Homework 1

1. Problem

(Maximum principle for polydisc) Show that for $P = P_r(a)$ and $z \in \overline{P}$ we have

$$|f(z)| \leqslant \|f\|_{\partial_0 P}\,, \quad \text{for all } f \in \mathcal{O}(P) \cap \mathscr{C}(\overline{P})\,.$$

2. Problem

(a) Show that if f_j is a sequence of holomorphic functions on an open set $U \subset \mathbb{C}^n$ which is uniformly convergent on each compact subset of U, then the limit is also holomorphic on U.

(b) On $\mathcal{O}(U)$ we introduce the family of seminorms p_j, $j \in \mathbb{N}$, $p_j(f) = \|f\|_{K_j}$, where K_j, $j \in \mathbb{N}$, is an exhaustion of U with compact sets. Show that $\mathcal{O}(U)$ endowed with this family of seminorms is a Fréchet space.

(b) Show **Montel's theorem**: bounded subsets of $\mathcal{O}(U)$ are relatively compact (a Fréchet space possessing this property is called a Montel space).

3. Problem

Let f be holomorphic at a point $a \in \mathbb{C}^n$. The *order of f at a* is defined by

$$\operatorname{ord}_a(f) = \begin{cases} \min\left\{|\alpha| : \partial^\alpha f(a) \neq 0\right\}, & \text{if } f \not\equiv 0\,, \\ \infty\,, \quad \text{if } f \equiv 0\,. \end{cases}$$

Let $P = P_r(a)$ and $f \in \mathcal{O}(P)$ be given. Let $k = \operatorname{ord}_a(f)$. Show that f has a unique representation of the form $f = \sum_{\ell=k}^{\infty} P_\ell$, with homogeneous polynomials P_ℓ of degree ℓ in $(z-a)$ such that $\sum_{\ell=k}^{\infty} P_\ell$ converges in $\mathcal{O}(P)$.

4. Problem

(Schwarz Lemma) Let f be holomorphic function in the polydisc $P_r(a) \subset \mathbb{C}^n$, where $r > 0$. Assume that f vanishes of order k at a. Show that

$$|f(z)| \leqslant \left(\frac{|z|}{r}\right)^k \|f\|_{P_r(a)}\,, \quad z \in P_r(a)\,.$$

Prove a similar theorem for a holomorphic function in a ball $B_r(a)$.

5. Problem

(a) A function $f \in \mathcal{O}(\mathbb{C}^n)$ is called an entire (holomorphic) function. Prove **Liouville's Theorem**: Every bounded entire function is constant.

(b) Let f be an entire function, and suppose that there exist a multi-index α and a constant $C > 0$ such that $|f(z)| \leqslant C|z^\alpha|$ for every $z \in \mathbb{C}^n$. Show that f is a polynomial of degree at most $|\alpha|$.

6. Problem

Let $P = P_{1,1}(0) \subset \mathbb{C}^2$ and $B = B_1(0) \subset \mathbb{C}^2$ be the polydisc and ball in $\mathbb{C}^2$.

(a) Show that the group of unitary matrices of rank two is a subgroup of the group of biholomorphic maps of B which leave the origin fixed.

(b) Show that for any $z \in P$ there exists a biholomorphic map $f : P \to P$ with $f(z) = 0$.

(c) Show that group of biholomorphic maps of P which leave invariant the origin is abelian.

(d) Deduce **Poincaré's Theorem**: the polydisc P and the unit ball B in $\mathbb{C}^2$ are not biholomorphic. Hence the Riemann mapping theorem does not generalize to higher dimensions.

7. Problem

Let $f : U \to V$ be a holomorphic map. Show that the natural pull-back map $f^* : \Omega^k(V) \to \Omega^k(U)$ induces maps $f^* : \Omega^{p,q}(V) \to \Omega^{p,q}(U)$.

8. Problem

Show that for the differential operators ∂ and $\overline{\partial}$ we have:

i) $d = \partial + \overline{\partial}$.

ii) $\partial^2 = 0$, $\overline{\partial}^2 = 0$ and $\partial\overline{\partial} + \partial\overline{\partial} = 0$.

iii) They satisfy the Leibniz rule, i.e.

$$\partial(\alpha \wedge \beta) = \partial\alpha \wedge \beta + (-1)^{p+q}\alpha \wedge \partial\beta\,, \quad \overline{\partial}(\alpha \wedge \beta) = \overline{\partial}\alpha \wedge \beta + (-1)^{p+q}\alpha \wedge \overline{\partial}\beta$$

for $\alpha \in \Omega^{p,q}(U)$ and $\beta \in \Omega^{r,s}(U)$.

9. Problem

Let $v \in \Omega_0^{0,1}(\mathbb{C}^n)$, $n > 1$, be a form with compact support and coefficients of class $\mathscr{C}^k$, $k \geqslant 0$. Assume that $\overline{\partial}v = 0$. Show that there exists $u \in \mathscr{C}_0^k(\mathbb{C}^n)$ such that $\overline{\partial}u = v$ (if $k = 0$ the equalities are satisfied in the sense of distributions).

10. Problem

(a) Calculate $\partial\overline{\partial}\log(1 + |z|^2)$ on $\mathbb{C}^n$.

(b) Let $P \subset \mathbb{C}^n$ be a polydisc and let $\alpha \in \Omega^{p,q}(P)$ be a d-closed form with $p, q \geq 1$. Show that there exists a form $\gamma \in \Omega^{p-1,q-1}(P)$ such that $\partial\overline{\partial}\gamma = \alpha$. Find such a γ for $\alpha = \sum_{j=1}^n dz_j \wedge d\overline{z}_j$.