Homework 2

1. Problem

Prove the following generalization of the argument principle. Let f be meromorphic in the domain G with zeros $a_1, \dots$ and poles $b_1, \dots$. Let $g \in \mathcal{O}(G)$. Let $\Gamma \subset G$ be a cycle homologous to zero with $a_j, b_k \notin |\Gamma|$. Then

$$\frac{1}{2\pi i}\int_\Gamma g(z)\frac{f(z)}{f'(z)}dz = \sum_j n(\Gamma, a_j)\nu(a_j)g(a_j) + \sum_k n(\Gamma, b_k)\operatorname{ord}_{b_k}(f)g(b_k).$$

2. Problem

Consider the variables $w_1, \dots, w_n$ and the elementary symmetric functions $\sigma_1, \dots, \sigma_n$, defined by

$$\prod_{j=1}^n (X - w_j) = X^n - \sigma_1 X^{n-1} + \ldots + (-1)^k \sigma_k X^{n-k} + \ldots + (-1)^n \sigma_n\,.$$

Consider the symmetric functions $S_m = \sum_{j=1}^n w_j^m$.

(a) Use the expansion of the logarithm on both sides of the equality

$$\sum_{j=1}^n \log(1 - w_j X) = \log\left(1 + \sum_{k=1}^n (-1)^k \sigma_k X^k\right)$$

to prove that

$$-\sum_{k=1}^\infty \frac{s_k}{k} X^k = \sum_{p=1}^\infty \frac{(-1)^{p-1}}{p}\left(\sum_{k=1}^n (-1)^k \sigma_k X^k\right)^p.$$

Deduce that

$$S_k = \sum_{\substack{p_1,\dots,p_n \geq 0,\\ p_1+2p_2+\ldots+np_n=k}} \frac{(-1)^{p_1+\ldots+p_n+k} k(p_1+\ldots+p_n-1)!}{p_1!\ldots p_n!}\sigma_1^{p_1}\ldots\sigma_n^{p_n}$$

and $S_k \in \mathbb{Z}[\sigma_1, \dots, \sigma_n]$.

(b) Show that

$$\sum_{k=0}^n (-1)^k \sigma_k X^k = \exp\left(-\sum_{k=1}^\infty \frac{S_k}{k} X^k\right) = \sum_{p=0}^\infty \frac{(-1)^p}{p!}\left(\sum_{k=1}^\infty \frac{S_k}{k} X^k\right)^p$$

and deduce that

$$\sigma_k = \sum_{\substack{p_1,\dots,p_n \geq 0,\\ p_1+2p_2+\ldots+np_n=k}} \frac{(-1)^{k+p_1+\ldots+p_n}}{p_1!\ldots p_n!}\,\frac{S_1^{p_1}\ldots S_k^{p_k}}{1^{p_1}2^{p_2}\ldots n^{p_n}}$$

and $\sigma_k \in \mathbb{Q}[S_1, \dots, S_k]$ (**Newton's formulas**).

3. Problem

Let $f\colon \mathbb{C}^n \to \mathbb{C}^m, f = (f_1, \dots, f_m)$ be a holomorphic map.

(a) Show that f is real-differentiable and its differential $df(a)\colon \mathbb{C}^n \to \mathbb{C}^m$ is $\mathbb{C}$-linear with matrix representation in the canonical bases

$$J_f(z) = \left(\frac{\partial f_i}{\partial z_j}(z)\right)_{\substack{1\le i\le m,\\ 1\le j\le n}}.$$

(b) Write $f_j = u_j + \sqrt{-1}v_j$ and consider $f = (u_1, v_1, \dots u_m, v_m)$ as mapping from $\mathbb{R}^{2n}$ to $\mathbb{R}^{2m}$. Denote by

$$J_f^{\mathbb{R}}(z) = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial y_1} & \cdots & \frac{\partial u_1}{\partial y_n} \\ \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial y_1} & \cdots & \frac{\partial v_1}{\partial y_1} \\ \vdots & \vdots & & \vdots \\ \frac{\partial v_m}{\partial x_1} & \frac{\partial v_m}{\partial y_1} & \cdots & \frac{\partial v_m}{\partial y_n} \end{pmatrix}$$

the real Jacobi matrix of f. Asumme $n = m$. Show that

$$\det J_f^{\mathbb{R}}(z) = \left|\det J_f(z)\right|^2.$$

(c) Let $G \subset \mathbb{C}^n, D \subset \mathbb{C}^m$ be open sets and $f\colon G \to D, f = (f_1, \dots, f_m)$ and $g\colon D \to \mathbb{C}$ be holomorphic. Show that $g \circ f$ is holomorphic and

$$\frac{\partial (g\circ f)}{\partial z_j}(z) = \sum_{k=1}^{m} \frac{\partial g}{\partial w_k}(f(z))\frac{\partial f_k}{\partial z_j}(z), \qquad 1 \le j \le n.$$

Deduce that if $h\colon D \to \mathbb{C}^\ell$ is holomorphic, then

$$J_{h\circ f}(z) = J_h(f(z))J_f(z), \qquad z \in G.$$

(d) Let $f\colon G \to \mathbb{C}^n, G \subset \mathbb{C}^n$ be of class $\mathcal{C}^\ell$. Show that

$$\det J_f^{\mathbb{R}} = \det \begin{pmatrix} \left(\frac{\partial f_k}{\partial z_j}\right) & \left(\frac{\partial f_k}{\partial \bar{z}_j}\right) \\ \left(\frac{\partial \bar{f}_k}{\partial z_j}\right) & \left(\frac{\partial \bar{f}_k}{\partial \bar{z}_j}\right) \end{pmatrix}$$

4. Problem

Prove the holomorphic implicit mapping theorem by using the implicit mapping theorem from real analysis.

5. Problem

Prove the constant rank theorem : Let $f\colon G \to \mathbb{C}^m$ holomorphic, where $G \subset \mathbb{C}^n$ is open. Assume J_f has constant rank k in G. Show that for any $a \in G$ there exists a neighbourhood U_a of a, a neighbourhood $V_{f(a)}$ of $f(a)$, biholomorphic maps $\Phi\colon U_a \to P^n_\varepsilon(0) \subset \mathbb{C}^n$, $\Psi\colon V_{f(a)} \to P^m_\varepsilon(0) \subset \mathbb{C}^m$ such that $\Psi \circ f \circ \Phi^{-1}\colon P^n_\varepsilon(0) \to P^m_\varepsilon(0)$ has the form $(z_1, \dots, z_n) \mapsto (z_1, \dots, z_k, 0, \dots, 0)$. Show that $\{z \in G \,|\, f(z) = f(a)\}$ is a submanifold of dimension $n-k$ of G for any $a \in G$, and $f(U_a)$ is a submanifold of dimension k of $V_{f(a)}$.

6. Problem

Let $G \subset \mathbb{C}^n$ be open. A map $f\colon G \to \mathbb{C}^m$ is called holomorphic embedding if f is a holomorphic immersion and $f\colon G \to f(G)$ is a homeomorphism. Show that if f is a holomorphic embedding, then $f(G)$ is a complex submanifold of $\mathbb{C}^m$.

7. Problem

(a) Let $h \in {}_{n-1}\mathcal{O}_0[z_n]$ be a Weierstrass polynomial and $f \in {}_n\mathcal{O}_0$. Assume that $fh \in {}_{n-1}\mathcal{O}_0[z_n]$. Show that $f \in {}_{n-1}\mathcal{O}_0[z_n]$.

(b) Show that the Weierstrass Division Theorem implies the Weierstrass Preparation Theorem.

(c) Let $f \in {}_n\mathcal{O}_0$ be of order k. Show that the ${}_{n-1}\mathcal{O}_0$-module ${}_n\mathcal{O}_0/(f)$ is finitely generated and free of rank k and if f is regular of order k in z_n, then a basis of ${}_n\mathcal{O}_0/(f)$ is given by $1, z_n, \dots, z_n^{k-1}$.

8. Problem

(a) Let $f \in {}_n\mathcal{O}_0$ have finite order in z_n and let $h \in {}_{n-1}\mathcal{O}_0[z_n]$ be the Weierstrass polynomial of f (according to the Preparation Theorem). Then the injection ${}_{n-1}\mathcal{O}_0[z_n] \to {}_n\mathcal{O}_0$ induces a $\mathbb{C}$-algebra isomorphism ${}_{n-1}\mathcal{O}_0[z_n]/{}_{n-1}\mathcal{O}_0[z_n]h \to {}_n\mathcal{O}_0/{}_n\mathcal{O}_0 h$.

(b) Show that a Weierstrass polynomial $h \in {}_{n-1}\mathcal{O}_0[z_n]$ is prime in ${}_{n-1}\mathcal{O}_0[z_n]$ if and only if it is prime in ${}_n\mathcal{O}_0$.

(c) Let $h \in {}_{n-1}\mathcal{O}_0[z_n]$ be a Weierstrass polynomial. Assume that h is reducible in ${}_{n-1}\mathcal{O}_0[z_n]$. Show that the factors of its decomposition are Weierstrass polynomials up to units in ${}_n\mathcal{O}_0$.

9. Problem

Prove **Hensel's Lemma**: Let $h \in {}_{n-1}\mathcal{O}_0[z_n]$ be a monic polynomial of degree ≥ 1 and let $h(0', \zeta) = (\zeta - c_1)^{k_1} \dots (\zeta - c_m)^{k_m}$. Then there exist monic polynomials $h_1, \dots, h_m \in {}_{n-1}\mathcal{O}_0[z_n]$ of degree $k_1, \dots, k_m$ such that $h = h_1 \dots h_m$ and $h_j(0', \zeta) = (\zeta - c_j)^{k_j}$, $1 \leq j \leq m$.

10. Problem

Prove in detail Proposition 2.4.20.