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Semi-classical properties of Berezin–Toeplitz operators with $\mathscr{C}^k$-symbol

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We obtain the semi-classical expansion of the kernels and traces of Toeplitz operators with $\mathscr{C}^k$-symbol on a symplectic manifold. We also give a semi-classical estimate of the distance of a Toeplitz operator to the space of self-adjoint and multiplication operators. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4870869]

I. INTRODUCTION

The purpose of this paper is to extend some of the semiclassical results about the Berezin-Toeplitz quantization to the case of Toeplitz operators with $\mathscr{C}^k$-symbol.

A fundamental problem in mathematical physics is to find relations between classical and quantum mechanics. On one side, we have symplectic manifolds and Poisson algebras, on the other Hilbert spaces and selfadjoint operators. The goal is to establish a dictionary between these theories such that given a quantum system depending on a parameter, to obtain a classical system when the parameter approaches a so called semiclassical limit, in such a way that properties of the quantum system are controlled up to first order by the underlying classical system. It is very interesting to go the other way, namely, to quantize a classical system, that is, introduce a quantum system whose semiclassical limit is the given classical system.

The aim of the geometric quantization theory of Kostant[12] and Souriau[22] is to relate the classical observables (smooth functions) on a phase space (a symplectic manifold) to the quantum observables (bounded linear operators) on the quantum space (holomorphic sections of a line bundle). Berezin-Toeplitz quantization is a particularly efficient version of the geometric quantization theory.[1,2,6,10,13] Toeplitz operators and more generally Toeplitz structures were introduced in geometric quantization by Berezin[2] and Boutet de Monvel-Guillemin.[5] Using the analysis of Toeplitz structures,[5] Bordemann-Meinrenken-Schlichenmaier,[3] and Schlichenmaier[20] proved the existence of the asymptotic expansion for the composition of Toeplitz operators in the Kähler case when we twist a trivial bundle $E = \mathbb{C}$.

In Refs. 15 and 16, Ma-Marinescu have extended the Berezin-Toeplitz quantization to symplectic manifolds and orbifolds by using as quantum space the kernel of the Dirac operator acting on powers of the prequantum line bundle twisted with an arbitrary vector bundle. In Ref. 18, Ma-Marinescu calculated the first coefficients of the kernel expansions of Toeplitz operators and of their composition.

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Let us review shortly the results from Refs. 15–17. We consider a compact symplectic manifold X with symplectic form ω and a Hermitian line bundle (L, h^L, ∇^L) whose curvature satisfies the prequantization condition $\frac{\sqrt{-1}}{2\pi}R^L = \omega$. Let (E, h^E, ∇^E) be a Hermitian vector bundle on X with Hermitian connection ∇^E. Let J be a ω-compatible almost-complex structure and g^{TX} be a J-invariant metric on TX. For any $p \in \mathbb{N}$, let $L^p := L^{\otimes p}$ be the pth tensor power of L, $\Omega^{0,\bullet}(X, L^p \otimes E)$ be the space of smooth anti-holomorphic forms with values in $L^p \otimes E$ with norm induced by h^L, h^E, and g^{TX}, and $P_p : \Omega^{0,\bullet}(X, L^p \otimes E) \to \mathrm{Ker}(D_p)$ be the orthogonal projection on the kernel of the Dirac operator D_p.

To any $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, we associate a sequence of linear operators

$$T_{f,\,p} : \Omega^{0,\bullet}(X, L^p \otimes E) \to \Omega^{0,\bullet}(X, L^p \otimes E), \quad T_{f,\,p} = P_p\, f\, P_p\,, \tag{1.1}$$

where for simplicity we denote by f the operator of multiplication with f. For any $f, g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, the product $T_{f,p}\,T_{g,p}$ has an asymptotic expansion

$$T_{f,\,p}\,T_{g,\,p} = \sum_{k=0}^{\infty} T_{C_k(f,\,g),\,p}\, p^{-k} + \mathscr{O}(p^{-\infty}) \tag{1.2}$$

in the sense of (4.2), where C_k are bidifferential operators of order $\le 2r$, satisfying $C_0(f, g) = fg$ and if $f, g \in \mathscr{C}^\infty(X)$, $C_1(f, g) - C_1(g, f) = \sqrt{-1}\,\{f, g\}$. Here, $\{\,\cdot\,,\cdot\,\}$ is the Poisson bracket on $(X, 2\pi\omega)$ (cf. (3.31)). We deduce from (1.2),

$$[T_{f,\,p}\,, T_{g,\,p}] = \frac{\sqrt{-1}}{p} T_{\{f,g\},\,p} + \mathcal{O}(p^{-2})\,. \tag{1.3}$$

Moreover, the norm of the Toeplitz operators allows to recover the sup-norm of the classical observable $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$

$$\lim_{p\to\infty} \|T_{f,\,p}\| = \|f\|_\infty := \sup_{x\in X} |f(x)|\,, \tag{1.4}$$

and $\|\cdot\|$ is the operator norm. Thus, the Poisson algebra $(\mathscr{C}^\infty(X), \{\cdot,\cdot\})$ is approximated by the operator algebras of Toeplitz operators (for $E = \mathbb{C}$) in the norm sense as $p \to \infty$; the role of the Planck constant is played by $\hbar = 1/p$. This is the so-called semi-classical limit process.

All these papers consider the case of a smooth observable. The assumption that the symbol is $\mathscr{C}^\infty$ is quite restrictive and analysts studying Toeplitz operators normally do not require this. While most results on Berezin-Toeplitz quantization are for smooth symbols, some progress has been made towards understanding what is happening with non-smooth symbols, in particular in work of Coburn and co-authors (see, e.g., Refs. 7 and 8). It was remarked recently by Polterovich[19] that it is interesting to study the Berezin-Toeplitz quantization also for the case of continuous observables. A specific example of a situation where it would be helpful to know how to quantize non-smooth observables is quantization of the universal Teichmüller space in work of Sergeev (see, in particular, Ref. 21). We will extend in this paper the relations (1.2), (1.3), (1.4) for $\mathscr{C}^k$-symbols f, g. Moreover, we consider the question of how far is a Toeplitz operator from being self-adjoint or a multiplication with a function.

In this paper, we shall use the kernel calculus of Toeplitz operators developed in Refs. 15–18 which lends itself very well to handling less regular symbols.

The plan of the paper is as follows. In Sec. II, we recall the Bergman kernel expansion.[9,15] Section III is devoted to the kernel expansion of the Toeplitz operators. In Sec. IV, we explain the expansion of a product of Toeplitz operators. In Sec. V, we study the asymptotics of the norm of Toeplitz operators. Finally, in Sec. VI we consider the semi-classical estimates of the distance of a Toeplitz operator to various spaces (self-adjoint operators, constant multiples of the identity, multiplication operators).

II. QUANTIZATION OF SYMPLECTIC MANIFOLDS

We will briefly describe in this section the study of the Toeplitz operators and Berezin-Toeplitz quantization for symplectic manifolds. For details, we refer the reader to Refs. 15 and 16 and to the surveys in Refs. 13 and 17. We recall in Sec. II A the definition of the spin^c Dirac operator and formulate the spectral gap property for prequantum line bundles. In Sec. II B, we state the asymptotic expansion of the Bergman kernel.

A. Spectral gap of the spin^c Dirac operator

We will first show that in the general symplectic case the kernel of the spin^c operator is a good substitute for the space of holomorphic sections used in Kähler quantization.

Let (X, ω) be a compact symplectic manifold, $\dim_{\mathbb{R}} X = 2n$, with compatible almost complex structure $J : TX \to TX$. Set $T^{(1,0)}X = \{u \in TX \otimes_{\mathbb{R}} \mathbb{C} : Ju = \sqrt{-1}u\}$ and $T^{(0,1)}X = \{u \in TX \otimes_{\mathbb{R}} \mathbb{C} : Ju = -\sqrt{-1}u\}$. Let g^{TX} be a J-compatible Riemannian metric. The Riemannian volume form of g^{TX} is denoted by dv_X.

We do not assume that $g^{TX}(u, v) = \omega(u, Jv)$ for $u, v \in TX$. We relate g^{TX} with ω by means of the skew-adjoint linear map $\boldsymbol{J} : TX \longrightarrow TX$ which satisfies the relation

$$\omega(u, v) = g^{TX}(\boldsymbol{J}u, v) \quad \text{for} \quad u, v \in TX. \tag{2.1}$$

Then J commutes with $\boldsymbol{J}$, and $J = \boldsymbol{J}(-\boldsymbol{J}^2)^{-\frac{1}{2}}$.

Let (L, h^L, ∇^L) be a Hermitian line bundle on X, endowed with a Hermitian metric h^L and a Hermitian connection ∇^L, whose curvature is $R^L = (\nabla^L)^2$. We assume that the *prequantization condition*

$$\omega = \frac{\sqrt{-1}}{2\pi} R^L \tag{2.2}$$

is fulfilled. Let (E, h^E, ∇^E) be a Hermitian vector bundle on X with Hermitian metric h^E and Hermitian connection ∇^E. We will be concerned with asymptotics in terms of high tensor powers $L^p \otimes E$, when $p \to \infty$, that is, we consider the semi-classical limit $\hbar = 1/p \to 0$.

Let us denote by

$$\boldsymbol{E} := \Lambda^{\bullet}(T^{*(0,1)}X) \otimes E \tag{2.3}$$

the bundle of anti-holomorphic forms with values in E. The metrics g^{TX}, h^L, and h^E induce a L^2-scalar product on $\Omega^{0,\bullet}(X, L^p \otimes E) = \mathscr{C}^\infty(X, L^p \otimes \boldsymbol{E})$ by

$$\langle s_1, s_2 \rangle = \int_X \langle s_1(x), s_2(x) \rangle_{L^p \otimes \boldsymbol{E}} \, dv_X(x)\,, \tag{2.4}$$

whose completion is denoted $\big(L^2(X, L^p \otimes \boldsymbol{E}), \|\cdot\|_{L^2}\big)$.

Let $\nabla^{\det}$ be the connection on $\det(T^{(1,0)}X)$ induced by the projection of the Levi-Civita connection ∇^{TX} on $T^{(1,0)}X$. Let us consider the Clifford connection ∇^{Cliff} on $\Lambda^{\bullet}(T^{*(0,1)}X)$ associated to ∇^{TX} and to the connection $\nabla^{\det}$ on $\det(T^{(1,0)}X)$ (see, e.g., Sec. 1.3 in Ref. 15). The connections ∇^L, ∇^E, and ∇^{Cliff} induce the connection

$$\nabla_p = \nabla^{\text{Cliff}} \otimes \text{Id} + \text{Id} \otimes \nabla^{L^p \otimes E} \quad \text{on } \Lambda^{\bullet}(T^{*(0,1)}X) \otimes L^p \otimes E.$$

The *spin^c Dirac operator* is defined by

$$D_p = \sum_{j=1}^{2n} \mathbf{c}(e_j) \nabla_{p, e_j} : \Omega^{0,\bullet}(X, L^p \otimes E) \longrightarrow \Omega^{0,\bullet}(X, L^p \otimes E)\,, \tag{2.5}$$

where $\{e_j\}_{j=1}^{2n}$ local orthonormal frame of TX and $\mathbf{c}(v) = \sqrt{2}(v^*_{1,0} \wedge - i_{v_{0,1}})$ is the Clifford action of $v \in TX$. Here, we use the decomposition $v = v_{1,0} + v_{0,1}$, $v_{1,0} \in T^{(1,0)}X$, $v_{0,1} \in T^{(0,1)}X$ and $v^*_{1,0} \in T^{*(0,1)}X$ is the dual of $v_{1,0}$.

Remark 2.1. Let us assume for a moment that (X, J) is a complex manifold (i.e., J is integrable) and the bundles L, E are holomorphic and ∇^L, ∇^E are the Chern connections. If $g^{TX}(u, v) = \omega(u, Jv)$ for $u, v \in TX$ (thus (X, g^{TX}) is Kähler), then

$$D_p = \sqrt{2}\big(\overline{\partial} + \overline{\partial}^*\big), \quad \overline{\partial} = \overline{\partial}^{L^p \otimes E}\,, \tag{2.6}$$

so $\mathrm{Ker}(D_p) = H^0(X, L^p \otimes E)$ for $p \gg 1$. The following result (Theorem 2.2) shows that for a general symplectic manifold $\mathrm{Ker}(D_p)$ has all semi-classical properties of $H^0(X, L^p \otimes E)$.

Note that if (X, J) is a complex manifold but g^{TX} is not necessarily associated to ω by $g^{TX}(u, v) = \omega(u, Jv)$, we can still work with the operator $\widetilde{D}_p := \sqrt{2}\big(\overline{\partial} + \overline{\partial}^*\big)$ instead of D_p (cf. Theorem 1.5.5 in Ref. 15), although $\widetilde{D}_p$ is only a modified Dirac operator. Theorem 2.2 and the results which follow remain valid for $\widetilde{D}_p$, so that, for p large, the quantum space will be $H^0(X, L^p \otimes E)$ $(= \mathrm{Ker}(\widetilde{D}_p)$ for $p \gg 1)$.

Let us return now to our general situation of a compact symplectic manifold (X, ω), endowed with a ω-compatible almost complex structure J and J-compatible Riemannian metric g^{TX}. Let

$$\mu_0 = \inf\left\{R^L_x(u, \overline{u})/|u|^2_{g^{TX}} : u \in T^{(1,0)}_x X \setminus \{0\}, x \in X\right\}. \tag{2.7}$$

By (2.2) we have $\mu_0 > 0$.

Theorem 2.2 (Theorems 1.1 and 2.5 in Ref. 14, Theorem 1.5.5 in Ref. 15)**.** *There exists $C > 0$ such that for any $p \in \mathbb{N}$ and any $s \in \bigoplus_{k>0} \Omega^{0,k}(X, L^p \otimes E)$, we have*

$$\|D_p s\|^2_{L^2} \geqslant (2\mu_0 p - C)\|s\|^2_{L^2}. \tag{2.8}$$

Moreover, the spectrum of D^2_p verifies

$$\mathrm{Spec}(D^2_p) \subset \big\{0\big\} \cup \big[2\mu_0 p - C, +\infty\big[. \tag{2.9}$$

The proof of Theorem 2.2 is based on a direct application of the Lichnerowicz formula for D^2_p. By the Atiyah-Singer index theorem, we have for $p \gg 1$,

$$\dim \mathrm{Ker}(D_p) = \int_X \mathrm{Td}(T^{(1,0)}X)\,\mathrm{ch}(L^p \otimes E) = \mathrm{rk}(E)\,\frac{p^n}{n!}\int_X \omega^n + O(p^{n-1}), \tag{2.10}$$

where Td is the Todd class and ch is the Chern character. Theorem 2.2 shows the forms in $\mathrm{Ker}(D_p)$ concentrate asymptotically in the L^2 sense on their zero-degree component and (2.10) shows that $\dim \mathrm{Ker}(D_p)$ is a polynomial in p of degree n, as in the holomorphic case.

B. Off-diagonal asymptotic expansion of Bergman kernel

We recall that a bounded linear operator T on $L^2(X, L^p \otimes \boldsymbol{E})$ is called Carleman operator (see, e.g., Ref. 11) if there exists a kernel $T(\cdot\,,\,\cdot)$ such that $T(x, \cdot) \in L^2(X, (L^p \otimes \boldsymbol{E})_x \otimes (L^p \otimes \boldsymbol{E})^*)$ and

$$(TS)(x) = \int_X T(x, x')S(x')dv_X(x'), \quad \text{for all } S \in L^2(X, L^p \otimes \boldsymbol{E}). \tag{2.11}$$

Let us introduce the orthogonal projection

$$P_p : L^2(X, L^p \otimes \boldsymbol{E}) \longrightarrow \mathrm{Ker}(D_p),$$

called the Bergman projection in analogy to the Kähler case. It is a Carleman operator whose integral kernel is called *Bergman kernel*. Set $d_p := \dim \mathrm{Ker}(D_p)$. Let $\{S^p_i\}_{i=1}^{d_p}$ be any orthonormal basis of $\mathrm{Ker}(D_p)$ with respect to the inner product (2.4). Then

$$P_p(x, x') = \sum_{i=1}^{d_p} S^p_i(x) \otimes (S^p_i(x'))^* \in (L^p \otimes \boldsymbol{E})_x \otimes (L^p \otimes \boldsymbol{E})^*_{x'}. \tag{2.12}$$

The *Toeplitz operator* with symbol $f \in L^\infty(X, \mathrm{End}(E))$ is defined by

$$T_{f,p} : L^2(X, L^p \otimes \boldsymbol{E}) \to L^2(X, L^p \otimes \boldsymbol{E}), \quad T_{f,p} = P_p f P_p, \tag{2.13}$$

where the action of f is the pointwise multiplication by f. The map which associates to $f \in L^\infty(X, \mathrm{End}(E))$ the family of bounded operators $\{T_{f,p}\}_p$ on $L^2(X, L^p \otimes \boldsymbol{E})$ is called the *Berezin-Toeplitz quantization*. Note that $T_{f,p}$ is a Carleman operator with smooth integral kernel given by

$$T_{f,\,p}(x,x')=\int_X P_p(x,x'')f(x'')P_p(x'',x')\,dv_X(x'')\,. \tag{2.14}$$

The existence of the spectral gap expressed in Theorem 2.2 allows us to *localize* the behavior of the Bergman kernel and of the kernel of Toeplitz operators.

Let a^X be the injectivity radius of (X, g^{TX}). We denote by $B(x, \varepsilon) = B^X(x, \varepsilon)$ and $B(0, \varepsilon) = B^{T_xX}(0, \varepsilon)$ the open balls in X and T_xX with center x and radius ε, respectively. Then the exponential map $T_xX \ni Z \to \exp_x^X(Z) \in X$ is a diffeomorphism from $B^{T_xX}(0, \varepsilon)$ onto $B^X(x, \varepsilon)$ for $\varepsilon \le a^X$. From now on, we identify $B^{T_xX}(0, \varepsilon)$ with $B^X(x, \varepsilon)$ via the exponential map for $\varepsilon \le a^X$. Throughout what follows, ε runs in the fixed interval $]0, a^X/4[$.

By Ref. 9, Proposition 4.1, we have the *far off-diagonal* behavior of the Bergman kernel:

Proposition 2.3. For any $\ell, m \in \mathbb{N}$ and $\varepsilon > 0$, there exists $C_{\ell,m,\varepsilon} > 0$ such that for any $p \ge 1$, $x, x' \in X$, $d(x, x') > \varepsilon$,

$$\left|P_p(x,x')\right|_{\mathscr{C}^m(X\times X)} \leqslant C_{\ell,m,\varepsilon}\, p^{-l}\,. \tag{2.15}$$

The $\mathscr{C}^m$ norm (2.15) is induced by ∇^L, ∇^E, h^L, h^E, and g^{TX}.

Let $\pi : TX \times_X TX \to X$ be the natural projection from the fiberwise product of TX on X. Let $\nabla^{\mathrm{End}(\boldsymbol{E})}$ be the connection on $\mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E)$ induced by ∇^{Cliff} and ∇^E.

Let us elaborate on the identifications we use in the sequel. Let $x_0 \in X$ be fixed and consider the diffeomorphism $B^{T_{x_0}X}(0, 4\varepsilon) \ni Z \mapsto \exp_{x_0}^X(Z) \in B^X(x_0, 4\varepsilon)$ for $\varepsilon \le a^X/4$. We denote the pull-back of the vector bundles L, E, and $L^p \otimes \boldsymbol{E}$ via this diffeomorphism by the same symbols.

(i) There exist trivializations of L, E, and $L^p \otimes \boldsymbol{E}$ over $B^{T_{x_0}X}(0, 4\varepsilon)$ given by unit frames which are parallel with respect to ∇^L, ∇^E, and $\nabla^{L^p\otimes \boldsymbol{E}}$ along the curves $\gamma_Z : [0, 1] \to B^{T_{x_0}X}(0, 4\varepsilon)$ defined for every $Z \in B^{T_{x_0}X}(0, 4\varepsilon)$ by $\gamma_Z(u) = \exp_{x_0}^X(uZ)$.

(ii) With the previous trivializations, $P_p(x, x')$ induces a smooth section

$$B^{T_{x_0}X}(0,4\varepsilon)\ni Z, Z' \mapsto P_{p,\,x_0}(Z,Z')$$

of $\pi^*(\mathrm{End}(\boldsymbol{E}))$ over $TX \times_X TX$, which depends smoothly on x_0.

(iii) $\nabla^{\mathrm{End}(\boldsymbol{E})}$ induces naturally a $\mathscr{C}^m$-norm with respect to the parameter $x_0 \in X$.

(iv) By (2.1), $\boldsymbol{J}$ is an element of $\mathrm{End}(T^{(1,0)}X)$. Consequently, we can diagonalize $\boldsymbol{J}_{x_0}$, i.e., choose an orthonormal basis $\{w_j\}_{j=1}^n$ of $T_{x_0}^{(1,0)}X$ such that

$$\boldsymbol{J}_{x_0}\omega_j=\frac{\sqrt{-1}}{2\pi}a_j(x_0)w_j\,,\quad \text{for all } j=1,2,\dots,n\,, \tag{2.16}$$

where $0 < a_1(x_0) \le a_2(x_0) \le \dots \le a_n(x_0)$. The vectors $\{e_j\}_{j=1}^{2n}$ defined by

$$e_{2j-1}=\tfrac{1}{\sqrt{2}}(w_j+\overline{w}_j)\quad\text{and}\quad e_{2j}=\tfrac{\sqrt{-1}}{\sqrt{2}}(w_j-\overline{w}_j)\,,\quad j=1,\dots,n\,, \tag{2.17}$$

form an orthonormal frame of $T_{x_0}X$. The diffeomorphism

$$\mathbb{R}^{2n}\ni(Z_1,\dots,Z_{2n})\longmapsto\sum_i Z_ie_i\in T_{x_0}X \tag{2.18}$$

induces coordinates on $T_{x_0}X$, which we use throughout the paper. In these coordinates, we have $e_j = \partial/\partial Z_j$. The complex coordinates $z = (z_1, \dots, z_n)$ on $T_{x_0}X$ are given by $z_j = Z_{2j-1} + \sqrt{-1}Z_{2j}$, $j = 1, \dots, n$.

(v) If dv_{TX} is the Riemannian volume form on $(T_{x_0}X, g^{T_{x_0}X})$, there exists a smooth positive function $\kappa_{x_0} : T_{x_0}X \to \mathbb{R}$, $Z \mapsto \kappa_{x_0}(Z)$ defined by

$$dv_X(Z)=\kappa_{x_0}(Z)\,dv_{TX}(Z),\quad \kappa_{x_0}(0)=1, \tag{2.19}$$

where the subscript x_0 of $\kappa_{x_0}(Z)$ indicates the base point $x_0 \in X$. By (4.1.101) in Ref. 15, we have

$$\kappa_{x_0}(Z)=1+O(|Z|^2)\,. \tag{2.20}$$

(vi) Let $\Theta_p : L^2(X, L^p \otimes \boldsymbol{E}) \longrightarrow L^2(X, L^p \otimes \boldsymbol{E})$ be a sequence of continuous linear operators with smooth kernel $\Theta_p(\cdot\,,\cdot)$ with respect to dv_X (e.g., $\Theta_p = T_{f,\,p}$). In terms of our basic trivialization, $\Theta_p(x, y)$ induces a family of smooth sections $Z, Z' \mapsto \Theta_{p,\,x_0}(Z, Z')$ of $\pi^*\,\mathrm{End}(\boldsymbol{E})$ over $\{(Z, Z') \in TX \times {}_X TX : |Z|, |Z'| < \varepsilon\}$, which depends smoothly on x_0.

We denote by $|\Theta_{p,\,x_0}(Z, Z')|_{\mathscr{C}^m(X)}$ the $\mathscr{C}^m$ norm with respect to the parameter $x_0 \in X$. We say that

$$\Theta_{p,\,x_0}(Z, Z') = \mathcal{O}(p^{-\infty}), \quad p \to \infty,$$

if for any $\ell, m \in \mathbb{N}$, there exists $C_{\ell,m} > 0$ such that $|\Theta_{p,\,x_0}(Z, Z')|_{\mathscr{C}^m(X)} \leqslant C_{\ell,\,m}\, p^{-\ell}$.

We denote by $\det_{\mathbb{C}}$ for the determinant function on the complex bundle $T^{(1,\,0)}X$ and set $|\boldsymbol{J}_{x_0}| = (-\boldsymbol{J}_{x_0}^2)^{1/2}$. Note that

$$\det{}_{\mathbb{C}} |\boldsymbol{J}_{x_0}| = \prod_{i=1}^{n} \frac{a_i(x_0)}{2\pi}, \tag{2.21}$$

where $a_i(x_0)$ were defined in (2.16). Let

$$\begin{aligned} \mathscr{P}_{x_0}(Z, Z') &= \det{}_{\mathbb{C}}|\boldsymbol{J}_{x_0}| \exp\Big(-\frac{\pi}{2}\big\langle|\boldsymbol{J}_{x_0}|(Z - Z'), (Z - Z')\big\rangle - \pi\sqrt{-1}\big\langle \boldsymbol{J}_{x_0}Z, Z'\big\rangle\Big) \\ &= \frac{1}{(2\pi)^n}\prod_{i=1}^{n} a_i(x_0)\ \exp\Big(-\frac{1}{4}\sum_i a_i(x_0)\big(|z_i|^2 + |z_i'|^2 - 2z_i\overline{z}_i'\big)\Big). \end{aligned} \tag{2.22}$$

We recall that $\mathscr{P}_{x_0}(Z, Z')$ is actually the kernel of the orthogonal projection $\mathscr{P}_{x_0}$ from $L^2(\mathbb{R}^{2n})$ onto the Bargmann-Fock space, see Sec. 2 in Ref. 16. Especially, $\mathscr{P}_{x_0}^2 = \mathscr{P}_{x_0}$ so that

$$\mathscr{P}_{x_0}(Z, Z') = \int_{\mathbb{R}^{2n}} \mathscr{P}_{x_0}(Z, Z'')\mathscr{P}_{x_0}(Z'', Z')\, dv(Z''). \tag{2.23}$$

Fix $k \in \mathbb{N}$ and $\varepsilon' \in\,]0, a^X[$. Let $\big\{Q_{r,\,x_0} \in \mathrm{End}(\boldsymbol{E})_{x_0}[Z, Z'] : 0 \leqslant r \leqslant k, x_0 \in X\big\}$ be a family of polynomials in Z, Z', which is smooth with respect to the parameter $x_0 \in X$. We say that

$$p^{-n}\Theta_{p,x_0}(Z, Z') \cong \sum_{r=0}^{k}(Q_{r,\,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')p^{-r/2} + \mathcal{O}(p^{-(k+1)/2}), \tag{2.24}$$

on $\{(Z, Z') \in TX \times {}_X TX : |Z|, |Z'| < \varepsilon'\}$ if there exist a decomposition

$$\begin{aligned} p^{-n}\Theta_{p,x_0}(Z, Z')\kappa_{x_0}^{1/2}(Z)\kappa_{x_0}^{1/2}(Z') - \sum_{r=0}^{k}(Q_{r,\,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')p^{-r/2} \\ = R_{p,k,x_0}(Z, Z') + \mathcal{O}(p^{-\infty}), \end{aligned} \tag{2.25}$$

where R_{p,k,x_0} satisfies the following estimate on $\{(Z, Z') \in TX \times {}_X TX : |Z|, |Z'| < \varepsilon'\}$: for every $m, m' \in \mathbb{N}$ there exist $C_{k,\,m,\,m'} > 0, M > 0$ such that for all $p \in \mathbb{N}^*$

$$\begin{aligned} \sup_{|\alpha|+|\alpha'|\leqslant m'} \left|\frac{\partial^{\alpha+\alpha'}}{\partial Z^\alpha \partial Z'^{\alpha'}} R_{p,k,x_0}(Z, Z')\right|_{\mathscr{C}^m(X)} \\ \leqslant C_{k,\,m,\,m'}\, p^{(m'-k-1)/2}(1 + \sqrt{p}\,|Z| + \sqrt{p}\,|Z'|)^M e^{-C_0\sqrt{p}\,|Z-Z'|}. \end{aligned} \tag{2.26}$$

We consider the orthogonal projection

$$I_{\mathbb{C}\otimes E} : \boldsymbol{E} = \Lambda^\bullet(T^{*(0,1)}X) \otimes E \longrightarrow \mathbb{C} \otimes E. \tag{2.27}$$

By Theorem 4.18′ in Ref. 9, we have the following *off diagonal expansion* of the Bergman kernel:

Theorem 2.4. *Let* $\varepsilon \in\,]0, a^X/4[$. *There exist a smooth family relative to the parameter* $x_0 \in X$

$$\Big\{J_{r,\,x_0}(Z, Z') \in \mathrm{End}(\boldsymbol{E}_{x_0})[Z, Z'] : r \in \mathbb{N},\ x_0 \in X\Big\}, \quad \deg J_{r,\,x_0} \leqslant 3r,$$

of polynomials J_{r,x_0} *having the same parity as r, and whose coefficients are polynomials in* R^{TX}, $R^{T^{(1,0)}X}$, R^E *(and* R^L*) and their derivatives of order* $\leq r - 1$ (resp. $\leq r$) *such that*

$$p^{-n}P_{p,x_0}(Z,Z') \cong \sum_{r=0}^{k}(J_{r,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z,\sqrt{p}Z')p^{-\frac{r}{2}} + \mathcal{O}(p^{-\frac{k+1}{2}})\,, \tag{2.28}$$

on the set $\{(Z,Z') \in TX \times_X TX : |Z|, |Z'| < 2\varepsilon\}$. *Moreover,* $J_{0,x_0}(Z,Z') = I_{\mathbb{C}\otimes E}$.

By taking $Z = Z' = 0$ in (2.28), we obtain the diagonal expansion of the Bergman kernel. Namely, for any $k \in \mathbb{N}$, $P_p(x,x)$ has an expansion in the $\mathscr{C}^\infty$-topology

$$P_p(x,x) = \sum_{r=0}^{k}\boldsymbol{b}_r(x)p^{n-r} + \mathscr{O}(p^{n-k-1})\,, \quad \boldsymbol{b}_r \in \mathscr{C}^\infty(X,\mathrm{End}(\boldsymbol{E}))\,, \tag{2.29}$$

and by Theorem 2.4 and (2.22), we get

$$\boldsymbol{b}_0(x_0) = \det{}_{\mathbb{C}}|\boldsymbol{J}_{x_0}|\, I_{\mathbb{C}\otimes E} \in \mathrm{End}(\boldsymbol{E}_{x_0})\,. \tag{2.30}$$

Let us remark that if $g^{TX}(u,v) = \omega(u,Jv)$ for $u, v \in TX$, then $a_i = 2\pi$, so $\boldsymbol{b}_0(x) = I_{\mathbb{C}\otimes E}$.

III. EXPANSION OF THE KERNELS AND TRACES OF TOEPLITZ OPERATORS

For a smooth symbol $f \in \mathscr{C}^\infty(X,\mathrm{End}(E))$ we know by Lemma 4.6 in Ref. 16 and Lemma 7.2.4 in Ref. 15 that the kernel of the associated Toeplitz operator $T_{f,p}$ as in (2.13) has for any $\ell \in \mathbb{N}$ an expansion on the diagonal in the $\mathscr{C}^\infty$-topology,

$$T_{f,p}(x,x) = \sum_{r=0}^{\ell}\boldsymbol{b}_{r,f}(x)p^{n-r} + \mathscr{O}(p^{n-\ell-1})\,, \quad \boldsymbol{b}_{r,f} \in \mathscr{C}^\infty(X,\mathrm{End}(\boldsymbol{E}))\,, \tag{3.1}$$

where

$$\boldsymbol{b}_{0,f}(x) = \boldsymbol{b}_0(x)f(x)\,. \tag{3.2}$$

Note as an aside, that the coefficients $\boldsymbol{b}_{r,f}$, $r = 0, 1, 2$, were calculated in Theorem 0.1 in Ref. 18, if (X,ω) is Kähler and the bundles L, E are holomorphic. In Lemma 4.6 in Ref. 16, we actually established the off-diagonal expansion of the Toeplitz kernel. We wish to study here the asymptotic behavior of the Toeplitz kernel for a less regular symbol *f*. Let us begin with the analogue of Lemma 4.2 in Ref. 16.

Lemma 3.1. Let $f \in L^\infty(X,\mathrm{End}(E))$. *For every* $\varepsilon > 0$ *and every* $\ell, m \in \mathbb{N}$, *there exists* $C_{\ell,m,\varepsilon} > 0$ *such that*

$$|T_{f,p}(x,x')|_{\mathscr{C}^m(X\times X)} \leqslant C_{\ell,m,\varepsilon}\,p^{-\ell}\,, \quad \text{for all } p \geqslant 1,\ (x,x') \in X\times X \text{ with } d(x,x') > \varepsilon, \tag{3.3}$$

where the $\mathscr{C}^m$*-norm is induced by* ∇^L, ∇^E, and h^L, h^E, g^{TX}. *Moreover, there exists* $C > 0$ *such that for all* $p \geq 1$ *and all* $(x,x') \in X \times X$ *with* $d(x,x') \leq \varepsilon$,

$$|T_{f,p}(x,x')|_{\mathscr{C}^m(X\times X)} \leqslant Cp^{n+\frac{m}{2}}e^{-\frac{1}{2}C_0\sqrt{p}\,d(x,x')} + O(p^{-\infty}). \tag{3.4}$$

Proof. Due to (2.14) and (2.15), (3.3) holds if we replace $T_{f,p}$ by P_p . Moreover, from (2.28), for any $m \in \mathbb{N}$, there exists $C_m > 0$ such that

$$|P_p(x,x')|_{\mathscr{C}^m(X\times X)} < Cp^{n+\frac{m}{2}}\,, \quad \text{for all } (x,x') \in X\times X. \tag{3.5}$$

These two facts and formula (2.14) imply (3.3). By Theorem 2.4, for $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| < \varepsilon$ we have

$$\sup_{|\alpha|+|\alpha'|\leqslant m} \left| \frac{\partial^{\alpha+\alpha'}}{\partial Z^{\alpha}\partial Z'^{\alpha'}} P_{p,x_0}(Z,Z') \right| \qquad (3.6)$$

$$\leqslant C\, p^{n+\frac{m}{2}} (1+\sqrt{p}\,|Z|+\sqrt{p}\,|Z'|)^M\, e^{-C_0\sqrt{p}\,|Z-Z'|} + O(p^{-\infty})\,.$$

Using (2.14) and (3.6) by taking $x_0 = x$, we get (3.4). □

In order to formulate our results for a family of functions (observables), we consider equicontinuous and uniformly bounded families. The results are of course valid for individual functions, too.

Definition 3.2. We denote by ∇^E the connection on $(T^*X)^{\otimes k} \otimes \mathrm{End}(E)$ induced by the Levi-Civita connection ∇^{TX} and ∇^E. Let $\mathscr{A}^0 \subset \mathscr{C}^0(X, \mathrm{End}(E))$ be a subset which is equicontinuous on X, and $\mathscr{A}^1 \subset \mathscr{C}^1(X, \mathrm{End}(E))$ be a subset such that $\nabla^E \mathscr{A}^1 \subset \mathscr{C}^0(X, \mathrm{End}(E))$ is uniformly bounded. Let $\mathscr{A}^2 \subset \mathscr{C}^2(X, \mathrm{End}(E))$ be a subset such that $\nabla^E \mathscr{A}^2 \subset \mathscr{C}^0(X, \mathrm{End}(E))$ is uniformly bounded, and $\nabla^E\nabla^E \mathscr{A}^2 \subset \mathscr{C}^0(X, \mathrm{End}(E))$ is equicontinuous on X. Let $\mathscr{A}^k_\infty$ be a subset of $\mathscr{A}^k$ (for $k = 0, 1, 2$) which is uniformly bounded.

Theorem 3.3. *Let (X, ω) be a compact symplectic manifold, $(L, h^L, \nabla^L) \to X$ a prequantum line bundle, $(E, h^E, \nabla^E) \to X$ be an auxiliary vector bundle. We have as $p \to \infty$*

$$p^{-n} T_{f,p}(x,x) = f(x)\boldsymbol{b}_0(x) + o(1)(\|f\|_{\mathscr{C}^0}+1)\,, \ \textit{uniformly for } f \in \mathscr{A}^0, x \in X\,, \qquad (3.7)$$

$$p^{-n} T_{f,p}(x,x) = f(x)\boldsymbol{b}_0(x) + O(p^{-1/2})(\|f\|_{\mathscr{C}^0}+1)\,, \ \textit{uniformly for } f \in \mathscr{A}^1, x \in X\,, \qquad (3.8)$$

$$p^{-n} T_{f,p}(x,x) = f(x)\boldsymbol{b}_0(x) + \boldsymbol{b}_{1,f}(x)p^{-1} + o(p^{-1})(\|f\|_{\mathscr{C}^0}+1)\,, \qquad (3.9)$$

$$\textit{uniformly for } f \in \mathscr{A}^2, x \in X.$$

In particular, the remainders $o(1)$, $O(p^{-1/2})$, $o(p^{-1})$ do not depend on f.

Proof. We start by proving (3.7). Recall that we trivialized L, E by a unit frame over $B^{T_xX}(0, 4\varepsilon)$ which is parallel with respect to ∇^L, ∇^E along the geodesics starting in x. With this trivialization, the section $f \in \mathrm{End}(E)$ induces a section

$$B^{T_xX}(0, 4\varepsilon) \ni Z \mapsto f_x(Z).$$

We denote by $f(x) \in \mathrm{End}(E)|_U$ the endomorphism obtained by parallel transport of $f(x) \in \mathrm{End}(E_x)$ into the neighboring fibers $\mathrm{End}(E_{x'})$, $x' \in U = B(0, 4\varepsilon)$.

Let $\delta > 0$ be given. Since $\mathscr{A}^0$ is uniformly continuous on X, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$ and for all $x' \in B(x, \varepsilon)$ we have $|f(x') - f(x)| \leq \delta$ for any $f \in \mathscr{A}^0$.

By (2.14) and (3.3) we have

$$T_{f,\,p}(x,x) = \int_{B(x,\varepsilon)} P_p(x,x') f(x') P_p(x',x)\, dv_X(x') + O(p^{-\infty})\|f\|_{\mathscr{C}^0}\,. \qquad (3.10)$$

We write now $f(x') = f(x) + (f(x') - f(x)) \in \mathrm{End}(E_{x'})$ and split accordingly the last integral in a sum of two integrals. From Theorem 2.4, the first one is

$$\int_{B(x,\varepsilon)} P_p(x,x') f(x) P_p(x',x)\, dv_X(x')$$

$$= p^{2n} \int_{B(x,\varepsilon)} \sum_{i+j=0}^{1} (J_{i,x}\mathscr{P}_x)(0,\sqrt{p}Z') f(x) (J_{j,x}\mathscr{P}_x)(\sqrt{p}Z',0) dZ' + O(p^{n-1})\|f\|_{\mathscr{C}^0}. \qquad (3.11)$$

Note that $J_{0,x_0} = I_{\mathbb{C}\otimes E}$ and $J_{1,x_0}(0, Z')$ is a polynomial on Z' with odd degree, thus

$$\int_{\mathbb{R}^{2n}} (J_{1,x}\mathscr{P}_x)(0, Z') f(x)(J_{0,x}\mathscr{P}_x)_x(Z', 0) dZ' = \int_{\mathbb{R}^{2n}} (J_{0,x}\mathscr{P}_x)(0, Z') f(x)(J_{1,x}\mathscr{P}_x)(Z', 0) dZ' = 0. \tag{3.12}$$

From (2.30), (3.11), and (3.12), we get

$$\begin{aligned}\int_{B(x,\varepsilon)} P_p(x, x') f(x) P_p(x', x)\, dv_X(x') &= f(x)\mathscr{P}_x(0, 0) p^n + O(p^{n-1})\|f\|_{\mathscr{C}^0} \\ &= f(x)\boldsymbol{b}_0(x) p^n + O(p^{n-1})\|f\|_{\mathscr{C}^0}.\end{aligned} \tag{3.13}$$

Note that if *f* is a function, then we have

$$\int_{B(x,\varepsilon)} P_p(x, x') f(x) P_p(x', x)\, dv_X(x') = f(x)\Big(P_p(x, x') + O(p^{-\infty})\Big).$$

The second one can be estimated by

$$\begin{aligned}\Big|\int_{B(x,\varepsilon)} P_p(x, x')\big(f(x') - f(x)\big) P_p(x', x)\, dv_X(x')\Big| &\leqslant \delta \int_{B(x,\varepsilon)} |P_p(x, x')||P_p(x', x)|\, dv_X(x') \\ &= \delta \int_{B(0,\varepsilon)} |P_{p,x}(0, Z')||P_{p,x}(Z', 0)|\, dv_X(Z').\end{aligned} \tag{3.14}$$

We use now the off-diagonal expansion from Theorem 2.4. By (2.28) we have

$$\begin{aligned}&P_{p,x}(Z, 0) = p^n\Big(\textstyle\prod_{j=1}^n \frac{a_j}{2\pi}\, e^{-\frac{1}{4}p\sum_{j=1}^n a_j|z_j|^2} + p^{-1/2}R_{p,x}(Z, 0) + O(p^{-\infty})\Big)\kappa_x^{-\frac{1}{2}}(Z) \\ &\quad\text{where}\quad |R_{p,x}(Z, 0)| \leqslant C(1 + \sqrt{p}\,|Z|)^M\, e^{-C_0\sqrt{p}\,|Z|},\end{aligned} \tag{3.15}$$

so in order to estimate the last integral in (3.14), by (2.19), we have to estimate

$$\int_{B(0,\varepsilon)} p^{2n}\Big|\textstyle\prod_{j=1}^n \frac{a_j}{2\pi}\, e^{-\frac{1}{4}p\sum_{j=1}^n a_j|z'_j|^2} + p^{-1/2}R_{p,x}(Z', 0) + O(p^{-\infty})\Big|^2 dZ'.$$

By using the change of variables $\sqrt{p}Z' = Y$ we see that

$$p\int_{\mathbb{C}} e^{-\pi p|Z'|^2}\, dZ' = 1, \quad p^n\int_{\mathbb{C}^n} (1 + \sqrt{p}\,|Z'|)^M\, e^{-C_0\sqrt{p}\,|Z'|}\, dv_X(Z') = O(1),$$

hence

$$\int_{B(0,\varepsilon)} |P_{p,x}(0, Z')||P_{p,x}(Z', 0)|\, dv_X(Z') = O(p^n),$$

so there exists $C > 0$ such that for any $x \in X$, $f \in \mathscr{A}^0$, $p \in \mathbb{N}$, we have

$$\Big|\int_{B(x,\varepsilon)} P_p(x, x')\big(f(x') - f(x)\big) P_p(x', x)\, dv_X(x')\Big| \leqslant C\delta\, p^n. \tag{3.16}$$

From (3.10), (3.13), and (3.16), we get (3.7).

By Taylor's formula, there exist $C > 0$, $\varepsilon > 0$, such that for $|Z| \le \varepsilon$, $f \in \mathscr{A}^1$, we have

$$f_x(Z) - f_x(0) = R(Z), \quad |R(Z)| \leqslant C|Z|. \tag{3.17}$$

We repeat the proof above by plugging this expression in the integral from (3.16), we observe that only

$$\int_{B(0,\varepsilon)} |P_{p,x}(0, Z')||Z'||P_{p,x}(Z', 0)|\, dv_X(Z')$$

contributes to the subleading term. But then the change of variables $\sqrt{p}Z' = Y$ introduces a factor $p^{-1/2}$, whereof (3.8) follows.

Finally, for any $\delta > 0$ there exists $\varepsilon > 0$, such that for $|Z| \leq \varepsilon$, $f \in \mathscr{A}^2$,

$$f_x(Z) - f_x(0) = \sum_j \partial_j f_x(0) Z_j + \sum_{j,k} \partial_{jk} f_x(0) Z_j Z_k + R(Z)\,, \quad |R(Z)| \leqslant \delta |Z|^2\,. \tag{3.18}$$

Taking into account the proof of the asymptotic expansion (3.1) from Lemma 4.6 in Ref. 16, we see that (3.9) holds. □

Remark 3.4. In the same vein, we show that in the conditions of Theorem 3.3, we have for $f \in \mathscr{C}^k(X, \mathrm{End}(E))$, $k \in \mathbb{N}$, as $p \to \infty$,

$$p^{-n} T_{f,p}(x,x) = \sum_{r=0}^{\lfloor k/2 \rfloor} \boldsymbol{b}_{r,f}(x) p^{-r} + R_{k,p}(x)\,, \text{ uniformly on } X, \tag{3.19}$$

where $\boldsymbol{b}_{r,f}$ are the universal coefficients from (3.1) and

$$R_{k,p} = \begin{cases} o(p^{-k/2})\,, & \text{for } k \text{ even}\,, \\ O(p^{-k/2})\,, & \text{for } k \text{ odd}\,. \end{cases} \tag{3.20}$$

Here, $\lfloor a \rfloor$ denotes the integer part of $a \in \mathbb{R}$.

We recall that by (4.79) in Ref. 16 and (7.4.6) in Ref. 15, for any $f, g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, the kernel of the composition $T_{f,p} \circ T_{g,p}$ has for all $\ell \in \mathbb{N}$ an asymptotic expansion on the diagonal in the $\mathscr{C}^\infty$-topology,

$$(T_{f,\,p} \circ T_{g,\,p})(x,x) = \sum_{r=0}^{\ell} \boldsymbol{b}_{r,\,f,\,g}(x) p^{n-r} + \mathscr{O}(p^{n-\ell-1})\,, \quad \boldsymbol{b}_{r,\,f,\,g} \in \mathscr{C}^\infty(X, \mathrm{End}(\boldsymbol{E}))\,. \tag{3.21}$$

The coefficients $\boldsymbol{b}_{r,\,f,\,g}$, $r = 0, 1, 2$, were calculated in Theorem 0.2 in Ref. 18 in the case of a Kähler manifold (X, ω) and of holomorphic bundles L and E. We give here the analogue of the expansion (3.21) in the case of $\mathscr{C}^k$ symbols.

Theorem 3.5. *Let $m \in \mathbb{N}$ and $f_1, \dots, f_m \in L^\infty(X, \mathrm{End}(E))$. Write*

$$p^{-n}(T_{f_1,p} \dots T_{f_m,p})(x,x) = f_1(x) \dots f_m(x) \boldsymbol{b}_0(x) + R_p(x)\,. \tag{3.22}$$

We have as $p \to \infty$, uniformly in $x \in X$,

$$R_p(x) = o(1), \text{ uniformly on } f_i \in \mathscr{A}^0_\infty\,, \tag{3.23}$$

$$R_p(x) = O(p^{-1/2}), \text{ uniformly on } f_i \in \mathscr{A}^1_\infty\,, \tag{3.24}$$

$$R_p(x) = O(p^{-1}), \text{ uniformly on } f_i \in \mathscr{A}^2_\infty. \tag{3.25}$$

Proof. To prove (3.23) let $\delta > 0$ be given. Choose $\varepsilon > 0$ such that for $x' \in B(x, \varepsilon)$ we have $|f_j(x') - f_j(x)| \leq \delta$, $1 \leq j \leq m$, where $f_j(x) \in \mathrm{End}(\boldsymbol{E}_{x'})$ is the parallel transport of $f_j(x) \in \mathrm{End}(\boldsymbol{E}_x)$ as in the previous proof. By (2.13) we have $T_{f_1,p} \dots T_{f_m,p} = P_p f_1 P_p f_2 \dots P_p f_m P_p$ hence

$$\begin{aligned} &(T_{f_1,p} \dots T_{f_m,p})(x,x) \\ &= \int_{X^m} P_p(x,x_1) f_1(x_1) P_p(x_1,x_2) f_2(x_2) \dots f_m(x_m) P_p(x_m,x) \textstyle\prod_{i=1}^m dv_X(x_i) \\ &= I + O(p^{-\infty})\,, \end{aligned} \tag{3.26}$$

where

$$I = \int_{\substack{|Z_i|\leqslant\varepsilon \\ 1\leqslant i\leqslant m}} P_{p,x}(0, Z_1) f_{1,x}(Z_1) P_{p,x}(Z_1, Z_2) f_{2,x}(Z_2) \dots f_{m,x}(Z_m) P_{p,x}(Z_m, 0) \prod_{i=1}^m dv_X(Z_i).$$

We write now

$$I = I_0 + \sum_{j=1}^m I_j$$

with

$$\begin{aligned} I_0 &= \int_{\substack{|Z_i|\leqslant\varepsilon \\ 1\leqslant i\leqslant m}} P_p(0, Z_1) f_{1,x}(0) P_p(Z_1, Z_2) f_{2,x}(0) \dots f_{m,x}(0) P_p(Z_m, 0) \prod_{i=1}^m dv_X(Z_i) \\ &= f_1(x) \dots f_m(x) \big(\mathscr{P}_x(0,0) p^n + O(p^{n-1})\big)\,, \end{aligned} \tag{3.27}$$

in the above second equation, we use the argument in (3.12), and for $1 \le j \le m$,

$$\begin{aligned} I_j = \int_{\substack{|Z_i|\leqslant\varepsilon \\ 1\leqslant i\leqslant m}} & P_p(0, Z_1) f_{1,x}(Z_1) \dots P_p(Z_{j-1}, Z_j) \big(f_{j,x}(Z_j) - f_{j,x}(0)\big) \\ & P_p(Z_j, Z_{j+1}) f_{j+1,x}(Z_{j+1}) \dots f_{m,x}(Z_m) P_p(Z_m, 0) \prod_{i=1}^m dv_X(Z_i). \end{aligned}$$

By (2.28), for $\le j \le m$, we have

$$|I_j| \leqslant C\delta \int_{\substack{|Z_i|\leqslant\varepsilon \\ 1\leqslant i\leqslant m}} |P_p(0, Z_1)| A_p |P_p(Z_m, 0)| \kappa_x^{1/2}(Z_1) \kappa_x^{1/2}(Z_m) \prod_{i=1}^m dZ_i, \tag{3.28}$$

where

$$A_p = p^{(m-1)n} \prod_{i=1}^{m-1} (1 + \sqrt{p}\,|Z_i| + \sqrt{p}\,|Z_{i+1}|)^M\, e^{-C_0 \sqrt{p}\,|Z_i - Z_{i+1}|}\,.$$

We plug now the expansion (3.15) for $P_p(0, Z_1)$ and $P_p(Z_m, 0)$ in (3.28) and estimate the exponential terms appearing there. By (2.7), we have

$$\begin{aligned} &\exp(-\tfrac{p}{4} \textstyle\sum_{j=1}^n a_j |Z_{1j}|^2 - C_0 \sqrt{p}\,|Z_1 - Z_2|) \\ &\qquad \leqslant \exp(-\tfrac{\mu_0}{8} p |Z_1|^2) \exp(-C(\sqrt{p}|Z_2| - \tfrac{1}{4})), \end{aligned}$$

since

$$\begin{aligned} \tfrac{\mu_0}{8} p |Z_1|^2 + C_0 \sqrt{p}\,|Z_1 - Z_2| &\geqslant C\big(p|Z_1|^2 + \sqrt{p}\,|Z_1 - Z_2|\big) \\ &\geqslant C\big(\sqrt{p}|Z_1| - \tfrac{1}{4} + \sqrt{p}\,|Z_1 - Z_2|\big) \geqslant C\big(\sqrt{p}|Z_2| - \tfrac{1}{4}\big)\,. \end{aligned}$$

We pair now one factor $e^{-\frac{C}{2}\sqrt{p}|Z_2|}$ with $e^{-C_0\sqrt{p}\,|Z_2 - Z_3|}$ and obtain

$$e^{-\frac{C}{2}\sqrt{p}|Z_2|}\, e^{-C_0\sqrt{p}\,|Z_2 - Z_3|} \leqslant e^{-C_3\sqrt{p}|Z_3|} = e^{-\frac{C_3}{2}\sqrt{p}|Z_3|}\, e^{-\frac{C_3}{2}\sqrt{p}|Z_3|}\,;$$

we pair further $e^{-\frac{C_3}{2}\sqrt{p}|Z_3|}$ with $e^{-C_0\sqrt{p}|Z_3-Z_4|}$ and so on. Finally, we obtain for the left-hand side of (3.28) the estimate

$$\int\limits_{\substack{|Z_i|\leqslant\varepsilon\\ 1\leqslant i\leqslant m}} |P_p(0,Z_1)|A_p|P_p(Z_m,0)|\kappa_x^{1/2}(Z_1)\kappa_x^{1/2}(Z_m)\prod_{i=1}^m dZ_i$$
$$\leqslant \int\limits_{\substack{|Z_i|\leqslant\varepsilon\\ 1\leqslant i\leqslant m}} \exp\Big(-\frac{\mu_0}{8}p|Z_1|^2 - C_2\sqrt{p}|Z_2| - \ldots - C_{m-1}\sqrt{p}|Z_{m-1}| - \frac{\mu_0}{4}p|Z_m|^2\Big)p^{2n}B_p\prod_{i=1}^m dZ_i, \tag{3.29}$$

where

$$B_p = p^{(m-1)n}(1+\sqrt{p}\,|Z_1|)^M(1+\sqrt{p}\,|Z_m|)^M\prod_{i=1}^{m-1}(1+\sqrt{p}\,|Z_i|+\sqrt{p}\,|Z_{i+1}|)^M .$$

Since the right-hand side integral in (3.29) converges we obtain that $|I_j| \le C'\delta p^n$, for some $C' > 0$. This completes the proof of (3.23).

To prove (3.24) and (3.25), we repeat the proof above by estimating $f_{j,\,x}(Z_j) - f_{j,\,x}(0)$ with the help of Taylor formulas (3.17) and (3.18). As in the proof of Theorem 3.3, we obtain the remainders $O(p^{-1/2})$ and $O(p^{-1})$, respectively, due to the change of variables $\sqrt{p}Z = Y$. □

Remark 3.6. In the same vein, we show that if $f, g \in \mathscr{C}^k(X, \mathrm{End}(E))$, $k \in \mathbb{N}$, we have as $p \to \infty$,

$$(T_{f,\,p} \circ T_{g,\,p})(x,x) = \sum_{r=0}^{\lfloor k/2\rfloor} \boldsymbol{b}_{r,\,f,\,g}(x)p^{n-r} + R_{k,p}(x)\,,\ \text{uniformly on } X, \tag{3.30}$$

where $\boldsymbol{b}_{r,f,g}$ are the universal coefficients from (3.21) and

$$R_{k,p} = \begin{cases} o(p^{-k/2})\,, & \text{for } k \text{ even}\,,\\ O(p^{-k/2})\,, & \text{for } k \text{ odd}\,. \end{cases}$$

We will now consider traces of Toeplitz operators.

Theorem 3.7. *Let $f \in L^\infty(X, \mathrm{End}(E))$. Then for any $k \in \mathbb{N}$ we have as $p \to \infty$,*

$$\mathrm{Tr}(T_{f,p}) = \sum_{r=0}^{k} \boldsymbol{t}_{r,f}\,p^{n-r} + O(p^{n-k-1})\,, \quad \textit{with } \boldsymbol{t}_{r,f} = \int_X \mathrm{Tr}[\boldsymbol{b}_r f]\,dv_X\,. \tag{3.31}$$

Proof. By (2.29), we infer

$$\begin{aligned}\mathrm{Tr}(T_{f,p}) = \mathrm{Tr}(P_p f P_p) = \mathrm{Tr}(P_p f) &= \int_X \mathrm{Tr}\left[P_p(x,x)f(x)\right]dv_X(x)\\ &= \sum_{r=0}^{k} p^{n-r}\int_X \mathrm{Tr}\left[\boldsymbol{b}_r(x)f(x)\right]dv_X(x) + O(p^{n-k-1})\,.\end{aligned} \tag{3.32}$$

□

Theorem 3.8. *Let $f_1, \ldots, f_m \in L^\infty(X, \mathrm{End}(E))$. Write*

$$p^{-n}\,\mathrm{Tr}(T_{f_1,p}\ldots T_{f_m,p}) = \int_X \mathrm{Tr}\left[f_1\ldots f_m\right]\frac{\omega^n}{n\,!} + R_p\,. \tag{3.33}$$

Then as $p \to \infty$,

$$R_p = \begin{cases} o(1)\,, & \text{uniformly on } f_i \in \mathscr{A}^0_\infty\,, \\ O(p^{-1/2})\,, & \text{uniformly on } f_i \in \mathscr{A}^1_\infty\,, \\ O(p^{-1})\,, & \text{uniformly on } f_i \in \mathscr{A}^2_\infty\,. \end{cases} \tag{3.34}$$

Proof. We have

$$\operatorname{Tr}(T_{f_1,p}\dots T_{f_m,p}) = \int_X \operatorname{Tr}(T_{f_1,p}\dots T_{f_m,p})(x,x)\,dv_X\,,$$

and we apply Theorem 3.5 together with the dominated convergence theorem. □

When (X, J, ω) is a compact Kähler manifold, $g^{TX}(\cdot,\cdot) = \omega(\cdot, J\,\cdot)$, $E = \mathbb{C}$ with the trivial metric, and each $f_i \in \mathscr{C}^\infty(X)$, then (3.33) appears in p. 292 in Ref. 3 and Theorem 4.2 in Ref. 4 with $R_p = O(p^{-1})$.

IV. EXPANSION OF A PRODUCT OF TOEPLITZ OPERATORS

We consider in this section the expansion of the composition of two Toeplitz operators at the operator level. We recall first the situation for Toeplitz operators with smooth symbols. A *Toeplitz operator* is a sequence $\{T_p\} = \{T_p\}_{p\in\mathbb{N}}$ of linear operators $T_p : L^2(X, L^p \otimes \boldsymbol{E}) \longrightarrow L^2(X, L^p \otimes \boldsymbol{E})$ with the properties:

(i) For any $p \in \mathbb{N}$, we have $T_p = P_p\,T_p\,P_p$.
(ii) There exist a sequence $g_l \in \mathscr{C}^\infty(X, \operatorname{End}(E))$ such that for all $k \geq 0$ there exists $C_k > 0$ such that for all $p \in \mathbb{N}^*$, we have

$$\Big\| T_p - P_p\Big(\sum_{l=0}^{k} p^{-l} g_l\Big) P_p \Big\| \leqslant C_k\, p^{-k-1}, \tag{4.1}$$

where $\|\cdot\|$ denotes the operator norm on the space of bounded operators.

We write symbolically

$$T_p = P_p\Big(\sum_{l=0}^{\infty} p^{-l} g_l\Big) P_p + \mathcal{O}(p^{-\infty}). \tag{4.2}$$

Let $f, g \in \mathscr{C}^\infty(X, \operatorname{End}(E))$. By Theorem 1.1 in Ref. 16, the product of the Toeplitz operators $T_{f,p}$ and $T_{g,p}$ is a Toeplitz operator, more precisely, it admits the asymptotic expansion in the sense of (4.2)

$$T_{f,\,p} \circ T_{g,\,p} = \sum_{r=0}^{\infty} p^{-r}\, T_{C_r(f,g),\,p} + \mathcal{O}(p^{-\infty}), \tag{4.3}$$

where C_r are bidifferential operators with smooth coefficients of total degree $2r$ (cf. Lemma 4.6 and (4.80) in Ref. 16). We have $C_0(f, g) = fg$ and if $f, g \in \mathscr{C}^\infty(X)$,

$$C_1(f,g) - C_1(g,f) = \sqrt{-1}\{f,g\}\,\operatorname{Id}_E\,. \tag{4.4}$$

In the case of a Kähler manifold (X, ω), the operators C_0, C_1, C_2 were calculated in Theorem 0.1 in Ref. 18.

We study now the expansion of the product of two Toeplitz operators with $\mathscr{C}^k$ symbols.

Theorem 4.1. *Let* $k \in \mathbb{N}$ *and* $f, g \in \mathscr{C}^k(X, \operatorname{End}(E))$. *Then for* $m \in \{0, \dots, \lfloor k/2 \rfloor\}$, *we have*

$$T_{f,\,p} \circ T_{g,\,p} = \sum_{r=0}^{m} p^{-r}\, T_{C_r(f,g),p} + R_{m,p}\,, \tag{4.5}$$

where $C_r(f, g)$ are the universal coefficients from (4.3) and $R_{m,\,p}$ satisfies the following estimates:

$$R_{m,p} = \begin{cases} o(p^{-k/2}), & \text{for } m = \lfloor k/2 \rfloor, \\ O(p^{-m-1}), & \text{for } m < \lfloor k/2 \rfloor, \end{cases} \tag{4.6}$$

in the operator norm sense.

In order to prove this theorem, we need to develop some machinery from Ref. 16 concerning a criterion for a sequence of operators to be a (generalized) Toeplitz operator. For this purpose, we refine the condition from (vi) in Sec. II.

Let $\Theta_p : L^2(X, L^p \otimes \boldsymbol{E}) \longrightarrow L^2(X, L^p \otimes \boldsymbol{E})$ be a sequence of continuous linear operators with smooth kernel $\Theta_p(\cdot, \cdot)$ with respect to dv_X. Fix $k \in \mathbb{N}$ and $\varepsilon' \in]0, a^X[$. Let

$$\left\{ Q_{r,\,x_0} \in \mathrm{End}(\boldsymbol{E})_{x_0}[Z, Z'] : 0 \leqslant r \leqslant k, x_0 \in X \right\}$$

be a family of polynomials in Z, Z', such that $Q_{r,\,x_0}$ is of class $\mathscr{C}^{k-r}$ with respect to the parameter $x_0 \in X$. We say that

$$p^{-n}\Theta_{p,x_0}(Z, Z') \cong \sum_{r=0}^{k} (Q_{r,\,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')p^{-r/2} + \mathcal{O}(p^{-\frac{k+1}{2}}), \tag{4.7}$$

on $\{(Z, Z') \in TX \times_X TX : |Z|, |Z'| < \varepsilon'\}$ if there exist a decomposition (2.25) and (2.26) holds for $m = m' = 0$.

We say that

$$p^{-n}\Theta_{p,x_0}(Z, Z') \cong \sum_{r=0}^{k} (Q_{r,\,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')p^{-r/2} + \boldsymbol{o}(p^{-\frac{k}{2}}), \tag{4.8}$$

if there exist a decomposition (2.25) where R_{p,k,x_0} satisfies the following estimate: for any $\delta > 0$, there exists $\varepsilon > 0$, $C_k > 0$, $M > 0$ such that for all $(Z, Z') \in TX \times_X TX$ with $|Z|, |Z'| < \varepsilon$ and $p \in \mathbb{N}^*$,

$$\left| R_{p,k,x_0}(Z, Z') \right|_{\mathscr{C}^0(X)} \leqslant \delta\, p^{-k/2}(1 + \sqrt{p}\,|Z| + \sqrt{p}\,|Z'|)^M e^{-C_0\sqrt{p}\,|Z-Z'|}. \tag{4.9}$$

We have the following analogue of Lemma 4.6 in Ref. 16.

Lemma 4.2. Let $f \in \mathscr{C}^k(X, \mathrm{End}(E))$. *There exists a family*

$$\left\{ Q_{r,\,x_0}(f) \in \mathrm{End}(\boldsymbol{E})_{x_0}[Z, Z'] : 0 \leqslant r \leqslant k, x_0 \in X \right\}$$

such that

(a) $Q_{r,\,x_0}(f)$ *are polynomials with the same parity as* r,
(b) $Q_{r,\,x_0}(f)$ *is of class* $\mathscr{C}^{k-r}$ *with respect to the parameter* $x_0 \in X$,
(c) *There exists* $\varepsilon \in]0, a^X/4[$ *such that for any* $m \in \{0, 1, \dots, k\}$, $x_0 \in X$, $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| < \varepsilon/2$, *we have*

$$p^{-n}T_{f,\,p,\,x_0}(Z, Z') \cong \sum_{r=0}^{m} (Q_{r,\,x_0}(f)\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')p^{-\frac{r}{2}} + \mathcal{R}_{m,p}, \tag{4.10}$$

where

$$\mathcal{R}_{m,p} = \begin{cases} \mathcal{O}(p^{-\frac{m+1}{2}}), & \text{if } m \leqslant k-1, \\ \boldsymbol{o}(p^{-\frac{m}{2}}), & \text{if } m = k, \end{cases}$$

in the sense of (4.7) and (4.8). *The coefficients* $Q_{r,\,x_0}(f)$ *are expressed by*

$$Q_{r,\,x_0}(f) = \sum_{r_1+r_2+|\alpha|=r} \mathscr{K}\left[J_{r_1,\,x_0}, \frac{\partial^\alpha f_{x_0}}{\partial Z^\alpha}(0)\frac{Z^\alpha}{\alpha!}J_{r_2,\,x_0} \right]. \tag{4.11}$$

Especially,

$$Q_{0,x_0}(f) = f(x_0)\, I_{\mathbb{C}\otimes E}. \tag{4.12}$$

Proof. We just have to modify the proof of Lemma 4.6 in Ref. 16 in what concerns the Taylor formula for f_{x_0}

$$f_{x_0}(Z) = \sum_{|\alpha|\leqslant m} \frac{\partial^\alpha f_{x_0}}{\partial Z^\alpha}(0)\frac{Z^\alpha}{\alpha!} + \mathscr{R}_m(Z)\,, \quad \mathscr{R}_m(Z) = \begin{cases} O(|Z|^{m+1})\,, & \text{if } m \leqslant k-1\,,\\ o(|Z|^m)\,, & \text{if } m = k\,,\end{cases}$$

thus

$$f_{x_0}(Z) = \sum_{|\alpha|\leqslant m} p^{-|\alpha|/2}\,\frac{\partial^\alpha f_{x_0}}{\partial Z^\alpha}(0)\frac{(\sqrt{p}Z)^\alpha}{\alpha!} + \mathscr{R}_{m,p}(Z)\,, \tag{4.13}$$

where

$$\mathscr{R}_{m,p}(Z) = \begin{cases} p^{-\frac{m+1}{2}}\,O(|\sqrt{p}Z|^{m+1})\,, & \text{if } m \leqslant k-1\,,\\ o(p^{-\frac{m}{2}})O(|\sqrt{p}Z|^m)\,, & \text{if } m = k\,.\end{cases}$$

The last line just means that there exists $C > 0$ such that for any $\delta > 0$, there exists $\varepsilon > 0$ such that for all $|Z| \leq \varepsilon$ and all $p \in \mathbb{N}$, we have $|\mathscr{R}_{m,p}(Z)| \leqslant C\delta p^{-\frac{m}{2}}|\sqrt{p}Z|^m$. □

Lemma 4.3. Let $\mathscr{T}_p : L^2(X, L^p \otimes \boldsymbol{E}) \longrightarrow L^2(X, L^p \otimes \boldsymbol{E})$ *be a sequence of continuous linear operators with smooth kernel* $\mathscr{T}_p(\cdot,\cdot)$ *with respect to* dv_X*. Assume that in the sense of* (4.8),

$$p^{-n}\,\mathscr{T}_{p,x_0}(0, Z') \cong \boldsymbol{o}(1)\,,\ \ p \longrightarrow \infty\,. \tag{4.14}$$

Then there exists $C > 0$ *such that for every* $\delta > 0$, *there exists* p_0 *such that for every* $p > p_0$ *and* $s \in L^2(X, L^p \otimes \boldsymbol{E})$ *we have*

$$\|\mathscr{T}_p\, s\|_{L^2} \leqslant C\delta\|s\|_{L^2}\,, \quad \|\mathscr{T}_p^* s\|_{L^2} \leqslant C\delta\|s\|_{L^2}\,. \tag{4.15}$$

Proof. By the Cauchy-Schwarz inequality, we have

$$\big\|\mathscr{T}_p\, s\big\|^2_{L^2} \leqslant \int_X \Big(\int_X \big|\mathscr{T}_p(x,y)\big|dv_X(y)\Big)\Big(\int_X \big|\mathscr{T}_p(x,y)\big||s(y)|^2 dv_X(y)\Big)dv_X(x)\,. \tag{4.16}$$

We split then the inner integrals into integrals over $B^X(x,\varepsilon')$ and $X\backslash B^X(x,\varepsilon')$ and use the fact that the kernel of $\mathscr{T}_p$ has the growth $\mathcal{O}(p^{-\infty})$ outside the diagonal. By (4.14), there exists $C' > 0$ such that for every $\delta > 0$, there exists p_0 such that for every $p > p_0$ and $x \in X$,

$$\begin{aligned}\int_X \big|\mathscr{T}_p(y,x)\big|dv_X(y) &\leqslant \int_{B^X(y,\varepsilon')} Cp^n\delta(1+\sqrt{p}d(y,x))^M e^{-C_0\sqrt{p}d(y,x)}dv_X(y) + O(p^{-\infty})\\ &= O(1)\delta + O(p^{-\infty}), \end{aligned}\tag{4.17}$$

$$\int_X \big|\mathscr{T}_p(x,y)\big|dv_X(y) \leqslant C\delta\,.$$

Combining (4.16) and (4.17) and Fubini's theorem, we obtain

$$\begin{aligned}\big\|\mathscr{T}_p\, s\big\|^2_{L^2} &\leqslant C\delta\int_X \Big(\int_X \big|\mathscr{T}_p(x,y)\big||s(y)|^2 dv_X(y)\Big)dv_X(x)\\ &= C\delta\int_X \Big(\int_X \big|\mathscr{T}_p(x,y)\big|dv_X(x)\Big)|s(y)|^2 dv_X(y)\\ &\leqslant (C\delta)^2\int_X |s(y)|^2 dv_X(y).\end{aligned}\tag{4.18}$$

This proves the first estimate of (4.15). The second one follows by taking the adjoint. The proof of Lemma 4.3 is completed. □

Proof of Theorem 4.1. First, it is obvious that $P_p T_{f,p} T_{g,p} P_p = T_{f,p} T_{g,p}$. Lemmas 3.1 and 4.2 imply that for $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| < \varepsilon/4$

$$(T_{f,\,p} \circ T_{g,\,p})_{x_0}(Z, Z') = \int_{T_{x_0}X} T_{f,\,p,\,x_0}(Z, Z'')\rho\Big(\frac{4|Z''|}{\varepsilon}\Big) T_{g,\,p,\,x_0}(Z'', Z')\kappa_{x_0}(Z'')\,dv_{TX}(Z'') + \mathscr{O}(p^{-\infty}). \quad (4.19)$$

Case $k = 0$. By (4.19), we deduce as in the proof of Lemma 4.2, that for $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| < \varepsilon/4$, we have

$$p^{-n}(T_{f,\,p} \circ T_{g,\,p})_{x_0}(Z, Z') \cong (Q_{0,\,x_0}(f, g)\mathscr{P}_{x_0})(\sqrt{p}\,Z, \sqrt{p}\,Z') + \boldsymbol{o}(1), \quad (4.20)$$

where

$$Q_{0,\,x_0}(f, g) = \mathscr{K}[Q_{0,\,x_0}(f), Q_{0,\,x_0}(g)] = f(x_0)g(x_0)\,. \quad (4.21)$$

We conclude by Lemma 4.3 that $T_{f,p} \circ T_{g,p} - T_{fg,p} = o(1)$, as $p \to \infty$.

Case $k = 1$. By (4.19) and the Taylor formula (4.13) for $m = k = 1$ we deduce as in the proof of Lemma 4.2 an estimate analogous to (4.20) with $\boldsymbol{o}(1)$ replaced by $\boldsymbol{o}(p^{-1/2})$, so we obtain $T_{f,p} \circ T_{g,p} - T_{fg,p} = o(p^{-1/2})$, as $p \to \infty$.

Case $k \geq 2$. We obtain now that for $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| < \varepsilon/4$ and for every $m \in \{0, 1, \ldots, k\}$ we have

$$p^{-n}(T_{f,\,p} \circ T_{g,\,p})_{x_0}(Z, Z') \cong \sum_{r=0}^{m}(Q_{r,\,x_0}(f, g)\mathscr{P}_{x_0})(\sqrt{p}\,Z, \sqrt{p}\,Z')p^{-\frac{r}{2}} + \mathcal{R}_{m,p}, \quad (4.22)$$

where

$$\mathcal{R}_{m,p} = \begin{cases} \mathcal{O}(p^{-\frac{m+1}{2}}), & \text{if } m \leqslant k-1, \\ \boldsymbol{o}(p^{-\frac{m}{2}}), & \text{if } m = k, \end{cases}$$

in the sense of (4.7) and (4.8).

Note that for $f, g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, by Lemma 4.2 and (4.3), we know that

$$\sum_{r=0}^{\lfloor l/2 \rfloor} Q_{l-2r}(C_r(f, g)) = Q_l(f, g). \quad (4.23)$$

As C_r are bidifferential operators with smooth coefficients of total degree $2r$ defined in (4.3), thus for $f, g \in \mathscr{C}^k(X, \mathrm{End}(E))$, (4.23) still holds for $l \leq k$. Lemma 4.3 and (4.23) imply that Theorem 4.1 holds. □

Corollary 4.4. Let $f, g \in \mathscr{C}^k(X, \mathrm{End}(E))$, $k \in \mathbb{N}$. Then as $p \to \infty$, in the operator norm sense, we have

$$T_{f,\,p} \circ T_{g,\,p} = T_{fg,p} + R_{0,p}\,, \quad R_{0,p} = \begin{cases} o(1)\,, & \textit{for } k = 0\,, \\ o(p^{-1/2})\,, & \textit{for } k = 1\,. \end{cases} \quad (4.24)$$

If $k \geq 2$, then

$$T_{f,\,p} \circ T_{g,\,p} = T_{fg,p} + p^{-1}T_{C_1(f,g),p} + R_p\,, \quad R_p = \begin{cases} o(p^{-1})\,, & \textit{for } k = 2\,, \\ o(p^{-3/2})\,, & \textit{for } k = 3\,, \\ O(p^{-2})\,, & \textit{for } k = 4\,. \end{cases} \quad (4.25)$$

By (4.4) and (4.25), we get

Corollary 4.5. Let $f, g \in \mathscr{C}^2(X)$. Then the commutator of the operators $T_{f,p}$, $T_{g,p}$ satisfies

$$\left[T_{f,\,p}\,,T_{g,\,p}\right]=\frac{\sqrt{-1}}{p}T_{\{f,g\},\,p}+R_p\,,\;\;p\to\infty\,, \tag{4.26}$$

where $\{\,\cdot\,,\cdot\,\}$ is the Poisson bracket on $(X, 2\pi\omega)$ and R_p satisfies the estimates from (4.25).

The Poisson bracket $\{\,\cdot\,,\cdot\,\}$ on $(X, 2\pi\omega)$ is defined as follows. For $f, g \in \mathscr{C}^2(X)$, let ξ_f be the Hamiltonian vector field generated by f, which is defined by $2\pi i_{\xi_f}\omega = df$. Then $\{f, g\} := \xi_f(dg)$.

V. ASYMPTOTICS OF THE NORM OF TOEPLITZ OPERATORS

For $f \in L^\infty(X, \mathrm{End}(E))$, we denote the essential supremum of f by

$$\|f\|_\infty=\operatorname{ess\,sup}_{x\in X}|f(x)|_{\mathrm{End}(E)}\,.$$

Note that the operator norm $\|T_{f,p}\|$ of $T_{f,p}$ satisfies

$$\|T_{f,p}\|\leqslant\|f\|_\infty\,. \tag{5.1}$$

Theorem 5.1. *Let (X, ω) be a compact symplectic manifold and let $(L, h^L, \nabla^L) \to X$ be a prequantum line bundle satisfying (2.2). Let $(E, h^E, \nabla^E) \to X$ be a twisting Hermitian vector bundle. Let $f \in L^\infty(X, \mathrm{End}(E))$ and assume that there exists $x_0 \in X$ such that $\|f\|_\infty = |f(x_0)|_{\mathrm{End}(E)}$ and f is continuous in x_0. Then the norm of $T_{f,p}$ satisfies*

$$\lim_{p\to\infty}\|T_{f,\,p}\|=\|f\|_\infty\,. \tag{5.2}$$

Proof. By hypothesis there exist $x_0 \in X$ and $u_0 \in E_{x_0}$, with $|u_0|_{h^E} = 1$, such that $|f(x_0)(u_0)| = \|f\|_\infty$. Let us trivialize the bundles L, E in normal coordinates over a neighborhood U of x_0, and let e_L be the unit frame of L which trivialize L. In these normal coordinates, we take the parallel transport of u_0 and obtain a nowhere vanishing section e_E of E over U. Denote by $f(x_0) \in \mathrm{End}(E)|_U$ the endomorphism obtained by parallel transport of $f(x_0) \in \mathrm{End}(E_{x_0})$.

Let $\delta > 0$ be fixed. Since f is continuous, there exists $\varepsilon > 0$ such that for all $x \in B(x_0, 2\varepsilon)$, $u \in E_x$, with $|u|_{h^E} = 1$, we have

$$|f(x)u-f(x_0)u|\leqslant\delta. \tag{5.3}$$

Since x_0 is fixed let us denote for simplicity $a_i = a_i(x_0)$ and set

$$|Z|_a:=\frac{1}{2}\Big(\sum_{i=1}^n a_i|z_i|^2\Big)^{\frac12}\,,$$

$$\rho\in\mathscr{C}^\infty(X)\,,\;\operatorname{supp}\rho\subset B(x_0,\varepsilon)\,,\;\rho=1\text{ on }B(x_0,\varepsilon/2).$$

Define sections

$$S^p=S^p_{x_0,\,u_0}=p^{\frac n2}\sqrt{\det\nolimits_{\mathbb{C}}|\boldsymbol{J}_{x_0}|}\,P_p\big(\rho e^{-p|Z|_a^2}e_L^{\otimes p}\otimes e_E\big)\in\mathrm{Ker}(D_p)\,. \tag{5.4}$$

Our goal is to prove the following.

Proposition 5.2. There exists $C > 0$ (independent of δ) such that for $p \gg 1$,

$$\left\|T_{f,p}S^p-\rho f(x_0)S^p\right\|_{L^2}\leqslant C\delta\|S^p\|_{L^2}\,. \tag{5.5}$$

Moreover, for $p \to \infty$,

$$\left\|\rho f(x_0)S^p\right\|_{L^2}=\|f\|_\infty+O(p^{-\frac12}). \tag{5.6}$$

We start by showing that S^p are *peak sections*, i.e., satisfies the properties in Lemma 5.3 below.

Lemma 5.3. The following expansions hold as $p \to \infty$:

$$S^p(Z) = p^{\frac{n}{2}}\sqrt{\det_{\mathbb{C}}|\boldsymbol{J}_{x_0}|}\, e^{-p|Z|_a^2}\Big(1 + \sum_{r=1}^{k} Q_r(\sqrt{p}\,Z)p^{-\frac{r}{2}}\Big)u_0 + O\Big(p^{\frac{n-k-1}{2}} e^{-C_0\sqrt{p}\,|Z|}\big(1+\sqrt{p}\,|Z|\big)^{2M}\Big) + O(p^{-\infty}), \quad Z \in B(0,\, \varepsilon/2) \tag{5.7}$$

for some constants $C_0, M > 0$ *and polynomials* Q_r *with values in* $\operatorname{End}(\boldsymbol{E}_{x_0})$,

$$S^p = O(p^{-\infty}),\ \ \textit{uniformly on any compact set } K \textit{ such that } x_0 \notin K, \tag{5.8}$$

$$\|S^p\|^2_{L^2} = \int_X |S^p|^2 dv_X = 1 + O(p^{-1}). \tag{5.9}$$

Proof. By (5.4), we have for $x \in X$,

$$S^p(x) = p^{\frac{n}{2}}\sqrt{\det_{\mathbb{C}}|\boldsymbol{J}_{x_0}|}\int_{B(x_0,\varepsilon)} P_p(x, x')(\rho e^{-p|Z|_a^2} e_L^{\otimes p} \otimes e_E)(x')\, dv_X(x'). \tag{5.10}$$

We deduce from (2.15) that for $p \to \infty$,

$$S^p(x) = O(p^{-\infty}),\ \ \text{uniformly on } X \setminus B(x_0,\, 2\varepsilon). \tag{5.11}$$

For $Z \in B(0, 2\varepsilon)$, by (2.19) and (5.10), we have

$$S^p(Z) = p^{\frac{n}{2}}\sqrt{\det_{\mathbb{C}}|\boldsymbol{J}_{x_0}|}\int_{B(0,\varepsilon)} P_{p,x_0}(Z, Z')\widetilde{\kappa}(Z')e^{-p|Z'|_a^2}u_0\, dZ', \tag{5.12}$$

where we have denoted $\widetilde{\kappa} = \rho\kappa_{x_0}$. We wish to obtain an expansion of S^p in powers of p, so we apply Theorem 2.4. By (2.28) (see (2.25)), we have

$$P_{p,x_0}(Z, Z')\kappa_{x_0}^{\frac{1}{2}}(Z)\kappa_{x_0}^{\frac{1}{2}}(Z') = \sum_{r=0}^{k}(J_{r,\,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')p^{n-\frac{r}{2}} + R_{p,k,x_0}(Z, Z') + O(p^{-\infty}). \tag{5.13}$$

By (2.20), we write the Taylor expansion of $\varphi(Z, Z') = \kappa_{x_0}^{-\frac{1}{2}}(Z)\kappa_{x_0}^{-\frac{1}{2}}(Z')\widetilde{\kappa}(Z')$ in the form

$$\begin{aligned}\varphi(Z, Z') &= 1 + \sum_{1<|\alpha|+|\beta|\leqslant k} \partial_Z^\alpha\partial_{Z'}^\beta\varphi(0,0)\frac{Z^\alpha}{\alpha!}\frac{Z'^\beta}{\beta!} + O(|(Z, Z')|^{k+1})\\ &= 1 + \sum_{1<|\alpha|+|\beta|\leqslant k} p^{-(|\alpha|+|\beta|)/2}\,\partial_Z^\alpha\partial_{Z'}^\beta\varphi(0,0)\frac{(\sqrt{p}Z)^\alpha}{\alpha!}\frac{(\sqrt{p}Z')^\beta}{\beta!} + p^{-(k+1)/2}O(|\sqrt{p}(Z, Z')|^{k+1})\end{aligned} \tag{5.14}$$

and multiply it with the right-hand side of (5.13). We obtain in this way an expansion in powers of $p^{1/2}$ of $P_{p,x_0}(Z, Z')\widetilde{\kappa}(Z')$

$$P_{p,\,x_0}(Z, Z')\widetilde{\kappa}(Z') = \sum_{r=0}^{k}(\widetilde{J}_{r,\,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')p^{n-\frac{r}{2}} + p^n\widetilde{R}_{p,k}(Z, Z'), \tag{5.15}$$

for some polynomials $\widetilde{J}_{r,\,x_0} \in \operatorname{End}(\boldsymbol{E}_{x_0})[Z, Z']$, $\widetilde{J}_{0,\,x_0} = I_{\mathbb{C}\otimes E}$, and a rest $\widetilde{R}_{p,k}(Z, Z')$ satisfying appropriate estimates corresponding to (2.26).

We apply now to $P_{p,x_0}(Z, Z')$ the off-diagonal expansion (5.15) and integrate

$$\int_{B(0,\varepsilon)} P_{p,x_0}(Z, Z')\widetilde{\kappa}(Z')e^{-p|Z'|_a^2}u_0\, dZ' = \sum_{r=0}^{k} I_r\, p^{-\frac{r}{2}} + I'_k, \tag{5.16}$$

where

$$I_r = \int_{B(0,\varepsilon)} (\widetilde{J}_{r,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z,\sqrt{p}Z')p^n e^{-p|Z'|_a^2}u_0\,dZ'\,,$$
$$I'_k = \int_{B(0,\varepsilon)} p^n \widetilde{R}_{p,k}(Z,Z')e^{-p|Z'|_a^2}u_0\,dZ'. \tag{5.17}$$

The norms $Z \mapsto |Z|$ and $Z \mapsto |Z|_a$ are equivalent so by the exponential decay of $\mathscr{P}_{x_0}$ we have, as $p \to \infty$,

$$I_r = \int_{|Z|_a \leqslant 2\varepsilon} (\widetilde{J}_{r,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z,\sqrt{p}Z')p^n e^{-p|Z'|_a^2}u_0\,dZ' + O(p^{-\infty})\,. \tag{5.18}$$

We deal first with I_0. Let $|Z|_a \leq \varepsilon/2$. Then by (2.22),

$$\int\limits_{|Z'|_a\leqslant 2\varepsilon} \mathscr{P}_{x_0}(\sqrt{p}Z,\sqrt{p}Z')p^n e^{-p|Z'|_a^2}\,dZ' = \int\limits_{\mathbb{C}^n} \mathscr{P}_{x_0}(\sqrt{p}Z,\sqrt{p}Z')p^n e^{-p|Z'|_a^2}\,dZ' + O(e^{-Cp}).$$

By using $\mathscr{P}_{x_0}$ is a projector operator, we get from (2.22) and (2.23),

$$\begin{aligned}&\int\limits_{\mathbb{R}^{2n}} \mathscr{P}_{x_0}(\sqrt{p}Z,\sqrt{p}Z')p^n e^{-p|Z'|_a^2}\,dZ'\\ &= (\det\nolimits_{\mathbb{C}}|\boldsymbol{J}_{x_0}|)^{-1}\int\limits_{\mathbb{R}^{2n}} \mathscr{P}_{x_0}\Big(\sqrt{p}\,Z,\sqrt{p}\,Z'\Big)\mathscr{P}_{x_0}\Big(\sqrt{p}\,Z',0\Big)p^n\,dZ'\\ &= (\det\nolimits_{\mathbb{C}}|\boldsymbol{J}_{x_0}|)^{-1}\,\mathscr{P}_{x_0}\big(\sqrt{p}\,Z,0\big) = e^{-p|Z|_a^2}\,,\end{aligned} \tag{5.19}$$

where the first and third equalities follow from (2.22) and the second from (2.23). We obtain thus

$$\begin{aligned}I_0 &= \int_{|Z'|_a\leqslant 2\varepsilon} (\widetilde{J}_{0,x_0}\mathscr{P}_{x_0})(\sqrt{p}Z,\sqrt{p}Z')p^n e^{-p|Z'|_a^2}u_0\,dZ' + O(p^{-\infty})\\ &= e^{-p|Z|_a^2}\,e_E + O(p^{-\infty})\,.\end{aligned} \tag{5.20}$$

In a similar manner, we show that as $p \to \infty$,

$$I_r = e^{-p|Z|_a^2}\,Q_r(\sqrt{p}Z)\,e_E + O(p^{-\infty})\,. \tag{5.21}$$

Taking into account the definition of $\widetilde{R}_{p,k}(Z,Z')$, (2.26) and (5.17) we obtain in the same vein, as $p \to \infty$,

$$I'_k = O\left(p^{\frac{-k-1}{2}}e^{-C_0\sqrt{p}\,|Z|}\left(1+\sqrt{p}\,|Z|\right)^{2M}\right) + O(p^{-\infty})\,. \tag{5.22}$$

Combining (5.20)–(5.22) we get (5.7). From (5.11) and (5.7), we deduce immediately (5.8).

Note that by (2.19), (5.8), we get

$$\|S^p\|_{L^2}^2 = \int_X \left|S^p(x)\right|^2 dv_X(x) = \int_{B(0,\,2\varepsilon)} \left|S^p(Z)\right|^2 \kappa_{x_0}(Z)dZ + O(p^{-\infty})\,. \tag{5.23}$$

By (2.20) we have

$$\begin{aligned}&\int_{B(0,\,2\varepsilon)} p^n e^{-2p|Z|_a^2}\kappa_{x_0}(Z)\,dZ = \int_{B(0,\,2\varepsilon\sqrt{p})} e^{-2p|Z|_a^2}\kappa_{x_0}(Z/\sqrt{p})\,dZ\\ &= \int_{\mathbb{R}^{2n}} e^{-\sum_{j=1}^n \frac{1}{2}p\,a_j|Z_j|^2}\,dZ + O(p^{-1}) = \textstyle\prod_{j=1}^n \frac{2\pi}{a_j} + O(p^{-1})\,.\end{aligned} \tag{5.24}$$

Further

$$\int_{\mathbb{R}^{2n}} \left|p^{\frac{n}{2}}e^{-p|Z|_a^2}Q_r(\sqrt{p}Z)\right|^2\,dZ < \infty\,, \tag{5.25}$$

and

$$\int_{B(0,\,2\varepsilon)} \left| p^{\frac{n-k-1}{2}} e^{-C_0\sqrt{p}|Z|} \left(1+\sqrt{p}\,|Z|\right)^{2M} \right|^2 dZ = O(p^{-k-1})\,. \tag{5.26}$$

From (5.7)–(5.25) we obtain (5.9). The proof of Lemma 5.3 is completed. □

Lemma 5.4. We have as $p \to \infty$

$$T_{f,p}S^p = O(p^{-\infty}) \text{ uniformly on } X \setminus B(x_0, 2\varepsilon)\,. \tag{5.27}$$

Proof. Due to Lemma 3.1 and (5.8), as $p \to \infty$, we have

$$\begin{aligned} T_{f,p}S^p(x) &= \int_{B(x_0,\varepsilon)} T_{f,\,p}(x,x')S^p(x')\,dv_X(x') + O(p^{-\infty}) \\ &= O(p^{-\infty})\,, \end{aligned} \tag{5.28}$$

uniformly for $x \in X\backslash B(x_0, 2\varepsilon)$. □

Lemma 5.5. As $p \to \infty$, we have

$$\int_{X\setminus B(x_0,2\varepsilon)} \left|T_{f,p}S^p - \rho f(x_0)S^p\right|^2 dv_X = O(p^{-\infty})\,. \tag{5.29}$$

Proof. This follows immediately from (5.8) and Lemma 5.4. □

Lemma 5.6. For $p \gg 1$, we have

$$\int_{B(x_0,2\varepsilon)} \left|T_{f,p}S^p - \rho f(x_0)S^p\right|^2 dv_X \leqslant C^2\delta^2 \|S^p\|^2_{L^2}\,. \tag{5.30}$$

Proof. We have $P_pS^p = S^p$, since $S^p \in \operatorname{Ker}(D_p)$. Thus, $T_{f,\,p}S^p = P_p(fS^p)$. Hence,

$$(T_{f,p}S^p)(x) = \int_X P_p(x,x')f(x')S^p(x')\,dv_X(x')\,.$$

Let us split

$$(T_{f,p}S^p)(x) - (\rho f(x_0)S^p)(x) = g_p(x) + h_p(x), \tag{5.31}$$

where

$$\begin{aligned} g_p(x) &:= \int_X P_p(x,x')\big[f(x') - \rho f(x_0)\big]S^p(x')\,dv_X(x'), \\ h_p(x) &:= \int_X P_p(x,x')\rho f(x_0)S^p(x')\,dv_X(x') - (\rho f(x_0)S^p)(x). \end{aligned}$$

Set

$$R_{p,1,x_0}(Z,Z') := P_{p,x_0}(Z,Z')\kappa_{x_0}(Z') - p^n(J_0\mathscr{P}_{p,x_0})(\sqrt{p}Z,\sqrt{p}Z').$$

By (2.26) we have

$$\left|R_{p,1,x_0}(Z,Z')\right| \leqslant C\,p^{n-\frac{1}{2}}(1+\sqrt{p}\,|Z|+\sqrt{p}\,|Z'|)^M\,e^{-C_0\sqrt{p}\,|Z-Z'|} + O(p^{-\infty}). \tag{5.32}$$

Set

$$I_{1,p}(Z) = \int_{B(0,2\varepsilon)} R_{p,1,x_0}(Z,Z')\rho f(x_0) S^p(Z')\kappa(Z')\,dZ',$$
$$S_0^p = \sqrt{\det|\boldsymbol{J}_{x_0}|}\,p^{\frac{n}{2}} e^{-p|Z|_a^2} u_0, \tag{5.33}$$
$$I_{2,p}(Z) = \int_{B(0,2\varepsilon)} p^n (J_{0,x_0}\mathscr{P})(\sqrt{p}Z,\sqrt{p}Z')\rho f(x_0)(S^p - S_0^p)(Z')\,dZ'.$$

We have

$$h_p(Z) = I_{1,p}(Z) + I_{2,p}(Z) + \rho f(x_0)\big(S^p - S_0^p\big)(Z') + O(p^{-\infty}). \tag{5.34}$$

Estimates (5.7) and (5.32) entail

$$\big|I_{1,p}(Z)\big| \leqslant C\,p^{\frac{n-1}{2}}(1+\sqrt{p}\,|Z|)^{2M} e^{-C_0\sqrt{p}\,|Z|} + O(p^{-\infty})\,. \tag{5.35}$$

By (5.7),

$$\big|I_{2,p}(Z)\big| \leqslant p^{\frac{n-1}{2}}\, e^{-C\sqrt{p}\,|Z|}\,. \tag{5.36}$$

By (5.7), (5.25), (5.34)–(5.36), we obtain as $p \to \infty$,

$$\begin{aligned}\Big(\int_{B(x_0,2\varepsilon)} |h_p(x)|^2\,dv_X(x)\Big)^{\frac12} &\leqslant \Big(\int_{B(0,2\varepsilon)} |I_{1,p}(Z)|^2\,dv_X(Z)\Big)^{\frac12} + O(p^{-\infty})\\ &+ \Big(\int_{B(0,2\varepsilon)} |I_{2,p}(Z)|^2\,dv_X(Z)\Big)^{\frac12} + \Big(\int_{B(0,2\varepsilon)} \big|\rho f(x_0)\big(S^p - S_0^p\big)(Z)\big|^2\,dv_X(Z)\Big)^{\frac12}\\ &= O(p^{-\frac12})\,.\end{aligned} \tag{5.37}$$

Moreover, for g_p from (5.31), we get by (5.3) that for $Z \in T_{x_0}X$, $|Z| \le 2\varepsilon$ we have

$$\big|g_p(Z)\big| \leqslant \delta \int_{B(0,4\varepsilon)} \widetilde{g}(Z,Z')\,dv_X(Z') + O(p^{-\infty}),$$

where

$$\widetilde{g}(Z,Z') = \big(p^n e^{-C\sqrt{p}|Z-Z'|}(1+\sqrt{p}Z+\sqrt{p}Z')^M\big) p^{\frac{n}{2}} e^{-p|Z|_a^2}(1+\sqrt{p}|Z'|)^{M'},$$

hence

$$\big|g_p(Z)\big| \leqslant C\delta\big(p^{\frac{n}{2}}\, e^{-C\sqrt{p}\,|Z|} + O(p^{-\infty})\big). \tag{5.38}$$

From (5.9) and (5.38), we infer

$$\int_{B(x_0,2\varepsilon)} |g_p(x)|^2\,dv_X(x) \leqslant C_1^2\delta^2 \|S^p\|_{L^2}^2. \tag{5.39}$$

Now (5.31), (5.37), and (5.39) yield the desired estimate (5.30). □

Lemmas 5.5 and 5.6 yield (5.5). By (2.20), (5.7), and (5.33), similar to (5.24), we have for $p \to \infty$,

$$\begin{aligned}\big\|\rho f(x_0) S^p\big\|_{L^2}^2 &= \int_{B(0,\varepsilon)} \big|f(x_0)\rho S^p(Z)\big|^2 \kappa(Z)\,dZ\\ &= \int_{B(0,\varepsilon/2)} \big|f(x_0) S_0^p(Z)\big|^2\,dZ + O(p^{-1})\\ &= |f(x_0)u_0|^2 \int_{B(0,\varepsilon/2)} p^n \det\nolimits_{\mathbb{C}}|\boldsymbol{J}_{x_0}| e^{-2p|Z|_a^2}\,dZ + O(p^{-1})\\ &= \|f\|_\infty^2 + O(p^{-1}).\end{aligned} \tag{5.40}$$

This completes the proof of Proposition 5.2. □

Remark 5.7. If we improve the regularity of the section f in Theorem 5.1, the convergence speed in (5.2) improves accordingly (by improving Lemma 5.6):

(a) If $f \in \mathscr{C}^1(X, \mathrm{End}(E))$, then there exists $C > 0$ such that

$$\|f\|_\infty - \frac{C}{\sqrt{p}} \leqslant \|T_{f,p}\| \leqslant \|f\|_\infty.$$

The estimate does not improve even if f is a function.

(b) Assume that in Theorem 5.1, (X, J, ω) is Kähler, $(L, h^L, \nabla^L) \to X$ is a prequantum holomorphic line bundle (where ∇^L is the Chern connection) satisfying (2.2) and $(E, h^E, \nabla^E) \to X$ is a holomorphic Hermitian vector bundle with the Chern connection ∇^E. The Kähler assumption implies that $J_1(Z, Z') = 0$ (cf. (4.1.102) in Ref. 15). Using (2.20) and

$$\int_{\mathbb{C}^n} e^{-\frac{\pi}{2}p|Z|^2} Z_j \, dZ = 0,$$

we deduce that for $f \in \mathscr{C}^2(X, \mathrm{End}(E))$ there exists $C > 0$ such that

$$\|f\|_\infty - \frac{C}{p} \leqslant \|T_{f,p}\| \leqslant \|f\|_\infty.$$

For $f \in \mathscr{C}^1(X, \mathrm{End}(E))$, we have the same estimate as in (a), which cannot be improved.

Remark 5.8. Theorem 5.1 holds also for large classes of non-compact manifolds, see Sec. 7.5 in Ref. 15, Sec. 5 in Ref. 16, and Sec. 2.8 in Ref. 17.

VI. HOW FAR IS $T_{f,p}$ FROM BEING SELF-ADJOINT OR MULTIPLICATION OPERATOR

In this section, we continue to work in the setting of Sec. II A. Let (X, ω) be a $2n$-dimensional connected compact symplectic manifold, and $(L, h^L, \nabla^L) \to X$ be a prequantum line bundle satisfying (2.2). We assume in the following that the vector bundle E is trivial of rank one ($E = \mathbb{C}$). To avoid lengthy formulas, let us denote

$$\mathcal{H}_p = \ker(D_p). \tag{6.1}$$

Let $\mathscr{C}^0(X)$ denote the space of continuous complex-valued functions on X. We shall denote by $\mathscr{C}^0(X, \mathbb{R})$ the space of continuous real-valued functions on X. For $f, g \in \mathscr{C}^0(X)$ set

$$\langle f, g\rangle = \int_X f(x)\overline{g(x)}\, \frac{\omega^n}{n!} \cdot \tag{6.2}$$

Let $L^2(X)$ be the completion of $\mathscr{C}^0(X)$ with respect to the norm $\|f\| = \sqrt{\langle f, f\rangle}$ and let $L^2(X, \mathbb{R})$ be the subspace of $L^2(X)$ that consists of (equivalence classes of) real-valued functions. By a slight abuse of notation, we denote by $\mathbb{C} \subset \mathscr{C}^0(X)$ the one-dimensional subspace of $\mathscr{C}^0(X)$ that consists of constant functions. Note: $L^2(X, \mathbb{R})$ and $\mathbb{C}$ are closed subspaces of $L^2(X)$. For $f \in L^2(X)$ the orthogonal projection of f onto $\mathbb{C}$ (with respect to the inner product $\langle\cdot,\cdot\rangle$) is the constant function

$$⨍_X f\, \frac{\omega^n}{n!} := \frac{\langle f, 1\rangle}{\langle 1, 1\rangle} = \frac{1}{\mathrm{vol}(X)} \int_X f\, \frac{\omega^n}{n!} \quad \text{with } \mathrm{vol}(X) = \int_X \frac{\omega^n}{n!}, \tag{6.3}$$

and the orthogonal projection of f onto $L^2(X, \mathbb{R})$ (with respect to the inner product $\mathrm{Re}\langle\cdot,\cdot\rangle$) is $\mathrm{Re}(f)$.

Denote by $(\cdot,\cdot)_{HS}$ the Hilbert-Schmidt inner product on $\mathrm{End}(\mathcal{H}_p)$: for $A, B \in \mathrm{End}(\mathcal{H}_p)$

$$(A, B)_{HS} = \mathrm{Tr}(AB^*),$$

where B^* is the adjoint of B. Note that the operator norm does not exceed the Hilbert-Schmidt norm. The inner product on the underlying real vector space $\mathrm{End}_{\mathbb{R}}(\mathcal{H}_p)$ is given by

$$(A, B)_{\mathbb{R}} = \mathrm{Re}\ \mathrm{Tr}(AB^*).$$

Denote by $\mathrm{Herm}(\mathcal{H}_p)$ the subspace of $\mathrm{End}_{\mathbb{R}}(\mathcal{H}_p)$ that consists of self-adjoint (Hermitian) operators. Denote by $\mathbb{C}\mathrm{Id}_{\mathcal{H}_p}$ the subspace of $\mathrm{End}(\mathcal{H}_p)$ that consists of constant multiples of the identity operator.

We shall use $\operatorname{dist}(v, V)$ to denote the distance between an element v of a normed vector space and a closed subspace V. For example, for $f \in L^2(X)$

$$\operatorname{dist}(f, \mathbb{C})^2 = \int_X |f|^2 \frac{\omega^n}{n!} - \frac{1}{\operatorname{vol}(X)} \left| \int_X f \frac{\omega^n}{n!} \right|^2 . \tag{6.4}$$

It is clear that for any $f \in L^\infty(X)$ we have

$$T^*_{f,p} = T_{\overline{f},p} , \tag{6.5}$$

hence for $f \in L^\infty(X, \mathbb{R})$ the operator $T_{f,p}$ is self-adjoint. We denote M_f the pointwise multiplication by f. On the other hand, if $f \in L^\infty(X)$ is constant, then $T_{f,p} = M_f$.

Theorem 6.1, stated below, addresses, informally speaking, the following issues: given $f \in \mathscr{C}^0(X)$,

(1) how far f is from being real-valued should be related to how far $T_{f,p}$ is from being self-adjoint (in $\operatorname{End}_{\mathbb{R}}(\mathcal{H}_p)$),

(2) how far f is from being constant should be related to how far $T_{f,p}$ is from being a constant multiple of the identity operator (in $\operatorname{End}(\mathcal{H}_p)$).

Theorem 6.1. *Let $f \in L^\infty(X)$. Write*

$$\begin{aligned} p^{-n}\big[\operatorname{dist}(T_{f,p}, \operatorname{Herm}(\mathcal{H}_p))\big]^2 &= \big[\operatorname{dist}(f, L^2(X, \mathbb{R}))\big]^2 + R_{1,p} , \\ p^{-n}\big[\operatorname{dist}(T_{f,p}, \mathbb{C}\operatorname{Id}_{\mathcal{H}_p})\big]^2 &= \big[\operatorname{dist}(f, \mathbb{C})\big]^2 + R_{2,p} . \end{aligned} \tag{6.6}$$

Then $R_{i,p}$, $i = 1, 2$, satisfy as $p \to \infty$

$$R_{i,p} = \begin{cases} o(1) , & \text{uniformly on } f \in \mathscr{A}^0_\infty , \\ O(p^{-1/2}) , & \text{uniformly on } f \in \mathscr{A}^1_\infty , \\ O(p^{-1}) , & \text{uniformly on } f \in \mathscr{A}^2_\infty . \end{cases} \tag{6.7}$$

Proof. We consider first the case $i = 1$. Let $A \in \operatorname{End}(\mathcal{H}_p)$. The orthogonal projection of A onto $\operatorname{Herm}(\mathcal{H}_p)$ is $\frac{1}{2}(A + A^*)$. We have

$$\begin{aligned} \big[\operatorname{dist}(A, \operatorname{Herm}(\mathcal{H}_p))\big]^2 &= \big(\tfrac{1}{2}(A - A^*), \tfrac{1}{2}(A - A^*)\big)_{\mathbb{R}} = \tfrac{1}{4} \operatorname{Tr}\big[(A - A^*)(A^* - A)\big] \\ &= \tfrac{1}{4}\big[- \operatorname{Tr}(A^2) - \operatorname{Tr}((A^*)^2) + \operatorname{Tr}(AA^*) + \operatorname{Tr}(A^*A)\big]. \end{aligned} \tag{6.8}$$

We apply the previous formula for $A = T_{f,p}$ by using Theorem 3.8 and (6.5), we get as $p \to \infty$

$$\begin{aligned} p^{-n}\big[\operatorname{dist}(A, \operatorname{Herm}(\mathcal{H}_p))\big]^2 &= \tfrac{1}{4} \int_X \big(- f^2 - \overline{f}^2 + 2f\overline{f}\,\big) \frac{\omega^n}{n\,!} + R_{1,p} \\ &= \int_X \big|\operatorname{Im}(f)\big|^2 \frac{\omega^n}{n\,!} + R_{1,p}, \end{aligned} \tag{6.9}$$

where $R_{1,p}$ satisfies (6.7). This proves the assertion of the theorem for $i = 1$, since

$$\big[\operatorname{dist}(f, L^2(X, \mathbb{R}))\big]^2 = \int_X \big|f - \operatorname{Re}(f)\big|^2 \frac{\omega^n}{n\,!}. \tag{6.10}$$

We consider now the case $i = 2$. For $A \in \operatorname{End}(\mathcal{H}_p)$, the orthogonal projection of A onto $\mathbb{C}\operatorname{Id}_{\mathcal{H}_p}$ is $\alpha\operatorname{Id}_{\mathcal{H}_p}$, and $\alpha = \frac{1}{\dim \mathcal{H}_p} \operatorname{Tr}(A)$. Therefore,

$$\begin{aligned} \big[\operatorname{dist}(A, \mathbb{C}\operatorname{Id}_{\mathcal{H}_p})\big]^2 &= (A - \alpha\operatorname{Id}_{\mathcal{H}_p}, A - \alpha\operatorname{Id}_{\mathcal{H}_p}) = \operatorname{Tr}\big[(A - \alpha\operatorname{Id}_{\mathcal{H}_p})(A^* - \bar{\alpha}\operatorname{Id}_{\mathcal{H}_p})\big] \\ &= \operatorname{Tr}(AA^* - \alpha A^* - \bar{\alpha}A + \alpha\bar{\alpha}\operatorname{Id}_{\mathcal{H}_p}) \\ &= \operatorname{Tr}(AA^*) - \frac{1}{\dim \mathcal{H}_p} \operatorname{Tr}(A) \operatorname{Tr}(A^*), \end{aligned} \tag{6.11}$$

since $\mathrm{Tr}(\mathrm{Id}_{\mathcal{H}_p}) = \dim \mathcal{H}_p$. Note that by the Atiyah-Singer index formula (2.10), we have

$$\dim \mathcal{H}_p = p^n \operatorname{vol}(X) + O(p^{n-1}). \tag{6.12}$$

We apply formula (6.11) for $A = T_{f,p}$ by using Theorem 3.8, (6.4), (6.5), and (6.12) to get

$$\begin{aligned} p^{-n}\Big[\operatorname{dist}(T_{f,p}\,, \mathbb{C}\mathrm{Id}_{\mathcal{H}_p})\Big]^2 &= \int_X |f|^2 \,\frac{\omega^n}{n\,!} - \frac{1}{\operatorname{vol}(X)}\bigg|\int_X f\,\frac{\omega^n}{n\,!}\bigg|^2 + R_{2,p} \\ &= \Big[\operatorname{dist}(f, \mathbb{C})\Big]^2 + R_{2,p}, \end{aligned} \tag{6.13}$$

where $R_{2,\,p}$ satisfies (6.7) as $p \to \infty$. □

It is also intuitively clear that how far *f* is from being constant (i.e., how far *df* is from zero) should be related to how far $T_{f,\,p}$ is from $M_f P_p$. This is addressed in the next proposition. The main point here is to estimate the Hilbert-Schmidt norm of the difference $T_{f,\,p} - M_f P_p$ *uniformly* for $f \in \mathscr{C}^1(X)$.

Proposition 6.2. We suppose that $g^{TX}(\cdot,\cdot) = \omega(\cdot, J\ \cdot)$. *Let* $\lambda > 0$ *be the lowest positive eigenvalue of the Laplace operator* $\Delta_{g^{TX}}$ *acting on functions. Then for any* $\varepsilon > 0$, *there exists* $p_0 > 0$ *such that for any* $p \geq p_0$, $f \in \mathscr{C}^1(X)$, *we have*

$$p^{-n}\|T_{f,p} - M_f P_p\|^2_{HS} \leqslant \lambda^{-1}(1+\varepsilon)\|df\|^2_{L^2}. \tag{6.14}$$

Proof. Denote for simplicity $\Delta = \Delta_{g^{TX}}$. The Hodge decomposition of *f* has the form $f = f_1 + f_2$, where $f_1 \in \operatorname{Ker}(\Delta)$ is the harmonic component of *f* and $f_2 \in (\operatorname{Ker}\Delta)^{\perp}$. We have $\operatorname{Ker}(\Delta) = \mathbb{C}$ and $f_1 = \fint_X f\, dv_X$. Moreover,

$$\|f_2\|^2_{L^2} \leqslant \lambda^{-1}\|df_2\|^2_{L^2} = \lambda^{-1}\|df\|^2_{L^2}. \tag{6.15}$$

Now

$$T_{f,p} - M_f P_p = T_{f_2,p} - M_{f_2} P_p\,. \tag{6.16}$$

As $\mathrm{Tr}[T_{f_2,p}T^*_{f_2,p}] > 0$, we get by using (2.12) and (6.16) that

$$\begin{aligned} \|T_{f,p} - M_f P_p\|^2_{HS} &= \mathrm{Tr}\Big[T_{f_2,p}T^*_{f_2,p} + f_2 P_p \overline{f}_2 - T_{f_2,p}\overline{f}_2 - f_2 P_p \overline{f}_2 P_p\Big] \\ &= \mathrm{Tr}\Big[P_p f_2 \overline{f}_2 P_p - T_{f_2,p}T^*_{f_2,p}\Big] \leqslant \mathrm{Tr}\Big[P_p |f_2|^2 P_p\Big] \\ &= \sum_{i=1}^{d_p} \|f_2 S_i^P\|^2_{L^2} = \int_X |f_2(x)|^2\, \mathrm{Tr}[P_p(x,x)]dv_X(x). \end{aligned} \tag{6.17}$$

By the argument after (2.29), for any $\varepsilon > 0$, there exists $p_0 > 0$ such that for $p \geq p_0$, we have

$$\int_X |f_2(x)|^2\, \mathrm{Tr}[P_p(x,x)]dv_X(x) \leqslant (1+\varepsilon)p^n \|f_2\|^2_{L^2}. \tag{6.18}$$

By (6.15), (6.17), and (6.18), we get (6.14). □

Remark 6.3. The results in this paper hold in particular in the case of the Kähler quantization. Let us assume that (X, J, g^{TX}) is a compact Kähler manifold (i.e., *J* is integrable and $g^{TX}(u, v) = \omega(u, Jv)$ for $u, v \in TX$). Assume moreover that the bundles *L* and *E* are holomorphic and ∇^L, ∇^E are the Chern connections. Then by Remark 2.1, the quantum space are the spaces of global holomorphic sections of $L^p \otimes E$. We can even dispense of the Kähler condition $g^{TX}(u, v) = \omega(u, Jv)$ for $u, v \in TX$, see Remark 2.1.

To illustrate the kind of results, we obtain in the Kähler case let us formulate the following special case of Proposition 6.2.

Proposition 6.4. Assume that (X, J, g^{TX}) is a compact Kähler manifold and the bundles L and E are holomorphic. Let $\lambda > 0$ be the lowest positive eigenvalue of the Kodaira Laplace operator $\overline{\partial}^*\overline{\partial}$ acting on functions. Then for any $\varepsilon > 0$, there exists $p_0 > 0$ such that for any $p \geq p_0$, $f \in \mathscr{C}^1(X)$, we have

$$p^{-n}\|T_{f,p} - M_f P_p\|^2_{HS} \leqslant \lambda^{-1}(1+\varepsilon)\|\overline{\partial} f\|^2_{L^2}. \tag{6.19}$$

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