Chapter 1.

Selected Results in One Complex Variable

One of the most important problems in complex function theory is to construct holomorphic functions with specified properties. Let us consider some examples:

(1) Let $G \subset \mathbb{C}$ be a domain. The set of poles of a meromorphic fct is discrete and closed. We consider the following converse:
Let $D \subset G$ be a discrete, closed set $D = \{z_j : j \in J\}$ and for any $j \in J$ let $g_j(z) = \sum_{\ell=1}^{m_j} a_{j\ell}(z-z_j)^{-\ell} \neq 0$ be a finite principal part at z_j (endlicher Haupt-teil). We look for a mero fct $f \in \mathcal{M}(G)$ s.t. $P(f) = D$ and whose principal part at z_j is g_j, $j \in J$. The existence of such f is the object of the Mittag-Leffler theorem.

(2) Let $G \subset \mathbb{C}$ be a domain, $D \subset G$ as in (1). Consider a map $\delta : D \to \mathbb{Z}^*$. We look for a mero fct $f \in \mathcal{M}(G)$ such that $N(f) \cup P(f) = D$ and $o_z(f) = \delta(z)$ for any $z \in D$. This is posible due to Weierstrass' theorem.

(3) Let $f \in \mathcal{M}(G)$. Can we find $h, g \in \mathcal{O}(G)$ s.t. $f = g/h$? (Poincaré problem)

(4) Let $\{g_1, \dots, g_m\} \subset \mathcal{O}(G)$ without common zero in G. Can we find $\{f_1, \dots, f_m\} \subset \mathcal{O}(G)$ s.t. $1 = f_1 g_1 + \dots + f_m g_m$? This is a Bezout or division problem.

Proving the existence statements (1)-(4) are typical examples of <u>local to global</u> problems. That is, we know that the problems have a solution in a neighbourhood of each point of G. Thus we have proved (1)-(4) if we can show that: if there is a solution locally in the neighbourhood of each point, then there exist a global solution.

Note that a problem like (4) has an easy solution for the class of smooth (or continuous) functions. Could you explain why?

The function theory of one variable is full of methods of extension from local to global: Weierstrass infinite products, Mittag-Leffler series, Runge approximation theorem, analytic extension. More general devices are the $\bar{\partial}$-equation and sheaf theory (vanishing theorems).

In this lecture we show how the solution of the $\bar{\partial}$-equation is a unifying tool for proving (1)-(4).

1.1 The Cauchy-Riemann Equation

In the following $G \subset \mathbb{C}$ is a domain. Recall that $f: G \to \mathbb{C}$ is called holomorphic $:\Longleftrightarrow$ f is real-differentiable and satisfies the homogeneous Cauchy-Riemann equation

$$\frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0 \quad \text{in } G.$$

The inhomogeneous Cauchy-Riemann equation is

$$\frac{\partial f}{\partial \bar{z}} = g, \quad \text{for some given } g: G \to \mathbb{C}.$$

Let $G \Subset \mathbb{C}$ have piecewise $\mathcal{C}^1$-boundary. Let $\omega = a\,dz$ be a $(1,0)$-form with $a \in \mathcal{C}^1(\overline{G})$.

Then

$$d\omega = da \wedge dz = \left(\frac{\partial a}{\partial z}dz + \frac{\partial a}{\partial \bar z}d\bar z\right)\wedge dz = \frac{\partial a}{\partial \bar z}d\bar z \wedge dz$$

By Stokes

$$(1.1)\quad \int_{\partial G} a\,dz = \int_{\partial G}\omega = \int_G d\omega = \int_G \frac{\partial a}{\partial \bar z}d\bar z\wedge dz$$

This yields the Cauchy integral formula:

$$\int_{\partial G} a\,dz = 0 \quad \text{if } a \in \mathcal{C}^1(\overline{G}) \cap \mathcal{O}(G)$$

We get moreover the inhomogeneous Cauchy formula.

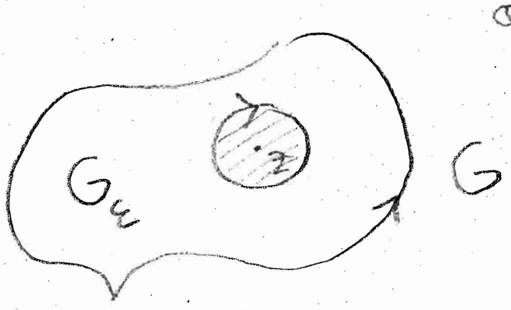
<u>1.1.1. Theorem</u> (Cauchy-Pompeiu)

Let $G \Subset \mathbb{C}$ have piecewise $\mathcal{C}^1$-boundary, $f \in \mathcal{C}^1(\overline{G})$. Then

$$f(z) = \frac{1}{2\pi i}\int_{\partial G}\frac{f(\zeta)}{\zeta - z}d\zeta + \frac{1}{2\pi i}\int_G \frac{\partial f}{\partial \bar\zeta}(\zeta)\frac{1}{\zeta - z}d\bar\zeta\wedge d\zeta$$

for any $z \in G$.

<u>Proof</u> Let $z \in G$, $\varepsilon > 0$ s.t. $\overline{B_\varepsilon(z)} \subset G$. Set $G_\varepsilon = G \setminus \overline{B_\varepsilon(z)}$. We apply the Stokes formula (1.1) on G_ε with $\partial G_\varepsilon = \partial G - \partial B_\varepsilon(z)$ for $\omega = \frac{f(\zeta)}{\zeta - z}d\zeta$. Then



$$\frac{\partial}{\partial \bar\zeta}\left(\frac{f(\zeta)}{\zeta - z}\right) = \frac{\partial f}{\partial \bar\zeta}(\zeta)\cdot\frac{1}{\zeta - z} \quad \text{so}$$

$$\int_{G_\varepsilon}\frac{\partial f}{\partial \bar\zeta}(\zeta)\frac{1}{\zeta - z}d\bar\zeta\wedge d\zeta = \int_{\partial G_\varepsilon}\omega = \int_{\partial G}\omega - \int_{\partial B_\varepsilon(z)}\omega =$$

$$= \int_{\partial G} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \int_0^{2\pi} \frac{f(z+\varepsilon e^{it})}{\varepsilon e^{it}}\, i\varepsilon e^{it}\, dt$$

Die Funktion $\overline{G} \ni \zeta \longmapsto \frac{1}{|\zeta - z|} \in [0,\infty]$ is integrable (Polarcoordinates around z) so the Lebesgue dominated convergence theorems yields

$$\int_{G_\varepsilon} d\omega = \int_G \mathbb{1}_{G_\varepsilon}\, d\omega \xrightarrow[\varepsilon \to 0]{} \int_G d\omega = \int_G \frac{\partial f}{\partial \bar\zeta}(\zeta)\, \frac{d\bar\zeta \wedge d\zeta}{\zeta - z}$$

On the other hand

$$i \int_0^{2\pi} f(z+\varepsilon e^{it})\, dt \xrightarrow[\varepsilon \to 0]{} 2\pi i f(z)$$

since f is continuous and

$$\left| \int_0^{2\pi} f(z+\varepsilon e^{it})\, dt - 2\pi f(z) \right| = \left| \int_0^{2\pi} \left(f(z+\varepsilon e^{it}) - f(z) \right) dt \right|$$

$$\leq 2\pi \sup_{t \in [0,2\pi]} \left| f(z+\varepsilon e^{it}) - f(z) \right| \longrightarrow 0 \text{ as } \varepsilon \to 0.$$

<u>1.1.2 Theorem</u> Let $f \in \mathcal{C}^1_0(\mathbb{C})$. Then there exists $u \in \mathcal{C}^1(\mathbb{C})$ with $\frac{\partial u}{\partial \bar z} = f$.

<u>Proof</u> Set $u : \mathbb{C} \to \mathbb{C}$

$$u(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar\zeta = -\frac{1}{\pi} \int_{[0,2\pi) \times [0,\infty)} f(z + re^{i\theta})\, e^{-i\theta}\, dr \wedge d\theta$$

Note that the integrand in the second integral is $\mathcal{C}^1$ w.r.t. z and has compact support thus u defines a $\mathcal{C}^1$ function. We have

$$\frac{\partial u}{\partial \bar z} = -\frac{1}{\pi}\frac{\partial}{\partial \bar z}\int_{[0,\infty)\times[0,2\pi)} f(z+re^{i\theta})e^{-i\theta}\,dr\wedge d\theta$$

$$= -\frac{1}{\pi}\int_{[0,\infty)\times[0,2\pi)} \frac{\partial}{\partial \bar z} f(z+re^{i\theta})e^{-i\theta}\,dr\wedge d\theta$$

$$= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial f}{\partial \bar\zeta}(\zeta)\frac{d\zeta\wedge d\bar\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_{B_R(0)} \frac{\partial f}{\partial \bar\zeta}(\zeta)\frac{d\zeta\wedge d\bar\zeta}{\zeta - z}$$

$$\underset{(1.2)}{=} f(z) - \frac{1}{2\pi i}\int_{\partial B_R(0)} \frac{f(\zeta)}{\zeta - z}\,d\zeta = f(z)$$

where R is chosen s.t. $\operatorname{supp} f \subset B_R(0)$. ∎

1.2. Runge, Mittag-Leffler, Weierstrass Theorems

These are all applications of the resolution of the $\bar\partial$-equation. Later we will also apply it to the Dolbeault lemma

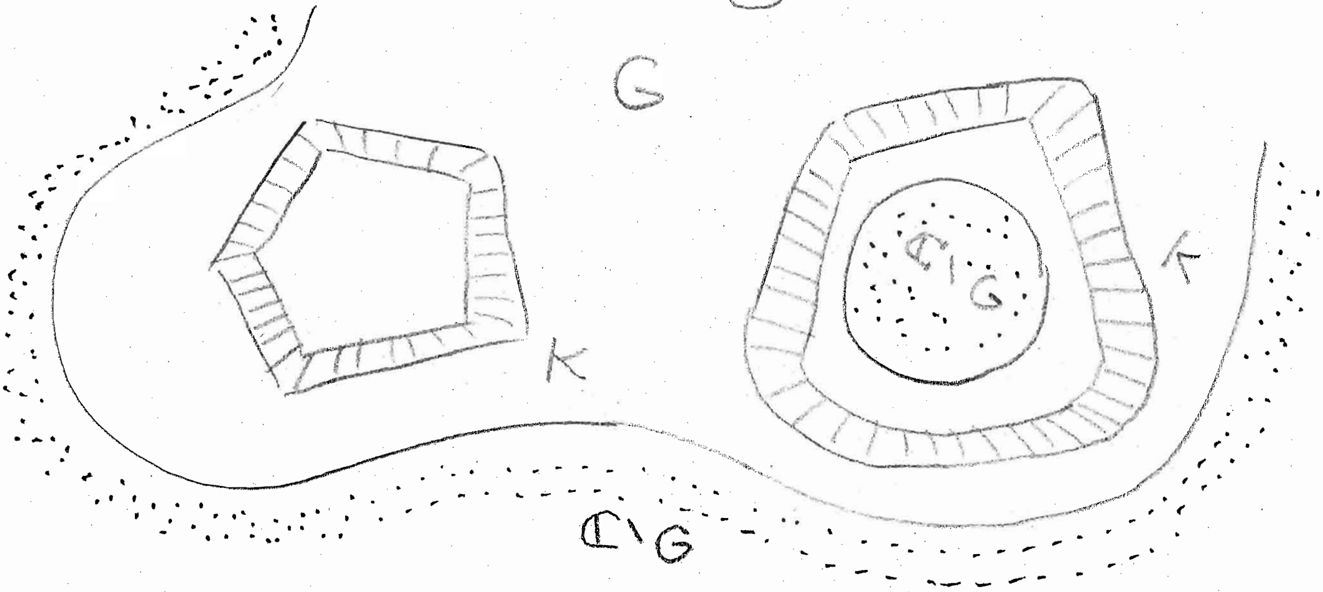
1.2.1 Theorem (Runge) Let $G \subset \mathbb{C}$ open, $K \subset G$ compact. The following are equivalent:

(a) Every fct which is holomorphic in a neighbourhood of K is the uniform limit over K of functions in $\mathcal{O}(G)$

(b) None of the connected components of $G\setminus K$ is relatively compact in G.

(c) For every $z \in G\setminus K$, there exists $f \in \mathcal{O}(G)$ s.t.

$$|f(z)| > \sup_K |f| = \|f\|_K$$



1.2.2 Definition Let $G \subset \mathbb{C}$ open, $K \subset G$ compact. The holomorphically convex hull of K in G is

$$\widehat{K}_G := \{ z \in G : \forall f \in \mathcal{O}(G), |f(z)| \le \|f\|_K \}$$

If there is no confusion, write $\widehat{K}$ instead of $\widehat{K}_G$. K is called holomorphically convex if $K = \widehat{K}$.

1.2.3 Proposition The set $\widehat{K}_G$ is the union of K and the connected components of $G \setminus K$ which are relatively compact in G.

Proof. See Hörmander, Th. 1.3.3. ▤

1.2.4. Proposition For every $G \subset \mathbb{C}$ open there exists an exhaustion of G by compact subsets which are holom convex in G.

Proof Let $(L_j)_{j \ge 1}$ be an exhaustion of G with compacts. Let $K_1 := \widehat{L_1}$. There exists $j_2 > 2$ s.t. $\mathring{L}_{j_2} \supset K_1$, hence $\mathring{L}_{j_2} \supset K_1 \cup L_2$. Set $K_2 = \widehat{L_{j_2}}$. There exists $j_3 > 3$ s.t. $\mathring{L}_{j_3} \supset K_2 \cup L_3$. Set $K_3 = \widehat{L_{j_3}}$ etc. □

Note that $\widehat{\widehat{K}} = \widehat{K}$, so $\widehat{K}$ is holom. convex.