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Christiaan Huygens and Contact Geometry

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> "— Oui, voilà le géomètre! Et ne crois pas que les géomètres n'aient pas à s'occuper des femmes!"
>
> Jean Giraudoux,
> *La guerre de Troie n'aura pas lieu*

1. Introduction

For me, the most evocative painting in the Mauritshuis in Den Haag has always been *Het meisje met de parel*, even before a novel and a film turned the girl into something of a pop icon. However, that museum is the home to another portrait that cannot fail to attract the attention of any scientifically interested visitor, and one where the identity of (some of) the portrayed, like in Vermeer's famous painting, is shrouded in mystery. I am speaking of Adriaan Hanneman's *Portret van Constantijn Huygens en zijn kinderen* (Figure 1). This family portrait depicts C. Huygens (1596–1687) — "the most versatile and the last of the true Dutch Renaissance virtuosos" (Encyclopaedia Britannica), whose most notable contributions lay in the fields of diplomacy and poetry —, together with his five children. Among them is Christiaan Huygens (1629–1695), who would go on to become one of the most famous mathematical scientists of his time, later to be characterised as "ein Junggeselle von hervorragendem Charakter und außergewöhnlicher Intelligenz" [11]. While I expound some of the mathematical themes of Christiaan Huygens' life and hint at their relation to modern contact geometry, I leave the reader to ponder the question just which of the four boys in the family portrait shows that intellectual promise, a question to which I shall return at the end of this article.

Figure 1

An inaugural lecture is not only an opportunity to present one's field of research to a wider public, it also allows one to reflect on the standing of mathematics within the general intellectual discourse. On an earlier occasion of this kind [5] I have not been overly optimistic in this respect, and I have no reason to qualify anything I said there. Still, it is worth remembering that there have been even more precarious times for mathematics. In [4] we read that

"The new Savilian professor [Baden Powell, Savilian professor of geometry at the University of Oxford 1827–1860] was shocked and dismayed by the low esteem accorded to mathematics in the University. He had been advised not to give an inaugural lecture on arrival, as he would almost surely not attract an audience."

Disclaimer. A foreigner, even one who has lived in the Netherlands for several years, is obviously carrying tulips to Amsterdam (or whatever the appropriate turn of phrase might be) when writing about Christiaan Huygens in a Dutch journal. Then again, from a visit to the Huygensmuseum Hofwijck in Voorburg near Den Haag I gathered that in the Netherlands the fame of Constantijn Huygens tends to outshine that of his second-eldest son. Be that as it may, this article is intended merely as a relatively faithful record of my inaugural lecture (with some mathematical details added) and entirely devoid of scholarly aspirations. I apologise for the fact that the transcription from the spoken to the written word has taken rather longer than anticipated.

2. The best slide for twins

Imagine that you are trying to connect two points A, B in a vertical plane by a slide along which a point mass M will move, solely under the influence of gravitation, in shortest time from A to B (see Figure 2). This is the famous brachistochrone problem (from Greek $\beta\rho\acute{\alpha}\chi\iota\sigma\tau o\varsigma$ = shortest, $\chi\rho\acute{o}\nu o\varsigma$ = time), posed by Johann Bernoulli in 1696 in rather more erudite language: "Datis in plano verticali duobus punctis A & B assignare Mobili M viam AMB, per quam gravitate sua descendens & moveri incipiens a puncto A, brevissimo tempore perveniat ad alterum punctum B." (*Problema novum ad cujus solutionem Mathematici invitantur*, Joh. Op. XXX (pars), [3, p. 212]).

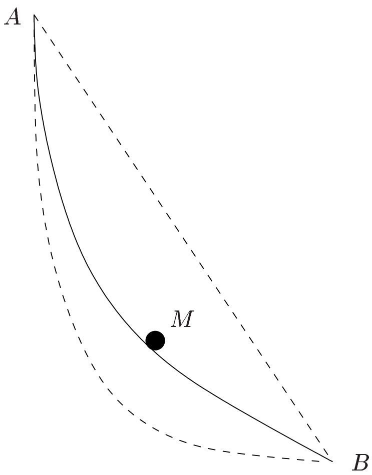

Figure 2

A related problem is to find the slide connecting the points A and B in such a way that one will reach the end-point B of the slide in the same amount of time, no matter where on the slide one starts. This is known as the tautochrone problem.

Rather surprisingly, it turns out that the solution to either question is one and the same curve, the so-called *cycloid*. This is obviously the best slide a doting uncle can build for his twin nephews: not only will their slide be faster than anybody else's; if both of them start at the same time at any two points of the slide, they will reach the bottom of the slide simultaneously. This gives them the chance and the time to fight over other things.

In 1697 Jacob Bernoulli responded to the challenge set by his brother concerning the brachistochrone with a paper bearing the beautiful title *Solutio Problematum Fraternorum, una cum Propositione reciproca aliorum*, Jac. Op. LXXV [3, pp. 271–282]. Johann's own solution appeared the same year (Joh. Op. XXXVII, [3, pp. 263–270]). The tautochrone problem had been solved by Christiaan Huygens as early as 1657, but the solution was not published until 1673 in his famous *Horologium Oscillatorium* [9], cf. [16].

3. The cycloid

The cycloid is the locus traced out by a point on the rim of a circle as that circle rolls along a straight line (Figure 3). Choose cartesian coordinates in the plane such that the circle rolls along the x–axis, with the point on the rim initially lying at the origin $(0, 0)$. Let a be the radius of the circle. When the circle has turned through an angle t, its centre lies at the point (at, a), and so a parametric description of the cycloid is given by

$$x(t) = a(t - \sin t), \quad y(t) = a(1 - \cos t).$$

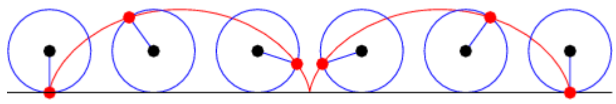

Figure 3 (from `http://mathworld.wolfram.com`)

The cycloidal slide is obtained by turning this curve upside down. It is convenient to effect this by reversing the direction of the y–coordinate, while keeping the parametric equations unchanged. Given two points $A = (0, 0)$ and $B = (b_1, b_2)$ with $b_1 > 0$, $b_2 \geq 0$ in the xy–plane, there is a unique radius a and angle $t_B \in (0, 2\pi]$ such that $A = (x(0), y(0))$ and $B = (x(t_B), y(t_B))$. There are various ways to see this, the following is due to Isaac Newton, cf. [3, p. 43]: Draw any cycloid starting at A, and let Q be its intersection with the straight line segment AB. Then expand the cycloid by a factor AB/AQ. (Here and below I use the same symbol AB to denote a curve or line segment between two points A, B, as well as the length of that segment, provided the meaning is clear from the context.)

For some of the reasonings below I shall assume implicitly that $t_B \leq \pi$, so that the cycloidal segment connecting A and B does not have any upward slope; this is equivalent to requiring $b_2 \geq 2b_1/\pi$.

The brachistochrone and tautochrone problems were two of the most challenging geometric questions of 17th century mathematics, attracting the attention of the most famous (and cantankerous) mathematicians of that time, including the Marquis de L'Hospital, Leibniz, and Newton. As a result, these problems were the source of acrimonious battles over priority — the publications of the Bernoulli brothers on this topic have even been published in a collection bearing the title *Streitschriften* [3]. This was not the only occasion when the cycloid was the object of desire in a mathematical quarrel, and so this curve has often been dubbed the 'Helen of Geometers'.

The following allusion to the tautochronous property of the cycloid in Herman Melville's *Moby Dick* [13, Chapter 96, The Try-Works] shows that there were happy times when the beauty of mathematics had to some degree entered popular consciousness: "[The try-pot[1]] is a place also for profound mathematical meditation. It was in the left hand try-pot of the *Pequod*[2], with the soapstone diligently circling around me, that I was first indirectly struck by the remarkable fact, that in geometry all bodies gliding along the cycloid, my soapstone for example, will descend from any point in precisely the same time."

4. The cycloidal pendulum

Besides the discovery of the true shape of Saturn's rings and one of its moons, namely Titan, Christiaan Huygens' most important scientific contributions are his theory of light, based on what has become known as Huygens' principle (discussed in the next section), and his development of a pendulum clock starting from his proof of the tautochronous property of the cycloid.

At the time of Huygens, pendulum clocks were built (as they usually are today) with a simple circular pendulum. The problem with such a pendulum is that its frequency depends on the amplitude of the oscillation. With regard to the pendulum clock in your living room this is no cause for concern, since there the amplitude stays practically constant. But arguably the most outstanding problem of applied mathematics at that time was to build a clock that was also reliable in more adverse conditions, say on a ship sailing through gale force winds. Why are such accurate clocks important?

As is wryly remarked in the introduction to the lavishly illustrated proceedings of the Longitude Symposium [2], "Traveling overseas, we now complain when delayed for an hour: we have forgotten that once there were problems finding continents". Indeed, how was it possible to determine your exact position at sea (or anywhere else, for that matter), prior to the days of satellite-based Global Positioning Systems? Mathematically the answer is simple (at least on a sunny day): Observe when the sun reaches its highest elevation. This will be noon local time. Moreover, the angle α of elevation will give you the latitude: If the axis of the earth's rotation were orthogonal to the plane in which the earth moves around the sun, that latitude would simply be $90° - \alpha$. In order to take the tilting of the earth's axis by $23°$ into account, one needs to adjust this by an angle that depends on the date, varying between $0°$ at the equinoxes and $\pm 23°$ at the solstices.

The longitude, on the other hand, cannot be determined from this observational data alone. Indeed, the actual value of the longitude at any given point is a matter of convention. The fact that the zero meridian passes through Greenwich is a consequence of the scientific achievements and geopolitical power of the British, not astronomy. However, if you keep a clock with you that shows accurate Greenwich time, and you bear in mind that the earth rotates by a full $360°$ in 24 hours, then multiplying the difference between your local time and that shown on the clock by $15°/h$ will determine your longitude relative to that of Greenwich.

All the practical problems involved in building such an accurate clock were first solved by John Harrison in 1759, cf. [2] and the thrilling account of Harrison's life in [14].

From a mathematical point of view, the question addressed by Huygens concerned the most interesting aspect of these practical problems: Is it possible to devise a pendulum whose frequency does not depend on the amplitude of the pendular motion? The hardest part of this question is to find the tautochronous curve, along which the pendulum mass should be forced to move. This Huygens established to be the cycloid. He further observed that one could make the pendulum move along a cycloid by restricting the swinging motion of the pendulum between appropriately shaped plates.

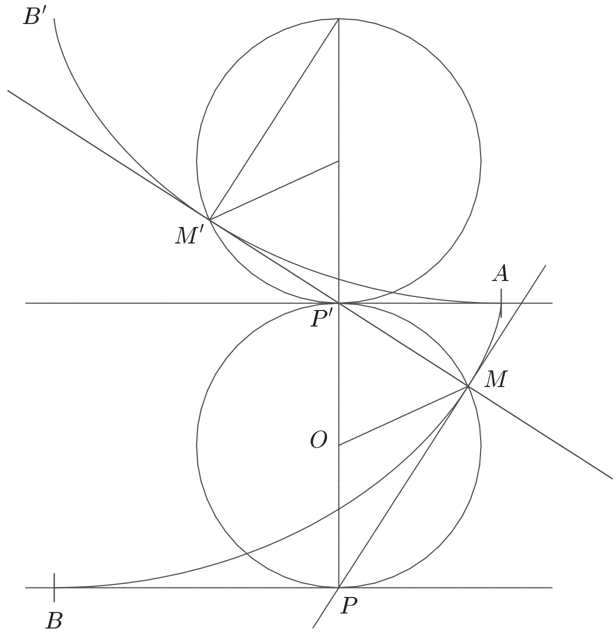

Figure 4

[1] A pot for trying oil from blubber.

[2] Captain Ahab's ship, named after an Indian people.

Take a look at Figure 4 (kindly provided by Manfred Lehn). Here AB is (half) a cycloid, along which the pendulum mass M, attached to the string $B'M$, is supposed to move. This means that we require this string to be tangent to the curve $B'A$ at the point M', and the length $B'M$ to equal $B'A$, the length of the pendulum. In other words, the cycloid AB is given by tightly unrolling (whence the title of [16]) a string from the curve $B'A$. If the pendulum is forced to swing between two plates shaped like $B'A$, then the pendulum mass will move along the cycloid, as desired.

Such a curve AB obtained by unrolling a string from a curve $B'A$ is called the *involute* of $B'A$ (and $B'A$ the *evolute* of AB). So the second question faced by Huygens was: Which curve has the cycloid as its involute? Rather miraculously, the answer is again: the cycloid. Here is the geometric proof: Let AB be the cycloid traced out by the point M as the lower circle in Figure 4 rolls to the left along the horizontal line between the two circles (with $M = A$ at $t = 0$), and $B'A$ the cycloid traced out by the point M' as the upper circle rolls to the right along a horizontal line through B' (with $M' = B'$ at $t = 0$). With the defining equations for the cycloids as in the previous section, the situation shown in the figure corresponds to $t = t_0$ for some $t_0 \in [0, \pi]$ for the lower circle and $t = \pi - t_0$ for the upper circle.

The velocity (with respect to the parameter t) of the point M can be split into two vector components of length a: one in horizontal direction, corresponding to the speed of the centre of the circle, and one in the direction tangent to the circle, corresponding to the angular speed of the rolling circle. An elementary consideration shows that the line MP bisects the angle between these two directions, and so this line constitutes the tangent line to the cycloid at M. Analogously, the line $M'P'$ is the tangent line to the cycloid $B'A$ at M'. By symmetry of the construction, the line $M'P'$ passes through M. In order to conclude that AB is the involute of $B'A$ it therefore suffices to show that the length of the cycloidal segment $M'A$ equals the length of the line segment $M'M$. Also observe that, by the theorem of Thales, the line $M'M$ is orthogonal to the tangent line MP at M ; this is a general phenomenon for an involute.

The angle $\angle MOP'$ equals t_0, hence $\angle MPP' = t_0/2$. This yields

$$M'M = 2P'M = 4a \sin \frac{t_0}{2}.$$

On the other hand, from the defining equations of the cycloid we have

$$\dot{x}^2 + \dot{y}^2 = a^2(1 - \cos t)^2 + a^2 \sin^2 t = 4a^2 \sin^2 \frac{t}{2},$$

whence

$$M'A = \int_{\pi - t_0}^{\pi} 2a \sin \frac{t}{2}\, dt = 4a \cos \frac{\pi - t_0}{2} = 4a \sin \frac{t_0}{2},$$

that is, $M'M = M'A$, which was to be shown.

Huygens did not stop at these theoretical considerations, but proceeded to construct an actual pendulum clock with cycloidal plates. The construction plan from Huygens' *Horologium Oscillatorium*, with the cycloidal plates indicated by 'FIG. II', is shown in Figure 5. A replica of this clock can be seen in the Huygensmuseum Hofwijck.

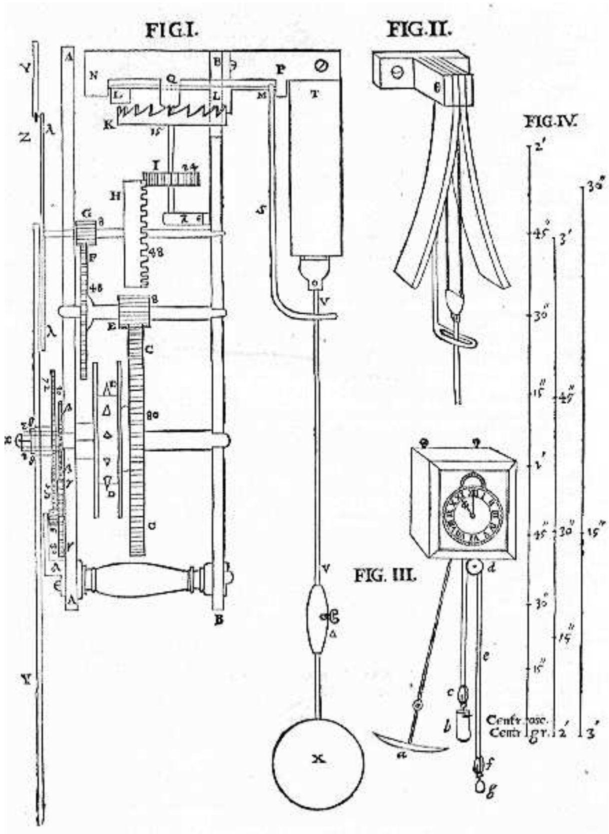

Figure 5

5. Geometric optics

Either of the following fundamental principles can be used to explain the propagation of light:

Fermat's Principle (1658) Any ray of light follows the path of shortest time.

Huygens' Principle (1690, [10]) Every point of a wave front is the source of an elementary wave. The wave front at a later time is given as the envelope of these elementary waves.

The simplest possible example is the propagation of light in a homogeneous and isotropic medium. Here we expect the rays of light to be straight lines. Figure 6 illustrates that this is indeed what the two principles predict. We merely need to observe that, in a homogeneous and isotropic medium, the curves of shortest time are the same as geometrically shortest curves, i.e. straight lines, and elementary waves are circular waves around their centre.

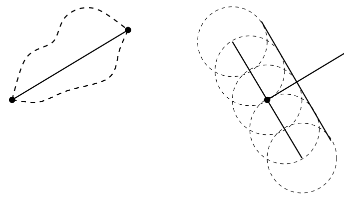

Figure 6

Whereas Fermat's principle can only be justified as an instance of nature's parsimony, cf. [8], Huygens' principle can be explained mechanistically from a particle theory of light, see Figure 7.

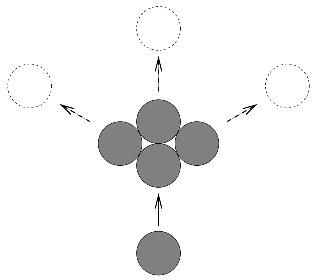

Figure 7

To illustrate the power of these principles, here are two further examples. The first is the law of reflection, which states that the angle of incidence equals the angle of reflection. Figure 8 shows how this follows from Fermat's principle: The path connecting A and B has the same length as the corresponding one connecting A and the mirror image B' of B, and for the latter the shortest (and hence quickest) path is given by the straight line.

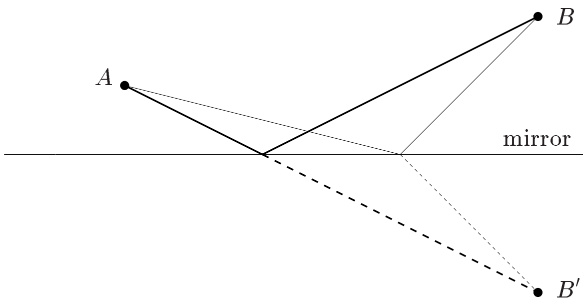

Figure 8

The explanation of the law of reflection from Huygens' principle is illustrated in Figure 9.

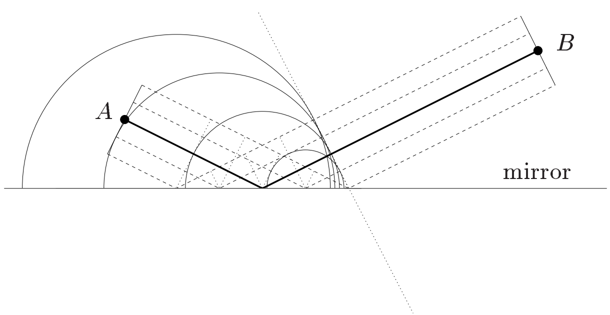

Figure 9

As a final application of the two principles, we turn to the law of refraction, also known as Snell's law after the Dutch astronomer and mathematician Willebrord van Roijen Snell (1580–1628), whose latinised name Snellius now adorns the Mathematical Institute of the Universiteit Leiden. Snell discovered this law in 1621; in print it appears for instance in Huygens' *Traité de la lumière*, with proofs based on either of the two principles. The law states that as a ray of light crosses the boundary between two (homogeneous and isotropic) optical media, the angle of incidence α_1 (measured relative to a line perpendicular to the separating surface) and the angle of refraction α_2 (see Figure 10) are related by

$$\frac{\sin\alpha_1}{v_1} = \frac{\sin\alpha_2}{v_2},$$

where v_1 and v_2 denote the speed of light in the respective medium.

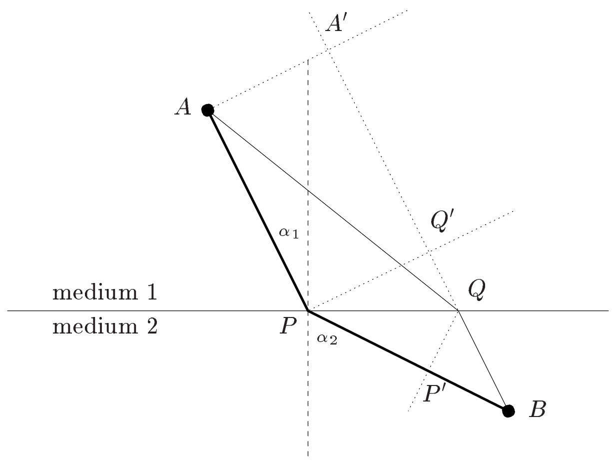

Figure 10

Figure 10 shows how to derive Snell's law from Fermat's principle. The path from A to B via P (drawn in bold) is supposed to be the one satisfying Snell's law. We need to show that it takes longer to travel along any other broken path from A to B via some Q different from P. We compute

$$\frac{PP'}{v_2} = \frac{PQ\sin\alpha_2}{v_2} = \frac{PQ\sin\alpha_1}{v_1} = \frac{QQ'}{v_1},$$

that is,

$$t(PP') = t(QQ'),$$

where $t(\cdot)$ denotes the amount of time it takes to travel along a certain line segment in the corresponding medium. Therefore

$$\begin{aligned} t(AQ) + t(QB) &> t(A'Q) + t(P'B) \\ &= t(A'Q') + t(Q'Q) + t(P'B) \\ &= t(AP) + t(PP') + t(P'B) \\ &= t(AP) + t(PB). \end{aligned}$$

Figure 11 indicates how Snell's law is implied by Huygens' principle.

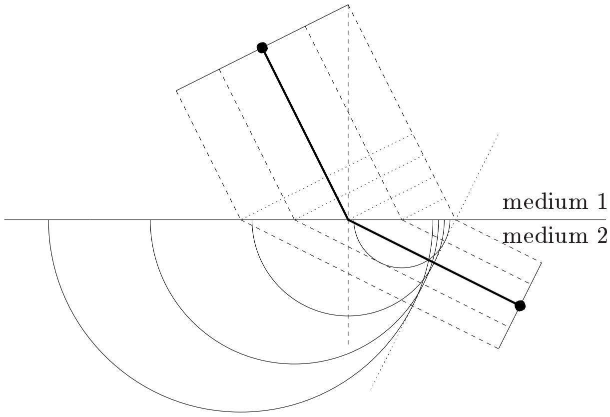

Figure11

6. Johann Bernoulli's solution of the brachistochrone problem

Jacob Bernoulli, in his response (cited in Section 2) to the fraternal challenge, developed a general method for dealing with problems of this kind, nowadays known as the calculus of variations. In the present section we shall be concerned with Johann's own solution, which nicely relates to the concepts of geometric optics discussed above.

When the mass M has reached a point (x, y) on the slide from $A = (0, 0)$ to B, with the y–coordinate oriented downwards, its speed has reached, under the influence of gravitation, the value

$$v = \sqrt{2gy},$$

where $g = 9.81m/s^2$ denotes the gravitational acceleration near the surface of the Earth. In order to determine which path the point M should follow so as to take the shortest time from A to B, we discretise the problem.

Imagine that the region between A and B is layered into finitely many horizontal slices, in each of which the speed of M stays constant. In particular, M should follow a straight line in each layer. As M passes from the ith to the $(i+1)$st layer, the angle α_i of incidence and α_{i+1} of 'refraction' should be related to the respective speeds v_i, v_{i+1} by Snell's law

$$\frac{v_{i+1}}{\sin \alpha_{i+1}} = \frac{v_i}{\sin \alpha_i},$$

for the fact that Snell's law is an instance of Fermat's principle guarantees this to yield the quickest path (Figure 12).

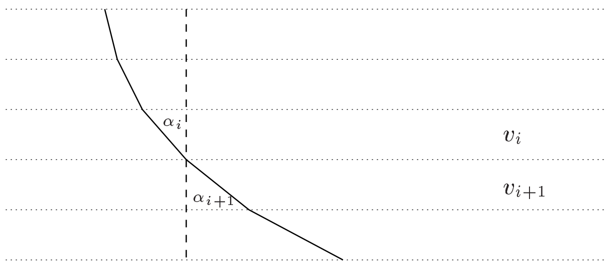

Figure 12

As we let the number of slices tend to infinity, the equation describing the brachistochrone becomes

$$\frac{v}{\sin \alpha} = c$$

for some constant c, see Figure 13. Bravely computing with infinitesimals, we have $\sin \alpha = dx/\sqrt{dx^2 + dy^2}$, whence

$$\sqrt{1 + (\frac{dy}{dx})^2} \cdot \sqrt{2gy} = c.$$

This can be written as

$$\frac{dx}{dy} = \sqrt{\frac{y}{2a - y}}$$

with $a = c^2/4g$. Substitute

$$y(t) = 2a \sin^2 \frac{t}{2} = a(1 - \cos t).$$

Then

$$\frac{dx}{dt} = \frac{dx}{dy} \cdot \frac{dy}{dt} = \sqrt{\frac{1 - \cos t}{1 + \cos t}} \cdot a \sin t = 2a \sin^2 \frac{t}{2} = y,$$

hence (with $x(0) = 0$)

$$x(t) = a(t - \sin t),$$

"ex qua concludo: curvam *Brachystochronam* esse *Cycloidem vulgarem*" [3, p. 266].

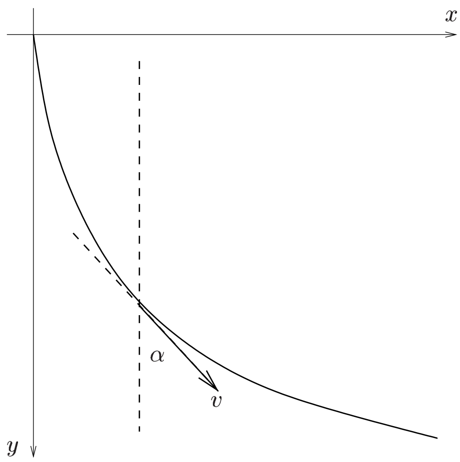

Figure 13

This is as good a point as any to recommend the wonderful textbook [7]. It contains an extensive discussion of both the brachistochrone and tautochrone problem in their historical context, and many other historical gems that so sadly are missing from our usual introductory courses on analysis, which tend to suffer from the dictate of efficiency and the haste to 'cover material'.

7. Elementary contact geometry

Here at last we come to the second part of this article's title. My modest aim is to convey a couple of basic notions of contact geometry and to show how they relate to some of the ideas discussed above. In doing so, I am aware of W. Thurston's warning that "one person's clear mental image is another person's intimidation" [15].

One of the fundamental notions of contact geometry is the so-called *space of (oriented) contact elements* of a given manifold. Let us first consider a concrete example, see Figure 14.

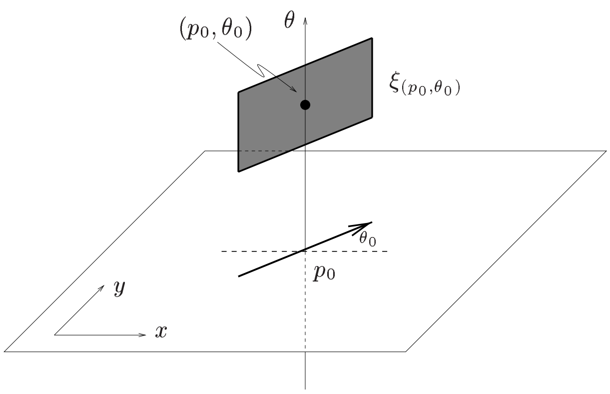

Figure 14

An oriented contact element to the 2–plane $\mathbb{R}^2$ at some point $p_0 \in \mathbb{R}^2$ is simply an oriented line passing through the point p_0. Such a line is uniquely determined by p_0 and an angle θ_0. We can think of this angle θ_0 as an element of the unit circle S^1, so the space of all contact elements of $\mathbb{R}^2$ can be identified with the product $\mathbb{R}^2 \times S^1$.

Let ∂_x, ∂_y denote the unit vectors in the coordinate directions of $\mathbb{R}^2$ at any given point $(x_0, y_0, \theta_0) \in \mathbb{R}^2 \times S^1$. They can be thought of as the velocity vectors of the curves

$$x \longmapsto (x_0 + x, y_0, \theta_0) \quad \text{resp.} \quad y \longmapsto (x_0, y_0 + y, \theta_0).$$

Similarly, we can speak of the velocity vector ∂_θ of the curve

$$\theta \longmapsto (x_0, y_0, \theta_0 + \theta).$$

In the local picture of Figure 14, where θ is measured along a real axis, this is once again simply the unit vector in the direction of the θ–coordinate.

We now specify a 2–plane $\xi_{(p_0,\theta_0)}$ at any point $(p_0, \theta_0) \in \mathbb{R}^2 \times S^1$ as the plane spanned by the vectors

$$\partial_\theta \quad \text{and} \quad \cos\theta_0\, \partial_x + \sin\theta_0\, \partial_y.$$

Alternatively, this plane is determined by the condition that it contain ∂_θ and that it project to the contact element at p_0 defined by θ_0. The collection of all these 2–planes is called the *natural contact structure* on the space of contact elements of $\mathbb{R}^2$.

This probably sounds esoteric or banal, depending on your education. It is anavoidable that at this point I shall have to assume a certain level of mathematical literacy. But I make no apology for continuing, whenever possible, also to address those who are beginning to feel just a little intimidated. The space of contact elements of $\mathbb{R}^2$ in fact has a very natural interpretation as a space of physical configurations. If you want to describe the position of a wheel of your bicycle, say, you should describe its position p and its direction, given by θ. Moreover, instantaneously the wheel can only roll in the direction in which it points at any given moment, so the motion of the wheel, interpreted as a curve in the 3–dimensional space of contact elements, will be tangent to the natural contact structure.

The next concept we want to introduce is that of a *contact transformation*. Such transformations play an important role in the geometric theory of differential equations. Most physicists first encounter them in their special incarnation as so-called Legendre transformations. For our purposes, we can define a contact transformation as a diffeomorphism ϕ of the space of contact elements $\mathbb{R}^2 \times S^1$ with the property that if a curve w passes through a point (p, θ) and is tangent to the 2–plane $\xi_{(p,\theta)}$ at that point, then the image curve $\phi \circ w$ will be tangent to $\xi_{\phi(p,\theta)}$ at $\phi(p, \theta)$.

Here is an example of a whole family of contact transformations: For $t \in \mathbb{R}$, define

$$\begin{array}{rccc} \phi_t: & \mathbb{R}^2 \times S^1 & \longrightarrow & \mathbb{R}^2 \times S^1 \\ & (x, y, \theta) & \longmapsto & (x - t\sin\theta, y + t\cos\theta, \theta). \end{array}$$

In order to verify that these are indeed contact transformations, consider a parametrised curve

$$s \longmapsto w(s) = (x(s), y(s), \theta(s)) \in \mathbb{R}^2 \times S^1, \quad s \in (-\varepsilon, \varepsilon),$$

for some small $\varepsilon > 0$ say, with tangent vector

$$w'(0) = (x'(0), y'(0), \theta'(0))$$

assumed to lie in $\xi_{w(0)}$. With $\pi\colon \mathbb{R}^2 \times S^1 \to \mathbb{R}^2$ denoting the natural projection, this is equivalent to saying that the tangent vector $(x'(0), y'(0))$ of the projected curve $\pi \circ w$ at the point $(x(0), y(0))$ lies in the line determined by $\theta(0)$, i.e. is a multiple of $(\cos\theta(0), \sin\theta(0))$.

The transformed curve is

$$\phi_t \circ w(s) = (x(s) - t\sin\theta(s), y(s) + t\cos\theta(s), \theta(s)).$$

Notice that the θ–coordinate remains unchanged under ϕ_t. We compute

$$\frac{d}{ds}(\phi_t \circ w)(s) = \\ (x'(s) - t\theta'(s)\cos\theta(s), y'(s) - t\theta'(s)\sin\theta(s), \theta'(s))$$

and observe that the $\mathbb{R}^2$–component of this vector at $s = 0$ does again lie in the line determined by $\theta(0)$.

This family ϕ_t of transformations is called the *geodesic flow* of $\mathbb{R}^2$. Here is why: In a general Riemannian manifold, geodesics are locally shortest curves. In $\mathbb{R}^2$ (with its euclidean metric), therefore, geodesics are simply the straight lines. Given a point $p \in \mathbb{R}^2$ and a direction $\theta \in S^1$ defining a contact element, let $\ell_{p,\theta}$ be the unique oriented line in $\mathbb{R}^2$ passing through the point p and positively orthogonal to the contact element θ. This line is parametrised by

$$t \longmapsto p + t(-\sin\theta, \cos\theta), \quad t \in \mathbb{R}.$$

Lo and behold, this is the same as $t \mapsto \pi \circ \phi_t(p, \theta)$. The θ–component of $\phi_t(p, \theta)$ encodes the direction orthogonal to this geodesic; in our case this component stays constant.

Great, I hear you say, but what does all that have to do with Huygens? Well, it turns out that we are but one simple step away from proving, with the help of contact geometry, the equivalence of the principles of Fermat and Huygens.

Let $\overline{f}$ be a wave front in $\mathbb{R}^2$, thought of as a parametrised curve $s \mapsto (x(s), y(s))$, $s \in (-\varepsilon, \varepsilon)$. For simplicity, we assume this to be regular, i.e.

$$\overline{f}'(s) = (x'(s), y'(s)) \neq (0,0) \text{ for all } s \in (-\varepsilon, \varepsilon).$$

Such a wave front lifts to a unique curve

$$s \longmapsto f(s) = (x(s), y(s), \theta(s))$$

in the space of contact elements subject to the requirement that $(x'(s), y'(s))$ be a positive multiple of $(\cos\theta(s), \sin\theta(s))$; this lift will be tangent to the natural contact structure. **Fermat's principle** says that light propagates along the geodesic rays (i.e. straight lines) orthogonal to the wave front $\overline{f}$, which translates into saying that the wave front at some later time t is given by $\pi \circ \phi_t \circ f$.

Next consider the curve

$$h\colon \theta \longmapsto (x(0), y(0), \theta).$$

This is simply the circle worth of all contact elements at the point $\pi \circ h \equiv (x(0), y(0))$. Under the geodesic flow and projected to $\mathbb{R}^2$, this becomes an elementary wave in the sense of Huygens: for each fixed $t \in \mathbb{R}$ the curve

$$\theta \longmapsto \pi \circ \phi_t \circ h(\theta) = (x(0), y(0)) + t(-\sin\theta, \cos\theta)$$

is a circle of radius t centred at $(x(0), y(0)$.

The curves h and f are both tangent to $\xi_{f(0)}$ at the point $f(0) = h(\theta(0))$. Since ϕ_t is a contact transformation, the transformed curves $\phi_t \circ h$ and $\phi_t \circ f$ will be tangent to $\xi_{\phi_t \circ f(0)}$ at $\phi_t \circ f(0)$. Then, by the definition of the natural contact structure, the transformed wave front $\pi \circ \phi_t \circ f$ and the elementary wave $\pi \circ \phi_t \circ h$ will be tangent to each other at the point $\pi \circ \phi_t \circ f(0)$ — this is **Huygens' principle**.

The general argument is entirely analogous: A contact element on a Riemannian manifold is a (cooriented) tangent hyperplane field. The space of all these contact elements once again carries a natural contact structure. A geodesic is uniquely determined by an initial point and a direction positively orthogonal to a contact element at that point. Like in the special case of $\mathbb{R}^2$ one can show that the geodesic flow preserves the natural contact structure on the space of contact elements, and this translates into the equivalence of the two principles of geometric optics. A quick proof of this general case is given in [6]; full details of that proof are meant to appear in a forthcoming book on contact topology.

8. The family portrait

It remains to identify the young Christiaan Huygens in Hanneman's family portrait. In the biography [1] (from an aptly named publishing company!), a whole chapter is devoted to this question, so we seem to be in muddy waters.

Since Christiaan was the second-eldest son, there is actually only a choice between the two boys at the top. My first guess was that Christiaan is the one on the left, who has arguably the most striking face. This intuitive feeling is confirmed by the catalogue of the Mauritshuis [12, p. 67] and by the afterword in [10]. Alas, it is wrong.

It appears that the confusion was started by an engraving of the printing carried out for a late 19th century edition of the collected works of Christiaan Huygens. Here Christiaan's name is placed at the upper left, contradicting an earlier engraving; the original painting does not associate names with the four boys. However, family iconography of the time demanded that the eldest son be placed to his father's right, i.e. on the left side of the portrait. This identification of the eldest brother Constantijn as the boy on the upper left, and thus Christiaan as the one on the right, seems to be confirmed by a comparison of the painting with other portraits from the same period.

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