case 2:* $\alpha \notin \mathbb{I}_\alpha, \mathbb{I}_\alpha \neq \emptyset \implies$ *Dynamical exponents satisfy nonlinear recursion* $z_\alpha = \min_{\beta \in \mathbb{I}_\alpha} \left[\left(1 + \frac{1}{z_\beta} \right) \right]$, *Scaling form* $\hat{S}_\alpha(p,t) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\left(-iv_\alpha pt - E_\alpha |p|^{z_\alpha} t \left[1 - iA_\alpha \tan\left(\frac{\pi z_\alpha}{2}\right) \mathrm{sgn}(p)\right]\right)}$ *(Lévy)*

A_α, E_α: *Determined by mode coupling coefficients and diffusion constants*

Theorem (cont')

Case 3a: $\alpha \in \mathbb{I}_\alpha$, *no diffusive mode* $\beta \in \mathbb{I}_\alpha \Longrightarrow$ *Dynamical exponent* $z_\alpha = 3/2$

Case 3b: $\alpha \in \mathbb{I}_\alpha$, *at least one diffusive mode* $\beta \in \mathbb{I}_\alpha \Longrightarrow$ *Dynamical exponent* $z_\alpha = 3/2$

b) Unique solution of non-linear recursion for dynamical exponents: Kepler ratios of Fibonacci numbers $z_\alpha = F_{i+1}/F_i$ *for some* $i(\alpha)$, *starting from* $z = 2$ *(coupling to diffusive mode), or* $z = 3/2$ *(coupling to KPZ mode) or Golden mean* $\phi = (1 + \sqrt{5})/2$ *(else)*

Numerically well-founded conjectures:

- Case 3a: Scaling form: Prähofer-Spohn function (KPZ), Case 3b: Scaling form: Unknown (modified KPZ) [Spohn, Stoltz (2015)]

- Fibonacci exponents exact, scaling forms universal

Fluctuating quantum hydrodynamics?

Consider only one conservation law and only steady state:

- Hydrodynamic (large-scale) theory for classical systems (one conservation law): Local equilibrium at density $\rho(x)$

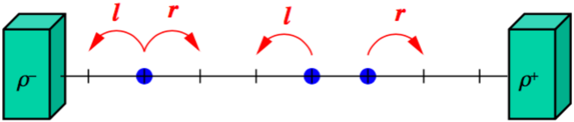
➔ Stationary current J and density profile $\rho(x)$ determined by D and boundary densities $\rho^{\pm}$ via stationary diffusion equation

$$\partial_x \left[D(\rho(x))\, \partial_x \rho(x)\right] = 0$$

- Fluctuating Hydrodynamics (Spohn, 1983):

➔ Long-range stationary density correlations

Example: Symmetric simple exclusion process $r = l = D(\rho)$



$J = -\,D\,(\rho^{+} - \rho^{-})/L, \qquad \rho(x) = \rho^{-} - J/D\, x \qquad C(x,y) = -\,1/L\, x\,(1-y)\,(\rho^{+} - \rho^{-})^2$

Boundary-driven XXZ-chain:

Maximal driving in XXZ quantum chain $\theta = \pi$:

Maximal driving in $\pm x$-direction, rotation-invariant around x-axis,
[Prosen (2011) and Buca and Prosen (2016)]

- Currents: $J^x = 4\pi^2/N^2$, $J^y = 0$ $J^z = 0$
- Magnetization profiles and correlations:

$$m^x(u) = \cos(\pi u), \quad m^y(u) = m^z(u) = 0$$

$$C^{xx}(u_1,u_2) = -1/N\, u_1(1-u_2)\, \pi^2\, C^{\mathrm{perp}}(u_1,u_2) \quad \rightarrow \textit{SSEP correlation}$$

$$C^{zz}(u_1,u_2) = C^{yy}(u_1,u_2) =: C^{\mathrm{perp}}(u_1,u_2) = \tfrac{1}{2}\sin(\pi u_1)\sin(\pi u_2)$$

$\Rightarrow$ Many open questions ...