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Differential geometry

Bergman kernels on punctured Riemann surfaces

Noyau de Bergman sur une surface de Riemann épointée

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ABSTRACT

In this paper, we consider a punctured Riemann surface endowed with a Hermitian metric that equals the Poincaré metric near the punctures and a holomorphic line bundle that polarizes the metric. We show that the Bergman kernel can be localized around the singularities and its local model is the Bergman kernel of the punctured unit disc endowed with the standard Poincaré metric. As a consequence, we obtain an optimal uniform estimate of the supremum norm of the Bergman kernel function, involving a fractional growth order of the tensor power.

RÉSUMÉ

On considère une surface de Riemann compacte munie d'une métrique hermitienne singulière égale à la métrique de Poincaré du disque épointé près d'un diviseur donné. On considère un fibré en droites holomorphe avec une métrique singulière qui polarise la métrique (singulière) sur la surface de Riemann. On donne l'asymptotique explicite près des singularités de la fonction de Bergman lorsque la puissance de fibré en droites tend vers l'infini.

Version française abrégée

On considère une surface de Riemann compacte et un sous-ensemble fini D représentant des singularités. On munit la surface d'une métrique hermitienne singulière égale à la métrique de Poincaré près de D. On considère un fibré en droites holomorphe L avec une métrique singulière qui polarise la métrique (singulière) sur la surface de Riemann. La fonction de Bergman associée est $B_p(x) = \sum_j |s_j(x)|^2$, avec $\{s_j\}$ une base orthonormée de l'espace de sections holomorphes de carré intégrable de L^p, la p-puissance tensorielle de L.

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On donne l'asymptotique explicite (en particulier près des singularités) de la fonction de Bergman B_p quand p tend vers l'infini. L'asymptotique uniforme pour des métriques singulières est, à notre connaissance, un résultat nouveau. Dans le cas arithmétique, l'espace de sections holomorphes de carré intégrable de L^p est l'espace de formes modulaires paraboliques de poids $2p$.

1. Localization of Bergman kernels near the singularities

In this note, we study the Bergman kernels of a singular Hermitian line bundle over a Riemann surface under the assumption that the curvature has singularities of Poincaré type at a finite set. Our first result shows that the Bergman kernel can be localized around the singularities and its local model is the Bergman kernel of the punctured disc endowed with the standard Poincaré metric. The proof follows the principle that the spectral gap of the Kodaira Laplacian implies the localization of the Bergman kernel [9]. By a detailed analysis of the local model, we deduce a sharp uniform estimate of the supremum norm of the Bergman kernels. Details and further extensions are developed in [3].

Let $\overline{\Sigma}$ be a compact Riemann surface, $D = \{a_1, \dots, a_N\} \subset \overline{\Sigma}$ be a finite set and let $\Sigma = \overline{\Sigma} \backslash D$ be the corresponding punctured Riemann surface. We consider a Hermitian form ω_Σ on Σ, a holomorphic line bundle L on $\overline{\Sigma}$, and a singular Hermitian metric h on L such that:

(α) h is smooth over Σ, and for all $j = 1, \dots, N$, there is a trivialization of L in the complex neighborhood $\overline{V_j}$ of a_j in $\overline{\Sigma}$, with associated coordinate z_j such that $|1|_h^2(z_j) = \big|\log(|z_j|^2)\big|$;

(β) there exists $\varepsilon > 0$ such that the (smooth) curvature R^L of h satisfies $\mathrm{i}R^L \geq \varepsilon\omega_\Sigma$ over Σ and $\mathrm{i}R^L = \omega_\Sigma$ on $V_j := \overline{V_j} \backslash \{a_j\}$; in particular, $\omega_\Sigma = \omega_{\mathbb{D}^*}$ in the local coordinate z_j on V_j and (Σ, ω_Σ) is complete.

Here $\omega_{\mathbb{D}^*}$ denotes the Poincaré metric on the punctured unit disc, normalized as follows:

$$\omega_{\mathbb{D}^*} := \frac{\mathrm{i}\, dz \wedge d\overline{z}}{|z|^2 \log^2(|z|^2)} \,. \tag{1}$$

For $p \geq 1$, let $h^p := h^{\otimes p}$ be the metric induced by h on $L^p|_\Sigma$, where $L^p := L^{\otimes p}$. We denote by $H^0_{(2)}(\Sigma, L^p)$ the space of $\boldsymbol{L}^2$-holomorphic sections of L^p relative to the metrics h^p and ω_Σ,

$$H^0_{(2)}(\Sigma, L^p) = \left\{ S \in H^0(\Sigma, L^p) : \|S\|^2_{\boldsymbol{L}^2} := \int_\Sigma |S|^2_{h^p}\, \omega_\Sigma < \infty \right\}, \tag{2}$$

endowed with the obvious inner product. The sections from $H^0_{(2)}(\Sigma, L^p)$ extend to holomorphic sections of L^p over $\overline{\Sigma}$, i.e. $H^0_{(2)}(\Sigma, L^p) \subset H^0\big(\overline{\Sigma}, L^p\big)$, (see [9, (6.2.17)]). In particular, the dimension d_p of $H^0_{(2)}(\Sigma, L^p)$ is finite.

We denote by $B_p \in C^\infty(\Sigma, \mathbb{R})$ the Bergman kernel function of the space $H^0_{(2)}(\Sigma, L^p)$, defined as follows: if $\{S^p_\ell\}^{d_p}_{\ell \geq 1}$ is an orthonormal basis of $H^0_{(2)}(\Sigma, L^p)$, then

$$B_p(x) = \sum_{\ell=1}^{d_p} |S^p_\ell(x)|^2_{h^p} \,. \tag{3}$$

Note that B_p is independent of the choice of basis (see [9, (6.1.10)]). Let $B^{\mathbb{D}^*}_p$ be the Bergman kernel function of $\big(\mathbb{D}^*, \omega_{\mathbb{D}^*}, \mathbb{C}, \big|\log(|z|^2)\big|^p |\cdot|\big)$.

The main result of this paper is a weighted estimate in the C^m-norm near the punctures for the global Bergman kernel function B_p compared to the Bergman kernel function $B^{\mathbb{D}^*}_p$ of the punctured disc, uniformly in the tensor powers of the given bundle.

Theorem 1.1. *Assume that $(\Sigma, \omega_\Sigma, L, h)$ fulfill conditions (α) and (β). Then the following estimate holds: for any $\ell, m \in \mathbb{N}$, and every $\delta > 0$, there exists $C = C(\ell, m, \delta) > 0$ such that for all $p \in \mathbb{N}^*$, and $z \in V_1 \cup \ldots \cup V_N$ with the local coordinate z_j,*

$$\left| B_p - B^{\mathbb{D}^*}_p \right|_{C^m}(z_j) \leq C p^{-\ell} \left|\log(|z_j|^2)\right|^{-\delta}. \tag{4}$$

Here the pointwise C^m-norm of a function is defined by using the Levi-Civita connection on (Σ, ω_Σ).

Remark 1. Theorem 1.1 admits a generalization to orbifold Riemann surfaces. Assume that $\overline{\Sigma}$ is a compact orbifold Riemann surface, and the finite set $D = \{a_1, \dots, a_N\} \subset \overline{\Sigma}$ does not meet the (orbifold) singular set of $\overline{\Sigma}$. Assume moreover that L is a holomorphic orbifold line bundle on $\overline{\Sigma}$. Let ω_Σ be an orbifold Hermitian form on Σ and h an orbifold Hermitian metric on L in the sense of [9, §5.4]. The proof of Theorem 1.1 can be modified to show: if conditions (α), (β) hold in this context, then (4) holds.

By [9, Theorems 6.1.1, 6.2.3], for any compact set $K \subset \Sigma$, we have the following expansion on K in any C^m-topology

$$\frac{1}{p} B_p(x) = \frac{1}{2\pi} + \sum_{j=1}^{\infty} \boldsymbol{b}_j(x)\, p^{-j} \qquad \text{as } p \to \infty. \tag{5}$$

Theorem 1.1 gives a precise description of B_p near the punctures, in terms of the Bergman kernel function of the Poincaré metric on the local model of the punctured unit disc in $\mathbb{C}$. Note that in the case of smooth metrics with positive curvature, the Bergman kernel can be localized, and its local model is the Euclidean space endowed with a trivial bundle of positive curvature, see [9, Sections 4.1.2–3]. This kind of localization is inspired from the analytic localization technique of Bismut–Lebeau [5] in local index theory. For the problem at hand here, we have to overcome difficulties linked to the presence of singularities.

From a study of the model Bergman kernel functions $B_p^{\mathbb{D}^*}$ on the punctured unit disc, we get the following ratio estimate as a corollary of Theorem 1.1 and Proposition 2.1.

Corollary 1.2. *Let $(\Sigma, \omega_\Sigma, L, h)$ be as in Theorem 1.1. Then*

$$\sup_{x\in\Sigma} B_p(x) = \sup_{\substack{x\in\Sigma \\ \sigma\in H^0_{(2)}(\Sigma, L^p)}} \frac{|\sigma(x)|^2_{h^p}}{\|\sigma\|^2_{\boldsymbol{L}^2}} = \left(\frac{p}{2\pi}\right)^{3/2} + \mathcal{O}(p) \qquad \text{as } p \to \infty. \tag{6}$$

It is, to our knowledge, the first example of a *uniform* $\boldsymbol{L}^\infty$ asymptotic description of the Bergman kernel function of a singular polarization. This is of particular interest in arithmetic situations.

Corollary 1.2 is also quite striking from a Kähler geometry point of view, as the supremum of the Bergman kernel is equivalent to $\left(\frac{p}{2\pi}\right)^n$ on compact polarized manifolds of complex dimension n.

2. Bergman kernels on the punctured unit disc

Let $p \in \mathbb{N}^*$, and $H^p_{(2)}(\mathbb{D}^*) := H^0_{(2)}\big(\mathbb{D}^*, \omega_{\mathbb{D}^*}, \mathbb{C}, \big|\log(|z|^2)\big|^p |\cdot|\big)$ be the space of holomorphic functions on $\mathbb{D}^*$ with finite $\boldsymbol{L}^2$-norm. For $p \geq 2$, the set

$$\left\{\left(\frac{\ell^p}{2\pi(p-1)!}\right)^{1/2} z^\ell : \ell \in \mathbb{N},\ \ell \geq 1\right\} \tag{7}$$

forms an orthonormal basis of $H^p_{(2)}(\mathbb{D}^*)$. We get in particular the Bergman kernel function of $H^p_{(2)}(\mathbb{D}^*)$ for all $p \geq 2$,

$$B_p^{\mathbb{D}^*}(z) = \frac{\big|\log(|z|^2)\big|^p}{2\pi(p-2)!} \sum_{\ell=1}^{\infty} \ell^{p-1} |z|^{2\ell}. \tag{8}$$

This readily provides the behavior of $B_p^{\mathbb{D}^*}$.

Proposition 2.1. *For any $0 < a < 1$ and $0 < \gamma < \frac{1}{2}$, there exists $c = c(a,\gamma) > 0$ such that for $m \in \mathbb{N}^*$,*

$$\left\| B_p^{\mathbb{D}^*}(z) - \frac{p-1}{2\pi} \right\|_{C^m(\{ae^{-p^\gamma} \leq |z| < 1\}, \omega_{\mathbb{D}^*})} = \mathcal{O}\big(e^{-cp^{1-2\gamma}}\big) \quad \text{as } p \to \infty, \tag{9}$$

and $\sup_{z\in\mathbb{D}^*} B_p^{\mathbb{D}^*}(z) = \left(\frac{p}{2\pi}\right)^{3/2} + \mathcal{O}(p)$ *as* $p \to \infty$.

3. Arithmetic case

We give an important example where Theorem 1.1 applies. If Γ is a geometrically finite Fuchsian group of the first kind, without elliptic elements, then $\Sigma := \Gamma\backslash\mathbb{H}$ can by compactified by finitely many points $D = \{a_1, \ldots, a_N\}$ into a compact Riemann surface $\overline{\Sigma}$ of genus g such that $2g - 2 + N > 0$ and $L = K_{\overline{\Sigma}} \otimes \mathscr{O}_{\overline{\Sigma}}(D)$ is ample, where $K_{\overline{\Sigma}}$ is the canonical line bundle on $\overline{\Sigma}$. The Kähler–Einstein metric ω_Σ is induced by the Poincaré metric on $\mathbb{H}$. Let h^{K_Σ} be the metric on K_Σ induced by ω_Σ. Let σ be the canonical section of $\mathscr{O}_{\overline{\Sigma}}(D)$. The singular metric $h^{\mathscr{O}_{\overline{\Sigma}}(D)}$ on $\mathscr{O}_{\overline{\Sigma}}(D)$ is defined by $|\sigma|^2_{h^{\mathscr{O}_{\overline{\Sigma}}(D)}} = 1$. The isomorphism

$$K_\Sigma \to K_\Sigma \otimes \mathscr{O}_{\overline{\Sigma}}(D)|_\Sigma = L|_\Sigma, \quad s \mapsto s \otimes \sigma$$

over Σ and the metrics h^{K_Σ} and $h^{\mathscr{O}_{\overline{\Sigma}}(D)}$ induce the metric h on $L|_\Sigma$. Then (Σ, ω_Σ) and the formal square root of (L, h) satisfy conditions (α) and (β).

Let $\mathcal{S}^{\Gamma}_{2p}$ be the space of cusp forms (Spitzenformen) of weight $2p$ of Γ endowed with the Petersson inner product. From the above construction, the isomorphism $\Phi : \mathcal{S}^{\Gamma}_{2p} \to H^0(\overline{\Sigma}, L^p)$, $f \to f\,dz^{\otimes p}$ in [11, Proposition 3.3, 3.4(b)] is an isometry. We can form the Bergman kernel function of $\mathcal{S}^{\Gamma}_{p}$ as in (3), denoted by B^{Γ}_{p}. We deduce from Corollary 1.2:

Corollary 3.1. *Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a geometrically finite Fuchsian group of the first kind without elliptic elements. Let B^{Γ}_{p} be the Bergman kernel function of cusp forms of weight $2p$. If Γ is cocompact then uniformly on $\Gamma\backslash\mathbb{H}$,*

$$B^{\Gamma}_{p}(x) = \frac{p}{\pi} + \mathcal{O}(1), \qquad \text{as } p \to \infty. \tag{10}$$

If Γ is not cocompact then

$$\sup_{x\in\Gamma\backslash\mathbb{H}} B^{\Gamma}_{p}(x) = \left(\frac{p}{\pi}\right)^{3/2} + \mathcal{O}(p), \qquad \text{as } p \to \infty. \tag{11}$$

Uniform estimates for $\sup_{x\in\Gamma\backslash\mathbb{H}} B^{\Gamma}_{p}(x)$ are relevant in arithmetic geometry and were proved in various degrees of generality and sharpness in [1,10,8,7]. In [7], it is proved that, in the cofinite but non-cocompact case, $\sup_{x\in\Gamma\backslash\mathbb{H}} B^{\Gamma}_{p}(x) = \mathcal{O}(p^{3/2})$ and the result is optimal, at least up to an additive term in the exponent of the form $-\varepsilon$ for any $\varepsilon > 0$. Estimate (11) gives the precise coefficient of the leading term $p^{3/2}$ and is sharp (by killing the "ε from below" from [7]). Estimate (10) is the consequence of the general expansion of the Bergman kernel on compact manifolds [12] (cf. [9] for a comprehensive study and complete references).

It turns out that Corollary 3.1 can be formulated so as to underline a certain uniformity in Γ, in the same fashion as in [7]:

Theorem 3.2. *Let $\Gamma_0 \subset \mathrm{PSL}(2,\mathbb{R})$ be a fixed Fuchsian subgroup of the first kind without elliptic elements and let $\Gamma \subset \Gamma_0$ be any subgroup of finite index. If Γ_0 is cocompact, then*

$$B^{\Gamma}_{p}(x) = \frac{p}{\pi} + \mathcal{O}_{\Gamma_0}(1), \qquad \text{as } p \to \infty. \tag{12}$$

If Γ_0 is not cocompact, then

$$\sup_{x\in\Gamma\backslash\mathbb{H}} B^{\Gamma}_{p}(x) = \left(\frac{p}{\pi}\right)^{3/2} + \mathcal{O}_{\Gamma_0}(p), \qquad \text{as } p \to \infty. \tag{13}$$

Here the implied constants in $\mathcal{O}_{\Gamma_0}(1)$, $\mathcal{O}_{\Gamma_0}(p)$ depend solely on Γ_0.

Note that (12) is a special case of a more general result implicit in [9, §6.1.2].

We consider a further extension of Theorem 3.2 to the case when the group Γ_0 has elliptic elements. Then the quotients $\Gamma\backslash\mathbb{H}$ are in general orbifolds. By using the result of Dai–Liu–Ma [6, (5.25)] on the Bergman kernel asymptotics on orbifolds and the orbifold version of Theorem 1.1, we obtain the following.

Theorem 3.3. *Let $\Gamma_0 \subset \mathrm{PSL}(2,\mathbb{R})$ be a fixed Fuchsian subgroup of the first kind. Let $\{x_j\}_{j=1}^{q}$ be the orbifold points of $\Gamma_0\backslash\mathbb{H}$ and U_{x_j} be a small neighborhood of x_j in $\Gamma_0\backslash\mathbb{H}$. Let $\Gamma \subset \Gamma_0$ be any subgroup of finite index and $\pi_\Gamma : \Gamma\backslash\mathbb{H} \to \Gamma_0\backslash\mathbb{H}$ be the natural projection. If Γ_0 is cocompact, then as $p \to \infty$*

$$B^{\Gamma}_{p}(x) = \frac{p}{\pi} + \mathcal{O}_{\Gamma_0}(1), \;\; \text{uniformly on } (\Gamma\backslash\mathbb{H}) \setminus \bigcup_{j=1}^{q} \pi_\Gamma^{-1}(U_{x_j}). \tag{14}$$

On each $\pi_\Gamma^{-1}(U_{x_j})$, we have, as $p \to \infty$,

$$B^{\Gamma}_{p}(x) = \left(1 + \sum_{\gamma\in\Gamma_{x_j^\Gamma}\setminus\{1\}} \exp\left(\mathrm{i}p\theta_\gamma - p(1-\mathrm{e}^{\mathrm{i}\theta_\gamma})|z|^2\right)\right)\frac{p}{\pi} + \mathcal{O}_{\Gamma_0}(1), \tag{15}$$

where $x_j^\Gamma \in \pi_\Gamma^{-1}(x_j)$ is in the same component of $\pi_\Gamma^{-1}(U_{x_j})$ as x, $\mathrm{e}^{\mathrm{i}\theta_\gamma}$ is the action of γ on the fiber of $K_{\Gamma\backslash\mathbb{H}}$ at x_j^Γ, and $z = z(x)$ is the coordinate of x in normal coordinates z centered at x_j^Γ in $\mathbb{H}$, and $\Gamma_y = \{\gamma \in \Gamma : \gamma y = y\}$ the stabilizer of y.

In particular, if $q_0 = \mathrm{lcm}\{|\Gamma_{0,x_j}| : j = 1,\dots,q\}$, $n_\Gamma = \max\{|\Gamma_y| : y \in \pi_\Gamma^{-1}(x_j),\, j = 1,\dots,q\}$, then

$$\sup_{x\in\Gamma\backslash\mathbb{H}} B^{\Gamma}_{q_0 p}(x) = n_\Gamma \frac{q_0 p}{\pi} + \mathcal{O}_{\Gamma_0}(1). \tag{16}$$

If Γ_0 is not cocompact, then as $p \to \infty$

$$\sup_{x \in \Gamma \backslash \mathbb{H}} B_p^\Gamma(x) = \left(\frac{p}{\pi}\right)^{3/2} + \mathcal{O}_{\Gamma_0}(p). \tag{17}$$

Here again, the implied constants in $\mathcal{O}_{\Gamma_0}(1)$, $\mathcal{O}_{\Gamma_0}(p)$ depend solely on Γ_0.

Theorems 3.2, 3.3 sharpen in an optimal way the main result of [7], which states that

$$\sup_{x \in \Gamma \backslash \mathbb{H}} B_p^\Gamma(x) = \begin{cases} \mathcal{O}_{\Gamma_0}(p) & \text{if } \Gamma_0 \text{ is cocompact,} \\ \mathcal{O}_{\Gamma_0}(p^{3/2}) & \text{if } \Gamma_0 \text{ is not cocompact.} \end{cases} \tag{18}$$

We obtain in this way the precise leading terms in (18).

4. Proof of Theorem 1.1

Let $\overline{\partial}^{L^p *}$ be the adjoint of the Dolbeault operator $\overline{\partial}^{L^p}$ on (L^p, h^p) over (Σ, ω_Σ). The Kodaira Laplacian on $\Omega_0^{(0,0)}(\Sigma, L^p)$ is defined by $\square_p := \overline{\partial}^{L^p *} \overline{\partial}^{L^p}$. It is essentially self-adjoint in $\boldsymbol{L}^2(\Sigma, L^p)$ and satisfies (see [9, Theorem 6.1.1]) for some $c > 0$ and $p \gg 1$,

$$\mathrm{Spec}(\square_p) \subset \{0\} \cup [cp, +\infty), \tag{19}$$

and for $p \gg 1$, $\ker(\square_p) = H^0_{(2)}(\Sigma, L^p)$.

To get the uniform estimate near the singularities, we need to establish a weighted elliptic estimate for Kodaira Laplacians on (L^p, h^p) over (Σ, ω_Σ) such that the estimate is uniform on p. Then we can combine the localization principle [9] implied by the spectral gap (19) and the weighted Sobolev embedding theorem in [4, §4] and [2, §4], to obtain that, for any $l \in \mathbb{N}^*$, $\gamma > 0$, there exists $C > 0$ such that for all $p \in \mathbb{N}^*$, and $z \in V_1 \cup \ldots \cup V_N$ with the local coordinate z_j,

$$\left| B_p - B_p^{\mathbb{D}^*} \right|_{C^m}(z_j) \le C p^{-\ell} \left|\log(|z_j|^2)\right|^\gamma. \tag{20}$$

To finally get (4), we use the following observation: the elements of $H^0_{(2)}(\Sigma, L^p)$ are, for $p \ge 2$, exactly the holomorphic sections of L^p over the whole $\overline{\Sigma}$ vanishing on the puncture divisor $D = \{a_1, \ldots, a_N\}$, in particular $B_p(z) = 0$ at D.

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